Relative Yamabe invariant and c-concordant metrics

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RELATIVE YAMABE INVARIANT AND C-CONCORDANT METRICS

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Abstract. We prove a surgery formula for the relative Yamabe invariant with several applications. In particular, we study a Yamabe invariant defined on the set of concordance classes of metrics.

Contents

1. Introduction

Let M be a compact n-manifold ($n \geq 3$) without boundary and a conformal class of metrics C on M. The *Yamabe constant* is defined as

$$\mu(M, C) = \inf \int_M \mathrm{Scal}_g dv_g.$$

where the infimum is taken over the metrics $g \in C$ such that $\mathrm{Vol}_g(M) = 1$. For any metric g on M and any $u \in C^\infty(M)$, let

$$J^n_{M,g}(u) = \frac{\int_M uL_g(u)dv_g}{\left(\int_M |u|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}$$

Here, L_g is the *conformal Laplacian* or *Yamabe operator* and is defined by

$$L_g = \frac{4(n-1)}{n-2}\Delta_g + \mathrm{Scal}_g.$$

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Then, it is well known that (see [LP87, Au98, He97])

$$\mu(M,[g]) = \inf_{u\in C^\infty(M);u\neq 0} J^n_{M,g}(u). \tag{1}$$

The Yamabe constant has been introduced by Yamabe in 1960 while attempting to find metrics of constant scalar curvature in a conformal class. He was able to show that the infimum in the definition of μ is always attained. Unfortunately, there was a gap in his proof which was repaired by Trüdinger (1968), Aubin (1976) and Schoen (1984). For a survey on this problem, the so-called *Yamabe problem*, the reader may refer to [LP87, Au98, He97].

Now, let

$$\sigma(M) = \sup \mu(M,C)$$

where the supremum runs over the sets of all conformal classes C of metrics. Aubin proved in [Au76] that for all C, $\mu(M,C) \leq \mu(\mathbb{S}^n) = n(n-1)\omega_n^{\frac{2}{n}}$ where $\mathbb{S}^n$ denotes the n-sphere S^n equipped with its standard metric. Its volume is denoted by ω_n. This implies that $\sigma(M) \leq \sigma(S^n) = \mu(\mathbb{S}^n)$ and hence, $\sigma(M)$ is well-defined and depends only on the differentiable manifold M. It is called the *Yamabe invariant*.

The critical points of the functional $g \to \int_M \mathrm{Scal}_g dv_g$ among the metrics g with $\mathrm{Vol}_g(M) = 1$ are Einstein metrics. Besides, one can check that $\sigma(M) > 0$ if and only if there exists a metric g on M with positive scalar curvature. Hence, the study of $\sigma(M)$ is connected to the difficult and still unsolved problems to determine all manifolds admitting einstein metrics and admitting metrics of positive scalar curvature. This explains why the Yamabe invariant has attracted so much interests in the last decades. For more informations and references on σ, the reader may consult [ADH08]. The Yamabe invariant turns out to be very difficult to compute explicitly and the value of σ is known only for very few manifolds (see again [ADH08]). A natural way to go further in its study is to use surgery techniques, whose power is demonstrated in the papers of Gromov-Lawson [GL80] and Schoen-Yau [SY79].

We focus here on the following surgery theorem, proved in [ADH08], which plays a central role in the whole paper:

Theorem 1.1. (Ammann-Dahl-Humbert; 2008) *Let (M,g) be a compact n-dimensional ($n \geq 3$) Riemannian manifold and let $M^\#$ be obtained from M by a surgery of dimension $k \in \{0,\cdots,n-3\}$. Then, there exists constants $\beta_{n,k} > 0$ with $\beta_{n,0} = +\infty$ depending only on n and k and metrics $(g_\theta)_{\theta>0}$ on $M^\#$ such that*

$$\lim_{\theta\to 0} \mu(M^\#,[g_\theta]) \geq \min(\mu(M,[g]),\beta_{n,k}).$$

In particular,

$$\sigma(M^\#) \geq \min(\sigma(M),\beta_n)$$

where $\beta_n = \min_{k\in\{0,\cdots,n-3\}} \beta_{n,k}$.

Now, let Ω be a $(n+1)$-manifold with boundary M. If g is a metric on Ω, we denote by ∂g the metric induced by g on M. If C is a conformal class, then $\partial C := \{\partial g | g \in C\}$. In particular, if g is a metric on Ω, $\partial[g] = [\partial g]$. Let us fix a

conformal class C (resp. $\bar{C}$) on M (resp. Ω) such that $\partial\bar{C} = C$. Again, we can define

$$\mu(\Omega, \bar{C}; M, C) = \inf \int_\Omega \mathrm{Scal}_g dv_g$$

where the infimum is taken over the metrics $g \in \bar{C}$ for which the boundary M is minimal and for which Ω has volume 1. This number is called the *relative Yamabe constant* and by Escobar [Es92]

$$\mu(\Omega, \bar{C}; M, C) \leq \mu(\mathbb{S}^{n+1}_+, \mathbb{S}^n) = 2^{\frac{2}{n+1}}\mu(\mathbb{S}^{n+1}) = 2^{\frac{2}{n+1}} n(n+1)\omega_{n+1}^{\frac{2}{n+1}}.$$

As in the case of manifolds without boundary, for all metric g on Ω such that $\partial g = h$ and all $u \in C^\infty(\Omega)$, we set

$$J^{n+1}_{\Omega,g}(u) = \frac{\int_\Omega uL_g(u)dv_g + \int_M H_g u^2 dv_h}{\left(\int_\Omega |u|^{\frac{2(n+1)}{n-1}} dv_g\right)^{\frac{n-1}{n+1}}}$$

where H_g is the mean curvature of the boundary M with respect to the metric g. Then, it is well known that

$$\mu(\Omega, [g]; M, [h]) = \inf_{u \in C^\infty(\Omega); u \neq 0} J^{n+1}_{\Omega,g}(u).$$

If in addition M is minimal for the metric g, then

$$\mu(\Omega, [g]; M, [h]) = \inf_{u \in C^\infty(\Omega); u \neq 0; \partial_\nu u = 0} J^{n+1}_{\Omega,g}(u), \tag{2}$$

where ν is the outer normal unit vector field on M.

Escobar [Es92] studied a Yamabe type problem concerning this conformal invariant. More precisely, he studied the problem of finding in a conformal class metrics with constant scalar curvature for which the boundary is minimal. He proved

Theorem 1.2. (Escobar; 1992) *Let (Ω, g) be a compact $(n+1)$-dimensional manifold for which the boundary M is minimal. Assume that*

$$\mu(\Omega, [g]; M, [\partial g]) < \mu(\mathbb{S}^{n+1}_+, \mathbb{S}^n).$$

Then, $\mu(\Omega, [g]; M, [\partial g])$ is attained. In other words, there exists a metric g' conformal to g for which $\mathrm{Vol}_{g'}(\Omega) = 1$, $\partial g' \in [\partial g]$, M is minimal in (Ω, g') and such that

$$\mu(\Omega, \bar{C}; M, C) = \int_\Omega \mathrm{Scal}_{g'} dv_{g'}.$$

One can verifies that the metric g' of the above theorem has constant scalar curvature. Since it is conformal to g, it has the form $g' = u^{\frac{4}{n-1}} g$ for some positive smooth function u on Ω that we can normalized by $\int_\Omega u^p dv_g = 1$ where $p := \frac{2(n+1)}{n-1}$. Then,

$$\mu(\Omega, [g]; M, [\partial g]) = J^{n+1}_{\Omega,g}(u).$$

Writing the Euler equation of u, we obtain:

$$\begin{cases} L_{g'}u = \mu(\Omega, \bar{C}; M, C)u^{p-1} & \text{on} \quad \mathring{\Omega} \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega = M. \end{cases}$$

Now, as in the case of manifolds without boundary, one defined the *relative Yamabe invariant* by

$$\sigma(\Omega; M, C) = \sup_{C'} \mu(\Omega, C'; M, C)$$

where the supremum runs over all conformal classes of metrics C' on Ω for which $\partial C' = C$. This invariant is related to the topology of the set of metrics with positive scalar curvature. It was studied for example by Akutagawa and Botvinnik. In particular, they proved in [AB02a] the following result:

Theorem 1.3. (Akutagawa, Botvinnik; 2002) *Let Ω_1, Ω_2 be $(n+1)$-dimensional manifolds with respective boundaries $M_1 \amalg M$ and $M_2 \amalg M$ (M_1 and M_2 are possibly empty). Let C_1, C_2, C be conformal classes of metrics respectively on M_1, M_2 and M and let Ω be the manifold with boundary $M_1 \amalg M_2$ obtained by gluing Ω_1 and Ω_2 along M. Assume that $\sigma(\Omega_i; M_i \amalg M, C_i \amalg C) > 0$ for $i = 1, 2$. Then, $\sigma(\Omega; M_1 \amalg M_2, C_1 \amalg C_2) > 0$ where by convention, $\sigma(\Omega; \emptyset) = \sigma(M)$.*

This paper aims to obtain a surgery formula for the relative Yamabe invariant similar to the one in Theorem 1.1 for the standard Yamabe invariant. More precisely, we prove

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \geq 2$ and (Ω, g) be a compact $(n+1)$-dimensional Riemannian manifold with boundary M. We set $h := \partial g$. Let also $k \in \{0, \cdots, n-2\}$ and $M^{\#}$ be obtained from M by a surgery of dimension k. We denote by $\Omega^{\#}$ the manifold with boundary $M^{\#}$ obtained from Ω by attaching the corresponding $(k+1)$-dimensional handle along M. Then, there exist some constants $\alpha_{n,k} > 0$ depending only on n and k and a sequence of metrics $(g_\theta)_{\theta>0}$ on $\Omega^{\#}$ such that, setting $h_\theta = \partial g_\theta$*

$$\lim_{\theta \to 0} \mu(\Omega^{\#}, [g_\theta]; M^{\#}, [h_\theta]) \geq \min(\mu(\Omega, [g]; M, [h]), \alpha_{n,k}). \tag{3}$$

If in addition, $n \geq 3$ and $k \leq n-3$, the metrics h_θ coincide with the metrics given by Theorem 1.1. In other words, there exists a constant $\beta_{n,k} > 0$ depending only on n and k (the same as in Theorem 1.1) such that

$$\lim_{\theta \to 0} \mu(M^{\#}, [h_\theta]) \geq \min(\mu(M, [h]), \beta_{n,k}). \tag{4}$$

Moreover, for $k = 0$, we can assume that

$$\alpha_{n,0} = \beta_{n,0} = +\infty. \tag{5}$$

This theorem is an equivalent of Theorem 1.1 for manifolds with boundary. Adapting such surgery results on manifolds with boundary has already be done and Theorem 1.4 is in the spirit of the results in [Ga87], [Da06] or [An08]. A first corollary of our theorem is:

Corollary 1.5. *Let $n \geq 2$ and let Ω be a $(n+1)$-dimensional compact manifold with boundary M and let $\Omega^\sharp$ be obtained by adding a $(k+1)$-dimensional handle on M for some $k \in \{0, \cdots, n-2\}$. Let C be a conformal class on M. We note $M^{\#} = \partial \Omega^{\#}$ which is obtained from M by a surgery of dimension k. Then there exists a conformal class $C^{\#}$ on $M^{\#}$ such that*

$$\sigma(\Omega^{\#}; M^{\#}, C^{\#}) \geq \min(\sigma(\Omega; M, C), \alpha_n)$$

where

$$\alpha_n := \min_{k \in \{0, \cdots n-2\}} \alpha_{n,k}$$

and where $\alpha_{n,k}$ is as in Theorem 1.4. If in addition, $n \geq 3$ and $k \neq n-2$, for all $\epsilon > 0$, we can choose $C^{\#}$ so that

$$\mu(M^{\#}, C^{\#}) \geq \min(\mu(M, C), \beta_n) - \epsilon$$

where

$$\beta_n := \min_{k \in \{0, \cdots n-3\}} \beta_{n,k}$$

and where $\beta_{n,k}$ is as in Theorem 1.1.

Among immediate consequence of Corollary 1.5, we can observe that since $\alpha_n, \beta_n > 0$, we obtain a new proof of main Theorem in [Ga87]. Note that there was a gap in the proof of Gajer which was repaired by Walsh [Wa08]. Another consequence of Corollary 1.5 is the main result concerning relative Yamabe invariant in Schwartz [Sc08] which in particular implies that handlebodies have maximal relative Yamabe invariant among manifolds with boundary.
We now explain a subtler consequence of our results. Let g, g' be metrics with positive scalar curvature on M. We say that that g, g' are *conformally concordant* or 0-*concordant* if there exists a metric G on $M \times [0,1]$ conformal to a metric with positive scalar curvature for which $\partial M \times [0,1]$ is minimal and such that $\partial[G] = [g] \amalg [g']$. It follows from [AB02a] that "to be concordant" is an equivalence relation. The set of equivalence classes is denoted by $\mathrm{Conc}_0(M)$.
We now define

$$\sigma'' := \left| \begin{array}{ccc} \mathrm{Conc}_0(M) & \to &]\infty, \sigma(M)] \\ C & \mapsto & \min\left(\sup_{g \in C}(\mu(M, [g])), \beta_n\right) \end{array} \right.$$

where β_n is as in Corollary 1.5 so that

$$\min(\sigma(M), \beta_n) = \sup_{C \in \mathrm{Conc}_0(M)} \sigma''(C).$$

A hard open question is to know whether σ is attained or not. A first step in this direction could be to study whether the supremum above is attained or not. This is the main motivation here to introduce σ''. As an application of Theorem 1.4, we prove in Section 3

Theorem 1.6. *Let M, N be compact n-manifolds such that N is obtained from M by a finite sequence of surgeries of dimension $k \in \{2, \cdots, n-3\}$. Then*

$$\sigma''(\mathrm{Conc}_0(M)) = \sigma''(\mathrm{Conc}_0(N)).$$

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2. Surgery

In this section, we prove Theorem 1.4. In this goal, we give some basic facts on the double of manifolds with boundary which will be used later. We also give the definitions of surgery and attachment of handles. The last Paragraph 2.5 is devoted to the proof of two lemmas which will be helpful in Section 3.

2.1. **The double of a manifold with boundary.** Let Ω be a compact $(n+1)$-dimensional manifold with boundary M. The *double of* M is the compact manifold without boundary $X := \Omega \cup_M \Omega$ obtained by gluing two copies of Ω along their common boundary. Let g be a metric on Ω and let $h := \partial g$ be the induced metric on the boundary M. Assume that g is a product metric near the boundary M, i.e. that g has the form $g = h + ds^2$, s being the distance to M. Then g extends naturally to a smooth metric $\bar{g} := g \cup g$ on X. We will need the following basic results:

Proposition 2.1. *Let $u \in C^\infty(\Omega)$, $u \geq 0$ be a non-negative function which satisfies:*

$$\begin{cases} L_g u = \lambda u^p & \text{on} \quad \overset{\circ}{\Omega} \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega = M, \end{cases}$$

for some $\lambda \in \mathbb{R}$ and some $p \geq 1$. The function $\bar{u} = u \cup u$ is smooth on X.

Proof. Just notice that $\bar{u} \in C^1(M)$ and satisfies $L_{\bar{g}} \bar{u} = \lambda \bar{u}^p$ weakly on X. Then, $\bar{u} \in C^\infty(X)$ by standard elliptic regularity theorems. □

Proposition 2.2. *We have*

$$2^{\frac{2}{n+1}} \mu(\Omega, [g]; M, [h]) = \inf J^{n+1}_{X,\bar{g}}(u)$$

where the infimum runs over the non-zero functions $u \in C^\infty(X)$ such that $\partial_\nu u \equiv 0$ on M, ν being any normal vector field on M.

The proof easily follows from (1), (2) and the fact that the mean curvature H_g vanishes on M.

2.2. **Surgeries and attachments of handles.** Let M be a n-dimensional manifold and let k be an integer such that $0 \leq k \leq n-1$. We assume that an embedding $f : S^k \times B^{n-k} \to \overset{\circ}{M}$ is given. Then, $M \setminus f(S^k \times B^{n-k})$ is a manifold whose boundary is diffeomophic to $S^k \times S^{n-k-1}$. We then construct

$$M^\#_f := (M \setminus f(S^k \times B^{n-k}) \cup_{f(S^k \times S^{n-k-1})} (\overline{B^{k+1}} \times S^{n-k-1}).$$

We say that $M^\#_f$ *is obtained from M by a surgery of dimension k along $f(S^k \times B^{n-k})$*. If M has a boundary, we say that $M^\#_f$ *is obtained from M by an interior surgery of dimension k* to emphasise the fact that nothing happens on the boundary.

Remark 2.3. Observe that

$$M = M^\#_f \setminus (B^{k+1} \times S^{n-k-1}) \cup_{f(S^k \times S^{n-k-1})} \overline{f(S^k \times B^{n-k})}.$$

In particular, M is obtained from $M^\#_f$ by a $(n-1-k)$-surgery that we will call the *dual surgery* of the surgery given by f.

Now, let Ω be a $(n+1)$-dimensional differentiable manifold whose boundary is M and attach the disk $D^{k+1} := \overline{B^{k+1}} \times \overline{B^{n-k}}$ along $f(S^k \times B^{n-k}) \subset M$ using f. Smoothing the corners, we get a new manifold

$$\Omega^\#_f := \Omega \cup_f (\overline{B^{k+1}} \times \overline{B^{n-k}})$$

with $\partial\Omega^\#_f = M^\#_f$. We say that $\Omega^\#_f$ *is obtained from Ω by attachment of a handle of dimension $k+1$*. The handle corresponding to the dual surgery of the one given

by f will be called the *the dual handle* of D^{k+1}

Assume that near the boundary M of Ω we have some trivialisation $\Omega \sim (M\times]-2,0])$. In other words, $M=\partial\Omega$ is identified to $M\times\{0\}$. We define the half-balls and the half-spheres

$$\begin{aligned} B^{m+1}_- \text{ (resp. } B^{m+1}_+) &:= \{(y_1,\cdots,y_{m+1})\in B^{m+1}\subset\mathbb{R}^{m+1} \mid y_{m+1}\leq 0(\text{ resp. } \geq 0)\} \\ S^m_- \text{ (resp. } S^m_+) &:= \{(y_1,\cdots,y_{m+1})\in S^{m}\subset\mathbb{R}^{m+1} \mid y_{m+1}\leq 0(\text{ resp. } \geq 0)\}. \end{aligned}$$

and we set

$$F : \left| \begin{array}{ccc} S^k\times\overline{B^{n+1-k}_-} & \to & M\times]-2,0] \\ (x,(y_1,\cdots,y_{n+1-k})) & \mapsto & (f(x,y'),y_{n+1-k}) \end{array} \right.$$

where $y'=(y_1,\cdots,y_{n-k})\in B^{n-k}$. Clearly, F is a smooth embedding in Ω such that $F_{/S^k\times B^{n-k}}=f$ where B^{n-k} is seen as a subset of B^{n+1-k}_- writing that $B^{n-k}=\{(y_1,\cdots y_{n+1-k})\in B^{n+1-k}_- | y_{n+1-k}=0\}$.
Set now

$$\tilde{\Omega}_F := (\Omega\setminus F(S^k\times\overline{B^{n+1-k}_-}))\cup(\overline{B^{k+1}}\times S^{n-k}_-)_{/\sim}$$

where $\sim$ means that we glue the boundaries. It is straightforward to see that $\tilde{\Omega}_F$ is diffeomorphic to $\Omega^{\#}_f$. In this way, attaching a handle is view as a "half-surgery" on Ω. This was also the viewpoint adopted by Ole Andersonn [An08] in his thesis.

2.3. **Connected sum along a submanifold of manifolds with boundary.** Assume first that (M_1,h_1) and (M_2,h_2) are Riemannian manifolds without boundary of dimension n and that W is a compact manifold of dimension k. Let embeddings $W\hookrightarrow M_1$ and $W\hookrightarrow M_2$ be given. We assume further that the normal bundles of these embeddings are trivial. Removing tubular neighborhoods of the images of W in M_1 and M_2, and gluing together these manifolds along their common boundary, we get a new compact manifold $M^{\#}:=M_1\cup_W M_2$, called *the connected sum of M_1 and M_2 along W*. Notice that $M^{\#}$ depends on the trivialisation of the normal bundles. Surgery as explained in last paragraph 2.2 is a special case of this construction: if $M_2=S^n$, $W=S^k$ and if $S^k\hookrightarrow S^n$ is the standard embedding, then $M^{\#}$ is obtained from M_1 from a k-dimensional surgery along $S^k\hookrightarrow M_1$. For more informations on this construction, see [ADH08].

In this paper, we need to adapt this construction to the case of manifolds with boundary. Let (Ω_1,g_1), (Ω_2,g_2) be $(n+1)$-dimensional Riemannian manifolds with respective boundaries M_1 and M_2. We denote by h_i $(i=1,2)$ the trace of g_i on M_i i.e. $\partial g_i=h_i$. Let W be a compact manifold of dimension k. If W embedds in $\overset{\circ}{\Omega}_1$ and $\overset{\circ}{\Omega}_2$, then we can proceed exactly as in the case of manifolds without boundary explained above and we obtain a new manifold $\Omega^{\#}:=\Omega_1\cup_W\Omega_2$ called again the connected sum of Ω_1 and Ω_2 along W. Obviously, $\partial\Omega^{\#}=M_1\amalg M_2$. In the case where $\Omega_2=S^n$, $W=S^k$ and if $S^k\hookrightarrow S^n$ is the standard embedding, then $\Omega^{\#}$ is obtained from Ω_1 by an interior k-dimensional surgery.
Now, assume that W embedds into the boundaries M_i of Ω_i. Let us make it precise now. We assume that some smooth embeddings $\bar{w}_i : W\times\mathbb{R}^{n+1-k}\to T\Omega_i$ $i=1,2$ are given. In what follows, we identify $\mathbb{R}^{n-k}$ with $\mathbb{R}^{n-k}\times\{0\}\subset\mathbb{R}^{n+1-k}$. We make the following additional assumptions of $\bar{w}_i$:

- First, we assume that $\bar{w}_i$ restricted to $W \times \mathbb{R}^{n-k}$ embedds in $TM_i \subset T\Omega_i$.
- Then, we want that $\bar{w}_i$ restricted to $W \times \{0\}$ maps to the zero section of TM_i (which we identify with M_i) and thus gives an embedding $W \hookrightarrow M_i \subset \Omega_i$. The image of this embedding is denoted by W_i'.
- Further we assume that $\bar{w}_i$ restrict to linear isomorphisms $\alpha_p : \{p\} \times \mathbb{R}^{n+1-k} \to N_{\bar{w}_i(p,0)}W_i'$ for all $p \in W$. Here NW_i' denotes the normal bundle of W_i' defined using g_i. In addition, we assume that α_p restricted to $\{p\} \times \mathbb{R}^{n-k}$ is an isomorphism onto $N_{\bar{w}_i(p,0)}W_i' \cap TM_i$. We can assume also that $\bar{w}_i(\{p\} \times (0, \cdots, 0, 1))$ denotes the outer normal unit vector at p.

Now, we set $w_i := \exp^{g_i} \circ \bar{w}_i$. This gives embeddings $w_i : W \times B_-^{n+1-k}(R_{\max}) \to \Omega_i$ for some $R_{\max} > 0$ and for $i = 1, 2$. We have $W_i' = w_i(W \times \{0\})$. We obtain a new manifold with boundary $\Omega^{\#}$ by gluing $\Omega_1 \setminus (w_1(W \times B_-^{n+1-k}(R_{\max})))$ and $\Omega_2 \setminus (w_2(W \times B_-^{n+1-k}(R_{\max})))$ along $w_i(W \times S^{n-k})$. This manifold is again called *the connected sum of* Ω_1 *and* Ω_2 *along* W. Let $M^{\#} := \partial\Omega$. Then, $M^{\#}$ is the connected sum of M_1 and M_2 along W as explained above.
In the special case that (Ω_2, g_2) is the half-sphere $\mathbb{S}_+^{n+1}$ (and hence M_2 is the standard n-dimensional sphere) and that $W = \mathbb{S}^k \subset \mathbb{S}^n = \partial\mathbb{S}_+^{n+1}$, then one can verify that the resulting manifold $\Omega^{\#}$ is obtained from Ω_1 by attachment of a $(k+1)$-dimensional handle as explained in paragraph 2.2 and hence, $M^{\#}$ is obtained from M_1 by a surgery of dimension k.
In what follows, we assume that the metrics g_i have a product form $h_i + ds_i^2$ near the boundaries M_i. We define the disjoint union

$$(\Omega, g) := (\Omega_1 \amalg \Omega_2, g_1 \amalg g_2),$$

$$(M, h) := (M_1 \amalg M_2, h_1 \amalg h_2)$$

and

$$W' := W_1' \amalg W_2'.$$

Let r_i be the function on Ω_i giving the distance to W_i' associated to the metric g_i. Since the metric g_i has the product form $h_i + ds_i^2$ near M_i, we have

$$r_i^2 = s_i^2 + (\mathrm{dist}_{h_i}(\cdot, W_i'))^2. \tag{6}$$

We also have $r_1 \circ w_1(p, x) = r_2 \circ w_2(p, x) = |x|$ for $p \in W$, $x \in B_-^{n+1-k}(R_{\max})$. Let r be the function on M defined by $r(x) := r_i(x)$ for $x \in M_i$, $i = 1, 2$. For $\epsilon > 0$ we set $U_i(\epsilon) := \{x \in M_i : r_i(x) < \epsilon\}$ and $U(\epsilon) := U_1(\epsilon) \cup U_2(\epsilon)$. For $0 < \epsilon < \theta$ we define

$$\Omega_\epsilon^{\#} := (\Omega_1 \setminus U_1(\epsilon)) \cup (\Omega_2 \setminus U_2(\epsilon))/\sim,$$

and

$$U_\epsilon^{\Omega^{\#}}(\theta) := (U(\theta) \setminus U(\epsilon))/\sim$$

where $\sim$ indicates that we identify $x \in \partial U_1(\epsilon)$ with $w_2 \circ w_1^{-1}(x) \in \partial U_2(\epsilon)$. Hence

$$\Omega_\epsilon^{\#} = (\Omega \setminus U(\theta)) \cup U_\epsilon^{\Omega^{\#}}(\theta).$$

We say that $\Omega_\epsilon^{\#}$ is obtained from M_1, M_2 (and $\bar{w}_1$, $\bar{w}_2$) *by a connected sum along* W *with parameter* ϵ.
The diffeomorphism type of $\Omega_\epsilon^{\#}$ is independent of ϵ, hence unless when the parameter ϵ is needed, we will usually write $\Omega^{\#} = \Omega_\epsilon^{\#}$.

2.4. **Surgery and Yamabe invariants.**

2.4.1. *Statement of the results.* First, we will need the following theorem due to Gromov-Lawson and Schoen-Yau ([GL80] and [SY79]) and which can be also deduced from the [ADH08]:

Theorem 2.4. *Let (Ω, G) be a compact Riemannian manifold of dimension greater than 3 with boundary M and let $\Omega^{\#}$ be obtained from M by an interior surgery of codimension at least 3. Assume that $\mu(\Omega, [G]; M, \partial[G]) > 0$. Then, there exists on $\Omega^{\#}$ a metric $G^{\#}$ equal to G in a neighborhood of $M = \partial\Omega = \partial\Omega^{\#}$ such that $\mu(\Omega^{\#}, [G^{\#}]; M, \partial[G]) > 0$.*

Let us deal now with the case where W embedds in the boundary. We prove the following result which in view of Paragraph 2.3 is stronger than Theorem 1.4.

Theorem 2.5. *Let $n \in \mathbb{N}$, $n \geq 2$ and (Ω_1, g_1), (Ω_2, g_2) be compact $(n+1)$-dimensional Riemannian manifolds with respective boundaries M_1 and M_2 and set $h_i = \partial g_i$. Let also W be a a compact manifold without boundary of dimension $k \in \{0, \cdots, n-2\}$ that embedds in M_i (see paragraph 2.3). Let $\Omega^{\#}$ be the connected sum of Ω_1 and Ω_2 along W and set $M^{\#} := \partial\Omega^{\#}$. Then, there exists some constants $\alpha_{n,k} > 0$ depending only on n and k with $\alpha_{n,0} = +\infty$ and a sequence of metrics $(g_\theta)_{\theta>0}$ on $\Omega^{\#}$ equal to $g = g_1 \amalg g_2$ except in a small neigbourhood of W (if U is a small neighborhood of W, we see $W \setminus U$ as embedded in $\Omega^{\#}$) such that, if we note $h_\theta = \partial g_\theta$,*

$$\lim_{\theta\to 0} \mu(\Omega^{\#}, [g_\theta]; M^{\#}, [h_\theta]) \geq \min(\mu(\Omega_1, g_1; M_1, [h_1]), \mu(\Omega_2, [g_2]; M_2, [h_2]), \alpha_{n,k}). \quad (7)$$

If in addition, $n \geq 3$ and $k \leq n-3$, the metrics $h_\theta := \partial g_\theta$ coincides with the metrics given by Theorem 2.3 in [ADH08]*. In other words, there exists a constant $\beta_{n,k} > 0$ (the same as in Theorem 1.1) with $\beta_{n,0} = +\infty$ such that*

$$\lim_{\theta\to 0} \mu(M^{\#}, [\partial g_\theta]) \geq \min(\mu(M_1, [h_1]), \mu(M_2, [h_2]), \beta_{n_k}. \quad (8)$$

Moreover, for $k = 0$, we have

$$\alpha_{n,0} = \beta_{n,0} = +\infty.$$

2.4.2. *Proof of Theorem 2.5.* We use the notations of Paragraph 2.3. We recall the notations $\Omega = \Omega_1 \amalg \Omega_2$, $M = M_1 \amalg M_2$, $W' = W'_1 \amalg W'_2$ and $g = g_1 \amalg g_2$. We also use the notation $h := \partial g$. If (g_m) is a sequence of metric which converges toward a metric g_∞ in $C^0(\Omega)$ and if Scal_{g_m} converges also in C^0 to Scal_{g_∞} then $\mu(\Omega, [g_m]; \partial\Omega, [\partial g_m]))$ tends to $\mu(\Omega, [g_\infty]; \partial\Omega, [\partial g_\infty])$ (see Proposition 4.31 of Bérard-Bergery in [Be8702] and Lemma 4.1 in [AB02a]). Theorem 4.6 in [AB02a] or the results of Carr [Ca88] then imply that we can choose a metric $\tilde{g}$ on Ω such that:

- $\partial\tilde{g} = \partial g = h$,
- $\tilde{g} = h + ds^2$ in a neighborhood of M (where $s = s_i$ on M_i with s_i defined as in the end of Paragraph 2.3),
- $\mu(\Omega, [\tilde{g}]; M, [h])$ is as close as desired to $\mu(\Omega, [g]; M, [h]) = \min(\mu(\Omega_1, [g_1]; M_1, [h_1]), \mu(\Omega_2, [g_2]; M_2, [h_2]))$.

Then, without loss of generality, we can replace g by $\tilde{g}$ so that the metric has now the above properties. The desired sequence (g_θ) of metrics will be constructed as in [ADH08]. We now explain how this construction can be adapted here. In the following, C denotes a constant that might change its value between lines. We

denote by h'_i the restriction of g_i to $TW' = T(W'_1 \amalg W'_2)$ over $W' \subset \Omega$. As already explained, the normal exponential map of W' defines a diffeomorphism

$$w_i : W \times B^{n+1-k}_-(R_{\max}) \to U_i(R_{\max}), \quad i = 1, 2,$$

which decomposes $U(R_{\max}) = U_1(R_{\max}) \amalg U_2(R_{\max})$ as a product $W' \times B^{n+1-k}_-(R_{\max})$. In general the Riemannian metric g does not have a corresponding product structure, and we introduce an error term T measuring the difference from the product metric. If r denotes the distance function to W', then the metric g can be written on $U(R_{\max}) \setminus W' \cong W' \times (0, R_{\max}) \times S^{n-k}_-$ as

$$g = h' + \xi^{n+1-k} + T = h' + dr^2 + r^2\sigma^{n-k} + T. \tag{9}$$

where h' is the restriction of g on TW', T is a symmetric $(2,0)$-tensor vanishing on W' (in the sense of sections of $(T^*\Omega \otimes T^*\Omega)|_{W'}$). Note that since g is a product near the boundary,

$$T(v, \cdot) = 0 \tag{10}$$

for all vector v normal to M. We also define the product metric

$$g' := h' + \xi^{n+1-k} = h' + dr^2 + r^2\sigma^{n-k}, \tag{11}$$

on $U(R_{\max}) \setminus W'$. Thus $g = g' + T$. We define $T_i := T|_{\Omega_i}$ for $i = 1, 2$.
For a fixed $R_0 \in (0, R_{\max})$ we choose a smooth positive function $F : \Omega \setminus W' \to \mathbb{R}$ such that

$$F(x) = \begin{cases} 1, & \text{if } x \in \Omega_i \setminus U_i(R_{\max}); \\ r_i(x)^{-1}, & \text{if } x \in U_i(R_0) \setminus W'. \end{cases}$$

Next we choose small numbers $\theta, \delta_0 \in (0, R_0)$ with $\theta > \delta_0 > 0$. Here "small" means that we first choose a sequence $\theta = \theta_j$ of small positive numbers tending to zero, such that all following arguments hold for all θ. Then we choose for any given θ a number $\delta_0 = \delta_0(\theta) \in (0, \theta)$ such that all arguments which need δ_0 to be small will hold,

For any $\theta > 0$ and sufficiently small δ_0 there is $A_\theta \in [\theta^{-1}, (\delta_0)^{-1})$ and a smooth function $f : U(R_{\max}) \to \mathbb{R}$ depending only on the coordinate $r = \mathrm{dist}_g(\cdot, W')$ such that

$$f(x) = \begin{cases} -\ln r(x), & \text{if } x \in U(R_{\max}) \setminus U(\theta); \\ \ln A_\theta, & \text{if } x \in U(\delta_0), \end{cases}$$

and such that

$$\left| r\frac{df}{dr} \right| = \left| \frac{df}{d(\ln r)} \right| \leq 1, \quad \text{and} \quad \left\| r\frac{d}{dr}\left(r\frac{df}{dr} \right) \right\|_{L^\infty} = \left\| \frac{d^2 f}{d^2(\ln r)} \right\|_{L^\infty} \to 0 \tag{12}$$

as $\theta \to 0$. We set $\epsilon = e^{-A_\theta}\delta_0$. We can and will assume that $\epsilon < 1$. Let $\Omega^\#$ be obtained from Ω by a connected sum along W with parameter ϵ, as described in Paragraph 2.3. In particular, $U^{\Omega^\#}_\epsilon(s) = U(s) \setminus U(\epsilon)/\sim$ for all $s \geq \epsilon$. On the set $U^{\Omega^\#}_\epsilon(R_{\max}) = U(R_{\max}) \setminus U(\epsilon)/\sim$ we define the variable t by

$$t := \begin{cases} -\ln r_1 + \ln \epsilon, & \text{on } U_1(R_{\max}) \setminus U_1(\epsilon); \\ \ln r_2 - \ln \epsilon, & \text{on } U_2(R_{\max}) \setminus U_2(\epsilon). \end{cases}$$

We can assume that $t : U_\epsilon^{\Omega^\#}(R_{\max}) \to \mathbb{R}$ is smooth. We choose a cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi = 0$ on $(-\infty, -1]$, $|d\chi| \leq 1$, and $\chi = 1$ on $[1, \infty)$. With these choices, we define

$$g_\theta := \begin{cases} F^2 g_i, & \text{on } \Omega_i \setminus U_i(\theta); \\ e^{2f(t)}(h_i' + T_i) + dt^2 + \sigma^{n-k}, & \text{on } U_i(\theta) \setminus U_i(\delta_0); \\ A_\theta^2 \chi(t/A_\theta)(h_2' + T_2) + A_\theta^2(1 - \chi(t/A_\theta))(h_1' + T_1) & \\ \quad + dt^2 + \sigma^{n-k}, & \text{on } U_\epsilon^{\Omega^\#}(\delta_0). \end{cases}$$

It remains to proves that the sequence (g_θ) satisfies the desired conclusions. Set $h_\theta := \partial g_\theta$. First of all, we prove that $M^\#$ is minimal for the metrics g_θ. Let $p \in M^\#$. Assume first that $p \in \Omega_i \setminus U_i(\theta)$. Note that the function F depends only on the coordinate r. We denote by ν the outer normal unit vector at p. Formula (6) then implies that $\partial_\nu r \equiv 0$ on $M \setminus W'$ and hence $\partial_\nu F(p) = 0$. This implies that the mean curvature vanishes at p. Assume now that $p \in U_i(\theta) \setminus U_i(\delta_0) \cup U_\epsilon^{\Omega^\#}(\delta_0) = U_\epsilon^{\Omega^\#}(\theta)$. Observe that by Relation (10), the metric g_θ has the form

$$g_\theta = H_1 + \alpha(r) H_2 + dt^2 + \sigma^{n-k}$$

where H_i are 2-forms satisfying $H_i(\nu, \cdot) \equiv 0$ and where $\alpha(r)$ is a function depending only on r and θ. Set $r_\theta := \mathrm{dist}_{h_\theta}(W', \cdot)$. Then, $dt^2 = \frac{1}{r^2} dr^2 = \frac{1}{r^2}(r_\theta^2 + ds^2)$. Since $\partial_\nu r \equiv 0$, we easily get that the mean curvature vanishes at p and hence $M^\#$ is minimal.

Assume for a while that $k \leq n-3$. Observe that since g is a product metric near M, the function $r = \mathrm{dist}_g(\cdot, W')$ coincides with $\mathrm{dist}_h(\cdot, W')$ on the boundary. Consequently, the metric h_θ on $M^\#$ is exactly the same than the one constructed in the proof of Theorem 2.3 of [ADH08] which then shows that Relation (8) holds. Let us come back to the general case $k \in \{0, \cdots, n-2\}$ and let us show Relation (7). Let us denote by $X_i := \Omega_i \cup_{M_i} \Omega_i$ $(i = 1, 2)$ (resp. $X^\# := \Omega^\# \cup_{M^\#} \Omega^\#$) the double of Ω_i (resp. $\Omega^\#$). Notice that $X^\#$ is the connected sum of X_1 and X_2 along W. We define on X_i (resp. $X^\#$) the metric $\bar{g}_i = g_i \cup g_i$ (resp. $\bar{g}_\theta = g_\theta \cup g_\theta$) as in Paragraph 2.1. Set also $X := X_1 \amalg X_2$ and $\bar{g} := \bar{g}_1 \amalg \bar{g}_2$. The manifold X is then the double of Ω. Clearly, we can assume that $\mu(\Omega^\#, [g_\theta]; M^\#, [h_\theta]) < \mu(\mathbb{S}_+^{n+1}, \mathbb{S}^n)$ otherwise the proof is done. By Theorem 1.2, there exists a function $u_\theta \in C^\infty(\Omega^\#)$, $u_\theta > 0$ normalized by

$$\int_{\Omega^\#} u_\theta^{\frac{2(n+1)}{n-1}} dv_{g_\theta} = 1$$

which satisfies

$$\begin{cases} L_{g_\theta} u_\theta = \lambda_\theta u_\theta^{\frac{n+3}{n-1}} & \text{on} \quad (\overset{\circ}{\Omega^\#}) \\ \partial_\nu u_\theta = 0 & \text{on} \quad M^\#, \end{cases}$$

where $\lambda_\theta = \mu(\Omega^\#, [g_\theta]; M^\#, [h_\theta])$. By possibly taking a subsequence, we can assume that $\lambda_\infty := \lim_{\theta \to 0} \lambda_\theta \in [-\infty, \mu(\mathbb{S}_+^{n+1}, \mathbb{S}^n)]$ exists.
Define

$$\bar{u}_\theta := \frac{u_\theta \cup u_\theta}{2^{\frac{n-1}{2(n+1)}}} = \frac{u_\theta \cup u_\theta}{\|u_\theta \cup u_\theta\|_{L^{\frac{2(n+1)}{n-1}}(X^\#)}}$$

on $X^\#$. Then,

$$\int_{X^\#} \bar{u}_\theta^{\frac{2(n+1)}{n-1}} dv_{\bar{g}_\theta} = 1.$$

Proposition 2.1 implies that $\bar{u}_\theta$ is smooth on $X^\#$ and satisfies

$$L_{\bar{g}_\theta}\bar{u}_\theta = 2^{-\frac{n-1}{2(n+1)}}\lambda_\theta(u_\theta \cup u_\theta)^{\frac{n+3}{n-1}} = 2^{\frac{2}{n+1}}\lambda_\theta \bar{u}_\theta^{\frac{n+3}{n-1}}.$$

The idea now is to see how the proof of Theorem 2.3 in [ADH08] can be adapted to this situation. The first observation is that the metric $(X^\#, \bar{g}_\theta)$ is construted from $(X, \bar{g})$ exactly in the same way than (N, g_θ) is constructed from (M, g) in [ADH08]. We deduce immediatly that

$$2^{\frac{2}{n+1}}\lambda_\infty \geq \min(\mu(X, [\bar{g}]), \beta_{n+1,k})$$

where $\beta_{n,k}$ is as in Theorem 1.1. The problem here is to get a lower bound of $\mu(X, [\bar{g}])$ in terms of $\mu(\Omega, [g]; M, [h])$ which seems difficult without additional assumptions. So we have to go through the proof in [ADH08] a little more deeply. Observe that it is divided in many cases. The only case which is an issue is Subcase II.1.2. Indeed, in other cases, we obtain that $2^{\frac{2}{n+1}}\lambda_\infty \geq \beta_{n+1,k}$ and we just set $\alpha_{n,k} := 2^{-\frac{2}{n+1}}\beta_{n+1,k}$ to get Theorem 2.5. So assume now that assumptions of Subcase II.1.2 occur. More precisely, we assume, using the notations of Paragraph (2.3) that:

$$\lim_{b\to 0}\limsup_{\theta\to 0}\sup_{U_\epsilon^{X^\#}(b)} \bar{u}_\theta = 0$$

where

$$U_\epsilon^{X^\#}(b) := U_\epsilon^{\Omega^\#}(b) \cup_{\partial U_\epsilon^{\Omega^\#}(b)\cap\partial\Omega^\#} U_\epsilon^{\Omega^\#}(b).$$

We then mimick the proof of [ADH08]. Let $d_0 > 0$. We can choose a $b > 0$ such that

$$\int_{X^\#\setminus U_\epsilon^{X^\#}(2b)} \bar{u}_\theta^{\frac{2(n+1)}{n-1}} dv_{\bar{g}_\theta} \geq 1 - d_0$$

and

$$\int_{U_\epsilon^{X^\#}(2b)\setminus U_\epsilon^{X^\#}(b)} \bar{u}_\theta^2 dv_{\bar{g}_\theta} \leq d_0.$$

Then we choose a cut-off function $\eta \in C^\infty(X^\sharp)$, $0 \leq \eta \leq 1$ depending only on t (clearly the function t can be naturally extended smoothly to $X^\#$) equal to 0 on $U_\epsilon^{X^\#}(b)$, equal to 1 on $X^\# \setminus U_\epsilon^{X^\#}(2b)$ and which satisfies $|d\xi|_{\bar{g}_\theta} \leq 2\ln(2)$. Then, as in [ADH08], we obtain that

$$J^{n+1}_{X,\bar{g}}(\chi\bar{u}_\theta) \leq \frac{2^{\frac{2}{n+1}}\lambda_\theta + |2^{\frac{2}{n+1}}\lambda_\theta|d_0 + 4(\ln(2))^2 a d_0}{(1-d_0)^{\frac{n-1}{n+1}}}$$

where $a = \frac{4n}{n-1}$. Since χ depends only on t and hence of r, observe that the function $\chi\bar{u}_\theta$ has normal derivative vanishing on the minimal hypersurface $M^\# \subset X^\#$. By Proposition 2.2, we obtain that

$$2^{\frac{2}{n+1}}\mu(\Omega, [g]; M, [h]) \leq J^{n+1}_{X,\bar{g}}(\chi\bar{u}_\theta)$$

and hence, letting d_0 tends to zero,

$$\mu(\Omega, [g]; M, [h]) \leq \lambda_\infty.$$

This proves Theorem 2.5.

2.5. **Surgery on cylinders.**

2.5.1. *Statements of the results.* Let M, N be a compact n-dimensional manifold without boundary. Assume that N is obtained from M by a surgery of dimension $k \in \{0, \cdots, n-1\}$ associated to an embedding $f : S^k \times B^{n-k} \hookrightarrow M$. Let $\Omega = M \times [0,1]$. Attaching on Ω two $(k+1)$-dimensional handles along $f(S^k \times B^{n-k}) \times \{0,1\}$, we get a new manifold Ω' whose boundary is $N \amalg N$ (see Paragraph 2). We prove:

Lemma 2.6. *The manifold $N \times [0,1]$ is obtained from Ω' by an interior $(k+1)$-dimensional surgery.*

Start again with $\Omega = M \times [0,1]$. Let Ω'' be obtained from attaching first a k-dimensional handle on Ω along $f(S^k \times B^{n-K}) \times \{0\}$ and then attaching the dual handle along $N \subset \left(\Omega \cup_{f(S^k \times B^{n-K}) \times \{0\}} \overline{B^{k+1}} \times \overline{B^{n-k}}\right)$. The new manifold Ω'' has a boundary $M \amalg M$. We prove

Lemma 2.7. *The manifold $M \times [0,1]$ is obtained from Ω'' by an interior $(n-k)$-dimensional surgery.*

Remark 2.8. If $k = 0$ Lemma 2.7, then by standard surgery theory, the interior n-dimensional surgery can be replaced by an interior surgery of dimension 1.

2.5.2. *Proof of Lemma 2.6.* The manifold Ω' is equal to

$$\Omega' = \left(B^{k+1} \times B^{n-k}\right) \cup_{f(S^k \times B^{n-k}) \times \{0\}} \Omega \cup_{f(S^k \times B^{n-k}) \times \{1\}} \left(B^{k+1} \times B^{n-k}\right).$$

We define

$$\begin{aligned} W \quad := \quad & \left(B^{k+1} \times B^{n-k}\left(\frac{1}{2}\right)\right) \cup_{f(S^k \times B^{n-k}\left(\frac{1}{2}\right)) \times \{0\}} \\ & \left(f\left(S^k \times B^{n-k}\left(\frac{1}{2}\right)\right) \times [0,1]\right) \cup_{f\left(S^k \times B^{n-k}\left(\frac{1}{2}\right)\right) \times \{1\}} \\ & \left(B^{k+1} \times B^{n-k}\left(\frac{1}{2}\right)\right) \subset (\overset{\circ}{\Omega'}). \end{aligned}$$

Let $m \in \mathbb{N}$. Observe that

$$B^{m+1} \cup_{S^m \times \{0\}} (S^m \times [0,1]) \cup_{S^m \times \{1\}} B^{m+1} \simeq S^{m+1}. \tag{13}$$

Here, $\simeq$ means diffeomorphic. Hence, $W \simeq S^{k+1} \times B^{n-k}$.
Define

$$\begin{aligned} W' \quad := \quad & \left(B^{k+2}_+ \times S^{n-k-1}\right) \cup_{B^{k+1} \times S^{n-k-1} \times \{0\}} \\ & \left(B^{k+1} \times S^{n-k-1} \times [0,1]\right) \cup_{S^{k+1} \times S^{n-k-1} \times \{1\}} \\ & \left(B^{k+2}_+ \times S^{n-k-1}\right). \end{aligned}$$

Note that $\partial B^{k+2}_+ = S^{k+1}_+ \cup_{S_k} B^{k+1}$ hence W' is well defined. For $m \in \mathbb{N}$, let us note that

$$B^{m+1}_+ \cup_{B^m \times \{0\}} (B^m \times [0,1]) \cup_{B^m \times \{1\}} B^{m+1}_+ \simeq B^{m+1}.$$

Hence, $W' \simeq B^{k+2} \times S^{n-k-1}$ and if we define

$$\Omega^{\#} := (\Omega' \setminus W) \cup W'$$

where we glue the boundaries, $\Omega^{\#}$ is obtained from Ω' by an interior $(k+1)$-dimensional surgery along W.
Define

$$\begin{aligned} H &:= \left(B^{k+1}\times B^{n-k}\right)\setminus\left(B^{k+1}\times B^{n-k}\left(\frac{1}{2}\right)\right)\cup_{B^{k+1}\times S^{n-k-1}\left(\frac{1}{2}\right)\simeq S^{k+1}_{+}\times S^{n-k-1}} \\ &\quad \left(B^{k+2}_{+}\times S^{n-k-1}\right) \\ &\simeq \left(B^{k+1}\times S^{n-k-1}\left(\frac{1}{2}\right)\times[\frac{1}{2},1]\right)\cup_{B^{k+1}\times S^{n-k-1}\simeq S^{k+1}_{+}\times S^{n-k-1}} \\ &\quad \left(B^{k+2}_{+}\times S^{n-k-1}\right). \end{aligned}$$

Since

$$\left(B^{k+1}\times[\frac{1}{2},1]\right)\cup_{B^{k+1}\times\{\frac{1}{2}\}\simeq S^{k+1}_{+}} B^{k+2}_{+}\simeq B^{k+2}_{+}$$

we see that

$$H\simeq B^{k+2}_{+}\times S^{n-k-1}.$$

Now observe that

$$\Omega^{\#}=H\cup_{B^{k+1}\times S^{n-k-1}\times\{0\}}(N\times[0,1])\cup_{B^{k+1}\times S^{n-k-1}\times\{1\}}H$$

It is not difficult to see that $\Omega^{\#}\simeq N\times[0,1]$. This proves Lemma 2.6.

2.5.3. *Proof of Lemma 2.7.* Let

$$H:=\left(B^{k+1}\times B^{n-k}\right)\cup_{B^{k+1}\times S^{n-k-1}}\left(B^{k+1}\times B^{n-k}\right).$$

We have

$$\partial H=\left(S^{k}\times B^{n-k}\right)\cup_{S^{k}\times S^{n-k-1}}\left(S^{k}\times B^{n-k}\right).$$

Since for all $m\in\mathbb{N}$, $n\geq 1$,

$$B^{m}\cup_{S^{m-1}}B_{m}\simeq S^{m}$$

(by smoothing the corners), we have

$$H\simeq B^{k+1}\times S^{n-k}\ \text{ and }\ \partial H\simeq S^{k}\times S^{n-k}.$$

By construction, Ω'' is equal to

$$\Omega''=\Omega\cup_{f(S^{k}\times B^{n-k})}H.$$

Now, we set

$$\begin{aligned} W &:= \left(B^{k+1}\left(\frac{1}{2}\right)\times B^{n-k}\right)\cup_{B^{k+1}\left(\frac{1}{2}\right)\times S^{n-k-1}}\left(B^{k+1}\left(\frac{1}{2}\right)\times B^{n-k}\right) \\ &\simeq B^{k+1}\left(\frac{1}{2}\right)\times S^{n-k}\subset\overset{\circ}{H}. \end{aligned}$$

We now perform a surgery on Ω'' along W to get a new manifold $\Omega^{\#}$. Then,

$$\Omega^{\#}=\Omega\cup_{f(S^{k}\times B^{n-k})}H^{\#} \tag{14}$$

where

$$\begin{aligned} H^{\#} &\simeq \left(B^{k+1}\times B^{n-k}\right)\setminus\left(B^{k+1}\left(\frac{1}{2}\right)\times B^{n-k}\right)\cup_{S^{k}\times S^{n-k}}\left(S^{k}\times B^{n-k+1}\right) \\ &\simeq \left(\left[\frac{1}{2},1\right]\times S^{k}\times S^{n-k}\right)\cup_{S^{k}\times S^{n-k}}\left(S^{k}\times B^{n-k+1}\right) \\ &\simeq S^{k}\times B^{n-k-1}. \end{aligned}$$

Note again that

$$\partial H^{\#} = \left(S^k \times B^{n-k}\right) \cup_{S^k \times S^{n-k-1}} \left(S^k \times B^{n-k}\right)$$

and the gluing in Formula (14) is along the first $\left(S^k \times B^{n-k}\right)$. Now, it is easy to see that $\Omega^{\#} \simeq \Omega$. This ends the proof of Lemma 2.7.

3. c-CONCORDANT METRICS

Let M be a compact manifold without boundary of dimension $n \geq 3$. Let $\mathcal{R}(M)$ be the set of all Riemannian metrics on M. For all $c \in \mathbb{R}$, we set

$$\mathcal{R}_c(M) = \left\{ g \in \mathcal{R}(M) \middle| \mu(M, [g]) > c \right\}.$$

Let g, h be Riemannian metrics on M and $c \geq 0\mathbb{R}$. We say that h, g *are c-concordant* if $\mu(M \times [0,1]; M \amalg M, [g] \amalg [h]) > 0$ and if $g, h \in \mathcal{R}_c(M)$. If M is oriented and if g, h have positive scalar curvature, then g, h are c-concordant if and only if $g, h \in \mathcal{R}_c(M)$ and there exists a metric G with positive scalar curvature on $M \times [0,1]$ such that the boundary is minimal (see Corollary D in [AB02b]). A consequence of Theorem 5.1 in [AB02a] is the fact that "to be c-concordant" is an equivalence relation. We denote by $\mathrm{Conc}_c(M)$ the set of equivalence classes of concordant metrics. For a metric h on a manifold P, we denote by $[h]_P^c$ its class in $\mathrm{Conc}_c(P)$. If $c, c' \in \mathbb{R}$ are such that $c \leq c'$ and if $h \in \mathcal{R}_{c'}(M) \subset \mathcal{R}_c(M)$, then we clearly have

$$[h]_M^{c'} = [h]_M^c \cap \mathcal{R}_{c'}(M). \tag{15}$$

Let g, h be Riemannian metrics in M. An important well-know fact is the following

$$g, h \text{ are in the same connected component of } \mathcal{R}_0(M) \Longrightarrow [g]_M^0 = [h]_M^0. \tag{16}$$

Lots of works aim to study the sets $\mathcal{R}_0(M)$ and $\mathrm{Conc}_0(M)$ ([Ca88, Ha88, Ha91, RS98, Ru02]). In particular, Gajer proved in [Ga93] very interesting results about the topology and the structures of these sets. The reader may also consult Dahl [Da06] for a nice study of the set of metrics with invertible Dirac operator on spin manifolds.

The goal of this section is to show how Theorem 1.4 can be applied to collect informations on $\mathrm{Conc}_c(M)$ and in particular to prove Theorem 1.6. For this, we need to introduce lds-relative manifolds.

Definition 3.1. Let M_1, M_2 be n-dimensional compact manifolds without boundary. We say that M_1, M_2 are *lds-relative* ("lds" for "low dimensional surgery") if M_2 can be obtained from M_1 with a finite sequence of surgeries of dimension $2 \leq k \leq n-3$.

Remark 3.2.
(1) Remark 2.3 obviously implies that "to be lds-relative" is an equivalence relation. We denote by Γ_n^{lds} the set of equivalence classes of lds-relative n-manifolds.
(2) Let M, N be two compact connected n-manifolds. Assume that there is a 2-connected bordism between M and N. Then, it follows from standard theory that M, N are lds-relative.

An immediate consequence of Theorem 1.1 is the following.

Proposition 3.3. *Let $\beta_n > 0$ be the positive constant defined as in Corollary 1.5. For all compact n-manifold without boundary M, we define $\bar{\sigma}(M) = \min(\sigma(M), \beta_n)$. Then,*

$$\bar{\sigma} : \Gamma_n^{\text{lds}} \longrightarrow]-\infty, n(n-1)\omega_n^{\frac{2}{n}}]$$

where ω_n denotes the volume of the standard n-dimensional sphere, is a well-defined map.

As an application of Theorem 1.4, we prove:

Proposition 3.4. *Let M, N be lds-relative n-manifolds. For all $c \leq \beta_n$ (β_n is as above), there are bijective maps*

$$\Theta_{M,N}^c : \text{Conc}_c(M) \to \text{Conc}_c(N)$$

such that $\Theta_{M,N}^C = (\Theta_{N,M}^c)^{-1}$. In addition, let $c, c' \in \mathbb{R}$ with $c \leq c'$ and let $h \in \mathcal{R}_{c'}(M) \subset \mathcal{R}_c(M)$. Then,

$$\Theta_{M,N}^{c'}([h]_M^{c'}) = \Theta_{M,N}^c([h]_N^c) \cap \mathcal{R}_{c'}(N). \tag{17}$$

Remark 3.5. Let M, N be compact n-manifolds without boundary and assume that N is obtained from M by a surgery of dimension 0. In particular, these condition are satisfied if $M = M_1 \amalg M_2$ and if $N = M_1 \# M_2$ is the connected sum of M_1 and M_2. One can verify that the proof of Proposition 3.4 can be mimicked, unless we use Remark 2.8 instead of Lemma 2.7 to obtain for all c an injective map $\Theta_{M,N}^c : \text{Conc}_c(M) \to \text{Conc}_c(N)$.

For $c = 0$, Proposition 3.4 was already known (see [Ga93]). The proof here is slightly different and uses only basic facts on surgery.

We now define

$$\sigma' := \left| \begin{array}{ccc} \text{Conc}_0(M) & \to &]\infty, \sigma(M)] \\ C & \mapsto & \sup_{g \in C} \mu(M, [g]). \end{array} \right.$$

Clearly,

$$\sup_{C \in \text{Conc}_0(M)} \sigma'(C) = \sigma(M).$$

Let also $\sigma'' := \min(\sigma', \beta_n)$. As an application of Proposition 3.4, we get Theorem 1.6 we recall here:

Corollary 3.6. *Assume that M, N are lds-relative, then*

$$\sigma''(\text{Conc}_0(M)) = \sigma''(\text{Conc}_0(N)).$$

3.1. **Proof of Proposition 3.4.** . We set

$$c_n = \min_{k \in \{0, \cdots, n-3\}} \beta_{n,k} > 0$$

where $\beta_{n,k}$ is the constant which appears in the statement of Theorem 1.4. We fix some $c < c_n$. Let M, N be some compact manifolds and let $g \in \mathcal{R}_c(M)$. Assume that N is obtained from M by a surgery of dimension $k \in \{0, \cdots, n-3\}$. By Theorem 1.1, there exists a sequence of metrics $(g_\theta)_{\theta > 0}$ on N such that for θ small enough (smaller than some $\epsilon > 0$), $g_\theta \in \mathcal{R}_c(N)$. We define

$$\Theta_{M,N}^c := \left| \begin{array}{ccc} \text{Conc}_c(M) & \to & \text{Conc}_c(N) \\ [g]_M^c & \mapsto & [g_\theta]_N^c. \end{array} \right.$$

We have to show that $\Theta^c(M,N)$ is well-defined and is a bijection if M and N are lds-relative. First, let us show that if $0 < \theta_1, \theta_2$ are small enough then

$$[g_{\theta_1}]^c_N = [g_{\theta_2}]^c_N. \tag{18}$$

Let $\Omega := M \times [0,1]$. We equip Ω with the product metric $G = g + dt^2$. We attach the $(k+1)$-dimensional handle to Ω along $M \times \{0\}$ related to the given surgery to obtain a manifold Ω_1 with $\partial\Omega_1 = N \amalg M$. By Theorem 1.4 applied with $g = G$, there exists a sequence of metrics (G^1_θ) on Ω_1 for which the boundary is minimal and such that for θ small, $\mu(\Omega_1, [G^1_\theta]; N \amalg M, \partial[G^1_\theta]) > 0$ and such that $\partial G^1_\theta \in \mathcal{R}_c(N \amalg M)$. By construction,

$$\partial G^1_\theta = g_\theta \amalg g.$$

We choose $\theta = \theta_1$ small enough so that these conditions are satisfied. Now, we attach the $(k+1)$-dimensional handle to Ω along $M \times \{1\}$ related to the given surgery to obtain a manifold Ω_2 with $\partial\Omega_2 = N \amalg N$. Again by Theorem 1.4 applied with $g = G^1_{\theta_1}$, we obtain a sequence of metrics (G^2_θ) on Ω_2 for which the boundary is minimal and such that for θ small, $\mu(\Omega_2, [G^2_\theta]; N \amalg N, \partial[G^2_\theta]) > 0$, such that $\partial G^2_\theta \in \mathcal{R}_c(N \amalg N)$ and by construction,

$$\partial G^2_\theta = g_{\theta_1} \amalg g_\theta.$$

Choose θ_2 small enough such these conditions are satisfied. Note that since the metrics G^1_θ is equal to G near $M \times \{1\}$, the number θ_2 does not depend on the choice of θ_1. Now, by Lemma 2.6, $N \times [0,1]$ is obtained from Ω_2 by a $(k+1)$-dimensional interior surgery on Ω_2. By Theorem 2.4, there exists a sequence of metrics (G_θ) on $N \times [0,1]$ equal to $G^2_{\theta_2}$ in a neighborhood of $N \amalg N$ such that $\mu(N \times [0,1], [G_\theta]; N \amalg N, \partial[G_\theta]) > 0$. Since $\partial G^2_{\theta_2} = g_{\theta_1} \amalg g_{\theta_2}$, we obtain that $\sigma(N \times [0,1]; N \amalg N, g_{\theta_1} \amalg g_{\theta_2}) > 0$. Since $\partial G^2_\theta \in \mathcal{R}_c(N \amalg N)$, we have that $g_{\theta_1}, g_{\theta_2} \in \mathcal{R}_c(N)$ and hence, these two metrics are c-concordant.

Now, let g, h be two metrics on M which are c-concordant and let G be a metric on $M \times [0,1]$ such that the boundary $M \amalg M$ is minimal, with $\partial G = g \amalg h$ and such that $\mu(M \times [0,1]; M \amalg M, [g] \amalg [h]) > 0$. Doing the same than above, we show that g_{θ_1} and h_{θ_2} are c-concordant on N if θ_1 and θ_2 are small enough.
This shows that $\Theta^c_{M,N}$ is well-defined. Now assume that M and N are lds-relative and consider the dual surgery from N to M. In the same way, we can construct

$$\Theta^c_{N,M} : \mathrm{Conc}_c(N) \to \mathrm{Conc}_c(M)$$

as above. We now prove that

$$\Theta^c_{N,M} \circ \Theta^c_{M,N} = Id_{\mathrm{Conc}_c(M)}. \tag{19}$$

Let $g \in \mathcal{R}_c(M)$. Define $\Omega := M \times [0,1]$ and let $G := g + dt^2$ and let Ω_1 be obtained as above equipped with a metric $G^1_{\theta_0}$ (θ_0 small enough) for which the boundary is minimal and such that $\partial G^1_{\theta_0} = g_{\theta_0} \amalg g \in \mathcal{R}_c(N \amalg M$ with $[g_{\theta_0}]^c_N = \Theta^c_{M,N}([g]^c_M)$ and such that $\mu(\Omega_1, [G^1_{\theta_0}]; N \amalg M, \partial[G^1_{\theta_0}]) > 0$. Now, we attach the $(n-k)$-dimensional handle on Ω_1 along N corresponding to the dual surgery from N to M. We get a new manifold Ω_3 such that $\partial\Omega_3 = M \times M$. We apply Theorem 1.4 with $g = G^1_{\theta_0}$ and we get a metric $G^3_{\theta_3}$ for which the boundary is minimal and such that

$$\partial G^3_{\theta_3} = (g_{\theta_0})_{\theta_3} \amalg g \in \mathcal{R}_c(M \amalg M)$$

with

$$[(g_{\theta_0})_{\theta_3}]^c_M = \Theta^c_{N,M}([g_{\theta_0}]^c_N) = \Theta^c_{N,M} \circ \Theta^c_{M,N}([g]^c_M). \tag{20}$$

By Lemma 2.7, $M \times [0,1]$ is obtained from Ω_3 by an interior $(n-k)$-dimensional surgery. Hence, by Theorem 2.4, there exists a metric G_θ on $M \times [0,1]$ equal to $G^3_{\theta_3}$ in a neighborhood of the boundary in a neighborhood of $M \amalg M$ such that $\mu(M \times [0,1], [G_\theta]; M \amalg M, \partial[G_\theta]) > 0$. Since $\partial G^3_{\theta_3} = (g_{\theta_0})_{\theta_3} \amalg g$, and since $(g_{\theta_0})_{\theta_3}, g \in \mathcal{R}_c(N)$, they are c-concordant. By (20), we obtain

$$[g]^c_M = [(g_{\theta_0})_{\theta_3}]^c_M = \Theta^c_{N,M} \circ \Theta^c_{M,N}([g]^c_M).$$

This proves Relation (19). In the same way, we prove that

$$\Theta^c_{M,N} \circ \Theta^c_{N,M} = Id_{\mathrm{Conc}_c(N)}.$$

We obtain that $\Theta^c_{M,N}$ is a bijective map whose inverse is $\Theta^c_{N,M}$.

To prove Relation (17), we fix $c \leq c'$ and $h \in \mathcal{R}_{c'}(M)$. In view of the definition of $\Theta^c_{M,N}$ and using Relation (15), we have for θ small enough

$$\Theta^{c'}_{M,N}([h]^{c'}_M) = [h_\theta]^{c'}_N = [h_\theta]^c_N \cap \mathcal{R}_{c'}(N) = \Theta^c_{M,N}([h_\theta]^c_M) \cap \mathcal{R}_{c'}(N).$$

The proof of Proposition 3.4 is now complete.

3.2. **Proof of Corollary 3.6.** Let $C \in \mathrm{Conc}_0(M)$, $C' := \Theta^0_{M,N}(C)$. Set $c := \sigma''(C)$ and $c' := \sigma''(C')$. We are done if we prove that

$$c = c'. \tag{21}$$

By definition of σ'', for all $\epsilon > 0$, $C \cap \mathcal{R}_{c-\epsilon} \neq \emptyset$. So let $h_\epsilon \in C \cap \mathcal{R}_{c-\epsilon}$. By Relation (15), $C \cap \mathcal{R}_{c-\epsilon} = [h_\epsilon]^{c-\epsilon}_M$. Relation (17) then leads to

$$\begin{aligned} C' \cap \mathcal{R}_{c-\epsilon}(N) &= \Theta^0_{M,N}(C) \cap \mathcal{R}_{c-\epsilon}(N) \\ &= \Theta^0_{M,N}([h_\epsilon]^0_M) \cap \mathcal{R}_{c-\epsilon}(N) \\ &= \Theta^{c-\epsilon}_{M,N}([h_\epsilon]^{c-\epsilon}_M) \\ &= \Theta^{c-\epsilon}_{M,N}(C \cap \mathcal{R}_{c-\epsilon}(M)) \end{aligned}$$

and consequently, $C' \cap \mathcal{R}_{c-\epsilon}(N) \neq \emptyset$ which implies $c' \geq c$. In the same way, since $\Theta^0_{M,N} = (\Theta^0_{N,M})^{-1}$, we have $c \geq c'$ and Relation (21) is proven. This ends the proof of Corollary 3.6.

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