
ON LONG TIME BEHAVIOR FOR SOLUTIONS OF SEMI-LINEAR HARMONIC OSCILLATOR WITH SMALL CAUCHY DATA ON \mathbb{R}^d .

by

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Abstract. — We consider the semi-linear harmonic oscillator

$$i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$

where M is a Hermite multiplier and g a smooth function globally of order at least three.

We prove that such a Hamiltonian equation admits, in a neighborhood of the origin, a Birkhoff normal form at any order and that, under generic conditions on M related to the non resonance of the linear part, this normal form is integrable when $d = 1$ and gives rise to simple dynamics (in particular bounded) when $d \geq 2$.

As a consequence we prove the almost global existence for solutions of the above equation with small Cauchy data.

Résumé (Comportement pour des temps grands des solutions de l'oscillateur harmonique semi-linéaire avec donnée de Cauchy petite sur \mathbb{R}^d .)

Dans cette article nous considérons l'oscillateur harmonique semi-linéaire :

$$i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$

où M est un multiplicateur de Hermite et g est une fonction régulière globalement d'ordre au moins trois.

Nous montrons qu'une telle équation admet, au voisinage de zéro, une forme normale de Birkhoff à n'importe quel ordre et que, sous des hypothèses génériques sur M liées à la non résonance de la partie linéaire, cette forme normale est complètement intégrable si $d = 1$ et donne lieu à une dynamique simple (et en particulier bornée) pour $d \geq 2$.

Ce résultat nous permet de démontrer l'existence presque globale des solutions de l'équation non linéaire ci-dessus avec donnée initiale petite.

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1. Introduction, statement of the results

The aim of this paper is to study long-time existence and behavior for semi-linear harmonic oscillator equations of type

$$(1.1) \quad \begin{cases} i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}) \\ \psi|_{t=0} = \psi_0 \end{cases}$$

on the whole space \mathbb{R}^d . Here M is a Hermite multiplier (see Section 3 for a precise definition), g is a smooth function, globally of order $p \geq 3$ and $\partial_2 g$ denotes the partial derivative of g with respect to the second variable.. Let

$$(1.2) \quad \begin{aligned} \tilde{H}^s &= \{f \in H^s(\mathbb{R}^d, \mathbb{C}) \mid x \mapsto |x|^\alpha \partial^\beta f \in L^2(\mathbb{R}^d) \\ &\text{for any } \alpha, \beta \in \mathbb{N}^d \text{ satisfying } 0 \leq |\alpha| + |\beta| \leq s \} \end{aligned}$$

where $H^s(\mathbb{R}^d, \mathbb{C})$ is the standard Sobolev space on \mathbb{R}^d . Let $T = -\Delta + x^2$, denotes the quantum harmonic oscillator on \mathbb{R}^d ($d \geq 1$). We note that, for any $s \geq 0$, the domain of $T^{s/2}$ is \tilde{H}^s .

If $\psi_0 \in \tilde{H}^s$ is small, say of norm ϵ , local existence theory implies that (1.1) admits a unique solution in \tilde{H}^s defined on intervals of length $c\epsilon^{-p+2}$. Our goal is to prove that for M outside an exceptional subset, given any integer $r \geq 1$ and providing that s is large enough and ϵ is small enough, the solution extends over an interval of length $c\epsilon^{-r}$. Furthermore we control the norm of the solution in \tilde{H}^s -norm ($d \geq 1$) and localize the solution in the neighborhood of a torus (only in the case $d = 1$, cf. Theorem 3.4 and Theorem 3.10).

Precisely we have

Theorem 1.1. — *Let $r \in \mathbb{N}$ be an arbitrary integer. For generic M , there is $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, for any $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$ with $g(z, \bar{z}) \in \mathbb{R}$ vanishing at least at order 3 at the origin, there are $\epsilon_0 > 0$, $c > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, for any ψ_0 in \tilde{H}^s with $\|\psi_0\|_s \leq \epsilon$, the Cauchy problem (1.1) with Cauchy datum ψ_0 has a unique solution*

$$\psi \in C^1((-T_\epsilon, T_\epsilon), \tilde{H}^s)$$

with $T_\epsilon \geq c\epsilon^{-r}$. Moreover there is $C > 0$ such that, for any $t \in (-T_\epsilon, T_\epsilon)$, one has

$$(1.3) \quad \|\psi(t, \cdot)\|_{\tilde{H}^s} \leq C\epsilon .$$

To prove this result we use the Birkhoff normal form theory. This technique has been developed by Bourgain [Bou96], Bambusi [Bam03], Bambusi-Grébert [BG06] for semilinear PDEs (typically semilinear Schrödinger equation or semilinear wave equation) on the one dimensional torus and by Bambusi-Delort-Grébert-Szeftel [BDGS07] for the semilinear Klein-Gordon equation on the sphere S^d (or a Zoll manifold). These cases were concerned with compact domains. In our work the domain is \mathbb{R}^d , the potential x^2 guarantees that the spectrum remains pure point but the free modes of the harmonic oscillator are not so well localized.

For general reference on Hamiltonian PDEs and their perturbations, see the recent monographies [Cra00, Kuk00, Bou05, KP03].

Let us describe roughly the general method. Consider a Hamiltonian system whose Hamiltonian function decomposes in a quadratic part, H_0 (associated to the linear part of the equation), and a perturbative nonlinear part P (at least cubic): $H = H_0 + P$. We assume that H_0 is diagonal in a Hilbert basis $(\phi_j)_{j \geq 1}$ of the phase space \mathcal{P} (for the harmonic oscillator, this basis is given by the Hermite functions and $\mathcal{P} = \ell^2 \times \ell^2$) $H_0 = \sum_j \omega_j \xi_j \eta_j$ for $(\xi, \eta) \in \mathcal{P}$ and where $\omega = (\omega_j)_{j \geq 1}$ is the vector of free frequencies (the eigenvalues of the linear part). The heuristic idea could be resumed as follows: if the free modes do not interact linearly (i.e. if ω is non resonant) and if they do not interact too much via the nonlinear term then the system will remain close to an integrable one, up to a nonlinear term of very high order and thus the solutions will exist and will be under control during a very long time. More precisely, by a Birkhoff normal form approach we prove (cf. Theorem 2.20) that $H \sim H'_0 + P'$ where H'_0 is no more quadratic but remains integrable (in the case $d = 1$) and P' is at least of order r , where r can be chosen arbitrarily large as soon as we work in a sufficiently small neighborhood of the origin.

To guarantee the second condition, i.e. that the free modes do not interact too much via the nonlinear term, we have to control the integral of the product of three or more modes:

$$(1.4) \quad a_j = \int_D \phi_{j_1}(x) \cdots \phi_{j_k}(x) dx$$

where D is the working domain (\mathbb{R}^d in our case) and j is a multi-index in \mathbb{N}^k , k being smaller than the order r fixed. It turns out that in our case the control of such an expression cannot be as good as in the cases on compact domains studied previously.

Let us consider ordered multi-index j , i.e. such that $j_1 \geq j_2 \geq \cdots \geq j_k$. In [BDGS07, Gré07, Bam07] the following control was used: there exists

$\nu > 0$ and for any $N \geq 1$ there exists $C_N > 0$ such that for all ordered j

$$(1.5) \quad |a_j| \leq C_N j_3^\nu \left(\frac{j_3}{j_3 + j_1 - j_2} \right)^N.$$

In the case of the harmonic oscillator this estimate is false (cf [Wan08] where an equivalent is computed for four modes) and we only are able to prove the following: there exists $\nu > 0$ and for any $N \geq 1$ there exists $C_N > 0$ such that for all ordered j

$$(1.6) \quad |a_j| \leq C_N \frac{j_3^\nu}{j_1^{1/24}} \left(\frac{\sqrt{j_2 j_3}}{\sqrt{j_2 j_3} + j_1 - j_2} \right)^N.$$

The difference could seem minimal but it is technically important:

$\sum_{j_1} \left(\frac{j_3}{j_3 + j_1 - j_2} \right)^\mu \sim C j_3$ for an uniform constant C providing $\mu > 1$ and similarly $\sum_{j_1} \left(\frac{\sqrt{j_2 j_3}}{\sqrt{j_2 j_3} + j_1 - j_2} \right)^\mu \sim C \sqrt{j_2 j_3}$ for $\mu > 1$. In the first case, the extra term j_3 can be absorbed by changing the value of ν in (1.5) ($\nu' = \nu + 1$). This is not possible in the second case. In some sense the perturbative nonlinearity is no more short range (cf. [Wan08]).

Actually in the case studied in [Bou96, Bam03, BG06] the linear modes (i.e. the eigenfunctions of the linear part) are localized around the exponential e^{ikx} that are the eigenfunctions of the Laplacian on the torus. In particular the product of eigenfunctions is close to an other eigenfunction which makes much more simpler the control of (1.4). In the harmonic oscillator case, the eigenfunctions are no more localized and the product of eigenfunctions has no more simple properties. Notice that in the case of the semi-linear Klein-Gordon equation on the sphere, the control of (1.4) was also more complicated to obtain but an estimate of type (1.5) has been proved in [DS04] for the Klein-Gordon equation on Zoll manifold.

In the normal form point of view, the substitution of (1.5) by (1.6) has the following consequence:

Consider, \mathcal{T}^ν , the class of formal polynomials

$$Q(\xi, \eta) \equiv Q(z) = \sum_{l=0}^k \sum_{j \in \mathbb{N}^l} a_j z_{j_1} \dots z_{j_l}$$

with coefficients a_j satisfying (1.5) (resp. (1.6)). In [Gré07] it is proved that if a polynomial P is in the class \mathcal{T}^ν then its hamiltonian vector field X_P is regular from⁽¹⁾ $\mathcal{P}_s = \ell_s^2 \times \ell_s^2$ to \mathcal{P}_s for all s large enough (depending on

⁽¹⁾here $\ell_s^2 = \{(z_l) \mid \sum l^{2s} |z_l|^2 < \infty\}$ and correspond to function $\psi = \sum z_l \phi_l$ in \tilde{H}^s .

ν). Here we prove that if $P \in \mathcal{T}^\nu$ then X_P is regular from \mathcal{P}_s to $\mathcal{P}_{s'}$ for all $s' < s - 1/2 + 1/24$ and s large enough. This loss of regularity would be of course dramatic for an iterative procedure. However at the principle of the iteration the nonlinearity P is regular in the sense that X_P maps continuously \mathcal{P}_s to \mathcal{P}_s for s large enough (essentially because the space \tilde{H}^s is an algebra). Furthermore we prove that the canonical transform that we construct at each step of the iteration preserves the regularity. Actually, at each iteration, when we compute the canonical transformation as the time 1 flow of a Hamiltonian χ , the resolution of the so called homological equation lets appear an extra term in (1.6) for the coefficient of the polynomial χ :

$$(1.7) \quad |a_j| \leq C_N \frac{j_3^\nu}{j_1^{1/24}(1+j_1-j_2)} \left(\frac{\sqrt{j_2 j_3}}{\sqrt{j_2 j_3} + j_1 - j_2} \right)^N.$$

Using such an estimate on the coefficients (the class⁽²⁾ denoted $\mathcal{T}^{\nu,+}$ in Section 2.2), we prove in Proposition 2.12 that X_χ is regular from \mathcal{P}_s to \mathcal{P}_s for all s large enough. Further we prove in Proposition 2.16 that the Poisson bracket of a polynomial in \mathcal{T}^ν with a polynomial in $\mathcal{T}^{\nu,+}$ is in $\mathcal{T}^{\nu'}$ for some ν' larger than ν . So an iterative procedure is possible in \mathcal{P}_s .

Our article is organized as follows: in Section 2 we state and prove a specific Birkhoff normal form theorem adapted to the loss of regularity that we explained above. In Section 3, we apply this theorem to the $1-d$ semi-linear harmonic oscillator equation (Subsection 3.1) and we generalize it to cover the multidimensional case (Subsection 3.2).

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2. The Birkhoff normal form

2.1. The abstract model. — To begin with, we give an abstract model of infinite dimensional Hamiltonian system. In section 3 we will verify that the nonlinear harmonic oscillator that can be described in this abstract framework. We work in the phase space $\mathcal{P}_s \equiv \mathcal{P}_s(\mathbb{C}) := \ell_s^2(\mathbb{C}) \times \ell_s^2(\mathbb{C})$ where, for $s \in \mathbb{R}_+$, $\ell_s^2(\mathbb{C}) := \{(a_j)_{j \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{j \geq 1} j^{2s} |a_j|^2\}$ is a Hilbert space for the standard norm: $\|a\|_s^2 = \sum_{j \geq 1} |j|^{2s} |a_j|^2$. In the whole paper, we denote $\bar{\mathbb{N}} = \mathbb{N} \setminus \{0\}$ and $\bar{\mathbb{Z}} = \mathbb{Z} \setminus \{0\}$. Finally we denote $\mathcal{P}_s(\mathbb{R}) := \{(\xi, \bar{\xi}) \in \mathcal{P}_s(\mathbb{C})\}$ the "real" part of

⁽²⁾Actually in section 2.2, instead of \mathcal{T}^ν and $\mathcal{T}^{\nu,+}$, we consider more general classes $\mathcal{T}^{\nu,\beta}$ and $\mathcal{T}^{\nu,\beta,+}$ where the parameter β plays the rule of the exponent $1/24$ in (1.6) and (1.7)

$\mathcal{P}_s(\mathbb{C})$. We shall denote a general point of \mathcal{P}_s by $z = (\xi, \eta)$ with $z = (z_j)_{j \in \bar{\mathbb{Z}}}$, $\xi = (\xi_j)_{j \in \bar{\mathbb{N}}}$, $\eta = (\eta_j)_{j \in \bar{\mathbb{N}}}$ and the correspondence: $z_j = \xi_j$, $z_{-j} = \eta_j$ for all $j \in \bar{\mathbb{N}}$.

Definition 2.1. — Let $s \geq 0$, we denote by \mathcal{H}^s the space of Hamiltonian functions H defined on a neighborhood \mathcal{U} of the origin in $\mathcal{P}_s \equiv \mathcal{P}_s(\mathbb{C})$, satisfying $H(\xi, \bar{\xi}) \in \mathbb{R}$ (we say that H is real) and

$$H \in C^\infty(\mathcal{U}, \mathbb{C}) \quad \text{and} \quad X_H \in C^\infty(\mathcal{U}, \mathcal{P}_s).$$

In particular the Hamiltonian vector fields of functions F, G in \mathcal{H}^s are in $\ell_s^2(\mathbb{C}) \times \ell_s^2(\mathbb{C})$ and we can define their Poisson bracket by

$$\{F, G\} = i \sum_{j \geq 1} \frac{\partial F}{\partial \xi_j} \frac{\partial G}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j} \frac{\partial G}{\partial \xi_j}.$$

The following simple property of functions belonging in \mathcal{H}^s will be very useful :

Lemma 2.2. — Let $P \in \mathcal{H}^s$ such that P vanishes up to order $r + 1$ at the origin, that is :

$$\forall k \leq r + 1, \forall j \in \bar{\mathbb{Z}}^k, \frac{\partial^k P}{\partial z_{j_1} \dots \partial z_{j_k}}(0) = 0$$

Then there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for $z \in \mathcal{P}_s$ satisfying $\|z\|_s \leq \varepsilon_0$, we have

$$\|X_P(z)\|_s \leq C \|z\|_s^r.$$

Proof. — This estimate comes from the fact that X_P is a C^∞ function from a neighborhood of 0 in \mathcal{P}_s to \mathcal{P}_s : using the Taylor formula for X_P at order r , and since the map $z \mapsto D^r X_P(z)$ is continuous, and thus locally bounded, from a neighborhood of the origin in \mathcal{P}_s to the space of r -linear continuous applications from \mathcal{P}_s^r to \mathcal{P}_s , we obtain directly the above estimate. \square

Our model of integrable system is the harmonic oscillator

$$H_0 = \sum_{j \geq 1} \omega_j \xi_j \eta_j$$

where $\omega = (\omega_j)_{j \geq 1} \in \mathbb{R}^{\bar{\mathbb{N}}}$ is the frequencies vector. We will assume that these frequencies grow at most polynomially, i.e. that there exist $C > 0$ and $\bar{d} \geq 0$ such that for any $j \in \bar{\mathbb{N}}$,

$$(2.1) \quad |\omega_j| \leq C |j|^{\bar{d}},$$

in such a way that H_0 be well defined on \mathcal{P}_s for s large enough. The perturbation term is a real function, $P \in \mathcal{H}^s$, having a zero of order at least 3 at the origin. Our Hamiltonian function is then given by

$$H = H_0 + P$$

and the Hamilton's equations read,

$$(2.2) \quad \begin{cases} \dot{\xi}_j = -i\omega_j \xi_j - i \frac{\partial P}{\partial \eta_j}, & j \geq 1 \\ \dot{\eta}_j = i\omega_j \eta_j + i \frac{\partial P}{\partial \xi_j}, & j \geq 1. \end{cases}$$

Our theorem will require essentially two hypotheses: one on the perturbation P (see Definition 2.5 in the next section) and one on the frequencies vector ω that we describe now.

For $j \in \bar{\mathbb{Z}}^k$ with $k \geq 3$, we define $\mu(j)$ as the third largest integer between $|j_1|, \dots, |j_k|$. Then we set $S(j) := |j_{i_1}| - |j_{i_2}|$ where $|j_{i_1}|$ and $|j_{i_2}|$ are respectively the largest integer and the second largest integer between $|j_1|, \dots, |j_k|$. In particular, if the multi-index j is ordered i.e. if $|j_1| \geq \dots \geq |j_k|$ then

$$\mu(j) := |j_3| \text{ and } S(j) = |j_1| - |j_2|.$$

In [Bam03, BG06, Gré07, Bam07] the non resonance condition on ω reads

Definition 2.3. — A frequencies vector $\omega \in \mathbb{R}^{\bar{\mathbb{N}}}$ is **non resonant** if for any $r \in \bar{\mathbb{N}}$, there are $\gamma > 0$ and $\delta > 0$ such that for any $j \in \bar{\mathbb{N}}^r$ and any $1 \leq i \leq r$, one has

$$(2.3) \quad \left| \omega_{j_1} + \dots + \omega_{j_i} - \omega_{j_{i+1}} - \dots - \omega_{j_r} \right| \geq \frac{\gamma}{\mu(j)^\delta}$$

except if $\{j_1, \dots, j_i\} = \{j_{i+1}, \dots, j_r\}$.

In the harmonic oscillator case⁽³⁾, we are able to verify a slightly refined condition

Definition 2.4. — A frequencies vector $\omega \in \mathbb{R}^{\bar{\mathbb{N}}}$ is **strongly non resonant** if for any $r \in \bar{\mathbb{N}}$, there are $\gamma > 0$ and $\delta > 0$ such that for any $j \in \bar{\mathbb{N}}^r$ and any $1 \leq i \leq r$, one has

$$(2.4) \quad \left| \omega_{j_1} + \dots + \omega_{j_i} - \omega_{j_{i+1}} - \dots - \omega_{j_r} \right| \geq \gamma \frac{1 + S(j)}{\mu(j)^\delta}$$

except if $\{j_1, \dots, j_i\} = \{j_{i+1}, \dots, j_r\}$.

⁽³⁾actually in all the cases where the frequencies vector is non resonant as in definition 2.3 and satisfies the asymptotic: $\omega_l \sim l^n$ with $n \geq 1$.

2.2. Polynomial structure. — For $j \in \bar{\mathbb{Z}}^k$ with $k \geq 3$, we have already defined $\mu(j)$ and $S(j)$ we now introduce

$$B(j) = |j_{i_2} j_{i_3}|^{1/2}$$

where $|j_{i_2}|$ and $|j_{i_3}|$ are respectively the second largest integer and the third largest integer between $|j_1|, \dots, |j_k|$. We also define

$$(2.5) \quad A(j) = \frac{B(j)}{B(j) + S(j)}.$$

We also introduce

$$(2.6) \quad C(j) = |j_{i_1}|.$$

In particular, if the multi-index j is ordered i.e. if $|j_1| \geq \dots \geq |j_k|$ then

$$A(j) = \frac{|j_2 j_3|^{1/2}}{|j_2 j_3|^{1/2} + |j_1| - |j_2|}$$

and

$$C(j) = |j_1|.$$

The symplectic variables (ξ_j, η_j) will be denoted using the same superscript:

$$\begin{cases} z_j = \xi_j & \text{if } j \geq 1 \\ z_j = \eta_j & \text{if } j \leq -1 \end{cases}$$

Definition 2.5. — Let $k \geq 3$, $\beta \in (0, +\infty)$ and $\nu \in [0, +\infty)$ and let

$$(2.7) \quad Q(\xi, \eta) \equiv Q(z) = \sum_{j \in \bar{\mathbb{Z}}^k} a_j z_{j_1} \dots z_{j_i}$$

be a formal polynomial homogeneous of degree k on $\mathcal{P}_s(\mathbb{C})$. Q is in the class $\mathcal{T}_k^{\nu, \beta}$ if for any $N \geq 1$ there exists a constant $c_N > 0$ such that for all $j \in \bar{\mathbb{Z}}^k$

$$(2.8) \quad |a_j| \leq c_N \frac{\mu(j)^\nu}{C(j)^\beta} A(j)^N.$$

We will also use

Definition 2.6. — Let $k \geq 3$, $\beta \in [0, +\infty)$ and $\nu \in [0, +\infty)$ and let

$$Q(\xi, \eta) \equiv Q(z) = \sum_{j \in \bar{\mathbb{Z}}^k} a_j z_{j_1} \dots z_{j_i}$$

be a formal polynomial homogeneous of degree k on $\mathcal{P}_s(\mathbb{C})$. Q is in the class $\mathcal{T}_k^{\nu, \beta, +}$ if for any $N \geq 1$ there exists a constant $c_N > 0$ such that for all $j \in \bar{\mathbb{Z}}^k$

$$(2.9) \quad |a_j| \leq c_N \frac{\mu(j)^\nu}{C(j)^\beta (1 + S(j))} A(j)^N.$$

Remark 2.7. — Notice that the formula (2.7) does not give a unique representation of polynomials on \mathcal{P}_s . However, since the estimates (2.8) and (2.9) are symmetric with respect to the order of the indexes j_1, \dots, j_k , this non uniqueness does not affect the definitions 2.5 and 2.6.

Remark 2.8. — In the estimate (2.8), the numerator allows an increasing behaviour with respect to $\mu(j)$ that will be useful to control the small divisors. The denominator imposes a slight decreasing behaviour with respect to the largest index $C(j)$ and a high decreasing behaviour for monomials having their two modes of largest indexes that are not of the same order. This control is slightly better in $\mathcal{T}_k^{\nu, \beta, +}$.

Remark 2.9. — We will see in Proposition 2.12 that, when $\beta > 1/2$, $\mathcal{T}_k^{\nu, \beta} \subset \mathcal{H}^s$ for $s \geq \nu + 1$. Unfortunately β is not that large in the harmonic oscillator case where we only prove that we can take $\beta = 1/24$. Thus $P \in \mathcal{T}_k^{\nu, \beta}$ is not sufficient to ensure that $P \in \mathcal{H}^s$. Nevertheless, a polynomial in $\mathcal{T}_k^{\nu, \beta}$ is well defined and continuous on a neighborhood of the origin in $\mathcal{P}_s(\mathbb{C})$ for s large enough (cf. Proposition 2.12). In [Gré07, Bam07] instead of estimate (2.8) we had

$$(2.10) \quad |a_j| \leq C_N \frac{\mu(j)^{N+\nu}}{(\mu(j) + S(j))^N}.$$

which is actually better than (2.8). It turns out that a polynomial that satisfies (2.10) belongs to \mathcal{H}^s .

The best constants c_N in (2.8) define a family of semi-norms for which $\mathcal{T}_k^{\nu, \beta}$ is a Fréchet space.

Definition 2.10. — Let $\nu \geq 0$ and $\beta \geq 0$. A function P is in the class $\mathcal{T}^{\nu, \beta}$ if

- there exists $s_0 \geq 0$ such that, for any $s \geq s_0$ there exists a \mathcal{U}_s a neighborhood of the origin in \mathcal{P}_s such that $P \in C^\infty(\mathcal{U}_s, \mathbb{C})$.
- P has a zero of order at least 3 in 0.
- for each $k \geq 3$ the Taylor's expansion of degree k of P at zero belongs to $\otimes_{l=3}^k \mathcal{T}_l^{\nu, \beta}$.

We now define the class of polynomials in *normal form*:

Definition 2.11. — Let $k = 2m$ be an even integer, a formal polynomial Z homogeneous of degree k on \mathcal{P}_s is in normal form if it reads

$$(2.11) \quad Z(z) = \sum_{j \in \bar{\mathbb{N}}^m} b_j z_{j_1} z_{-j_1} \cdots z_{j_m} z_{-j_m}$$

i.e. Z depends only on the actions $I_l := z_l z_{-l} = \xi_l \eta_l$.

The aim of the Birkhoff normal form theorem is to reduce a given Hamiltonian of the form $H_0 + P$ with P in \mathcal{H}^s to a Hamiltonian of the form $Z + R$ where Z is in normal form and R remains very small in the sense that it has a zero of high order at the origin.

We now review the properties of polynomials in the class $\mathcal{T}^{\nu, \beta}$.

Proposition 2.12. — Let $k \in \bar{\mathbb{N}}$, $\nu \in [0, +\infty)$, $\beta \in [0, +\infty)$, $s \in \mathbb{R}$ with $s > \nu + 1$, and let $P \in \mathcal{T}_{k+1}^{\nu, \beta}$. Then

- (i) P extends as a continuous polynomial on $\mathcal{P}_s(\mathbb{C})$ and there exists a constant $C > 0$ such that for all $z \in \mathcal{P}_s(\mathbb{C})$

$$|P(z)| \leq C \|z\|_s^{k+1}$$

- (ii) For any $s' < s + \beta - \frac{1}{2}$, the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_{s'}(\mathbb{C})$. Furthermore, for any $s_0 \in (\nu + 1, s]$, there is $C > 0$ such that for any $z \in \mathcal{P}_s(\mathbb{C})$

$$(2.12) \quad \|X_P(z)\|_{s'} \leq C \|z\|_s \|z\|_{s_0}^{(k-1)}.$$

- (iii) Assume moreover that $P \in \mathcal{T}_{k+1}^{\nu, \beta, +}$ with $\beta > 0$, then the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_s(\mathbb{C})$. Furthermore, for any $s_0 \in (\nu + 1, s]$, there is $C > 0$ such that for any $z \in \mathcal{P}_s(\mathbb{C})$

$$(2.13) \quad \|X_P(z)\|_s \leq C \|z\|_s \|z\|_{s_0}^{(k-1)}.$$

- (iv) Assume finally that $P \in \mathcal{T}_{k+1}^{\nu, \beta}$ and P is in normal form in the sense of definition 2.11, then the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_s(\mathbb{C})$. Furthermore, for any $s_0 \in (\nu, s]$, there is $C > 0$ such that for any $z \in \mathcal{P}_s(\mathbb{C})$

$$(2.14) \quad \|X_P(z)\|_s \leq C \|z\|_s \|z\|_{s_0}^{(k-1)}.$$

Note that, when $0 < \beta \leq 1/2$, assertion (ii) does not guarantee that the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_s(\mathbb{C})$ which will be of course an important fact in the sequel. Thus in this case

assertions (iii) and (iv) are important. Naturally, when $\beta > 1/2$, assertions (iii) and (iv) remain useless.

Proof. — (i) Let P be an homogeneous polynomial of degree $k + 1$ in $\mathcal{T}_{k+1}^{\nu, \beta}$ and write for $z \in \mathcal{P}_s(\mathbb{C})$

$$(2.15) \quad P(z) = \sum_{j \in \mathbb{Z}^{k+1}} a_j z_{j_1} \dots z_{j_{k+1}} .$$

One has, using first (2.8) and then $A(j) \leq 1$, $C(j) \geq 1$,

$$\begin{aligned} |P(z)| &\leq C \sum_{j \in \mathbb{Z}^{k+1}} \frac{\mu(j)^\nu}{C(j)^\beta} A(j)^N \prod_{i=1}^{k+1} |z_{j_i}| \\ &\leq C \sum_{j \in \mathbb{Z}^{k+1}} \frac{\mu(j)^\nu}{\prod_{i=1}^{k+1} |j_i|^s} \prod_{i=1}^{k+1} |j_i|^s |z_{j_i}| \\ &\leq C \sum_{j \in \mathbb{Z}^{k+1}} \frac{1}{\prod_{i=1}^{k+1} |j_i|^{s-\nu}} \prod_{i=1}^{k+1} |j_i|^s |z_{j_i}| \\ &\leq C \left(\sum_{l \in \mathbb{Z}} \frac{1}{|l|^{2s-2\nu}} \right)^{\frac{k+1}{2}} \|z\|_s^{k+1} \end{aligned}$$

where, in the last inequality, we used $k + 1$ times the Cauchy-Schwarz inequality. Since $s > \nu + 1/2$, the last sum converges and the first assertion is proved.

(ii) We have to estimate the derivative of polynomial P with respect to any of its variables. Thanks to (2.8), given any N , we get

$$\left| \frac{\partial P}{\partial z_l} \right| \leq C_N (k+1) \sum_{j \in \mathbb{Z}^k} \frac{\mu(j, l)^\nu}{C(j, l)^\beta} A(j, l)^N |z_{j_1}| \dots |z_{j_k}| ,$$

where the quantities $\mu(j, l)$, $C(j, l)$ and $A(j, l)$ are computed for the $k+1$ -uple made of j_1, \dots, j_k, l . We have

$$\begin{aligned}
 \|X_P(z)\|_{s'}^2 &\leq C \sum_{l \in \bar{\mathbb{Z}}} \left(\sum_{j \in \bar{\mathbb{Z}}^k} \frac{|l|^{s'} \mu(j, l)^\nu}{C(j, l)^\beta} A(j, l)^N |z_{j_1}| \dots |z_{j_k}| \right)^2 \\
 &\leq C(k!)^2 \sum_{l \in \bar{\mathbb{Z}}} \left(\sum_{j \in \bar{\mathbb{Z}}_>^k} \frac{|l|^{s'} \mu(j, l)^\nu}{C(j, l)^\beta} A(j, l)^N |z_{j_1}| \dots |z_{j_k}| \right)^2 \\
 (2.16) \quad &\leq C' \|z\|_{s_0}^{2(k-3)} \sum_{l \in \bar{\mathbb{Z}}} \left(\sum_{|j_1| \geq |j_2| \geq |j_3|} \frac{|l|^{s'} \mu(j, l)^\nu}{C(j, l)^\beta} A(j, l)^N |z_{j_1}| |z_{j_2}| |z_{j_3}| \right)^2,
 \end{aligned}$$

where $\bar{\mathbb{Z}}_>^k$ denotes the set of *ordered* k -uples (j_1, \dots, j_k) such that $|j_1| \geq |j_2| \geq \dots \geq |j_k|$. We used in the last inequality the following result :

Lemma 2.13. — *Given any $s \geq 0$, $s_0 > \frac{1}{2}$ and $z \in \ell_{s+s_0}^2$ we have*

$$\sum_{j \in \bar{\mathbb{Z}}} |j|^s |z_j| \leq C_{s_0} \|z\|_{s+s_0}.$$

Proof. — This result is a trivial consequence of Cauchy-Schwarz inequality :

$$\sum_{j \in \bar{\mathbb{Z}}} |j|^s |z_j| = \sum_{j \in \bar{\mathbb{Z}}} \frac{1}{|j|^{s_0}} |j|^{s+s_0} |z_j| \leq \left(\sum_{j \in \bar{\mathbb{Z}}} \frac{1}{|j|^{2s_0}} \right)^{\frac{1}{2}} \|z\|_{s+s_0}.$$

□

We give now two technical lemmas which allow to estimate $A(j, l)$.

Lemma 2.14. — *Given any ordered k -uple $j \in \bar{\mathbb{Z}}_>^k$ and $l \in \bar{\mathbb{Z}}$, we have*

$$|l|A(j, l) \leq 2|j_1|.$$

Proof. — It is straightforward if $|l| \leq 2|j_1|$, since $A(j, l) \leq 1$. If not, the order is the following : $|l| > 2|j_1| > |j_1| \geq |j_2|$ and

$$|l|A(j, l) = \frac{l\sqrt{|j_1 j_2|}}{\sqrt{|j_1 j_2|} + |l| - |j_1|} \leq \frac{|l|\sqrt{|j_1 j_2|}}{|l|/2} \leq 2|j_1|,$$

and the lemma is proved. □

Lemma 2.15. — Given any ordered k -uple $j \in \bar{\mathbb{Z}}_>^k$ and $l \in \bar{\mathbb{Z}}$ we have

$$A(j, l) \leq \tilde{A}(j_1, j_2, l) := \begin{cases} 2 \frac{|j_2|}{|l+|j_1|-|j_2||} & \text{if } |l| \leq |j_2|, \\ 2 \frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|+||j_1|-|l||}} & \text{if } |l| \geq |j_2|. \end{cases}$$

Proof. — If $|l| > 2|j_1|$; $A(j, l)$ reads :

$$A(j, l) = \frac{\sqrt{|j_1 j_2|}}{\sqrt{|j_1 j_2|} + |l| - |j_1|}.$$

We can write :

$$\begin{aligned} \sqrt{|j_1 j_2|} + |l| - |j_1| &= \sqrt{|lj_2|} + |l| - |j_1| - \sqrt{|j_2|}(\sqrt{|l|} - \sqrt{|j_1|}) \\ &= \sqrt{|lj_2|} + |l| - |j_1| - \sqrt{|j_2|} \frac{|l| - |j_1|}{\sqrt{|l|} + \sqrt{|j_1|}} \\ &\geq \sqrt{|lj_2|} + |l| - |j_1| - \sqrt{|j_1|} \frac{|l| - |j_1|}{\sqrt{|l|} + \sqrt{|j_1|}} \\ &\geq \sqrt{|lj_2|} + \frac{\sqrt{2}}{\sqrt{2} + 1} (|l| - |j_1|) \end{aligned}$$

Hence,

$$A(j, l) \leq \frac{1 + \sqrt{2}}{\sqrt{2}} \frac{\sqrt{|j_1 j_2|}}{\sqrt{|j_1 j_2|} + |l| - |j_1|} \leq 2 \frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|} + |l| - |j_1|}.$$

If $|j_2| \leq |l| \leq 2|j_1|$, then $B(j, l)^2 = |j_2| \min(|l|, |j_1|) \in \left[\frac{|lj_2|}{2}, |lj_2| \right]$, therefore

$$A(j, l) \leq \frac{\sqrt{|lj_2|}}{1/\sqrt{2} \sqrt{|lj_2|} + ||l| - |j_1||} \leq 2 \frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|} + ||l| - |j_1||}.$$

Finally, if $|l| \leq |j_2|$ we get

$$A(j, l) = \frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|} + |j_1| - |j_2|} \leq 2 \frac{|j_2|}{|l| + |j_1| - |j_2|},$$

and this ends the proof of Lemma 2.15. \square

To continue with the proof of assertion (ii) of Proposition 2.12, we define $0 < \varepsilon < s - s' - \frac{1}{2}$, and $N = s + 1 + \varepsilon$. In view of (2.16), we may decompose :

$$(2.17) \quad \|X_P(z)\|_{s'}^2 \leq C \sum_{l \in \bar{\mathbb{Z}}} (T_1(l) + T_2(l))^2,$$

with

$$T_1(l) = \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| > |l|} \frac{|l|^{s'} |\mu(j, l)|^\nu}{\max(|j_1|, |l|)^\beta} A(j, l)^N |z_{j_1}| |z_{j_2}| |z_{j_3}|$$

$$T_2(l) = \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| \leq |l|} \frac{|l|^{s'} |\mu(j, l)|^\nu}{\max(|j_1|, |l|)^\beta} A(j, l)^N |z_{j_1}| |z_{j_2}| |z_{j_3}|.$$

Since $A(j, l) \leq 1$ and $N > \frac{1}{2} + s' + \varepsilon$, we may estimate $T_1(l)$ using Lemmas 2.14 and 2.15 :

$$\begin{aligned} T_1(l) &\leq C \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| > |l|} |j_1|^{s'} |j_2|^\nu \tilde{A}(j_1, j_2, l)^{\frac{1}{2} + \varepsilon} |z_{j_1}| |z_{j_2}| |z_{j_3}| \\ &\leq C \|z\|_{s_0} \sum_{|j_1| \geq |j_2|, |j_2| > |l|} \frac{1}{|l|^{\frac{1}{2} + \varepsilon}} |j_1|^{s - \frac{1}{2} - \varepsilon} |z_{j_1}| |j_2|^{\nu + \frac{1}{2} + \varepsilon} |z_{j_2}| \\ &\leq C \|z\|_{s_0} \frac{1}{|l|^{\frac{1}{2} + \varepsilon}} \|z\|_s \|z\|_{\nu + 1 + 2\varepsilon}, \end{aligned}$$

hence $T_1(l)$ is a ℓ^2 sequence, whose ℓ^2 -norm is bounded above by $C \|z\|_{s_0}^2 \|z\|_s$ if $s_0 > \nu + 1 + 2\varepsilon$. Concerning $T_2(l)$, using Lemmas 2.14 and 2.15, we obtain

$$\begin{aligned} T_2(l) &\leq C \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| \leq |l|} \frac{1}{|l|^{s - s' + \beta}} |j_1|^s |j_2|^\nu \tilde{A}(j_1, j_2, l)^{N - s} |z_{j_1}| |z_{j_2}| |z_{j_3}| \\ &\leq C \|z\|_{s_0} \frac{1}{|l|^{s - s' + \beta}} \sum_{|j_1| \geq |j_2|, |j_2| \leq |l|} \left(\frac{\sqrt{|l j_2|}}{1 + ||j_1| - |l||} \right)^{1 + \varepsilon} |j_1|^s |z_{j_1}| |j_2|^\nu |z_{j_2}| \\ &\leq C \|z\|_{s_0} \frac{1}{|l|^{s - s' + \beta - (1 + \varepsilon)/2}} \sum_{j_2 \in \bar{\mathbb{Z}}} |j_2|^{\nu + (1 + \varepsilon)/2} |z_{j_2}| \sum_{j_1 \in \bar{\mathbb{Z}}} \frac{|j_1|^s |z_{j_1}|}{(1 + ||j_1| - |l||)^{1 + \varepsilon}}. \end{aligned}$$

The last sum in j_1 is a convolution product of the ℓ^2 sequence $|j_1|^s |z_{j_1}|$ and the ℓ^1 sequence $\frac{1}{(1 + |j_1|)^{1 + \varepsilon}}$. Choosing $\varepsilon > 0$ in such a way that $s - s' + \beta - (1 + \varepsilon)/2 > 0$, the sequence $T_2(l)$ is in ℓ^2 , with a norm bounded by

$$\|T_2\| \leq C \|z\|_{s_0} \|z\|_{\nu + (1 + \varepsilon)/2} \|z\|_s \leq C \|z\|_{s_0}^2 \|z\|_s,$$

with $s_0 > \nu + (1 + \varepsilon)/2$. Collecting the estimates for T_1 and T_2 , we obtain the announced inequality.

(iii) We define $0 < \varepsilon < 1/12$ and $N = s + \frac{1}{2} + \varepsilon$. We have, as in (ii), this first estimate

$$(2.18) \quad \|X_P(z)\|_s^2 \leq C \|z\|_{s_0}^{2(k-3)} \sum_{l \in \mathbb{Z}} \left(\sum_{|j_1| \geq |j_2| \geq |j_3|} \frac{|l|^s \mu(j, l)^\nu}{C(j, l)^\beta (1 + S(j, l))} A(j, l)^N |z_{j_1}| |z_{j_2}| |z_{j_3}| \right)^2.$$

As in (ii), we may also cut the sum on j_1 , j_2 and j_3 into two pieces, $T_1^+(l)$ collecting all the terms with $|j_2| > |l|$ and $T_2^+(l)$ collecting those with $|j_2| \leq |l|$. Following (ii), since $C(j, l) \geq 1$ and $1 + S(j, l) \geq 1$, we obtain for T_1^+ :

$$\begin{aligned} T_1^+(l) &\leq C \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| > |l|} |l|^s |j_2|^\nu A(j, l)^N |z_{j_1}| |z_{j_2}| |z_{j_3}| \\ &\leq C \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| > |l|} |l|^{1/2+\varepsilon} |j_1|^{s-1/2-\varepsilon} |j_2|^\nu A(j, l)^{N-(s-1/2-\varepsilon)} |z_{j_1}| |z_{j_2}| |z_{j_3}| \\ &\leq C \|z\|_{s_0} \sum_{|j_1| \geq |j_2|, |j_2| > |l|} |l|^{1/2+\varepsilon} |j_1|^{s-1/2-\varepsilon} |j_2|^\nu \tilde{A}(j_1, j_2, l)^{N-(s-1/2-\varepsilon)} |z_{j_1}| |z_{j_2}| \\ &\leq C \|z\|_{s_0} \sum_{|j_1| \geq |j_2|, |j_2| > |l|} |l|^{s-N} |j_1|^{s-1/2-\varepsilon} |j_2|^{\nu+N-(s-1/2-\varepsilon)} |z_{j_1}| |z_{j_2}| \\ &\leq C \|z\|_{s_0} \frac{1}{|l|^{\frac{1}{2}+\varepsilon}} \sum_{|j_1| \geq |j_2|, |j_2| > |l|} |j_1|^{s-1/2-\varepsilon} |j_2|^{\nu+N-(s-1/2-\varepsilon)} |z_{j_1}| |z_{j_2}| \\ &\leq C \|z\|_{s_0} \frac{1}{|l|^{\frac{1}{2}+\varepsilon}} \|z\|_s \|z\|_{\nu+1+2\varepsilon}, \end{aligned}$$

hence $T_1^+(l)$ is a ℓ^2 sequence, whose ℓ^2 -norm is bounded above by $C \|z\|_{s_0}^2 \|z\|_s$ if $s_0 > \nu + 1 + 2\varepsilon$.

The estimate on T_2^+ will need all factors assigned in the definition of $\mathcal{T}^{\nu,+}$:

$$\begin{aligned} T_2^+(l) &\leq C \sum_{|j_1| \geq |j_2| \geq |j_3|, |j_2| \leq |l|} \frac{|j_1|^s |j_2|^\nu}{\max(j_1, l)^\beta (1 + ||j_1| - |l||)} \tilde{A}(j_1, j_2, l)^{N-s} |z_{j_1}| |z_{j_2}| |z_{j_3}| \\ &\leq C \|z\|_{s_0} \sum_{|j_1| \geq |j_2|, |j_2| \leq |l|} \left(\frac{\sqrt{|l j_2|}}{1 + ||j_1| - |l||} \right)^\varepsilon \frac{1}{|l|^\beta (1 + ||j_1| - |l||)} |j_1|^s |z_{j_1}| |j_2|^\nu |z_{j_2}| \\ &\leq C \|z\|_{s_0} \frac{1}{|l|^{\beta-\varepsilon/2}} \sum_{j_2 \in \mathbb{Z}} |j_2|^{\nu+\varepsilon/2} |z_{j_2}| \sum_{j_1 \in \mathbb{Z}} \frac{|j_1|^s |z_{j_1}|}{(1 + ||j_1| - |l||)^{1+\varepsilon}}. \end{aligned}$$

Once again, the last sum in j_1 is a convolution product of the ℓ^2 sequence $|j_1|^s |z_{j_1}|$ and the ℓ^1 sequence $\frac{1}{(1+|j_1|)^{1+\varepsilon}}$. Choosing $\varepsilon > 0$ in such a way that

$\beta - \varepsilon/2 > 0$, the sequence $T_2^+(l)$ is in ℓ^2 , with a norm bounded by

$$\|T_2\| \leq C \|z\|_{s_0} \|z\|_{\nu+(1+\varepsilon)/2} \|z\|_s \leq C \|z\|_{s_0}^2 \|z\|_s,$$

with $s_0 > \nu + (1 + \varepsilon)/2$. Collecting the estimates for T_1^+ and T_2^+ , we obtain the announced inequality.

(iv) Let $k + 1 = 2m$. As in (ii), we obtain

(2.19)

$$\|X_P\|_s^2 \leq C \sum_{l \in \mathbb{Z}} \left(\sum_{j \in \mathbb{N}_{>}^{m-1}} l^s z_l \frac{\mu(j, j, l)^\nu}{C(j, j, l)^\beta} |z_{j_1}| |z_{-j_1}| \cdots |z_{j_{m-1}}| |z_{-j_{m-1}}| \right)^2,$$

using the same convention for $\mu(j, j, l)$ and $C(j, j, l)$ as for $\mu(j, l)$ and $C(j, l)$: as an example, $\mu(j, j, l)$ is the third biggest integer between $|j_1|, |j_1|, \dots, |j_{m-1}|, |j_{m-1}|, |l|$ and $|l|$, that is, if j is ordered, either $\mu(j, j, l) = |j_1|$, and in this case $C(j, j, l) = |l|$, or $\mu(j, j, l) = |l|$ and in this case $C(j, j, l) = |j_1|$. Notice that $A(j, j, l) = 1$ does not help in this case. The sum over j can be decomposed into two parts :

$$\begin{aligned} & \sum_{j \in \mathbb{N}_{>}^{m-1}, j_1 \leq l} l^s z_l \frac{\mu(j, l)^\nu}{C(j, l)^\beta} |z_{j_1}| |z_{-j_1}| \cdots |z_{j_{m-1}}| |z_{-j_{m-1}}| \\ & \leq \sum_{j \in \mathbb{N}_{>}^{m-1}, j_1 \leq l} l^s z_l \frac{j_1^\nu}{l^\beta} |z_{j_1}| |z_{-j_1}| \cdots |z_{j_{m-1}}| |z_{-j_{m-1}}| \\ & \leq l^{s-\beta} z_l \sum_{j_1} j_1^\nu |z_{j_1}| |z_{-j_1}| \|z\|_0^{2(m-2)} \\ & \leq l^{s-\beta} z_l \|z\|_{\nu/2}^2 \|z\|_0^{2(m-2)}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{j \in \mathbb{N}_{>}^{m-1}, j_1 > l} l^s z_l \frac{\mu(j, l)^\nu}{C(j, l)^\beta} |z_{j_1}| |z_{-j_1}| \cdots |z_{j_{m-1}}| |z_{-j_{m-1}}| \\ & \leq l^s z_l \sum_{j \in \mathbb{N}_{>}^{m-1}} j_1^{\nu-\beta} |z_{j_1}| |z_{-j_1}| \cdots |z_{j_{m-1}}| |z_{-j_{m-1}}| \\ & \leq l^s z_l \|z\|_{(\nu-\beta)/2}^2 \|z\|_0^{2(m-2)} \end{aligned}$$

Inserting these two estimates in (2.19) we get (2.13). \square

The second essential property satisfied by polynomials in \mathcal{T}_k^ν is captured in the following

Proposition 2.16. — Let $k_1, k_2 \geq 2$, $\nu_1, \nu_2 \geq 0$ and $\beta > 0$. The map $(P, Q) \mapsto \{P, Q\}$ defines a continuous map from $\mathcal{T}_{k_1+1}^{\nu_1, \beta, +} \times \mathcal{T}_{k_2+1}^{\nu_2, \beta}$ to $\mathcal{T}_{k_1+k_2}^{\nu', \beta}$ for $\nu' = 2(\nu_1 + \nu_2) + 1$.

Proof. — We assume that $P \in \mathcal{T}_{k_1+1}^{\nu_1, \beta, +}$ and $Q \in \mathcal{T}_{k_2+1}^{\nu_2, \beta}$ are homogeneous polynomials and we write

$$P(z) = \sum_{j \in \bar{\mathbb{Z}}^{k_1+1}} a_j z_{j_1} \cdots z_{j_{k_1+1}}$$

and

$$Q(z) = \sum_{i \in \bar{\mathbb{Z}}^{k_2+1}} b_i z_{i_1} \cdots z_{i_{k_2+1}}.$$

In view of the symmetry of the estimate (2.8) with respect to the involved indices, one easily obtains

$$\{P, Q\}(z) = \sum_{(j, i) \in \bar{\mathbb{Z}}^{k_1+k_2}} c_{j, i} z_{j_1} \cdots z_{j_{k_1}} z_{i_1} \cdots z_{i_{k_2}}$$

with

$$|c_{j, i}| \leq c_{N, N'} \sum_{l \in \bar{\mathbb{Z}}} \frac{\mu(j, l)^{\nu_1}}{C(j, l)^\beta (1 + S(j, l))} A(j, l)^N \frac{\mu(i, l)^{\nu_2}}{C(i, l)^\beta} A(i, l)^{N'}.$$

Therefore it remains to prove that, for each $M \geq 1$, there exist $N, N' \geq 1$, $C > 0$ such that for all $j \in \bar{\mathbb{Z}}^{k_1}$ and all $i \in \bar{\mathbb{Z}}^{k_2}$,

$$(2.20) \quad \sum_{l \in \bar{\mathbb{Z}}} \frac{\mu(j, l)^{\nu_1}}{C(j, l)^\beta (1 + S(j, l))} A(j, l)^N \frac{\mu(i, l)^{\nu_2}}{C(i, l)^\beta} A(i, l)^{N'} \leq C \frac{\mu(j, i)^{\nu'}}{C(i, j)^\beta} A(j, i)^M$$

with $\nu' = 2(\nu_1 + \nu_2) + 1$.

In order to simplify the notations, and because it does not change the estimates of (2.20), we will assume $k_1 = k_2 = k$. We can also assume by symmetry that

- all the indices are positive: $j_1, \dots, j_k, i_1, \dots, i_k \geq 1$.
- j and i are ordered: $j_1 \geq \dots \geq j_k$ and $i_1 \geq \dots \geq i_k$.

We begin by two technical lemmas whose proofs are postponed at the end of this proof.

Lemma 2.17. — There is a constant $C > 0$ such that for any $j \in \bar{\mathbb{Z}}^{k_1}$, $i \in \bar{\mathbb{Z}}^{k_2}$ and $l \in \bar{\mathbb{Z}}$ we have

$$(2.21) \quad A(j, l)^2 A(i, l)^2 \leq CA(i, j).$$

Lemma 2.18. — *There is a constant $C > 0$ such that for any $j \in \bar{\mathbb{Z}}^{k_1}$, $i \in \bar{\mathbb{Z}}^{k_2}$ and $l \in \bar{\mathbb{Z}}$ we have*

$$(2.22) \quad \max(\mu(j, l)A(i, l)^2, \mu(i, l)A(j, l)^2) \leq C\mu(i, j)^2.$$

Using these lemmas, in order to prove (2.20), it suffices to prove

$$\sum_{l \in \bar{\mathbb{Z}}} \frac{1}{C(j, l)^\beta (1 + S(j, l))} \frac{A(i, l)^2}{C(i, l)^\beta} \leq C \frac{\mu(j, i)}{C(i, j)^\beta}.$$

Noticing that $C(i, l)C(j, l) \geq C(i, j)l$, it remains to verify that

$$\sum_{l \in \bar{\mathbb{Z}}} \frac{A(i, l)^2}{(1 + S(j, l))l^\beta} \leq C\mu(j, i).$$

Now we decompose the sum in two parts, $I_1 = \sum_{l > j_2}$ and $I_2 = \sum_{l \leq j_2}$. For the first sum we have

$$I_1 = \sum_{l > j_2} \frac{A(i, l)^2}{(1 + S(j, l))l^\beta} \leq \sum_{l \in \bar{\mathbb{Z}}} \frac{1}{(1 + |l - j_1|)l^\beta} \leq C$$

while for the second one

$$I_2 = \sum_{l \leq j_2} \frac{A(i, l)^2}{(1 + S(j, l))l^\beta} \leq \sum_{l \leq j_2} \frac{A(i, l)^2}{l^\beta}.$$

In this last sum, if $j_2 < \mu(i, j)$, then

$$I_2 \leq j_2 \leq \mu(i, j).$$

On the contrary, if $\mu(i, j) \leq j_2$, then we decompose the I_2 sum in two parts, $I_{2,1} = \sum_{l < 2i_1}$ and $I_{2,2} = \sum_{l \geq 2i_1}$. Since $i_1 \leq \mu(i, j) = \max(i_1, j_3)$ we have

$$I_{2,1} = \sum_{l < 2i_1} \frac{A(i, l)^2}{l^\beta} \leq 2i_1 \leq 2\mu(i, j).$$

Finally, when $l \geq 2i_1$ we have $S(i, l) \geq l/2$ and $B(i, l)^2 = i_1 i_2 \leq i_2 l/2 \leq \mu(i, j)l/2$ and thus $A(i, l) \leq \sqrt{2\mu(i, j)l^{-1/2}}$ which leads to

$$I_{2,2} = \sum_{2i_1 \leq l \leq j_2} \frac{A(i, l)^2}{l^\beta} \leq C\mu(i, j) \sum_{l \in \bar{\mathbb{N}}} \frac{1}{l^{1+\beta}} \leq C\mu(i, j).$$

□

Proof of lemma 2.17 — The estimate (2.21) being symmetric with respect to i and j , we can assume that $j_1 \geq i_1$. We consider three cases depending of the position of l with respect to i_1 and j_1 .

First case $l \geq j_1$:

We have $S(i, l) = |i_1 - l| \geq |i_1 - j_1| \geq S(i, j)$ and $B(i, l) = (i_1 i_2)^{1/2} \leq B(i, j)$. Therefore

$$A(i, l) = \frac{B(i, l)}{B(i, l) + S(i, l)} \leq \frac{B(i, l)}{B(i, l) + S(i, j)} \leq \frac{B(i, j)}{B(i, j) + S(i, j)} = A(i, j),$$

and using $A(j, l) \leq 1$, (2.21) is proved.

Second case $l \leq i_1$:

Similarly as in the first case, we have $S(j, l) \geq S(i, j)$ and $B(j, l) = (j_2 \max(j_3, l))^{1/2} \leq (j_2 \max(j_3, i_1))^{1/2} \leq B(i, j)$ and thus

$$A(j, l) \leq A(i, j).$$

Third case $i_1 < l < j_1$:

That is the most complicated case and we have to distinguish whether $i_1 \geq j_2$ or not.

Subcase 1. $i_1 \geq j_2$:

We have $B(i, l) \leq B(i, j)$ thus if $S(i, l) = |i_1 - l| \geq \frac{1}{2}|i_1 - j_1| = \frac{1}{2}S(i, j)$ we obtain $A(j, l) \leq 2A(i, j)$ and (2.21) holds true. Now if $S(i, l) < \frac{1}{2}S(i, j)$ then $S(j, l) \geq \frac{1}{2}S(i, j)$ since $S(i, l) + S(j, l) \geq S(i, j)$. Furthermore, if $B(j, l) \leq B(i, j)$ then

$$A(j, l) = \frac{B(j, l)}{B(j, l) + S(j, l)} \leq 2 \frac{B(j, l)}{B(j, l) + S(i, j)} \leq 2 \frac{B(i, j)}{B(i, j) + S(i, j)} = 2A(i, j),$$

and (2.21) holds. If $B(j, l) > B(i, j)$, then using

$$B(j, l)^2 = j_2 l = j_2 i_1 + j_2 (l - i_1) \leq B(i, j)^2 + j_2 S(i, l) \leq B(i, j)^2 + \frac{1}{2} B(i, j) S(i, j),$$

we deduce

$$A(j, l)^2 \leq \frac{B(j, l)^2}{(B(i, j) + \frac{1}{2} S(i, j))^2} \leq 2 \frac{B(i, j)^2 + B(i, j) S(i, j)}{(B(i, j) + S(i, j))^2} \leq 2(A(i, j)^2 + A(i, j)),$$

thus (2.21) is also satisfied in this case, since $A(i, j) \leq 1$.

Subcase 2. $i_1 \leq j_2$:

We still have $B(i, l) \leq B(i, j)$ thus if furthermore $S(i, j) \leq 2S(i, l)$ then $A(i, l) \leq 2A(i, j)$ and (2.21) is true. So we assume $2S(i, l) < S(i, j)$ which implies $S(i, j) \leq 2S(j, l)$ since $S(i, l) + S(j, l) \geq S(i, j)$. If furthermore $l \leq j_3$, $B(j, l) = B(j) \leq B(i, j)$ and thus $A(j, l) \leq 2A(i, j)$ and (2.21) is again true. So we assume $j_3 \leq l$ and we have

$$B(j, l)^2 = l j_2 = i_1 j_2 + j_2 (l - i_1) \leq B(i, j)^2 + j_2 S(i, l).$$

If $S(i, l) \leq l/2$ then we deduce $B(j, l)^2 \leq 2B(j, l)^2$ and (2.21) is satisfied. It remains to consider the case $S(i, l) > l/2$ which implies $i_1 < l/2$ and thus

$$(2.23) \quad A(i, l) \leq \frac{i_1}{i_1 + l/2} \leq 2\frac{i_1}{l}.$$

Let $n \geq 1$ such that $\frac{l}{2^{n+1}} \leq i_1 \leq \frac{l}{2^n}$ we get from (2.23)

$$(2.24) \quad A(i, l) \leq \frac{1}{2^{n-1}}.$$

On the other hand

$$(2.25) \quad A(j, l) \leq 2\frac{(lj_2)^{1/2}}{(lj_2)^{1/2} + S(i, j)} \leq 2\frac{(lj_2)^{1/2}}{(i_1j_2)^{1/2} + S(i, j)}$$

and

$$(2.26) \quad A(i, j) \geq \frac{(i_1j_2)^{1/2}}{(i_1j_2)^{1/2} + S(i, j)} \geq \frac{1}{2^{n+1}} \frac{(lj_2)^{1/2}}{(i_1j_2)^{1/2} + S(i, j)}.$$

Combining (2.24), (2.25) and (2.26) we conclude

$$A(i, l)A(j, l) \leq 8A(i, j).$$

Proof of lemma 2.18 — The estimate (2.22) being symmetric with respect to i and j , we can assume $j_1 \geq i_1$. If furthermore $i_1 \geq j_2$ then one easily verifies that

$$\mu(i, l) \leq \mu(i, j) \text{ and } \mu(j, l) \leq \mu(i, j)$$

and estimates (2.22) is satisfied.

In the case $j_1 \geq j_2 \geq i_1$ we still have $\mu(i, l) \leq \mu(i, j)$ but $\mu(j, l)$ could be larger than $\mu(i, j)$. Actually if $\mu(j, l) \leq 2\mu(i, j)$, estimates (2.22) is still trivially satisfied. Therefore it remains to consider the case where $\mu(j, l) > 2\mu(i, j)$. Remark that in this case $i_1 \leq \mu(i, j) \leq \frac{\mu(j, l)}{2} \leq l/2$ and thus $S(i, l) = |i_1 - l| \geq l/2$ which leads to

$$A(i, l) \leq \frac{(i_1i_2)^{1/2}}{S(i, l)} \leq \frac{(2i_2)^{1/2}}{l^{1/2}} \leq \frac{(2\mu(i, j))^{1/2}}{l^{1/2}}.$$

Using this last estimates one gets

$$\mu(j, l)A(i, l)^2 \leq lA(i, l)^2 \leq 2\mu(i, j)^2.$$

We end this section with a corollary concerning Lie transforms associated to polynomials in $\mathcal{T}^{\delta, \beta, +}$.

Corollary 2.19. — *Let χ be a real homogeneous polynomial in $\mathcal{T}_l^{\delta, \beta, +}$ with $\delta \geq 0$, $\beta > 0$, $l \geq 3$ and take $s > s_1 := \delta + 3/2$. We denote by ϕ the Lie transform associated with χ .*

- (i) ϕ is an analytic canonical transformation from an open ball B_ϵ of center 0 and radius ϵ in \mathcal{P}_s into the open ball $B_{2\epsilon}$ in \mathcal{P}_s satisfying

$$(2.27) \quad \|\phi(z) - z\|_s \leq C_s \|z\|_s^2 \text{ for any } z \in B_\epsilon.$$

In particular if $F \in \mathcal{H}^s$ with $s > s_1$ then $F \circ \phi \in \mathcal{H}^s$. Furthermore, if F is real then $F \circ \phi$ is real too.

- (ii) Let $P \in \mathcal{T}_n^{\nu, \beta} \cap \mathcal{H}^s$, $\nu \geq 0$, $n \geq 3$ and fix $r \geq n$ an integer. Then

$$P \circ \phi = Q_r + R_r$$

where:

- Q_r is a polynomial of degree r belonging to $\mathcal{T}^{\nu', \beta} \cap \mathcal{H}^s$ with $\nu' = \nu + (r - n)(\beta + 1) + 2$,
- R_r is a real Hamiltonian in $\mathcal{T}^{\nu'', \beta} \cap \mathcal{H}^s$ with $\nu'' = \nu + (r - n + 1)(\beta + 1) + 2$, having a zero of order $r + 1$ at the origin.

Proof. — (i) Since $\chi \in \mathcal{T}_1^{\delta, \beta, +}$, by proposition 2.12 assertion (iii), $X_\chi \in \mathcal{H}^s$ for $s > s_1 = \delta + 3/2$. In particular, for $s > s_1$, the flow Φ_χ^t generated by the vector field X_χ transports an open neighborhood of the origin in \mathcal{P}_s into an open neighborhood of the origin in \mathcal{P}_s . Notice that since χ is real, Φ_χ^t transports the "real line" of \mathcal{P}_s , $\{(\xi, \bar{\xi}) \in \mathcal{P}_s\}$, into the "real line". Furthermore one has for $z \in \mathcal{P}_s$ small enough

$$\Phi_\chi^t(z) - z = \int_0^t X_\chi(\Phi_\chi^{t'}(z)) dt'$$

and since χ has a zero of order 3 one get by proposition 2.12 assertion (iii),

$$\|\Phi_\chi^t(z) - z\|_s \leq C_s \int_0^t \|\Phi_\chi^{t'}(z)\|_s^2 dt'.$$

Therefore by a classical continuity argument, there exists $\epsilon > 0$ such that the flow $B_\epsilon \ni z \mapsto \Phi_\chi^t(z) \in B_{2\epsilon}$ is well defined and smooth for $0 \leq t \leq 1$. Furthermore, the Lie transform $\phi = \Phi^1$ satisfies (2.27).

On the other hand, in view of the formula

$$X_{F \circ \phi}(q, p) = (D\phi(q, p))^{-1} X_F(\phi(q, p)),$$

we deduce that if $F \in \mathcal{H}^s$ with $s > s_1$ then $F \circ \phi \in \mathcal{H}^s$.

- (ii) By a direct calculus one has

$$\frac{d^k}{dt^k} P \circ \phi^t(z) \Big|_{t=0} = P^{(k)}(z)$$

where $P^{(k+1)} = \{P^{(k)}, \chi\}$ and $P^{(0)} = P$. Therefore applying the Taylor's formula to $P \circ \phi^t(z)$ between $t = 0$ and $t = 1$ we deduce

$$(2.28) \quad P \circ \phi(z) = \sum_{k=0}^{r-n} \frac{1}{n!} P^{(k)}(z) + \frac{1}{(r-n)!} \int_0^1 (1-t)^r P^{(r-n+1)}(\Phi^t(z)) dt.$$

Notice that $P^{(k)}(z)$ is a homogeneous polynomial of degree $n+k(l-2)$ and, by proposition 2.16, $P^{(k)}(z) \in \mathcal{T}^{\nu+k\delta+k+2,\beta} \cap \mathcal{H}^s$. Therefore (2.28) decomposes in the sum of a polynomial of degree r in $\mathcal{T}_r^{\nu',\beta}$ and a function in \mathcal{H}^s having a zero of degree $r+1$ at the origin. \square

2.3. The Birkhoff normal form theorem. —

Theorem 2.20. — *Assume that P is a real Hamiltonian belonging to \mathcal{H}^s for all s large enough and to the class $\mathcal{T}^{\nu,\beta}$ for some $\nu \geq 0$ and $\beta > 0$. Assume that ω is strongly non resonant (cf. definition 2.4) and satisfies (2.1) for some $\bar{d} \geq 0$. Then for any $r \geq 3$ there exists s_0 and for any $s \geq s_0$ there exists $\mathcal{U}_s, \mathcal{V}_s$ neighborhoods of the origin in \mathcal{P}_s and $\tau_s : \mathcal{V}_s \rightarrow \mathcal{U}_s$ a real analytic canonical transformation which is the restriction to \mathcal{V}_s of $\tau := \tau_{s_0}$ and which puts $H = H_0 + P$ in normal form up to order r i.e.*

$$H \circ \tau = H_0 + Z + R$$

with

- (a) Z is a real continuous polynomial of degree r with a regular vector field (i.e. $Z \in \mathcal{H}^s$) which only depends on the actions: $Z = Z(I)$.
- (b) $R \in \mathcal{H}^s$ is real and $\|X_R(z)\|_s \leq C_s \|z\|_s^r$ for all $z \in \mathcal{V}_s$.
- (c) τ is close to the identity: $\|\tau(z) - z\|_s \leq C_s \|z\|_s^2$ for all $z \in \mathcal{V}_s$.

Proof. — The proof is close to the proof of Birkhoff normal form theorem stated in [Gré07] or [Bam07], but we have to check the \mathcal{H}^s regularity of the Hamiltonian function iteratively constructed independently of the fact they belong to \mathcal{T}^ν (here $P \in \mathcal{T}^\nu$ does not imply $P \in \mathcal{H}^s$).

Having fixed some $r \geq 3$, the idea is to construct iteratively for $k = 2, \dots, r$, a neighborhood \mathcal{V}_k of 0 in \mathcal{P}_s (s large enough depending on r), a canonical transformation τ_k , defined on \mathcal{V}_k , an increasing sequence $(\nu_k)_{k=2, \dots, r}$ of positive numbers and real Hamiltonians $Z_k, P_{k+1}, Q_{k+2}, R_k$ such that

$$(2.29) \quad H_k := H \circ \tau_k = H_0 + Z_k + P_{k+1} + Q_{k+2} + R_k,$$

satisfying the following properties

- (i) Z_k is a polynomial of degree k in $\mathcal{T}^{\nu_k, \beta} \cap \mathcal{H}^s$ having a zero of order 3 at the origin and Z_k depends only on the (new) actions: $\{Z_k, I_j\} = 0$ for all $j \geq 1$.
- (ii) P_{k+1} is a homogeneous polynomial of degree $k+1$ in $\mathcal{T}_{k+1}^{\nu_k, \beta} \cap \mathcal{H}^s$.
- (iii) Q_{k+2} is a polynomial of degree $r+1$ in $\mathcal{T}^{\nu_k, \beta} \cap \mathcal{H}^s$ having a zero of order $k+2$ at the origin.
- (iv) R_k is a regular Hamiltonian belonging to \mathcal{H}^s and having a zero of order $r+2$ at the origin.

First we fix $s > \nu_r + 3/2$ to be sure to be able to apply Proposition 2.12 at each step (ν_r will be defined later on independently of s). Then we notice that (2.29) at order r proves Theorem 2.20 with $Z = Z_r$ and $R = P_{r+1} + R_r$ (notice that $Q_{r+2} = 0$). Actually, since $R = P_{r+1} + R_r$ belongs to \mathcal{H}^s and has a zero of order $r+1$ at the origin, we can apply Lemma 2.2 to obtain

$$(2.30) \quad \|X_R(z)\|_s \leq C_s \|(z)\|_s^r.$$

on $\mathcal{V} \subset \mathcal{V}_r$ a neighborhood of 0 in \mathcal{P}_s .

The Hamiltonian $H = H_0 + P$ has the form (2.29) for $k = 2$ with $\tau_2 = I$, $\nu_2 = \nu$, $Z_2 = 0$, P_3 being the Taylor's polynomial of P of degree 3, Q_4 being the Taylor's polynomial of P of degree $r+1$ minus P_3 and $R_2 = P - P_3 - Q_4$. We show now how to pass from step k to step $k+1$.

We look for τ_{k+1} of the form $\tau_k \circ \phi_{k+1}$, ϕ_{k+1} being a Lie transform associated to χ_{k+1} a homogeneous polynomial of degree $k+1$.

We decompose $H_k \circ \phi_{k+1}$ as follows

$$(2.31) \quad \begin{aligned} H_k \circ \phi_{k+1} &= H_0 + Z_k + \{H_0, \chi_{k+1}\} + P_{k+1} \\ &+ H_0 \circ \phi_{k+1} - H_0 - \{H_0, \chi_{k+1}\} \end{aligned}$$

$$(2.32) \quad + Z_k \circ \phi_{k+1} - Z_k$$

$$(2.33) \quad + P_{k+1} \circ \phi_{k+1} - P_{k+1}$$

$$(2.34) \quad + Q_{k+2} \circ \phi_{k+1}$$

$$(2.35) \quad + R_k \circ \phi_{k+1}.$$

Using the lemma 2.21 above, we choose χ_{k+1} in $\mathcal{T}_{k+1}^{\nu_k, \beta, +}$ in such way that

$$(2.36) \quad Z_{k+1} := Z_k + \{H_0, \chi_{k+1}\} + P_{k+1}$$

satisfies (i). By corollary 2.19, the Lie transform associated to χ_{k+1} is well defined and smooth on a neighborhood $\mathcal{V}_{k+1} \subset \mathcal{V}_k$ and satisfies for $z \in \mathcal{V}_{k+1}$

$$\|\phi_{k+1}(z) - z\|_s \leq C \|z\|_s^2.$$

Then from Proposition 2.16, Corollary 2.19 and formula (2.28), we get that (2.32), (2.33), (2.34) and (2.35) are regular Hamiltonians having a zero of order $k + 2$ at the origin. For instance concerning (2.32), one has by Taylor formula for any $z \in \mathcal{V}_{k+1}$

$$Z_k \circ \phi_{k+1}(z) - Z_k(z) = \{Z_k, \chi_{k+1}\}(z) + \int_0^1 (1-t) \{ \{Z_k, \chi_{k+1}\}, \chi_{k+1} \} (\Phi_{\chi_{k+1}}^t(z)) dt$$

and $\{Z_k, \chi_{k+1}\}$ is a polynomial having a zero of order $3 + \text{degree}(\chi_{k+1}) - 2 = k + 2$ while the integral term is a regular Hamiltonian having a zero of order $2k + 1$. So if $2k + 1 \geq r + 2$ this last term contributes to R_{k+1} and if not, we have to use a Taylor formula at a larger order.

Therefore the sum of (2.32), (2.33), (2.34) and (2.35) decomposes in $\tilde{P}_{k+2} + \tilde{Q}_{k+3} + \tilde{R}_{k+1}$ with \tilde{P}_{k+2} , \tilde{Q}_{k+3} and \tilde{R}_{k+1} satisfying respectively the properties (ii), (iii) and (iv) at rank $k + 1$ (with $\nu_{k+1} = k\nu'_k + \nu_k + k + 2$).

Concerning the term (2.31), one has to proceed differently since H_0 does not belongs to the \mathcal{H}^s .

First notice that by the homological equation (2.36) one has $\{H_0, \chi_{k+1}\} = Z_{k+1} - Z_k - P_{k+1}$. By construction Z_k and P_{k+1} belong to \mathcal{H}^s . On the other hand, by lemma 2.21, $Z_{k+1} \in \mathcal{T}_{k+1}^{\nu'_k, \beta}$ and is in normal form (i.e. depends only on the actions). Thus by Proposition 2.12 assertion (iv) one concludes that $Z_{k+1} \in \mathcal{H}^s$. Therefore we have prove that $\{H_0, \chi_{k+1}\} \in \mathcal{H}^s$.

Now we use the Taylor formula at order one to get

$$H_0 \circ \phi_{k+1}(z) - H_0(z) = \int_0^1 \{H_0, \chi_{k+1}\} (\Phi_{\chi_{k+1}}^t(z)) dt.$$

But we know from the proof of Corollary 2.19 that $\Phi_{\chi_{k+1}}^t : \mathcal{V}_{k+1} \rightarrow \mathcal{P}_s$ for all $t \in [0, 1]$. Therefore $H_0 \circ \phi_{k+1} - H_0 \in \mathcal{H}^s$ and thus (2.31) defines a regular Hamiltonian.

Finally we use again the Taylor formula and the Homological equation to write

$$H_0 \circ \phi_{k+1}(z) - H_0(z) - \{H_0, \chi_{k+1}\}(z) = \int_0^1 (1-t) \{Z_{k+1} - Z_k - P_{k+1}, \chi_{k+1}\} (\Phi_{\chi_{k+1}}^t(z)) dt$$

and, since $Z_{k+1} - Z_k - P_{k+1}$ belongs to $\mathcal{T}_{k+1}^{\nu'_k, \beta}$ and $\chi_{k+1} \in \mathcal{T}_{k+1}^{\nu'_k, \beta, +}$ we conclude by Proposition 2.16 that $H_0 \circ \phi_{k+1} - H_0 - \{H_0, \chi_{k+1}\} \in \mathcal{T}^{\nu_{k+1}, \beta}$. Then we use Corollary 2.19 to decompose it in $\hat{P}_{k+2} + \hat{Q}_{k+3} + \hat{R}_{k+1}$ with \hat{P}_{k+2} , \hat{Q}_{k+3} and \hat{R}_{k+1} satisfying respectively the properties (ii), (iii) and (iv) at rank $k + 1$. The proof is achieved defining $P_{k+2} = \hat{P}_{k+2} + \tilde{P}_{k+2}$, $Q_{k+3} = \hat{Q}_{k+3} + \tilde{Q}_{k+3}$ and $R_{k+1} = \hat{R}_{k+1} + \tilde{R}_{k+1}$. \square

Lemma 2.21. — *Let $\nu \in [0, +\infty)$ and assume that the frequencies vector of H_0 is strongly non resonant. Let Q be a homogeneous real polynomial of degree k in $\mathcal{T}_k^{\nu, \beta}$, there exist $\nu' > \nu$, homogeneous real polynomials Z and χ of degree k respectively in $\mathcal{T}_k^{\nu'}$ and $\mathcal{T}_k^{\nu', \beta, +}$ and satisfying*

$$(2.37) \quad \{H_0, \chi\} + Q = Z$$

and

$$(2.38) \quad \{Z, I_j\} = 0 \quad \forall j \geq 1.$$

Furthermore Z and χ both belong to \mathcal{H}^s for $s > \nu' + 1$.

Proof. — For $j \in \bar{\mathbb{N}}^{k_1}$ and $l \in \bar{\mathbb{N}}^{k_2}$ with $k_1 + k_2 = k$ we denote

$$\xi^{(j)} \eta^{(l)} = \xi_{j_1} \dots \xi_{j_{k_1}} \eta_{l_1} \dots \eta_{l_{k_2}}.$$

One has

$$\{H_0, \xi^{(j)} \eta^{(l)}\} = -i\Omega(j, l) \xi^{(j)} \eta^{(l)}$$

with

$$\Omega(j, l) := \omega_{j_1} + \dots + \omega_{j_{k_1}} - \omega_{l_1} - \dots - \omega_{l_{k_2}}.$$

Let $Q \in \mathcal{T}_k^{\infty, \nu}$

$$Q = \sum_{(j, l) \in \bar{\mathbb{N}}^k} a_{j, l} \xi^{(j)} \eta^{(l)}$$

where $(j, l) \in \bar{\mathbb{N}}^k$ means that $j \in \bar{\mathbb{N}}^{k_1}$ and $l \in \bar{\mathbb{N}}^{k_2}$ with $k_1 + k_2 = k$. Let us define

$$(2.39) \quad b_{j, l} = i\Omega(j, l)^{-1} a_{j, l}, \quad c_{j, l} = 0 \quad \text{when } \{j_1, \dots, j_{k_1}\} \neq \{l_1, \dots, l_{k_2}\}$$

and

$$(2.40) \quad c_{j, l} = a_{j, l}, \quad b_{j, l} = 0 \quad \text{when } \{j_1, \dots, j_{k_1}\} = \{l_1, \dots, l_{k_2}\}.$$

As ω is strongly non resonant, there exist γ and α such that

$$|\Omega(j, l)| \geq \gamma \frac{1 + S(j, l)}{\mu(j, l)^\alpha}$$

for all $(j, l) \in \bar{\mathbb{N}}^k$ with $\{j_1, \dots, j_{k_1}\} \neq \{l_1, \dots, l_{k_2}\}$. Thus, in view of Definitions 2.5 and 2.6, the polynomial

$$\chi = \sum_{(j, l) \in \bar{\mathbb{N}}^k} b_{j, l} \xi^{(j)} \eta^{(l)},$$

belongs to $\mathcal{T}_k^{\nu', +}$ while the polynomial

$$Z = \sum_{(j, l) \in \bar{\mathbb{N}}^k} c_{j, l} \xi^{(j)} \eta^{(l)}$$

belongs to $\mathcal{T}_k^{\nu'}$ with $\nu' = \nu + \alpha$. Notice that in this non resonant case, Z contains only monomials of degree even and thus has valuation at least 4. Furthermore by construction they satisfy (2.37) and (2.38). Note that the reality of Q

is equivalent to the symmetry relation: $\bar{a}_{jl} = a_{lj}$. Taking into account that $\Omega_{lj} = -\Omega_{jl}$, this symmetry remains satisfied for the polynomials χ and Z . Finally, χ and Z belong to \mathcal{H}^s as a consequence of proposition 2.12 (iii) and (iv) respectively. \square

3. Dynamical consequences

3.1. Nonlinear harmonic oscillator in one dimension. — The quantum harmonic oscillator $T = -\frac{d^2}{dx^2} + x^2$ is diagonalized in the Hermite basis $(\phi_j)_{j \in \bar{\mathbb{N}}}$:

$$\begin{aligned} T\phi_j &= (2j - 1)\phi_j, \quad j \in \bar{\mathbb{N}} \\ \phi_{n+1} &= \frac{H_n(x)}{\sqrt{2^n n!}} e^{-x^2/2}, \quad n \in \mathbb{N} \end{aligned}$$

where $H_n(x)$ is the n^{th} Hermite polynomial relative to the weight e^{-x^2} :

$$\int_{\mathbb{R}} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \sqrt{\pi} \delta_{nm}.$$

In this basis the Hermite multiplier is given by

$$(3.1) \quad M\phi_j = m_j \phi_j$$

where $(m_j)_{j \in \bar{\mathbb{N}}}$ is a bounded sequence of real number. For any $k \geq 1$, we define the class

$$(3.2) \quad \mathcal{W}_k = \{(m_j)_{j \in \bar{\mathbb{N}}} \mid \text{for each } j, m_j = \frac{\tilde{m}_j}{j^k} \text{ with } \tilde{m}_j \in [-1/2, 1/2]\}$$

that we endow with the product probability measure. In this context the frequencies, i.e. the eigenvalues of $T + M = -d^2/dx^2 + x^2 + M$ are given by

$$\omega_j = 2j - 1 + m_j = 2j - 1 + \frac{\tilde{m}_j}{j^k}, \quad j \in \bar{\mathbb{N}}$$

and we denote for $(\xi, \eta) \in \mathcal{P}_s$ ($s > 1/2$),

$$H_0 = \sum_{j=1}^{\infty} \omega_j \xi_j \eta_j.$$

Proposition 3.1. — *There exists a set $F_k \subset \mathcal{W}_k$ whose measure equals 1 such that if $m = (m_j)_{j \in \mathbb{N}} \in F_k$ then the frequencies vector $(\omega_j)_{j \geq 1}$ is strongly non resonant (cf. definition 2.4).*

Proof. — First remark that it suffices to prove that the frequencies vector $(\omega_j)_{j \geq 1}$ is non resonant in the sense of definition 2.3. Actually if we prove that (2.3) is satisfied for given constants δ' and γ' then if $S(j) < r\mu(j)$

$$|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}| \geq \frac{\gamma'}{\mu(j)^{\delta'}} \geq \frac{\gamma'}{r+1} \frac{1+S(j)}{\mu(j)^{\delta'+1}}$$

and thus (2.4) is satisfied with $\delta = \delta' + 1$ and $\gamma = \frac{\gamma'}{r+1}$. Now if $S(j) \geq r\mu(j)$ then use

$$(3.3) \quad |\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}| \geq S(j) - (r-2)\mu(j),$$

to conclude that

$$|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}| \geq \frac{2}{r}S(j) \geq \frac{\gamma'}{r+1} \frac{1+S(j)}{\mu(j)^{\delta'+1}}$$

providing γ' is small enough.

The prove that there exists a set $F_k \subset \mathcal{W}_k$ whose measure equals 1 such that if $m = (m_j)_{j \in \mathbb{N}} \in F_k$ then the frequencies vector $(\omega_j)_{j \geq 1}$ is non resonant is exactly the same than the proof of Theorem 5.7 in [Gré07], we do not repeat it (see also [BG06]). \square

In equation (1.1) with $d = 1$, the Hamiltonian perturbation reads

$$(3.4) \quad P(\xi, \eta) = \int_{\mathbb{R}} g(\xi(x), \eta(x)) dx$$

where $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$, $\xi(x) = \sum_{j \geq 1} \xi_j \phi_j(x)$, $\eta(x) = \sum_{j \geq 1} \eta_j \phi_j(x)$ and $((\xi_j)_{j \geq 1}, (\eta_j)_{j \geq 1}) \in \mathcal{P}_s$. We first check that P belongs to \mathcal{H}^s for s large enough.

Lemma 3.2. — *Let P given by (3.4) with $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$ real i.e. $g(z, \bar{z}) \in \mathbb{R}$ and having a zero of order at least 3 at the origin. Then $P \in \mathcal{H}^s$ for all $s > 1/2$.*

Proof. — One computes

$$\frac{\partial P}{\partial \xi_j}(\xi, \eta) = \int_{\mathbb{R}} \partial_1 g(\xi(x), \eta(x)) \phi_j(x) dx$$

and

$$\frac{\partial P}{\partial \eta_j}(\xi, \eta) = \int_{\mathbb{R}} \partial_2 g(\xi(x), \eta(x)) \phi_j(x) dx$$

thus we get

$$\begin{aligned} \|X_P(\xi, \eta)\|_s^2 &= \sum_{j \geq 1} |j|^{2s} \left| \int_{\mathbb{R}} \partial_1 g(\xi(x), \eta(x)) \phi_j(x) dx \right|^2 \\ &\quad + \sum_{j \geq 1} |j|^{2s} \left| \int_{\mathbb{R}} \partial_2 g(\xi(x), \eta(x)) \phi_j(x) dx \right|^2. \end{aligned}$$

Therefore, to verify that $z \mapsto X_P(z)$ is a regular function from a neighborhood of the origin in \mathcal{P}_s into \mathcal{P}_s , it suffices to check that the functions $x \mapsto \partial_1 g(\xi(x), \eta(x))$ and $x \mapsto \partial_2 g(\xi(x), \eta(x))$ are in \tilde{H}^s when $\xi(x)$ and $\eta(x)$ are in \tilde{H}^s . So it remains to prove that functions of the type $x \mapsto |x|^i \partial_1^{l+1} \partial_2^m g(\xi(x), \eta(x)) (\xi^{(l_1)}(x))^{\alpha_1} \cdot (\xi^{(l_{k_1})}(x))^{\alpha_{k_1}} (\eta^{(m_1)}(x))^{\beta_1} \dots (\eta^{(m_{k_2})}(x))^{\beta_{k_2}}$ are in $L^2(\mathbb{R})$ for all $0 \leq i + l + m \leq s$, $0 \leq i + l_j \leq s$, $0 \leq i + m_j \leq s$. But this is true because

- g is a C^∞ function, ξ and η are bounded functions and thus $x \mapsto \partial_1^{l+1} \partial_2^m g(\xi(x), \eta(x))$ is bounded
- \tilde{H}^s is an algebra for $s > 1/2$ and thus $x \mapsto |x|^k \xi^{(l)}(x) \eta^{(m)}(x) \in L^2(\mathbb{R})$ for all $0 \leq k + l + m \leq s$.
- $|\partial_1 g(\xi(x), \eta(x))|, |\partial_2 g(\xi(x), \eta(x))| \leq C(|\xi(x)| + |\eta(x)|)^2$ for some uniform constant $C > 0$ and thus $x \mapsto |x|^k \partial_1 g(\xi(x), \eta(x)) \in L^2(\mathbb{R})$ for all $0 \leq k \leq s$.

□

That P belongs to the class $\mathcal{T}^{\nu, \beta}$ is directly in relation with the distribution of the ϕ_j 's. Actually we have

Proposition 3.3. — *Let $\nu > 1/8$ and $0 \leq \beta \leq \frac{1}{24}$. For each $k \geq 1$ and for each $N \geq 0$ there exists $c_N > 0$ such that for all $j \in \bar{\mathbb{N}}^k$*

$$(3.5) \quad \left| \int_{\mathbb{R}} \phi_{j_1} \dots \phi_{j_k} dx \right| \leq c_N \frac{\mu(j)^\nu}{C(j)^\beta} A(j)^N.$$

As a consequence, any P of the general form (3.4) is in the class \mathcal{T}^ν .

The proof will be done in the multidimensional case in the next section (cf. Proposition 3.6).

We can now apply our Theorem 2.20 to obtain

Theorem 3.4. — *Assume that $M \in F_m$ defined in Proposition 3.1 and that $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$ is real i.e. $g(z, \bar{z}) \in \mathbb{R}$ and has a zero of order at least 3 at the*

origin. For any $r \geq 3$ there exists $s_0(r)$ an integer such that for any $s \geq s_0(r)$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that if $\|\psi_0\|_{\tilde{H}^s} = \varepsilon < \varepsilon_0$ the equation

$$(3.6) \quad i\psi_t = (-\Delta + x^2 + M)\psi + \partial_3 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}, t \in \mathbb{R}$$

with Cauchy data ψ_0 has a unique solution $\psi \in C^1((-T_\varepsilon, T_\varepsilon), \tilde{H}^s)$ with

$$(3.7) \quad T_\varepsilon \geq C\varepsilon^{-r}.$$

Moreover

- (i) $\|\psi(t, \cdot)\|_{\tilde{H}^s} \leq 2\varepsilon$ for any $t \in (-T_\varepsilon, T_\varepsilon)$.
- (ii) $\sum_{j \geq 1} j^{2s} \|\xi_j(t)\|^2 - \|\xi_j(0)\|^2 \leq \varepsilon^3$ for any $t \in (-T_\varepsilon, T_\varepsilon)$
where $\|\xi_j(t)\|^2$, $j \geq 1$ are the actions of $\psi(t, \cdot) = \sum \xi_j(t) \phi_j$.
- (iii) there exists a torus $\mathcal{T}_0 \subset H^s$ such that,

$$\text{dist}_s(\psi(t, \cdot), \mathcal{T}_0) \leq C\varepsilon^{r_1/2} \quad \text{for } |t| \leq \varepsilon^{-r_2}$$

where $r_1 + 2r_2 = r + 3$ and dist_s denotes the distance on H^s associated with the norm $\|\cdot\|_s$.

Proof. — Let $\psi_0 = \sum \xi_j(0) \phi_j$ and denote $z_0 = (\xi(0), \bar{\xi}(0))$. Denote by $z(t)$ the solution of the Cauchy problem $\dot{z} = X_H(z)$, $z(0) = z_0$, where $H = H_0 + P$ is the Hamiltonian function associated to the equation (3.6) written in the Hermite decomposition $\psi(t) = \sum \xi_j(t) \phi_j$, $z(t) = (\xi(t), \bar{\xi}(t))$. We note that, since P is real, z remains a real point of \mathcal{P}_s for all t .

Then we denote by $z' = \tau^{-1}(z)$ where $\tau : \mathcal{V}_s \rightarrow \mathcal{U}_s$ is the transformation given by Theorem 2.20 (so that z' denotes the normalized coordinates) associated to the order $r + 2$ and $s \geq s_0(r + 2)$ given by the same Theorem. We note that, since the transformation τ is generated by a real Hamiltonian, $z'(t)$ is still a real point.

Let $\varepsilon_0 > 0$ such that $B_{2\varepsilon_0} \subset \mathcal{V}_s$ and take $0 < \varepsilon < \varepsilon_0$. We assume that $\|z(0)\|_s = \|\psi_0\|_{\tilde{H}^s} = \varepsilon$

For $z = (\xi, \eta) \in \mathcal{P}_s$ we define

$$N(z) := 2 \sum_{j=1}^{\infty} j^{2s} I_j(\xi, \eta)$$

where we recall that $I_j(\xi, \eta) = \xi_j \eta_j$. We notice that for a real point $z = (\xi, \bar{\xi}) \in \mathcal{P}_s$,

$$N(z) = \|z\|_s^2.$$

Thus in particular we have⁽⁴⁾

$$N(z(t)) = \|z(t)\|_s^2 \quad \text{and} \quad N(z'(t)) = \|z'(t)\|_s^2.$$

Using that Z depends only on the normalized actions, we have

$$(3.8) \quad \dot{N}(z') = \{N, H \circ \tau\} \circ \tau^{-1}(z) = \{N, R\}(z').$$

Therefore as far as $\|z(t)\|_s \leq 2\varepsilon$, and thus $z(t) \in \mathcal{V}_s$, by assertion (c) of Theorem 2.20, $\|z'(t)\|_s \leq C\varepsilon$ and using (3.8) and assertion (b) of Theorem 2.20 (at order $r+2$) we get

$$|N(z'(t)) - N(z'(0))| \leq \left| \int_0^t \{N, R\}(z'(t')) dt' \right| \leq Ct \|z'(t)\|_s^{r+3} \leq Ct\varepsilon^{r+3}.$$

In particular, as far as $\|z(t)\|_s \leq 2\varepsilon$ and $|t| \leq C\varepsilon^{-r}$

$$|N(z'(t)) - N(z'(0))| \leq C\varepsilon^3.$$

Therefore using again assertion (c) of Theorem 2.20, we obtain

$$|N(z(t)) - N(z(0))| \leq C\varepsilon^3$$

which, choosing ε_0 small enough, leads to $\|z(t)\|_s \leq 3/2 \varepsilon$ as far as $\|z(t)\|_s \leq 2\varepsilon$ and $|t| \leq C\varepsilon^{-r}$. Thus (3.7) and assertions (i) follow by a continuity argument.

To prove assertion (ii) we recall the notation $I_j(z) = I_j(\xi, \eta) = \xi_j \eta_j$ for the actions associated to $z = (\xi, \eta)$. Using that Z depends only on the actions, we have

$$\{I_j \circ \tau^{-1}, H\}(z) = \{I_j, H \circ \tau\} \circ \tau^{-1}(z) = \{I_j, R\}(z').$$

Therefore, we get in the normalized coordinates

$$\frac{d}{dt} I_j(\xi', \eta') = -i\xi'_j \frac{\partial R}{\partial \eta_j} + i\eta'_j \frac{\partial R}{\partial \xi_j}$$

and thus

$$\begin{aligned} \sum_j j^{2s} \left| \frac{d}{dt} I_j(\xi', \bar{\xi}') \right| &= \sum_j j^{2s} \left| -\xi'_j \frac{\partial R}{\partial \eta_j} + \bar{\xi}'_j \frac{\partial R}{\partial \xi_j} \right| \\ &\leq \left(\sum_j j^{2s} (|\xi'_j|^2 + |\bar{\xi}'_j|^2) \right)^{1/2} \left(\sum_j j^{2s} \left(\left| \frac{\partial R}{\partial \eta_j} \right|^2 + \left| \frac{\partial R}{\partial \xi_j} \right|^2 \right) \right)^{1/2} \end{aligned}$$

⁽⁴⁾That is precisely at this point that we need to work with real Hamiltonians. The Birkhoff normal form theorem is essentially algebraic and do hold for complex Hamiltonians.

which leads to

$$(3.9) \quad \sum_j j^{2s} \left| \frac{d}{dt} I_j(z') \right| \leq \|z'\|_s \|X_R(z')\|_s \leq \|z'\|_s^{r+3}.$$

Thus, recalling that $I_j(\xi', \bar{\xi}') = |\xi'_j|^2$ we get

$$(3.10) \quad \sum_{j \geq 1} j^{2s} \left| |\xi'_j(t)|^2 - |\xi'_j(0)|^2 \right| \leq \varepsilon^3 \text{ for any } |t| \leq C\varepsilon^{-r}.$$

On the other hand, one has using (i) and assertion (c) of theorem 2.20 that for any $|t| \leq C\varepsilon^{-r}$,

$$\sum_{j \geq 1} j^{2s} \left| |\xi_j(t)|^2 - |\xi'_j(t)|^2 \right| \leq \sum_{j \geq 1} j^{2s} (|\xi_j(t)| + |\xi'_j(t)|) |\xi_j(t) - \xi'_j(t)| \leq C\varepsilon^3.$$

Combining this last relation with (3.10) assertion (ii) follows.

To prove (iii), let $\bar{I}_j = I'_j(0)$ be the initial actions in the normalized coordinates and define the smooth torus

$$\Pi_0 := \{z \in \mathcal{P}_s : I_j(z) = \bar{I}_j, j \geq 1\}$$

and its image in H^s

$$\mathcal{T}_0 = \{u \in H^s : u = \sum \xi_j \phi_j \text{ with } \tau(\xi, \bar{\xi}) \in \Pi_0\}.$$

We have

$$(3.11) \quad d_s(z(t), \mathcal{T}_0) \leq \left[\sum_j j^{2s} \left| \sqrt{I'_j(t)} - \sqrt{\bar{I}_j} \right|^2 \right]^{1/2}$$

Notice that for $a, b \geq 0$,

$$\left| \sqrt{a} - \sqrt{b} \right| \leq \sqrt{|a - b|}.$$

Thus, using (3.9), we get

$$\begin{aligned} [d_s(z(t), \mathcal{T}_0)]^2 &\leq \sum_j j^{2s} |I'_j(t) - I'_j(0)| \\ &\leq |t| \sum_j j^{2s} |\dot{I}'_j(t)| \\ &\leq \frac{1}{\varepsilon^{r_1}} \|z'\|_s \|X_R(z')\|_s \\ &\leq C \frac{1}{\varepsilon^{r_1}} \varepsilon^{r+3} \leq C \varepsilon^{r+3-r_1}. \end{aligned}$$

which gives (ii). □

3.2. Multidimensional nonlinear harmonic oscillator. —

3.2.1. *model.* — The spectrum of the d -dimensional harmonic oscillator T is \mathbb{N}_d with

$$(3.12) \quad \mathbb{N}_d = \begin{cases} 2\mathbb{N} \setminus \{0, 2, \dots, d-2\} & \text{if } d \text{ is even} \\ 2\mathbb{N} + 1 \setminus \{1, 3, \dots, d-2\} & \text{if } d \text{ is odd} \end{cases}$$

For $j \in \mathbb{N}_d$ we denote the associated eigenspace E_j which dimension is

$$d_j = \#\{(i_1, \dots, i_d) \in (2\mathbb{N} + 1)^d \mid i_1 + \dots + i_d = j\}.$$

Finally we denote $\Phi_{j,l}$, $l = 1, \dots, d_j$, the eigenfunctions associated the eigenvalue j . Note that $\Phi_{j,l}$ is of the form $\Phi_{j,l} = \phi_{i_1} \otimes \dots \otimes \phi_{i_d}$.

The Hermite multiplier M is defined on the basis $(\Phi_{j,l})_{j \in \mathbb{N}_d, l=1, \dots, d_j}$ of $L^2(\mathbb{R}^d)$ by

$$(3.13) \quad M\Phi_{j,l} = m_{j,l}\Phi_{j,l}$$

where $(m_{j,l})_{j \in \mathbb{N}_d, l=1, \dots, d_j}$ is a bounded sequence of real numbers.

The linear part of (1.1) reads

$$H_0 = -\Delta + x^2 + M.$$

H_0 is still diagonalized by $(\Phi_{j,l})_{j \in \mathbb{N}_d, l=1, \dots, d_j}$ and thus the spectrum of H_0 is

$$(3.14) \quad \sigma(H_0) = \{j + m_{j,l} \mid j \in \mathbb{N}_d, l = 1, \dots, d_j\}$$

For simplicity, we will focus on the case $m_{j,l} = m_j$ for all $l = 1, \dots, d_j$. In this case we have $\sigma(H_0) = \{j + m_j \mid j \in \mathbb{N}_d\}$ and, as a consequence of proposition 3.1,

Proposition 3.5. — *There exists a set $F_k \subset \mathcal{W}_k$ whose measure equals 1 such that if $m = (m_j)_{j \in \bar{\mathbb{N}}} \in F_k$ then the frequencies vector $(\omega_{j,i})_{j \in \mathbb{N}_d, i=1, \dots, d_j}$ satisfies:*

for any $r \in \bar{\mathbb{N}}$, there are $\gamma > 0$ and $\delta > 0$ such that for any $j \in \mathbb{N}_d^r$, any $l \in \{1, \dots, d_{j_1}\} \times \dots \times \{1, \dots, d_{j_r}\}$ and any $1 \leq i \leq r$, one has

$$(3.15) \quad \left| \omega_{j_1, l_1} + \dots + \omega_{j_i, l_i} - \omega_{j_{i+1}, l_{i+1}} - \dots - \omega_{j_r, l_r} \right| \geq \gamma \frac{1 + S(j)}{\mu(j)^\delta}$$

except if $\{j_1, \dots, j_i\} = \{j_{i+1}, \dots, j_r\}$.

Concerning the product of eigenfunctions we have,

Proposition 3.6. — *Let $\nu > d/8$. For any $k \geq 1$ and any $N \geq 1$ there exists $c_N > 0$ such that for any $j \in \mathbb{N}_d^k$, any $l \in \{1, \dots, d_{j_1}\} \times \dots \times \{1, \dots, d_{j_k}\}$*

$$(3.16) \quad \left| \int_{\mathbb{R}^d} \Phi_{j_1, l_1} \dots \Phi_{j_k, l_k} dx \right| \leq c_N \frac{\mu(j)^\nu}{C(j)^{\frac{1}{24}}} A(j)^N.$$

We use the approach developed by Dario Bambusi in [Bam07] section 6.2. The basic idea lies in the following commutator lemma: Let A be a linear operator which maps $D(T^k)$ into itself and define the sequence of operators

$$A_N := [T, A_{N-1}], \quad A_0 := A$$

then ([Bam07] lemma 7) for any $j_1 \neq j_2$ in \mathbb{N}_d , any $0 \leq l_1 \leq d_1$, $0 \leq l_2 \leq d_2$ and any $N \geq 0$

$$|\langle A \Phi_{j_2, l_2}, \Phi_{j_1, l_1} \rangle| \leq \frac{1}{|j_1 - j_2|^N} |\langle A_N \Phi_{j_2, l_2}, \Phi_{j_1, l_1} \rangle|.$$

Let A be the multiplication operator by the function $\Phi = \Phi_{j_3, l_3} \dots \Phi_{j_k, l_k}$ then by an induction argument

$$A_N = \sum_{0 \leq |\alpha| \leq N} C_{\alpha, N} D^\alpha$$

where

$$C_{\alpha, N} = \sum_{0 \leq |\beta| \leq 2N - |\alpha|} V_{\alpha, \beta, N}(x) D^\beta \phi$$

and $V_{\alpha, \beta, N}$ are polynomials of degree less than $2N$. Therefore one gets

$$(3.17) \quad \begin{aligned} \left| \int_{\mathbb{R}^d} \Phi_{j_1, l_1} \dots \Phi_{j_k, l_k} dx \right| &\leq \frac{1}{|j_1 - j_2|^N} \|A_N \Phi_{j_2, l_2}\|_{L^2} \\ &\leq C \frac{1}{|j_1 - j_2|^N} \sum_{0 \leq |\alpha| \leq N} \sum_{0 \leq |\beta| \leq 2N - |\alpha|} \|V_{\alpha, \beta, N} D^\beta \phi D^\alpha \Phi_{j_2, l_2}\|_{L^2} \\ &\leq C \frac{1}{|j_1 - j_2|^N} \sum_{0 \leq |\alpha| \leq N} \sum_{0 \leq |\beta| \leq 2N - |\alpha|} \|\Phi_{j_2, l_2}\|_{|\alpha|} \|\Phi\|_{\nu_0 + |\beta|} \end{aligned}$$

where we used in the last estimate

$$\forall \nu_0 > d/2 \quad \|fg\|_{L^2} \leq C_{\nu_0} \|f\|_{\nu_0} \|g\|_{L^2}.$$

We now estimate $\|\Phi\|_{\nu_0 + |\beta|}$.

First notice that, since $T\Phi_{j, l} = j\Phi_{j, l}$, one has for all $s \geq 0$

$$(3.18) \quad \|\Phi_{j, l}\|_s \leq C j^{s/2}.$$

Then we recall that the Hermite eigenfunctions are uniformly bounded, and actually (see [Sze75] or [KT05])

$$(3.19) \quad \|\phi_j\|_{L^\infty} \leq Cj^{-1/12}$$

and thus, since $\Phi_{j,l} = \phi_{i_1} \otimes \cdots \otimes \phi_{i_d}$ with $i_1 + \cdots + i_d = j$, we deduce

$$(3.20) \quad \|\Phi_{j,l}\|_{L^\infty} \leq C_d j^{-1/12}$$

with $C_d = Cd^{1/12}$. Thus using the tame estimate (see for instance [Tay91])

$$\|uv\|_s \leq C(\|u\|_s \|v\|_{L^\infty} + \|v\|_s \|u\|_{L^\infty})$$

combined with (3.18) and (3.20), we get for $j_3 \geq \cdots \geq j_k$,

$$(3.21) \quad \|\Phi\|_s \leq Cj_3^{s/2}.$$

Inserting (3.18) and (3.21) in (3.17) we get

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \Phi_{j_1, l_1} \cdots \Phi_{j_k, l_k} dx \right| &\leq C \frac{1}{|j_1 - j_2|^N} \sum_{0 \leq |\alpha| \leq N} \sum_{0 \leq |\beta| \leq 2N - |\alpha|} j_2^{|\alpha|/2} j_3^{(\nu_0 + |\beta|)/2} \\ &\leq C \frac{1}{|j_1 - j_2|^N} \sum_{0 \leq |\alpha| \leq N} j_2^{|\alpha|/2} j_3^{\nu_0/2 + N - |\alpha|/2} \\ &\leq C \frac{1}{|j_1 - j_2|^N} j_3^{N + \nu_0/2} \left(\frac{j_2}{j_3} \right)^{N/2} \\ &= C \frac{1}{|j_1 - j_2|^N} j_3^{\nu_0/2} (j_2 j_3)^{N/2}. \end{aligned}$$

Now remark that if $\sqrt{j_2 j_3} \leq |j_1 - j_2|$ then the last estimate implies

$$(3.22) \quad \left| \int_{\mathbb{R}^d} \Phi_{j_1, l_1} \cdots \Phi_{j_k, l_k} dx \right| \leq C j_3^{\nu_0/2} \frac{(j_2 j_3)^{N/2}}{(\sqrt{j_2 j_3} + |j_1 - j_2|)^N} = C \mu(j)^{\nu/2} A(j)^N$$

while if $\sqrt{j_2 j_3} > |j_1 - j_2|$ then $A(j) \geq 1/2$ and thus (3.22) is trivially true.

On the other hand, using (3.20) one has

$$\left| \int_{\mathbb{R}^d} \Phi_{j_1, l_1} \cdots \Phi_{j_k, l_k} dx \right| \leq C j_1^{-1/12}.$$

Combining this estimate with (3.22) one gets for all $N \geq 1$

$$\left| \int_{\mathbb{R}^d} \Phi_{j_1, l_1} \cdots \Phi_{j_k, l_k} dx \right| \leq c_N \frac{\mu(j)^\nu}{C(j)^{\frac{1}{24}}} A(j)^N$$

with $\nu = \frac{\nu_0}{4}$. □

3.2.2. *Result.* — We follow the presentation of section 2 and only focus on the new feature.

We consider the phase space $\mathcal{Q}_s = \mathcal{L}_s \times \mathcal{L}_s$ with

$$\mathcal{L}_s = \{(a_{j,l})_{j \in \mathbb{N}_d, 1 \leq l \leq d_j} \mid \sum_{j \in \mathbb{N}_d} |j|^{2s} \sum_{l=1}^{d_j} |a_{j,l}|^2 < \infty\}$$

that we endow with the standart norm and the standart symplectic structure as for \mathcal{P}_s in section 2.1. Writing $\psi = \sum \xi_{j,l} \Phi_{j,l}$, $\bar{\psi} = \sum \eta_{j,l} \Phi_{j,l}$ with $(\xi, \eta) \in \mathcal{Q}_s$, the linear part of the multidimensional version of the linear part of (1.1) reads

$$H_0(\xi, \eta) = \frac{1}{2} \sum_{j \in \mathbb{N}_d} \sum_{l=1}^{d_j} \omega_{j,l} \xi_{j,l} \eta_{j,l}.$$

For $j \geq 1$, we define

$$J_j(\xi, \eta) = \sum_{l=1}^{d_j} \xi_{j,l} \eta_{j,l}.$$

Using notations of section 2.1, we define the class \mathbb{T}_k^ν of polynomials of degree k on \mathcal{Q}_s

$$Q(\xi, \eta) \equiv Q(z) = \sum_{m=0}^k \sum_{j \in \mathbb{N}_d^m} \sum_{l_1=1}^{d_{j_1}} \cdots \sum_{l_m=1}^{d_{j_m}} a_{j,l} z_{j_1, l_1} \cdots z_{j_m, l_m}$$

such that for each $N \geq 1$, there exists a constant $C > 0$ such that for all j, l

$$|a_{j,l}| \leq C \frac{\mu(j)^\nu}{C(j)^{1/24}} A(j)^N.$$

Then following definition 2.10 we define a corresponding class \mathbb{T}^ν of C^∞ Hamiltonians on \mathcal{Q}_s having their Taylor's polynomials in \mathbb{T}_k^ν . Similarly, following definition 2.1, we also define \mathcal{H}_d^s the class of real Hamiltonians P satisfying $P \in C^\infty(\mathcal{U}_s, \mathbb{C})$ and $X_P \in C^\infty(\mathcal{U}_s, \mathcal{Q}_s)$ for some $\mathcal{U}_s \subset \mathcal{Q}_s$ a neighborhood of the origin.

Adapting slightly the proof of theorem 2.20 we get

Theorem 3.7. — *Let P be a real Hamiltonian belonging in $\mathbb{T}^\nu \cap \mathcal{H}_d^s$ for some $\nu \geq 0$ and for all s sufficiently large and let ω be a weakly non resonant frequencies vector in the sense of (3.15). Then for any $r \geq 3$ there exists s_0 and for any $s \geq s_0$ there exists \mathcal{U}, \mathcal{V} neighborhoods of the origin in \mathcal{Q}_s and $\tau : \mathcal{V} \rightarrow \mathcal{U}$ a real analytic canonical transformation which puts $H = H_0 + P$ in normal form up to order r i.e.*

$$H \circ \tau = H_0 + Z + R$$

with

- (i) Z is a real continuous polynomial of degree r which commutes with all J_j , $j \geq 1$, i.e. $\{Z, J_j\} = 0$ for all $j \geq 1$.
- (ii) R is real and belongs to \mathcal{H}_d^s , furthermore $\|X_R(z)\|_s \leq C_s \|z\|_s^r$ for all $z \in \mathcal{V}_s$.
- (iii) τ is close to the identity: $\|\tau(z) - z\|_s \leq C_s \|z\|_s^2$ for all $z \in \mathcal{V}_s$.

Proof. — The only new point when comparing with theorem 2.20, is that in assertion (ii) we obtain $\{Z, J_j\} = 0$ for all $j \geq 1$ instead of $\{Z, I_j\} = 0$ for all $j \geq 1$. This is a consequence of a new definition of polynomial in normal form adapted to this multidimensional case. Actually instead of definition 2.11, we use

Definition 3.8. — Let $k = 2m$ be an even integer, a formal polynomial Z homogeneous of degree k on \mathcal{Q}_s is in normal form if it reads

$$Z(\xi, \eta) = \sum_{j \in \mathbb{N}_d^k} \sum_{l_1, l'_1=1}^{d_{j_1}} \dots \sum_{l_k, l'_k=1}^{d_{j_k}} a_{j, l, l'} \xi_{j_1, l_1} \eta_{j_1, l'_1} \dots \xi_{j_k, l_k} \eta_{j_k, l'_k}$$

for all $(\xi, \eta) \in \mathcal{Q}_s$.

One easily verifies that if Z is in normal form then Z commutes with each $J_j = \sum_{l=1}^{d_j} \xi_{j, l} \eta_{j, l}$ since for instance

$$\{\xi_{j_1, l_1} \eta_{j_1, l'_1}, \xi_{j_1, l_1} \eta_{j_1, l_1} + \xi_{j_1, l'_1} \eta_{j_1, l'_1}\} = 0.$$

On the other hand, we also verify following the lines of the proof of assertion (iv) of proposition 2.12, that a homogeneous polynomial of degree $k+1$ $Z \in \mathcal{T}^\nu$ which is in normal form satisfies $\|X_Z(z)\|_s \leq C \|z\|_s^k$ for all z in a neighborhood of the origin. In particular, if $Z \in \mathcal{T}^\nu$ is in normal form, it automatically belongs to \mathcal{H}_d^s (this point was crucial in the proof of theorem 2.20).

Then, in view of (3.15), we adapt lemma 2.21 where, and in particular (2.39) and (2.40), in such a way $\chi \in \mathcal{T}^{\nu,+}$ and Z is in normal form in the sense of definition 3.8. \square

Notice that the normal form $H_0 + Z$ is no more, in general, integrable. The dynamical consequences are the same as in Theorem 3.4 (i) and (ii) but we have to replace I_j by J_j in the second assertion. Actually the J_j play the role of almost actions: they are almost conserved quantities.

In equation (1.1), the Hamiltonian perturbation reads

$$(3.23) \quad P(\xi, \eta) = \int_{\mathbb{R}} g(\xi(x), \eta(x)) dx$$

where $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$, $\xi(x) = \sum_{j \geq 1} \xi_j \phi_j(x)$, $\eta(x) = \sum_{j \geq 1} \eta_j \phi_j(x)$ and $((\xi_j)_{j \geq 1}, (\eta_j)_{j \geq 1}) \in \mathcal{P}_s$. As in the one dimensional case (cf. Lemma 3.2), P belongs to \mathcal{H}_d^s for s large enough ($s > d/2$) and using Proposition 3.6, P belongs to the class \mathbb{T}^ν . So one has

Lemma 3.9. — *Let P given by (3.23) with $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$ real i.e. $g(z, \bar{z}) \in \mathbb{R}$ and having a zero of order at least 3 at the origin.. Then $P \in \mathcal{H}^s \cap \mathbb{T}^\nu$ for all $s > d/2$ and for $\nu > d/8$.*

Thus one can apply Theorem 3.7 to obtain

Theorem 3.10. — *Assume that $m \in F_k$ defined in proposition 3.5 and that $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$ is real i.e. $g(z, \bar{z}) \in \mathbb{R}$ and vanishing at least at order 3 at the origin. For each $r \geq 3$ and $s \geq s_0(r)$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any ψ_0 in \tilde{H}^s , any $\varepsilon \in (0, \varepsilon_0)$, the equation*

$$i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}$$

with Cauchy data ψ_0 has a unique solution $\psi \in C^1((-T_\varepsilon, T_\varepsilon), \tilde{H}^s)$ with

$$T_\varepsilon \geq c\varepsilon^{-r}.$$

Moreover for any $t \in (-T_\varepsilon, T_\varepsilon)$, one has

$$\|\psi(t, \cdot)\|_{\tilde{H}^s} \leq 2\varepsilon$$

and

$$\sum_{j \geq 1} j^{2s} |J_j(t) - J_j(0)| \leq \varepsilon^3$$

where $J_j(t) = \sum_{l=1}^{d_j} |\xi_{j,l}|^2$, $j \geq 1$ are the "pseudo-actions" of $\psi(t, \cdot) = \sum_{j,l} \xi_{j,l}(t) \Phi_{j,l}(\cdot)$.

Proof. — Just remark that as in the proof of theorem 3.4, defining $N(z) := 2 \sum_{j \in \mathbb{N}_d} j^{2s} J_j(\xi, \eta) = 2 \sum_{j \in \mathbb{N}_d} j^{2s} \sum_{l=1}^{d_j} \xi_{j,l} \eta_{j,l}$ one has $N(z) = \|z\|_s^2$ for all real point $z = (\xi, \bar{\xi})$. On the other hand, using that Z commutes with J_j , we have

$$\{N \circ \tau^{-1}, H\}(z) = \{N, H \circ \tau\} \circ \tau^{-1}(z) = \{N, R\}(z').$$

Therefore, we get in the normalized variables $|\dot{N}| \leq CN^{(r+1)/2}$ and the theorem follows as in the proof of Theorem 3.4. \square

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