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**CORRIGENDUM AND ADDENDUM TO ‘LINEARLY  
RECURRENT SUBSHIFTS HAVE A FINITE NUMBER OF  
NON-PERIODIC FACTORS’**

FABIEN DURAND

ABSTRACT. We prove that a subshift  $(X, T)$  is linearly recurrent if and only if it is a primitive and proper  $S$ -adic subshift. This corrects Proposition 6 in F. Durand (*Ergod. Th. & Dynam. Sys.* **20** (2000), 1061–1078).

1. INTRODUCTION AND DEFINITIONS

In this paper we freely use the definitions and the notations of [Du]. Proposition 6 in [Du] is false: There exist primitive  $S$ -adic subshifts that are not linearly recurrent (LR). We will give an example. Nevertheless the other part of this proposition is true: If a subshift is LR then it is primitive  $S$ -adic.

The author apologizes for the mistake. We correct this proposition with the next one, but before we have to give and to recall some definitions. Let  $A$  and  $B$  be two finite alphabets.

Let  $x$  be an element of  $A^{\mathbb{N}}$  or  $A^{\mathbb{Z}}$ . We call *occurrence* of  $u \in A^*$  in  $x$  every integer  $i$  such that  $x_{[i, i+|u|-1]} = x_i x_{i+1} \dots x_{i+|u|-1} = u$ . A *return word* to  $u \in A^*$  in  $x$  is a word  $w$  such that  $wu$  has an occurrence in  $x$ ,  $u$  is a prefix of  $wu$  and  $u$  has exactly 2 occurrences in  $wu$ . We say that  $x$  is *linearly recurrent* (LR) (with constant  $K \in \mathbb{N}$ ) if it is uniformly recurrent and if for all  $u$  having an occurrence in  $x$  and all return words,  $w$ , to  $u$  in  $x$  we have  $|w| \leq K|u|$ .

Let  $T$  be the shift transformation defined on  $A^{\mathbb{Z}}$ . We say the subshift  $(X, T)$  is generated by  $x$  if  $X$  is the set of the sequences  $z$  such that  $z_{[i, j]}$  has an occurrence in  $x$  for all intervals  $[i, j] \subset \mathbb{Z}$ . The subshift  $(X, T)$  is linearly recurrent if it is minimal and contains a LR sequence. We remark that if  $x \in A^{\mathbb{Z}}$  is linearly recurrent then  $x$  and  $x_{[0, +\infty)}$  generate the same LR subshift.

Let  $a$  be a letter of  $A$ ,  $S$  a finite set of morphisms  $\sigma$  from  $A(\sigma) \subset A$  to  $A^*$  and  $(\sigma_n : A_{n+1} \rightarrow A_n^*; n \in \mathbb{N})$  be a sequence of  $S^{\mathbb{N}}$  such that  $(\sigma_0 \sigma_1 \dots \sigma_n(aa \dots); n \in \mathbb{N})$  converges in  $A^{\mathbb{N}}$  to  $x$ . We will say that  $x$  is a  *$S$ -adic sequence on  $A$*  (generated by  $(\sigma_i; i \in \mathbb{N}) \in S^{\mathbb{N}}$  and  $a$ ). If there exists an integer  $s_0$  such that for all non-negative integers  $r$  and all  $b \in A_r$  and  $c \in A_{r+s_0+1}$ , the letter  $b$  has an occurrence in  $\sigma_{r+1} \sigma_{r+2} \dots \sigma_{r+s_0}(c)$ , then we say that  $x$  is a *primitive  $S$ -adic sequence* (with constant  $s_0$ ).

Let  $\sigma : A \rightarrow B^*$  be a morphism. We say  $\sigma$  is proper if there exist two letters  $r, l \in B$  such that for all  $a \in A$  the first letter of  $\sigma(a)$  is  $l$  and the last letter of  $\sigma(a)$  is  $r$ . We say the sequence  $x \in A^{\mathbb{N}}$  is a *proper  $S$ -adic sequence* if it is a  $S$ -adic sequence and the morphisms in  $S$  are proper. The subshift generated by a proper  $S$ -adic sequence is called *proper  $S$ -adic subshift*.

**Proposition 1.1.** *The subshift  $(X, T)$  is LR if and only if it is a primitive and proper  $S$ -adic subshift.*

## 2. COUNTEREXAMPLE TO THE PROPOSITION 6 OF [Du]

In this section we give a counterexample to Proposition 6 in [Du], i.e., a primitive  $S$ -adic subshifts that is not LR.

Let  $A = \{a, b, c\}$  be an alphabet, and,  $\sigma : A \rightarrow A^*$  and  $\tau : A \rightarrow A^*$  be two morphisms defined by

$$\begin{aligned}\sigma(a) &= acb, & \tau(a) &= abc, \\ \sigma(b) &= bab, & \tau(b) &= acb, \\ \sigma(c) &= cbc, & \tau(c) &= aac.\end{aligned}$$

We call  $\mathcal{M}$  the set of all finite composition of elements of  $S = \{\sigma, \tau\}$ . For each element  $\rho$  of  $\mathcal{M}$  there exists a unique  $n \in \mathbb{N}$  such that  $|\rho(a)| = |\rho(b)| = |\rho(c)| = 3^n$ . We set  $|\rho| = 3^n$ . We first give a lemma which proof is left to the reader.

**Lemma 2.1.** *Let  $z \in A^{\mathbb{N}}$  and  $n \in \mathbb{N}$ . The difference between two successive occurrences of the word  $ca$  in  $\sigma^n \tau(z)$  is greater than  $3^{n+1}$ .*

Let  $x \in A^{\mathbb{N}}$  be the primitive  $S$ -adic sequence defined by

$$x = \lim_{n \rightarrow +\infty} \sigma \tau \sigma^2 \tau \cdots \sigma^n \tau (aaa \dots).$$

We show  $x$  is not LR. Let  $n$  be an integer, we set  $\rho_n = \sigma \tau \sigma^2 \tau \cdots \sigma^n \tau$  and  $y = \lim_{l \rightarrow +\infty} \sigma^{n+2} \tau \sigma^{n+3} \tau \cdots \sigma^{n+l} \tau (aaa \dots)$ . We have  $x = \rho_n \sigma^{n+1} \tau(y)$ .

Since  $\sigma(a)$  is a prefix of  $y$ ,  $\tau \sigma(a) = abcaacacb$ ,  $\sigma^{n+1}(a) = au$  and  $\sigma^{n+1}(c) = vc$ , for some  $u, v \in A^*$ , then  $ca$  appears in  $\sigma^{n+1} \tau(y)$ . Let  $w$  be a return word to  $ca$  in  $\sigma^{n+1} \tau(y)$ . Hence  $wca$  ( $= caw'ca$  for some  $w'$ ) appears in  $x$ . The word  $\rho_n(ca)$  appears exactly twice in  $\rho_n(caw'ca)$  (the proof of this fact is left to the reader) hence  $\rho_n(w)$  is a return word to  $\rho_n(ca)$  in  $x$ .

Moreover from Lemma 2.1 we have  $|w| \geq 3^{n+2}$ . It implies  $x$  is not LR because

$$\frac{|\rho_n(w)|}{|\rho_n(ca)|} = \frac{|w||\rho_n|}{2|\rho_n|} \geq \frac{3^{n+2}}{2}.$$

Let  $z \in B^{\mathbb{N}}$  where  $B$  is a finite alphabet. We denote by  $L(z)$  the set of all words having an occurrence in  $z$ . For all  $n \in \mathbb{N}$  we define  $p_z(n)$  to be the number of distinct words of length  $n$  in  $L(z)$ . In [Du] it is proved that if  $z$  is LR with constant  $K$  then  $p_z(n) \leq Kn$  for all  $n \in \mathbb{N}$ . Even if  $x$  is not LR, there exists a constant  $C$  such that  $p_n(x) \leq Cn$  for all  $n \in \mathbb{N}$ . This is a consequence of the following proposition.

**Proposition 2.1.** *Let  $A$  be a finite alphabet,  $a$  be a letter of  $A$  and  $(\sigma_n : A_{n+1} \rightarrow A_n^+; n \in \mathbb{N})$  be a sequence of morphisms such that  $A_n \subset A$  for all  $n \in \mathbb{N}$ ,  $a \in \bigcap_{n \in \mathbb{N}} A_n$  and*

$$y = \lim_{n \rightarrow +\infty} \sigma_0 \sigma_1 \cdots \sigma_n (aaa \dots).$$

*Suppose moreover  $\inf_{c \in A_{n+1}} |\sigma_0 \sigma_1 \cdots \sigma_n(c)|$  tends to  $+\infty$  and there exists a constant  $D$  such that*

$$|\sigma_0 \sigma_1 \cdots \sigma_{n+1}(b)| \leq D |\sigma_0 \sigma_1 \cdots \sigma_n(c)|$$

*for all  $b \in A_{n+2}$  and  $c \in A_{n+1}$ , and all  $n \in \mathbb{N}$ . Then  $p_y(n) \leq D(\text{Card}(A))^2 n$ .*

*Proof.* This proof follows the lines of the proof of Proposition V.19 in [Qu].

Let  $n \geq 1$ . The sequence  $(\inf_{c \in A_{k+1}} |\sigma_0 \sigma_1 \cdots \sigma_k(c)|)_{k \in \mathbb{N}}$  is non-decreasing and tends to  $+\infty$ , hence there exists  $p \in \mathbb{N}$  such that

$$\inf_{c \in A_p} |\sigma_0 \sigma_1 \cdots \sigma_{p-1}(c)| \leq n \leq \inf_{c \in A_{p+1}} |\sigma_0 \sigma_1 \cdots \sigma_p(c)|.$$

From that, every word  $w \in L(y)$  of length  $n$  has an occurrence  $i$  in some  $\sigma_0 \sigma_1 \cdots \sigma_p(bc)$ , where  $b$  and  $c$  are two letters of  $A$ , with  $i \leq |\sigma_0 \sigma_1 \cdots \sigma_p(b)| - 1$ . Consequently

$$\begin{aligned} p_y(n) &\leq (\text{Card}(A))^2 \sup_{c \in A_{p+1}} |\sigma_0 \sigma_1 \cdots \sigma_p(c)| \\ &\leq (\text{Card}(A))^2 D \inf_{c \in A_p} |\sigma_0 \sigma_1 \cdots \sigma_{p-1}(c)| \leq D(\text{Card}(A))^2 n. \end{aligned}$$

This ends the proof.  $\square$

**Corollary 2.1.** *Let  $A$  be a finite alphabet,  $a$  be a letter of  $A$ ,  $l$  be a positive integer and  $(\sigma_n : A \rightarrow A^*; n \in \mathbb{N})$  be a sequence of morphisms of constant length  $l$  and*

$$y = \lim_{n \rightarrow +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(aaa \dots).$$

*Then  $p_y(n) \leq l(\text{Card}(A))^2 n$ .*

### 3. A SUFFICIENT CONDITION FOR A PRIMITIVE $S$ -ADIC SEQUENCE TO BE LR

This sufficient condition is given in the following lemma and will be used in the sequel.

**Lemma 3.1.** *Let  $S$  be a finite set of morphisms. Let  $x$  be a primitive  $S$ -adic sequence generated by  $(\sigma_i : A_{i+1} \rightarrow A_i^*; i \in \mathbb{N})$  and  $a$  (with constant  $s_0$ ). For all  $n \in \mathbb{N}$  suppose  $\lim_{l \rightarrow +\infty} \sigma_n \sigma_{n+1} \cdots \sigma_l(aaa \dots)$  exists and call it  $x^{(n)}$ . Let  $D_n$  be the largest difference between two consecutive occurrences of a word of length 2 in  $x^{(n)}$ .*

*If  $(D_n; n \in \mathbb{N})$  is bounded then  $x$  is LR.*

*Proof.* Let  $x = \lim_{n \rightarrow +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(aaa \dots)$ . It follows from Lemma 7 of [Du] that  $x$  is uniformly recurrent. We set  $S_k = \sigma_0 \cdots \sigma_k$  for all  $k \in \mathbb{N}$ . Let  $u$  be a non-empty word of  $L(x)$  such that  $|u| \geq \max\{|S_{s_0}(b)|; b \in A_{s_0+1}\}$ , and  $v$  be a return word to  $u$ . We denote by  $k_0$  the smallest positive integer  $k$  such that  $|u| < \min\{|S_k(b)|; b \in A_{k+1}\}$ ; we remark that  $k_0 \geq s_0 + 1$ . There exists a word of length 2,  $bc$ , of  $L(x^{(k_0+1)})$  such that  $u$  has an occurrence in  $S_{k_0}(bc)$ . The largest difference between two successive occurrences of  $bc$  in  $x^{(k_0+1)}$  is bounded by  $D = \max_{n \in \mathbb{N}} D_n$  (which does not depend on  $k_0$ ), hence we have

$$\begin{aligned} |v| &\leq D \max\{|S_{k_0}(d)|; d \in A_{k_0+1}\} \leq DK \min\{|S_{k_0}(d)|; d \in A_{k_0+1}\} \\ &\leq DK \max\{|S_{k_0-1}(d)|; d \in A_{k_0}\} \min\{|\sigma_{k_0}(d)|; d \in A_{k_0+1}\} \\ &\leq DK^2 \min\{|S_{k_0-1}(d)|; d \in A_{k_0}\} \min\{|\sigma_{k_0}(d)|; d \in A_{k_0+1}\} \\ &\leq DK^2 \min\{|\sigma_{k_0}(d)|; d \in A_{k_0+1}\} |u|, \end{aligned}$$

where  $K$  is the constant given by Lemma 8 of [Du], i.e.,  $K$  is such that for all integers  $r, s$  with  $s - r \geq s_0 + 1$  and all  $b, c$  of  $A_{s+1}$  we have  $|\sigma_r \cdots \sigma_s(b)| \leq K |\sigma_r \cdots \sigma_s(c)|$ . We set  $M = DK^2 \max\{|\sigma_i(d)|; i \in \mathbb{N}, d \in A_{i+1}\}$ . For all  $u$  of  $L(x)$  greater than  $\max\{|S_{s_0}(b)|; b \in A_{s_0+1}\}$  and all  $v$  in  $\mathcal{R}_u$  we have  $|v| \leq M|u|$ . Hence  $x$  is LR.  $\square$

#### 4. A NECESSARY AND SUFFICIENT CONDITION TO BE LR

In the original proof of Proposition 6 in [Du] we use the notion of return word. In the proof of Proposition 1.1 we will do the same but we will use an extension of this notion which was defined in [DHS]. We will take a sequence  $x$  belonging to  $X$  and, using these “new” return words, we will show that  $x^+ = x_{[0,+\infty)}$  is a primitive and proper  $S$ -adic sequence. The subshift  $(X, T)$  being minimal we see that it is generated by  $x^+$  and, consequently,  $(X, T)$  is a primitive and proper  $S$ -adic subshift.

Let  $A$  be a finite alphabet,  $x \in A^{\mathbb{Z}}$ , and,  $u$  and  $v$  two words of  $A^*$ . We say  $w \in A^*$  is a *return word to  $u.v$*  in  $x$  if there exist two consecutive occurrences  $j, k$  of  $uv$  in  $x$  such that  $w = x_{[j+|u|, k+|u|)}$ . It is immediate to check that a word  $w \in A^+$  is a return word to  $u.v$  in  $x$  if and only if:

- 1)  $uvw$  has an occurrence in  $x$ , and
- 2)  $v$  is a prefix of  $wv$  and  $u$  is a suffix of  $uw$ , and
- 3) the word  $uvw$  contains exactly two occurrences of the word  $uv$ .

We denote by  $\mathcal{R}_{x,u.v}$  the set of return words to  $u.v$  in  $x$ . If  $u$  is the empty word  $\epsilon$  then the return words to  $u.v$  are the return words to  $v$  defined in [Du] and we set  $\mathcal{R}_{x,u.v} = \mathcal{R}_{x,v}$ . The return words to  $u.v$  are different from the return words to  $\epsilon.uv$  but we have  $\#\mathcal{R}_{x,u.v} = \#\mathcal{R}_{x,\epsilon.uv} = \#\mathcal{R}_{x,uv}$ .

We suppose now that  $x$  is a uniformly recurrent sequence. It is easy to see that for all  $u, v \in L(x)$  the set  $\mathcal{R}_{x,u.v}$  is finite. It will be convenient to label the return words. We enumerate the elements  $w$  of  $\mathcal{R}_{x,u.v}$  in the order of the first appearance of  $uvw$  in  $x_{[-|u|, +\infty)}$ . This defines a bijective map  $\Theta_{x,u.v} : \mathcal{R}_{x,u.v} \rightarrow \mathcal{R}_{x,u.v} \subset A^+$  where  $\mathcal{R}_{x,u.v} = \{1, \dots, \#\mathcal{R}_{x,u.v}\}$ :  $u\Theta_{x,u.v}(k)v$  is the  $k$ -th word of the type  $uvw$ ,  $w \in \mathcal{R}_{x,u.v}$ , appearing in  $x_{[-|u|, +\infty)}$ .

We consider  $\mathcal{R}_{x,u.v}$  as an alphabet. The map  $\Theta_{x,u.v}$  defines a morphism from  $\mathcal{R}_{x,u.v}$  to  $A^*$  and the set  $\Theta_{x,u.v}(\mathcal{R}_{x,u.v}^*)$  consists of all concatenations of return words to  $u.v$ .

The following proposition is important in the proof of Proposition 1.1.

**Proposition 4.1** ([DHS]). *The map  $\Theta_{x,u.v} : \mathcal{R}_{x,u.v}^+ \rightarrow A^+$  is one to one.*

*Proof of Proposition 1.1.* Let  $S$  be a finite set of proper morphisms and suppose  $(X, T)$  is a primitive  $S$ -adic subshift generated by  $(\sigma_i : A_{i+1} \rightarrow A_i^*; i \in \mathbb{N}) \in S^{\mathbb{N}}$  and  $a$  (with constant  $s_0$ ). Let

$$x = \lim_{n \rightarrow +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(aaa\dots).$$

We prove that  $x$  is LR and consequently that the subshift it generates is LR.

As the morphisms are proper the limit  $\lim_{l \rightarrow +\infty} \sigma_n \sigma_{n+1} \cdots \sigma_l(aaa\dots)$  exists for all  $n \in \mathbb{N}$ . We call it  $x^{(n)}$  and we define  $D_n$  as in Lemma 3.1.

The composition of two proper morphism is again proper. Consequently, from the primitivity, we can suppose that  $s_0 = 0$  and that for all  $n \in \mathbb{N}$ , all  $a \in A_{n+1}$  and all  $b \in A_n$  the letter  $b$  appears in  $\sigma_n(a)$ .

Let  $n \in \mathbb{N}$  and set  $\tau = \sigma_n \sigma_{n+1}$ . It is a proper substitution. Let  $l$  and  $r$  be respectively the first and the last letter of the images of  $\sigma_n$ . Let  $y$  be a one-sided sequence of  $A_{n+2}^{\mathbb{N}}$  and  $z = \sigma_n \sigma_{n+1}(y)$ . The words of length 2 having an occurrence in  $z$  are exactly the words of length 2 having an occurrence in some  $\sigma_n(e)$ ,  $e \in A_{n+1}$ , and the word  $rl$ . On the other hand the letters of  $\sigma_{n+1}(y)$  appear

with gaps bounded by  $K_n = 2 \max\{|\sigma_{n+1}(e)|; e \in A_{n+2}\}$ . Consequently the words of length 2 of  $z$  appear with gaps bounded by  $K_n \max\{|\sigma_n(e)|; e \in A_{n+1}\}$  and, a fortiori,  $D_n \leq K_n \max\{|\sigma_n(e)|; e \in A_{n+1}\}$  for all  $n \in \mathbb{N}$ . Moreover  $S$  being finite the sequence  $(D_n; n \in \mathbb{N})$  is bounded. Lemma 3.1 implies  $(X, T)$  is LR.

We suppose now that  $(X, T)$  is LR. The periodic case is trivial hence we suppose that  $(X, T)$  is not periodic. From Proposition 5 in [Du] there exists  $K \geq 2$  such that

$$(\forall u \in L(X))(\forall w \in \mathcal{R}_u) (|u|/K \leq |w| \leq K|u|),$$

We set  $\alpha = K^2(K+1)$ . Let  $x = (x_n; n \in \mathbb{Z})$  be an element of  $X$ . It suffices to prove that  $x_{[0,+\infty)}$  is a primitive and proper  $S$ -adic sequence.

For each non-negative integer  $n$  we set  $u_n = x_{-\alpha^n} \cdots x_{-2}x_{-1}$ ,  $v_n = x_0x_1 \cdots x_{\alpha^n-1}$ ,  $\mathcal{R}_n = \mathcal{R}_{x, u_n.v_n}$ ,  $R_n = R_{x, u_n.v_n}$  and  $\Theta_n = \Theta_{x, u_n.v_n}$ . Let  $n$  be a positive integer and  $w$  be a return word to  $u_n.v_n$ . The word  $w$  is a concatenation of return words to  $u_{n-1}.v_{n-1}$ . The map  $\Theta_{n-1}$  being one to one (Proposition 4.1), this induces a map  $\lambda_n$  from  $R_n$  to  $R_{n-1}^*$  defined by  $\Theta_{n-1}\lambda_n = \Theta_n$ . We set  $\lambda_0 = \Theta_0$ . For each letter  $b$  of  $R_n$  we have  $|\Theta_{n-1}\lambda_n(b)| \leq K|u_nv_n| = 2K\alpha^n$ . Moreover each element of  $\mathcal{R}_{n-1}$  is greater than  $(2\alpha^{n-1})/K$  hence

$$|\lambda_n(b)| \leq \frac{K^2\alpha^n}{\alpha^{n-1}} = \alpha K^2.$$

By Proposition 5 of [Du] we have  $\#R_n = \#\mathcal{R}_{x, u_nv_n} \leq K(K+1)^2$ , consequently the set  $M = \{\lambda_n; n \in \mathbb{N}\}$  is finite. The definition of  $R_n$  implies that  $\Theta_n(1)x_0x_1 \cdots x_{\alpha^n-1}$  is a prefix of  $x_{[0,+\infty)}$  for all  $n \in \mathbb{N}$  and  $\lambda_0\lambda_1 \cdots \lambda_n(1) = \Theta_n(1)$ . Proposition 4 of [Du] implies that the length of  $\Theta_n(1)$  tends to infinity with  $n$  and

$$x = \lim_{n \rightarrow +\infty} \lambda_0\lambda_1 \cdots \lambda_n(11 \cdots).$$

Let  $n$  be an integer greater than 1. Each word of length  $2K\alpha^n$  has an occurrence in each word of length  $2K(K+1)\alpha^n$  (Proposition 5 of [Du]). Hence each element of  $\mathcal{R}_n$  has an occurrence in each word of length  $2K(K+1)\alpha^n$ . Let  $w$  be an element of  $\mathcal{R}_{n+1}$ , we have  $|w| \geq 2\alpha^{n+1}/K = 2K(K+1)\alpha^n$ . Therefore each element of  $\mathcal{R}_n$  has an occurrence in each element of  $\mathcal{R}_{n+1}$ . It means that if  $b$  belongs to  $R_{n+1}$  then each letter of  $R_n$  has an occurrence in  $\lambda_{n+1}(b)$ . Hence  $x$  is a primitive  $S$ -adic sequence.

It remains to show each  $\lambda_n$  is proper. Let  $w$  be a return word of  $\mathcal{R}_n$ . The word  $wv_{n-1}$  is a concatenation of return words to  $v_{n-1}$ . Let  $p \in \mathcal{R}_{n-1}$  be such that  $pv_{n-1}$  is a prefix of  $wv_{n-1}$  and consequently of  $wv_n$ . We know  $v_n$  is also a prefix of  $wv_n$  and

$$|v_n| = \alpha^n = \alpha^{n-1}(K^3 + K^2) \geq (2K+1)\alpha^{n-1} \geq |p| + |v_{n-1}|.$$

Consequently  $pv_{n-1}$  is a prefix of  $v_n$ . Let  $l \in R_{n-1}$  be such that  $\Theta_{n-1}(l) = p$ . Then  $l$  is the first letter of  $\lambda_n(c)$  for all  $c \in R_n$ .

In the same way there exists  $s \in \mathcal{R}_{n-1}$  and  $r \in R_{n-1}$  such that  $\Theta_{n-1}(r) = s$  and  $u_{n-1}s$  is a suffix of  $u_n$ . Hence  $r$  is the last letter of  $\lambda_n(c)$  for all  $c \in R_n$  and  $\lambda_n$  is proper.  $\square$

## 5. LR STURMIAN SEQUENCES

We give a correct proof of the next proposition (which is stated in [Du]) because the original proof used Proposition 6 in [Du].

**Proposition 5.1.** *A sturmian subshift  $(\Omega_\alpha, T)$  is LR if and only if the coefficients of the continued fraction of  $\alpha$  are bounded.*

*Proof.* Let  $0 < \alpha < 1$  be an irrational real number and  $[0 : i_1 + 1, i_2, i_3, \dots]$  be its continued fraction. From Proposition 9 in [Du] we know that  $\Omega_\alpha = \Omega(x)$  where

$$x = \lim_{k \rightarrow +\infty} \tau^{i_1} \sigma^{i_2} \tau^{i_3} \sigma^{i_4} \dots \tau^{i_{2k-1}} \sigma^{i_{2k}} (00\dots),$$

$\tau(0) = 0$ ,  $\tau(1) = 10$ ,  $\sigma(0) = 01$  and  $\sigma(1) = 1$ . We just have to prove that if the coefficients of the continued fraction of  $\alpha$  are bounded then the sequence  $x$  is LR. The other part of the proof is in [Du] and do not use Proposition 6 in [Du].

Let  $i$ ,  $j$  and  $k$  be in  $\mathbb{N}^*$ . We have

$$\tau^i \circ \sigma^j \circ \tau^k(0) = 0(10^i)^j \text{ and } \tau^i \circ \sigma^j \circ \tau^k(1) = 10^i(0(10^i)^j)^k.$$

Consequently if  $x$  belongs to  $\{0, 1\}^{\mathbb{N}}$  then the set of the words of length 2 having an occurrence in  $y = \tau^i \circ \sigma^j \circ \tau^k(x)$  is  $\{00, 01, 10\}$ . Moreover the difference of two successive occurrences of 01 (resp. 10) in  $y$  is less than  $i + 2$  (resp.  $i + 2$ ), and, the difference between two successive occurrences of 00 in  $y$  is less than  $2j + 3$  if  $i = 1$  and less than 3 if  $i \geq 2$ . Consequently the difference between two successive occurrences of a word of length 2 in  $y$  is bounded by  $2 \max\{i, j, k\} + 3$ .

The same bound can be found for  $\sigma^i \circ \tau^j \circ \sigma^k$ .

For all  $n \in \mathbb{N}$ , let  $x^{(n)}$  and  $D_n$  be defined as in Lemma 3.1. Hence, if the sequence  $(i_n; n \in \mathbb{N})$  is bounded by  $K$ , then  $D_n$  is bounded by  $2K + 3$  for all  $n \in \mathbb{N}$ . Lemma 3.1 ends the proof.  $\square$

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