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DISCRETE ASYMPTOTIC BEHAVIORS FOR SKEW-EVOLUTION SEMIFLOWS ON BANACH SPACES

CODRUȚA STOICA MIHAIL MEGAN

Abstract. The paper emphasizes asymptotic behaviors, as stability, instability, dichotomy and trichotomy for skew-evolution semiflows, defined by means of evolution semiflows and evolution cocycles and which can be considered generalizations for evolution operators and skew-product semiflows. The definitions are given in continuous time, but the unified treatment for the characterization of the studied properties in the nonuniform case is given in discrete time.

The property of trichotomy, introduced in finite dimension in [1] as a natural generalization for the dichotomy of linear time-varying differential systems, was studied in continuous time and from uniform point of view in [2] and in discrete time and from nonuniform point of view in [3], but for a particular case of one-parameter semiflows.

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1 Notations. Definitions

Let us consider (X, d) a metric space, V a Banach space and $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. We will denote $Y = X \times V$, $T = \{(t, t_0) \in \mathbf{R}^2, t \geq t_0 \geq 0\}$ and $\Delta = \{(m, n) \in \mathbf{N}^2, m \geq n\}$. Let $\mathcal{R} = \{R : \mathbf{R}_+ \rightarrow \mathbf{R}_+ \mid R \text{ nondecreasing, } R(0) = 0, R(t) > 0, \forall t > 0\}$. By I is denoted the identity operator on V . Let $P : Y \rightarrow Y$ be a projector given by $P(x, v) = (x, P(x)v)$, where $P(x)$ is a projection on $Y_x = \{x\} \times V$ and $x \in X$.

Definition 1.1 A mapping $\varphi : T \times X \rightarrow X$ is called *evolution semiflow* on X if following relations hold:

- (s₁) $\varphi(t, t, x) = x, \forall (t, x) \in \mathbf{R}_+ \times X$
- (s₂) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall t \geq s \geq t_0 \geq 0, x \in X$.

Definition 1.2 A mapping $\Phi : T \times X \rightarrow \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if:

- (c₁) $\Phi(t, t, x) = I$, the identity operator on $V, \forall (t, x) \in \mathbf{R}_+ \times X$
- (c₂) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall t \geq s \geq t_0 \geq 0, x \in X$.

Definition 1.3 The mapping $C : T \times Y \rightarrow Y$ defined by the relation

$$C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v), \quad (1.1)$$

where Φ is an evolution cocycle over an evolution semiflow φ , is called *skew-evolution semiflow* on Y .

Example 1.1 Let $V = \mathbb{R}^3$ endowed with the norm

$$\|(v_1, v_2, v_3)\| = |v_1| + |v_2| + |v_3|.$$

We denote $\mathcal{C} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ the set of all continuous functions $x : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, endowed with the topology of uniform convergence on compact subsets of \mathbb{R}_+ , metrizable relative to the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}, \quad \text{unde } d_n(x, y) = \sup_{t \in [0, n]} |x(t) - y(t)|.$$

If $x \in \mathcal{C}$ then for all $t \in \mathbb{R}_+$ we denote $x_t(s) = x(t + s)$, $x_t \in \mathcal{C}$. Let X the closure in \mathcal{C} of the set $\{f_t, t \in \mathbb{R}_+\}$, where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ is a nondecreasing function with the property $\lim_{t \rightarrow \infty} f(t) = l > 0$. Then (X, d) is a metric space and the mapping

$$\varphi : T \times X \rightarrow X, \quad \varphi(t, s, x)(\tau) = x(t - s + \tau)$$

is an evolution semiflow on X . The mapping $\Phi : T \times X \rightarrow \mathcal{B}(V)$, given by

$$\Phi(t, s, x)v = \left(e^{-2(t-s)x(0) + \int_0^t x(\tau) d\tau} v_1, e^{t-s + \int_0^t x(\tau) d\tau} v_2, e^{-(t-s)x(0) + 2 \int_0^t x(\tau) d\tau} v_3 \right)$$

is an evolution cocycle. Then $C = (\varphi, \Phi)$ is a skew-evolution semiflow.

Remark 1.1 If $C = (\varphi, \Phi)$ is a skew-evolution semiflow, then $C_\lambda = (\varphi, \Phi_\lambda)$, where $\lambda \in \mathbb{R}$ and

$$\Phi_\lambda : T \times X \rightarrow \mathcal{B}(V), \quad \Phi_\lambda(t, t_0, x) = e^{-\lambda(t-t_0)} \Phi(t, t_0, x), \quad (1.2)$$

is also a skew-evolution semiflow.

2 Nonuniform discrete exponential stability

Definition 2.1 A skew-evolution semiflow C is said to be

(s) *stable* if there exists a mapping $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0, x)v\| \leq N(s) \|\Phi(s, t_0, x)v\|, \quad (2.1)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$;

(es) *exponentially stable* if there exist a constant $\nu > 0$ and a mapping $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0, x)v\| \leq N(s) e^{-\nu(t-s)} \|\Phi(s, t_0, x)v\|, \quad (2.2)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Example 2.1 Let $X = \mathbb{R}_+$ and $V = \mathbb{R}$. We consider the continuous function

$$f : \mathbb{R}_+ \rightarrow [1, \infty), \quad f(n) = e^{2n} \quad \text{and} \quad f\left(n + \frac{1}{e^{n^2}}\right) = 1$$

and the mapping

$$\Phi_f : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_f(t, s, x)v = \frac{f(s)}{f(t)} e^{-(t-s)} v.$$

Then $C_f = (\varphi, \Phi_f)$ is a skew-evolution semiflow $Y = \mathbb{R}_+ \times \mathbb{R}$ over any evolution semiflow φ pe \mathbb{R}_+ . As

$$|\Phi_f(t, s, x)v| \leq f(s)e^{-(t-s)}|v|, \quad \forall (t, s, x, v) \in T \times Y,$$

it follows that C_f is exponentially stable.

Definition 2.2 A skew-evolution semiflow C has *exponential growth* if there exist the mappings $M, \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0, x)v\| \leq M(s)e^{\omega(s)(t-s)} \|\Phi(s, t_0, x)v\|, \quad (2.3)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

In discrete time, the exponential stability can be described as follows.

Proposition 2.1 *A skew-evolution semiflow C with exponential growth is exponentially stable if and only if there exist a constant $\mu > 0$ and a sequence of real numbers $(a_n)_{n \geq 0}$ with the property $a_n \geq 1, \forall n \geq 0$ such that*

$$\|\Phi(n, m, x)v\| \leq a_m e^{-\mu(n-m)} \|v\|, \quad (2.4)$$

for all $(n, m) \in \Delta$ and all $(x, v) \in Y$.

Proof. *Necessity.* It is obtained immediately if we consider in relation (2.2) $t = n, s = t_0 = m$ and if we define

$$a_m = N(m), \quad m \in \mathbb{N} \quad \text{and} \quad \mu = \nu > 0,$$

where the existence of $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and of ν is given by Definition 2.1.

Sufficiency. As a first step, if $t \geq t_0 + 1$, we denote $n = [t]$ and $n_0 = [t_0]$. Following relations hold

$$n \leq t < n + 1, \quad n_0 \leq t_0 < n_0 + 1, \quad n_0 + 1 \leq n.$$

We obtain

$$\|\Phi(t, t_0, x)v\| \leq$$

$$\begin{aligned} &\leq M(n)e^{\omega(n)(t-n)} \|\Phi(n, n_0 + 1, \varphi(n_0 + 1, t_0, x))\Phi(n_0 + 1, t_0, x)v\| \leq \\ &\leq a_n M^2(n) e^{2[\omega(n)+\mu]} e^{-\mu(t-t_0)} \|v\|, \end{aligned}$$

for all $(x, v) \in Y$, where functions M and ω are given by Definition 2.2. As a second step, for $t \in [t_0, t_0 + 1)$ we have

$$\|\Phi(t, t_0, x)v\| \leq M(t_0)e^{\omega(t_0)(t-t_0)} \|v\| \leq M(t_0)e^{\omega(t_0)+\mu} e^{-\mu(t-t_0)} \|v\|,$$

for all $(x, v) \in Y$.

Hence, C is exponentially stable. \square

Some characterizations for the exponential stability of skew-evolution semiflows in discrete time are given by the next results.

Theorem 2.1 *A skew-evolution semiflow C with exponential growth is exponentially stable if and only if there exist a mapping $R \in \mathcal{R}$, a constant $\rho > 0$ and a sequence of real numbers $(\alpha_n)_{n \geq 0}$ with the property $\alpha_n \geq 1$, $\forall n \geq 0$ such that*

$$\sum_{k=n}^m R \left(e^{\rho(k-n)} \|\Phi(k, n, x)v\| \right) \leq \alpha_n R(\|v\|), \quad (2.5)$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$.

Proof. *Necessity.* We consider $R(t) = t$, $t \geq 0$. Proposition 2.1 assures the existence of a constant $\nu > 0$ and of a sequence of real numbers $(a_n)_{n \geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$. We obtain for $\rho = \frac{\nu}{2} > 0$ and according to Proposition 2.1

$$\begin{aligned} \sum_{k=n}^m e^{\rho(k-n)} \|\Phi(k, n, x)v\| &\leq a_n \sum_{k=n}^m e^{\rho(k-n)} e^{-\nu(k-n)} \|\Phi(n, n, x)v\| = \\ &= a_n \|v\| \sum_{k=n}^m e^{-\frac{\nu}{2}(k-n)} \leq \alpha_n \|v\|, \forall m, n \in \Delta, \forall (x, v) \in Y, \end{aligned}$$

where we have denoted

$$\alpha_n = a_n e^{\frac{\nu}{2}n}, \quad n \in \mathbb{N}.$$

Sufficiency. Let $t \geq t_0 + 1$, $t_0 \geq 0$. We define $n = [t]$ and $n_0 = [t_0]$. We consider $C_{-\rho} = (\varphi, \Phi_{-\rho})$ given as in relation (1.2). It follows that

$$\begin{aligned} R(\|\Phi_{-\rho}(t, t_0, x)v\|) &= R(\|\Phi_{-\rho}(t, n, \varphi(n, t_0, x))\Phi_{-\rho}(n, t_0, x)v\|) \leq \\ &\leq R \left(M(n)e^{\omega(t-n)} \|\Phi_{-\rho}(n, t_0, x)v\| \right) \leq R(M(n)e^{\omega} \|\Phi_{-\rho}(n, t_0, x)v\|), \end{aligned}$$

for all $(x, v) \in Y$, where M and ω are given by Definition 2.2. We obtain further for $m \geq n$

$$\begin{aligned} \int_{t_0+1}^t R(\|\Phi_{-\rho}(\tau, t_0, x)v\|) d\tau &\leq \sum_{k=[t_0]+1}^m R(M(n)e^\omega \|\Phi_{-\rho}(k, t_0, x)v\|) \leq \\ &\leq \beta_n R(\|v\|), \end{aligned}$$

for all $(x, v) \in Y$, where $\beta_n = M(n)\alpha_n e^\omega$. Then there exist $N \geq 1$ and $\nu > 0$ such that

$$\|\Phi_{-\rho}(t, t_0, x)v\| \leq N e^{-\nu(t-t_0)} \|v\|, \quad \forall t \geq t_0 + 1, \quad \forall (x, v) \in Y.$$

On the other hand, for $t \in [t_0, t_0 + 1)$ we have

$$R(\|\Phi_{-\rho}(t, t_0, x)v\|) \leq R\left(Me^{\omega(t-t_0)} \|v\|\right) \leq R(Me^\omega \|v\|), \quad \forall (x, v) \in Y.$$

We obtain that $C_{-\rho}$ is stable, where

$$\Phi_{-\rho}(m, n, x) = e^{\rho(m-n)} \|\Phi(m, n, x)\|, \quad (m, n, x) \in \Delta \times X.$$

Hence, there exists a sequence $(a_n)_{n \geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$, such that

$$e^{\rho(m-n)} \|\Phi(m, n, x)v\| \leq a_n \|v\|, \quad \forall (m, n, x, v) \in \Delta \times Y,$$

which implies the exponential stability of C and ends the proof. \square

If we consider $R(t) = t^p$, $t \geq 0$, $p > 0$ we obtain

Corollary 2.1 *Let $p > 0$. A skew-evolution semiflow C is exponentially stable if and only if there exist a constant $\rho > 0$ and a sequence of real numbers $(\alpha_n)_{n \geq 0}$ with the property $\alpha_n \geq 1$, $\forall n \geq 0$ such that*

$$\sum_{k=n}^m e^{p\rho(k-n)} \|\Phi(k, n, x)v\|^p \leq \alpha_n \|v\|^p, \quad (2.6)$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$.

Theorem 2.2 *A skew-evolution semiflow C is exponentially stable if and only if there exist a function $R \in \mathcal{R}$, a constant $\gamma > 0$ a sequence of real numbers $(\beta_n)_{n \geq 0}$ with the property $\beta_n \geq 1$, $\forall n \geq 0$ such that*

$$\sum_{k=n}^m R\left(e^{\gamma(m-k)} \|\Phi(m, k, \varphi(k, n, x))^* v^*\|\right) \leq \beta_n R(\|v^*\|), \quad (2.7)$$

for all $(m, n) \in \Delta$ and all $(x, v^*) \in X \times V^*$.

Proof. *Necessity.* For $R(t) = t$, $t \geq 0$ and $\gamma = \frac{\nu}{2} > 0$ we obtain, according to Definition 2.1 and Proposition 2.1,

$$\sum_{k=n}^m e^{\frac{\nu}{2}(m-k)} \|\Phi(m, k, \varphi(k, n, x))^* v^*\| \leq a_n \|v^*\| \sum_{k=n}^m e^{-\frac{\nu}{2}(m-k)} \leq \beta_n \|v^*\|,$$

where we have denoted

$$\beta_n = \frac{a_n}{1 - e^{-\frac{\nu}{2}}}, \quad n \in \mathbb{N},$$

where $\nu > 0$ and the sequence of real numbers $(a_n)_{n \geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$ are given by Definition 2.1.

Sufficiency. Let $C_{-\gamma} = (\varphi, \Phi_{-\gamma})$ be given as in relation (1.2). For all $(k, n_0) \in \Delta$ we obtain

$$R(\|\Phi_{-\gamma}(k, n_0, x)v^*\|) \leq \beta_n R(\|v^*\|), \quad \forall x \in X, \forall v^* \in V^*.$$

Further we have

$$\begin{aligned} \sum_{k=n_0}^m R(\|\Phi_{-\gamma}(m, n_0, x)\|) &\leq \sum_{k=n_0}^m R(\|\Phi_{-\gamma}(m, k, x)\| \|\Phi_{-\gamma}(k, n_0, x)\|) \leq \\ &\leq \widetilde{M} \sum_{k=n_0}^m R(\|\Phi_{-\gamma}(m, k, x)\|) \leq \widetilde{M}^2, \end{aligned}$$

which implies

$$R(\|\Phi_{-\gamma}(m, n_0, x)\|) \leq \frac{\widetilde{M}^2}{m - n_0 + 1}, \quad \forall (m, n_0, x) \in \Delta \times X.$$

There exist $n_1 > n_0 + 1$ such that

$$\frac{\widetilde{M}^2}{n_1 - n_0 + 1} \leq R\left(\frac{1}{2}\right).$$

We obtain that $C_{-\gamma}$ is stable. According to the proof of Theorem 2.1 it follows that C is exponentially stable. \square

For $R(t) = t^p$, $t \geq 0$, $p > 0$ we obtain

Corollary 2.2 *Let $p > 0$. A skew-evolution semiflow C is exponentially stable if and only if there exist a constant $\gamma > 0$ and a sequence of real numbers $(\beta_n)_{n \geq 0}$ with $\beta_n \geq 1$, $\forall n \geq 0$ such that*

$$\sum_{k=n}^m e^{p\gamma(m-k)} \|\Phi(m, k, \varphi(k, n, x))^* v^*\|^p \leq \beta_n \|v^*\|^p, \quad (2.8)$$

for all $(m, n) \in \Delta$ and all $(x, v^*) \in X \times V^*$.

3 Nonuniform discrete exponential instability

Definition 3.1 A skew-evolution semiflow C is said to be

(is) *instable* if there exists a mapping $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$N(t) \|\Phi(t, t_0, x)v\| \geq \|\Phi(s, t_0, x)v\|, \quad (3.1)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$;

(eis) *exponentially instable* if there exist a mapping $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and a constant $\nu > 0$ such that

$$N(t) \|\Phi(t, t_0, x)v\| \geq e^{\nu(t-s)} \|\Phi(s, t_0, x)v\|, \quad (3.2)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Example 3.1 Let $X = \mathbb{R}_+$ and $V = \mathbb{R}$. We consider the function

$$f : \mathbb{R}_+ \rightarrow [1, \infty), \quad f(n) = 1 \text{ and } f\left(n + \frac{1}{e^{n^2}}\right) = e^{2n}$$

and the mapping

$$\Phi_f : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_f(t, s, x)v = \frac{f(s)}{f(t)} e^{(t-s)v}.$$

Then $C_f = (\varphi, \Phi_f)$ is a skew-evolution semiflow on $Y = \mathbb{R}_+ \times \mathbb{R}$ over every evolution semiflow φ on \mathbb{R}_+ . As

$$|\Phi_f(t, s, x)v| \geq \frac{1}{f(t)} e^{(t-s)} |v|, \quad \forall (t, s, x, v) \in T \times Y,$$

it follows that C_f is exponentially instable.

Definition 3.2 A skew-evolution semiflow C has *exponential decay* if there exist the mappings $M, \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(s, t_0, x)v\| \leq M(t) e^{\omega(t)(t-s)} \|\Phi(t, t_0, x)v\|, \quad (3.3)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

In order to describe the instability of skew-evolution semiflows in discrete time, we will consider first the next

Proposition 3.1 *A skew-evolution semiflow with exponential decay C is exponentially instable if and only if there exist a constant $\mu > 0$ and a sequence of real numbers $(a_n)_{n \geq 0}$ with the property $a_n \geq 1, \forall n \geq 0$ such that*

$$\|\Phi(n, n_0, x)v\| \leq a_n e^{-\mu(m-n_0)} \|\Phi(m, n_0, x)v\|, \quad (3.4)$$

for all $(m, n), (n, n_0) \in \Delta$ and all $(x, v) \in Y$.

Proof. *Necessity.* It is obtained if we consider in relation (3.2) $t = m$, $s = n$, $t_0 = n_0$ and if we define

$$a_m = N(m), \quad m \in \mathbb{N} \text{ și } \mu = \nu > 0,$$

where the existence of function $N : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and of constant ν is given by Definition 3.1.

Sufficiency. As a first step, we consider $t \geq t_0 + 1$ and we denote $n = [t]$ respectively $n_0 = [t_0]$. We obtain

$$n \leq t < n + 1, \quad n_0 \leq t_0 < n_0 + 1, \quad n_0 + 1 \leq n.$$

It follows that

$$\begin{aligned} & \|\Phi(t, t_0, x)v\| = \\ & = \|\Phi(t, n, \varphi(n, n_0 + 1, x))\Phi(n, n_0 + 1, \varphi(n_0 + 1, t_0, x))\Phi(n_0 + 1, t_0, x)v\| \geq \\ & \geq [M(t)]^{-1}e^{-\omega(t-n)}[M(t)]^{-1}e^{-\omega(n_0+1-t_0)}\|\Phi(n, n_0 + 1, x)v\| \geq \\ & \geq \frac{[M(t)]^{-2}}{\tilde{N}}e^{-2\omega}e^{\tilde{\nu}(n-n_0+1)}\|v\| \geq \frac{e^{\tilde{\nu}}}{[M(t)]^2\tilde{N}e^{2\omega}}e^{\tilde{\nu}(t-t_0)}\|v\|, \end{aligned}$$

for all $(x, v) \in Y$, where the existence of function M and of constant ω is assured by Definition 3.2.

As a second step, if we consider $t \in [t_0, t_0 + 1)$ we obtain

$$M\|\Phi(t, t_0, x)v\| \geq e^{-\omega(t-t_0)}\|v\| \geq e^{-(\tilde{\nu}+\omega)}e^{\tilde{\nu}(t-t_0)}\|v\|,$$

for all $(x, v) \in Y$.

Hence

$$N\|\Phi(t, t_0, x)v\| \geq e^{\nu(t-t_0)}\|v\|,$$

for all $(t, t_0, x, v) \in T \times Y$, where we have denoted

$$N = Me^{(\tilde{\nu}+\omega)} + M^2\tilde{N}e^{(-\tilde{\nu}+2\omega)} \text{ și } \nu = \tilde{\nu},$$

which proves the exponential instability of C . □

Theorem 3.1 *A skew-evolution semiflow C is exponentially instable if and only if there exist a function $R \in \mathcal{R}$, a constant $\rho < 0$ and a sequence of real numbers $(\alpha_n)_{n \geq 0}$ with the property $\alpha_n \geq 1$, $\forall n \geq 0$ such that*

$$\sum_{k=n}^m R\left(e^{-\rho(m-k)}\|\Phi(k, n, x)v\|\right) \leq R(\alpha_m\|\Phi(m, n, x)v\|), \quad (3.5)$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$.

Proof. *Necessity.* Let $R(t) = t$, $t \geq 0$. Proposition 3.1 assures the existence of a constant $\nu > 0$ and of a sequence of real numbers $(a_n)_{n \geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$. We obtain for $\rho = -\frac{\nu}{2} > 0$

$$\begin{aligned} \sum_{k=n}^m e^{-\rho(m-k)} \|\Phi(k, n, x)v\| &\leq a_m \sum_{k=n}^m e^{-\rho(m-k)} e^{-\nu(m-k)} \|\Phi(m, n, x)v\| = \\ &= a_m \|\Phi(m, n, x)v\| \sum_{k=n}^m e^{-\frac{\nu}{2}(k-n)} \leq \alpha_m \|\Phi(m, n, x)v\|, \end{aligned}$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$, where we have denoted

$$\alpha_m = \frac{1}{1 - e^{\rho}} a_m, \quad m \in \mathbb{N}.$$

Sufficiency. According to the hypothesis, if we consider $k = n$ we obtain

$$R\left(e^{-\rho(m-n)} \|\Phi(n, n, x)v\|\right) \leq R(a_m \|\Phi(m, n, x)v\|),$$

for all $(m, n, x, v) \in \Delta \times Y$, which implies, by means of the properties of function R , the exponential instability of C and ends the proof. \square

For $R(t) = t^p$, $t \geq 0$, $p > 0$ it is obtained

Corollary 3.1 *Let $p > 0$. A skew-evolution semiflow C is exponentially instable if and only if there exist a constant $\rho < 0$ and a sequence of real numbers $(\alpha_n)_{n \geq 0}$ with the property $\alpha_n \geq 1$, $\forall n \geq 0$ such that*

$$\sum_{k=n}^m e^{-p\rho(m-k)} \|\Phi(k, n, x)v\|^p \leq \alpha_m \|\Phi(m, n, x)v\|^p, \quad (3.6)$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$.

4 Nonuniform discrete exponential dichotomy

Definition 4.1 A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C = (\varphi, \Phi)$ if following relation

$$P(\varphi(t, s, x))\Phi(t, s, x) = \Phi(t, s, x)P(x), \quad (4.1)$$

holds for all $(t, s) \in T$ and all $x \in X$.

Definition 4.2 Two projectors P_1 and P_2 are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

- (d₁) projectors P_1 and P_2 are invariant on Y ;
- (d₂) for all $x \in X$, the projections $P_1(x)$ and $P_2(x)$ verify the relations

$$P_1(x) + P_2(x) = I \text{ and } P_1(x)P_2(x) = P_2(x)P_1(x) = 0.$$

Definition 4.3 A skew-evolution semiflow $C = (\varphi, \Phi)$ is called

(d) *dichotomic* if there exist some functions $N_1, N_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$, two projectors P_1 and P_2 compatible with C such that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| \quad (4.2)$$

$$\|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t) \|\Phi(t, t_0, x)P_2(x)v\|, \quad (4.3)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$;

(ed) *exponentially dichotomic* if there exist functions $N_1, N_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$, constants $\nu_1, \nu_2 > 0$ and two projectors P_1 and P_2 compatible with C such that

$$e^{\nu_1(t-s)} \|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| \quad (4.4)$$

$$e^{\nu_2(t-s)} \|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t) \|\Phi(t, t_0, x)P_2(x)v\| \quad (4.5)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark 4.1 An exponentially dichotomic skew-evolution semiflow is dichotomic.

Example 4.1 Let $X = \mathbb{R}_+$ and $V = \mathbb{R}^2$ endowed with the norm

$$\|(v_1, v_2)\| = |v_1| + |v_2|, \quad v = (v_1, v_2) \in V.$$

The mapping $\Phi : T \times X \rightarrow \mathcal{B}(V)$, given by

$$\Phi(t, s, x)(v_1, v_2) = (e^{t \sin t - s \sin s - 2t + 2s} v_1, e^{2t - 2s - 3t \cos t + 3s \cos s} v_2)$$

is an evolution cocycle over every evolution semiflow φ . We consider the projectors

$$P_1(x)(v_1, v_2) = (v_1, 0) \text{ and } P_2(x)(v_1, v_2) = (0, v_2).$$

As

$$t \sin t - s \sin s - 2t + 2s \leq -t + 3s, \quad \forall (t, s) \in T,$$

we obtain that

$$\|\Phi(t, s, x)P_1(x)v\| \leq e^{2s} e^{-(t-s)} |v_1|, \quad \forall (t, s, x, v) \in T \times Y.$$

Similarly, as

$$2t - 2s - 3t \cos t + 3s \cos s \geq -t - 5s, \quad \forall (t, s) \in T$$

it follows that

$$e^{6t} \|\Phi(t, s, x)P_2(x)v\| \geq e^{5(t-s)} |v_2|, \quad \forall (t, s, x, v) \in T \times Y.$$

The skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic with characteristics

$$N(u) = e^{6u} \text{ and } \nu = 1.$$

In what follows, let us denote

$$C_k(t, s, x, v) = (\varphi(t, s, x), \Phi_k(t, s, x)v), \quad \forall (t, t_0, x, v) \in T \times Y, \quad \forall k \in \{1, 2\},$$

where

$$\Phi_k(t, t_0, x) = \Phi(t, t_0, x)P_k(x), \quad \forall (t, t_0) \in T, \quad \forall x \in X, \quad \forall k \in \{1, 2\}.$$

In discrete time, we will describe the property of exponential dichotomy as given in the next

Proposition 4.1 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic if and only if there exist two projectors $\{P_k\}_{k \in \{1, 2\}}$, compatible with C , constants $\nu_1 \leq 0 \leq \nu_2$ and a sequence of real positive numbers $(a_n)_{n \geq 0}$ such that*

$$(d'_1) \quad \|\Phi(m, n, x)P_1(x)v\| \leq a_n \|P_1(x)v\| e^{\nu_1(m-n)} \quad (4.6)$$

$$(d'_2) \quad \|P_2(x)v\| \leq a_m \|\Phi(m, n, x)P_2(x)v\| e^{-\nu_2(m-n)} \quad (4.7)$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$.

Proof. *Necessity.* If we consider for C_1 in relation (4.4) of Definition 4.3 $t = m$, $s = t_0 = n$ and if we define

$$a_n = N(n), \quad n \in \mathbb{N},$$

relation (d'_1) is obtained.

Statement (d'_2) results from Definition 4.3 for C_2 if we consider in relation (4.5) $t = m$, $s = t_0 = n$ and

$$a_m = N(m), \quad m \in \mathbb{N}.$$

Sufficiency. It is obtained by means of Proposition 2.1 for C_1 , respectively of Proposition 3.1 for C_2 .

Hence, C is exponentially dichotomic. \square

Theorem 4.1 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic if and only if there exist two projectors $\{P_k\}_{k \in \{1, 2\}}$, compatible with C such that*

(ed'_1) there exist a constant $\rho_1 > 0$ and a sequence of real positive numbers $(\alpha_n)_{n \geq 0}$ such that

$$\sum_{k=n}^m e^{\rho_1(k-n)} \|\Phi(k, n, x)P_1(x)v\| \leq \alpha_n \|P_1(x)v\| \quad (4.8)$$

(ed'₂) there exist a constant $\rho_2 < 0$ and a sequence of real positive numbers $(\beta_n)_{n \geq 0}$ such that

$$\sum_{k=n}^m e^{-\rho_2(m-k)} \|\Phi(k, n, x)P_2(x)v\| \leq \beta_m \|\Phi(m, n, x)P_2(x)v\| \quad (4.9)$$

for all $(m, p), (p, n) \in \Delta$ and all $(x, v) \in Y$.

Proof. It is obtained by means of Theorem 2.1 and Theorem 3.1, by considering $R(t) = t$, $t \geq 0$. \square

5 Nonuniform discrete exponential trichotomy

Definition 5.1 Three projectors $\{P_k\}_{k \in \{1,2,3\}}$ are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

- (t₁) each projector P_k , $k \in \{1, 2, 3\}$ is invariant on Y ;
- (t₂) for all $x \in X$, the projections $P_0(x)$, $P_1(x)$ and $P_2(x)$ verify the relations

$$P_1(x) + P_2(x) + P_3(x) = I \text{ and } P_i(x)P_j(x) = 0, \forall i, j \in \{1, 2, 3\}, i \neq j.$$

Definition 5.2 A skew-evolution semiflow $C = (\varphi, \Phi)$ is called

- (t) *trichotomic* if there exist the functions $N_1, N_2, N_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and three projectors P_1, P_2 and P_3 compatible with C such that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| \quad (5.1)$$

$$\|\Phi(s, t_0, x)P_2(x)v\| \leq N_2(t_0) \|\Phi(t, t_0, x)P_2(x)v\| \quad (5.2)$$

$$\|\Phi(t, t_0, x)P_3(x)v\| \leq N_3(s) \|\Phi(s, t_0, x)P_3(x)v\| \quad (5.3)$$

$$\|\Phi(s, t_0, x)P_3(x)v\| \leq N_3(t) \|\Phi(t, t_0, x)P_3(x)v\| \quad (5.4)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$;

- (et) *exponentially trichotomic* if there exist the functions $N_1, N_2, N_3, N_4 : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$, constants $\nu_1, \nu_2, \nu_3, \nu_4$ with the properties

$$\nu_1 \leq \nu_2 \leq 0 \leq \nu_3 \leq \nu_4$$

and three projectors P_1, P_2 and P_3 compatible with C such that

$$\|\Phi(t, t_0, x)P_1(x)v\| \leq N_1(s) \|\Phi(s, t_0, x)P_1(x)v\| e^{\nu_1(t-s)} \quad (5.5)$$

$$N_4(t) \|\Phi(t, t_0, x)P_2(x)v\| \geq \|\Phi(s, t_0, x)P_2(x)v\| e^{\nu_4(t-s)} \quad (5.6)$$

$$N_2(t) \|\Phi(t, t_0, x)P_3(x)v\| \geq \|\Phi(s, t_0, x)P_3(x)v\| e^{\nu_2(t-s)} \quad (5.7)$$

$$\|\Phi(t, t_0, x)P_3(x)v\| \leq N_3(s) \|\Phi(s, t_0, x)P_3(x)v\| e^{\nu_3(t-s)} \quad (5.8)$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark 5.1 An exponentially trichotomic skew-evolution semiflow is trichotomic.

Example 5.1 Let us consider the example of C given in Example 1.1. We consider the projections

$$P_1(x)(v) = (v_1, 0, 0), \quad P_2(x)(v) = (0, v_2, 0), \quad P_3(x)(v) = (0, 0, v_3).$$

The skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic with characteristics

$$\nu_1 = \nu_2 = -x(0), \quad \nu_3 = x(0) \text{ and } \nu_4 = 1,$$

$$N_1(u) = e^{ux(0)}, \quad N_2(u) = e^{-2lu}, \quad N_3(u) = e^{2ux(0)} \text{ and } N_4(u) = e^{-lu}.$$

As in the case of dichotomy, we denote

$$C_k(t, s, x, v) = (\varphi(t, s, x), \Phi_k(t, s, x)v), \quad \forall (t, t_0, x, v) \in T \times Y, \quad \forall k \in \{1, 2, 3\},$$

where

$$\Phi_k(t, t_0, x) = \Phi(t, t_0, x)P_k(x), \quad \forall (t, t_0) \in T, \quad \forall x \in X, \quad \forall k \in \{1, 2, 3\}.$$

In discrete time, the trichotomy of a skew-evolution semiflow can be described as in the next

Proposition 5.1 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic if and only if there exist three projectors $\{P_k\}_{k \in \{1, 2, 3\}}$ compatible with C , constants $\nu_1, \nu_2, \nu_3, \nu_4$ with the property $\nu_1 \leq \nu_2 \leq 0 \leq \nu_3 \leq \nu_4$ and a sequence of positive real numbers $(a_n)_{n \geq 0}$ such that*

(t₁)

$$\|\Phi(m, n, x)P_1(x)v\| \leq a_p \|\Phi(p, n, x)P_1(x)v\| e^{\nu_1(m-p)} \quad (5.9)$$

(t₂)

$$\|\Phi(p, n, x)P_2(x)v\| \leq a_m \|\Phi(m, n, x)P_2(x)v\| e^{-\nu_4(m-p)} \quad (5.10)$$

(t₃)

$$\|\Phi(p, n, x)P_3(x)v\| \leq a_m \|\Phi(m, n, x)P_3(x)v\| e^{-\nu_2(m-p)} \quad (5.11)$$

(t₄)

$$a_p \|\Phi(p, n, x)P_3(x)v\| \geq \|\Phi(m, n, x)P_3(x)v\| e^{-\nu_3(m-p)} \quad (5.12)$$

for all $(m, p), (p, n) \in \Delta$ and all $(x, v) \in Y$.

Proof. *Necessity.* (t_1) is obtain if we consider for C_1 in relation (2.2) of Definition 2.1 $t = n$, $s = t_0 = m$ and if we define

$$a_p = N(p), \quad p \in \mathbb{N} \text{ and } \nu_1 = -\nu < 0.$$

(t_2) follows according to Definition 3.1 for C_2 if we consider in relation (3.2) $t = m$, $s = n$, $t_0 = n_0$ and

$$a_m = N(m), \quad m \in \mathbb{N} \text{ and } \nu_4 = \nu > 0.$$

(t_3) is obtained for C_3 out of relation (2.3) of Definition 2.2 for $t = m$, $s = p$, $t_0 = n$ and if we define

$$a_m = M(m) \text{ and } \nu_2 = -\omega(m) < 0, \quad m \in \mathbb{N}.$$

(t_4) follows for C_3 from relation (3.3) of Definition 3.2 for $t = m$, $s = p$, $t_0 = n$ and if we consider

$$a_p = M(p) \text{ and } \nu_3 = \omega(p) > 0, \quad p \in \mathbb{N}.$$

Sufficiency. Let $t \geq t_0 + 1$. We denote $n = [t]$, $n_0 = [t_0]$ and we obtained the relations

$$n \leq t < n + 1, \quad n_0 \leq t_0 < n_0 + 1, \quad n_0 + 1 \leq n.$$

According to (t_1) , we have

$$\begin{aligned} & \|\Phi_1(t, t_0, x)v\| \leq \\ & \leq M(n)e^{\omega(n)(t-n)} \|\Phi(n, n_0 + 1, \varphi(n_0 + 1, t_0, x))\Phi(n_0 + 1, t_0, x)P_1(x)v\| \leq \\ & \leq a_n M^2(n) e^{2[\omega(n)+\mu]} e^{-\mu(t-t_0)} \|P_1(x)v\|, \end{aligned}$$

for all $(x, v) \in Y$, where functions M and ω are given as in Definition 2.2.

For $t \in [t_0, t_0 + 1)$ we have

$$\begin{aligned} \|\Phi_1(t, t_0, x)v\| & \leq M(t_0)e^{\omega(t_0)(t-t_0)} \|P_1(x)v\| \leq \\ & \leq M(t_0)e^{\omega(t_0)+\mu} e^{-\mu(t-t_0)} \|P_1(x)v\|, \end{aligned}$$

for all $(x, v) \in Y$. Hence, relation (5.5) is obtained.

Let $t \geq t_0 + 1$ and $n = [t]$ respectively $n_0 = [t_0]$. It follows

$$n \leq t < n + 1, \quad n_0 \leq t_0 < n_0 + 1, \quad n_0 + 1 \leq n.$$

From (t_2) , it is obtained

$$\begin{aligned} & \|\Phi_2(t, t_0, x)v\| = \\ & = \|\Phi(t, n, \varphi(n, n_0 + 1, x))\Phi(n, n_0 + 1, \varphi(n_0 + 1, t_0, x))\Phi(n_0 + 1, t_0, x)P_2(x)v\| \geq \\ & \geq [M(t)]^{-1} e^{-\omega(t-n)} [M(t)]^{-1} e^{-\omega(n_0+1-t_0)} \|\Phi_2(n, n_0 + 1, x)v\| \geq \end{aligned}$$

$$\geq \frac{[M(t)]^{-2}}{\tilde{N}} e^{-2\omega} e^{\tilde{\nu}(n-n_0+1)} \|P_2(x)v\| \geq \frac{e^{\tilde{\nu}}}{[M(t)]^2 \tilde{N} e^{2\omega}} e^{\tilde{\nu}(t-t_0)} \|P_2(x)v\|,$$

for all $(x, v) \in Y$, where M and ω are given by Definition 3.2.

For $t \in [t_0, t_0 + 1)$ we have

$$M \|\Phi_2(t, t_0, x)v\| \geq e^{-\omega(t-t_0)} \|P_2(x)v\| \geq e^{-(\tilde{\nu}+\omega)} e^{\tilde{\nu}(t-t_0)} \|P_2(x)v\|,$$

for all $(x, v) \in Y$. It follows that

$$N \|\Phi_2(t, t_0, x)v\| \geq e^{\nu(t-t_0)} \|P_2(x)v\|,$$

for all $(t, t_0, x, v) \in T \times Y$, where we have denoted

$$N = M e^{(\tilde{\nu}+\omega)} + M^2 \tilde{N} e^{(-\tilde{\nu}+2\omega)} \text{ and } \nu = \tilde{\nu}$$

and which implies relation (5.6).

By a similar reasoning, from (t_3) relation (5.7) is obtained, and from (t_4) relation (5.8) follows.

Hence, the skew-evolution semiflow C is exponentially trichotomic. \square

Some characterizations in discrete time for the exponential trichotomy for skew-evolution semiflows are given in what follows.

Theorem 5.1 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic if and only if there exist three projectors $\{P_k\}_{k \in \{1,2,3\}}$ compatible cu C such that C_1 has exponential growth, C_2 has exponential decay and such that following relations hold*

(t'_1) *there exist a constant $\rho_1 > 0$ and a sequence of positive real numbers $(\alpha_n)_{n \geq 0}$ such that*

$$\sum_{k=n}^m e^{\rho_1(k-n)} \|\Phi(k, n, x)P_1(x)v\| \leq \alpha_n \|P_1(x)v\| \quad (5.13)$$

(t'_2) *there exist a constant $\rho_2 > 0$ and a sequence of positive real numbers $(\beta_n)_{n \geq 0}$ such that*

$$\sum_{k=n}^m e^{-\rho_2(k-n)} \|\Phi(k, n, x)P_2(x)v\| \leq \beta_m e^{-\rho_2(m-n)} \|\Phi(m, n, x)P_2(x)v\| \quad (5.14)$$

(t'_3) *there exist a constant $\rho_3 > 0$ and a sequence of positive real numbers $(\gamma_n)_{n \geq 0}$ such that*

$$\sum_{k=p}^m e^{-\rho_3(k-n)} \|\Phi(k, n, x)P_3(x)v\| \leq \gamma_n e^{-\rho_3(p-n)} \|\Phi(p, n, x)P_3(x)v\| \quad (5.15)$$

(t'_4) there exist a constant $\rho_4 > 0$ and a sequence of positive real numbers $(\delta_n)_{n \geq 0}$ such that

$$\sum_{k=p}^m e^{\rho_4(k-n)} \|\Phi(k, n, x)P_3(x)v\| \leq \delta_m e^{\rho_4(m-n)} \|\Phi(m, n, x)P_3(x)v\| \quad (5.16)$$

for all $(m, p), (p, n) \in \Delta$ and all $(x, v) \in Y$.

Proof. (t_1) \Leftrightarrow (t'_1) *Necessity.* As C is exponentially trichotomic, Proposition 5.1 assures the existence of a projector P , of a constant $\nu_1 \leq 0$ and of a sequence of positive real numbers $(a_n)_{n \geq 0}$ such (t_1) holds. Let $P_1 = P$. If we consider

$$\rho_1 = -\frac{\nu_1}{2} > 0,$$

we obtain

$$\begin{aligned} & \sum_{k=n}^m e^{\rho_1(k-n)} \|\Phi(k, n, x)P_1(x)v\| \leq \\ & \leq a_n \sum_{k=n}^m e^{\rho_1(k-n)} e^{\nu_1(k-n)} \|\Phi(n, n, x)P_1(x)v\| = \\ & = a_n \|P_1(x)v\| \sum_{k=n}^m e^{(\rho_1+\nu_1)(k-n)} \leq \alpha_n \|P_1(x)v\|, \forall m, n \in \mathbb{N}, \forall (x, v) \in Y, \end{aligned}$$

where we have denoted

$$\alpha_n = a_n e^{-\frac{\nu_1}{2}}, \quad n \in \mathbb{N}.$$

Sufficiency. As C_1 has exponential growth, there exist constants $M \geq 1$ and $r > 1$ such that relation

$$\|\Phi(n+p, n, x)v\| \leq Mr^p \|v\|, \quad (5.17)$$

holds for all $n, p \in \mathbb{N}$ and all $(x, v) \in Y$. If we denote $\omega = \ln r > 0$ the inequality can be written as follows

$$\|\Phi(m, n, x)\| \leq M e^{\omega(m-k)} \|\Phi(k, n, x)\|, \quad \forall (m, k), (k, n) \in \Delta, \quad x \in X.$$

We consider successively the $m - n + 1$ relations. By denoting $P = P_1$, we obtain

$$\|\Phi(m, n, x)P(x)v\| \leq \frac{M e^{\omega(m-n)}}{1 + e^{\rho_1} + \dots + e^{\rho_1(m-n)}} \alpha_n \|P(x)v\|,$$

for all $(m, n) \in \Delta$ and all $(x, v) \in Y$. If we define the constant

$$\nu_1 = \omega - \rho_1 \text{ pentru } \omega < \rho_1$$

and the sequence of nonnegative real numbers

$$a_n = M\alpha_n, \quad n \in \mathbb{N},$$

relation (t_1) is obtained.

Similarly can also be proved the other equivalences. \square

In order to characterize the exponential trichotomy by means of four projectors, we give the next

Definition 5.3 Four invariant projectors $\{R_k\}_{k \in \{1,2,3,4\}}$ that satisfies for all $(x, v) \in Y$ following relations:

$$(pc'_1) \quad R_1(x) + R_3(x) = R_2(x) + R_4(x) = I;$$

$$(pc'_2) \quad R_1(x)R_2(x) = R_2(x)R_1(x) = 0 \text{ si } R_3(x)R_4(x) = R_4(x)R_3(x);$$

$$(pc'_3) \quad \|[R_1(x) + R_2(x)]v\|^2 = \|R_1(x)v\|^2 + \|R_2(x)v\|^2;$$

$$(pc'_4) \quad \|[R_1(x) + R_3(x)R_4(x)]v\|^2 = \|R_1(x)v\|^2 + \|R_3(x)R_4(x)v\|^2;$$

$$(pc'_5) \quad \|[R_2(x) + R_3(x)R_4(x)]v\|^2 = \|R_2(x)v\|^2 + \|R_3(x)R_4(x)v\|^2,$$

are called *compatible* with the skew-evolution semiflow C .

Theorem 5.2 A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic if and only if there exist four projectors $\{R_k\}_{k \in \{1,2,3,4\}}$ compatible with C , constants $\mu > \nu > 0$ and a sequence of positive real numbers $(\alpha_n)_{n \geq 0}$ such that

$$(t''_1) \quad \|\Phi(m+p, m, x)R_1(x)v\| \leq \alpha_m \|R_1(x)v\| e^{-\nu p} \quad (5.18)$$

$$(t''_2) \quad \|R_2(x)v\| \leq \alpha_p \|\Phi(m+p, m, x)R_2(x)v\| e^{-\mu p} \quad (5.19)$$

$$(t''_3) \quad \|R_3(x)v\| \leq \alpha_p \|\Phi(m+p, m, x)R_3(x)v\| e^{\nu p} \quad (5.20)$$

$$(t''_4) \quad \|\Phi(m+p, m, x)R_4(x)v\| \leq \alpha_m \|R_4(x)v\| e^{\mu p} \quad (5.21)$$

for all $m, p \in \mathbb{N}$ and all $(x, v) \in Y$.

Proof. *Necessity.* As C is exponentially trichotomic, according to Proposition 5.1 there exist three projectors $\{P_k\}_{k \in \{1,2,3\}}$ compatible with C , constants $\nu_1 \leq \nu_2 \leq 0 \leq \nu_3 \leq \nu_4$ and a sequence of positive real numbers $(a_n)_{n \geq 0}$ such that relations (5.9)–(5.12) hold.

We will define the projectors

$$R_1 = P_1, \quad R_2 = P_2, \quad R_3 = I - P_1 \text{ and } R_4 = I - P_2,$$

such that

$$R_3R_4 = R_4R_3 = P_3.$$

Projectors R_1, R_2, R_3 și R_4 are compatible with C . Let us define

$$\mu = \nu_3 = \nu_4 > 0, \nu = -\nu_1 = -\nu_2 > 0 \text{ and } \alpha_n = a_n, n \in \mathbb{N}.$$

Hence, relations (5.18)–(5.21) hold.

Sufficiency. We consider the projectors

$$P_1 = R_1, P_2 = R_2, P_3 = R_3R_4.$$

These are compatible with C .

The statements of Proposition 5.1 follow if we consider

$$\nu_1 = \nu_2 = -\nu < 0, \nu_3 = \nu_4 = \mu > 0 \text{ and } a_n = \alpha_n, n \in \mathbb{N}.$$

Hence, C is exponentially trichotomic, which ends the proof. \square

References

- [1] S. Elaydi, O. Hajek, *Exponential trichotomy of differential systems*, J. Math. Anal. Appl. **129** (1988), 362–374
- [2] M. Megan, C. Stoica, *On exponential trichotomy for discrete time skew-evolution semiflows in Banach spaces*, Bul. Științ. Univ. Politeh. Timiș. Ser. Mat. Fiz. **51** (65), Fascicle 2 (2006), 25–32
- [3] M. Megan, C. Stoica, *Trichotomy for discrete skew-evolution semiflows in Banach spaces*, Ann. of Tiberiu Popoviciu Sem. on Funct. Eqs, Approx. and Conv. **5** (2007), 79–88

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