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EQUIVALENCES BETWEEN FUSION SYSTEMS OF FINITE GROUPS OF LIE TYPE

CARLES BROTO, JESPER M. MØLLER, AND BOB OLIVER

ABSTRACT. We prove, for certain pairs G, G' of finite groups of Lie type, that the p -fusion systems $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G')$ are equivalent. In other words, there is an isomorphism between a Sylow p -subgroup of G and one of G' which preserves p -fusion. This occurs, for example, when $G = \mathbb{G}(q)$ and $G' = \mathbb{G}(q')$ for a simple Lie “type” \mathbb{G} , and q and q' are prime powers, both prime to p , which generate the same closed subgroup of p -adic units. Our proof uses homotopy theoretic properties of the p -completed classifying spaces of G and G' , and we know of no purely algebraic proof of this result.

When G is a finite group and p is a prime, the fusion system $\mathcal{F}_p(G)$ is the category whose objects are the p -subgroups of G , and whose morphisms are the homomorphisms between subgroups induced by conjugation in G . If G' is another finite group, then $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G')$ are isotypically equivalent if there is an equivalence of categories between them which commutes, up to natural isomorphism of functors, with the forgetful functors from $\mathcal{F}_p(-)$ to the category of groups. Alternatively, $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G')$ are isotypically equivalent if there is an isomorphism between Sylow p -subgroups of G and of G' which is “fusion preserving” in the sense of Definition 1.2 below.

The goal of this paper is to use methods from homotopy theory to prove that certain pairs of fusion systems of finite groups of Lie type are isotypically equivalent. Our main result is the following theorem.

Theorem A. *Fix a prime p , a connected reductive integral group scheme \mathbb{G} , and a pair of prime powers q and q' both prime to p . Then the following hold, where “ \simeq ” always means isotypically equivalent.*

- (a) $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ if $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$ as subgroups of \mathbb{Z}_p^\times .
- (b) If \mathbb{G} is of type A_n , D_n , or E_6 , and τ is a graph automorphism of \mathbb{G} , then $\mathcal{F}_p(\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\tau\mathbb{G}(q'))$ if $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$ as subgroups of \mathbb{Z}_p^\times .
- (c) If the Weyl group of \mathbb{G} contains an element which acts on the maximal torus by inverting all elements, then $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ (or $\mathcal{F}_p(\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\tau\mathbb{G}(q'))$) for τ as in (b) if $\overline{\langle -1, q \rangle} = \overline{\langle -1, q' \rangle}$ as subgroups of \mathbb{Z}_p^\times .
- (d) If \mathbb{G} is of type A_n , D_n for n odd, or E_6 , and τ is a graph automorphism of \mathbb{G} of order two, then $\mathcal{F}_p(\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ if $\overline{\langle -q \rangle} = \overline{\langle q' \rangle}$ as subgroups of \mathbb{Z}_p^\times .

Here, in all cases, $\mathbb{G}(q)$ means the fixed subgroup of the field automorphism ψ^q acting on $\mathbb{G}(\overline{\mathbb{F}}_q)$, and $\tau\mathbb{G}(q)$ means the fixed subgroup of $\tau\psi^q$ acting on $\mathbb{G}(\overline{\mathbb{F}}_q)$.

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We remark here that this theorem does not apply when comparing fusion systems of $SO_n^\pm(q)$ and $SO_n^\pm(q')$ for even n , at least not when q or q' is a power of 2, since $SO_n(K)$ is not connected when K is algebraically closed of characteristic two. Instead, one must compare the groups $\Omega_n^\pm(-)$. For example, for even $n \geq 4$, $\Omega_n^+(4)$ and $\Omega_n^+(7)$ have equivalent 3-fusion systems, while $SO_n^+(4)$ and $SO_n^+(7)$ do not.

Points (a)–(c) of Theorem A will be proven in Proposition 3.2, where we deal with the more general situation where \mathbb{G} is reductive (thus including cases such as $\mathbb{G} = GL_n$). Point (d) will be proven as Proposition 3.3. In all cases, this will be done by showing that the p -completed classifying spaces of the two groups are homotopy equivalent. A theorem of Martino and Priddy (Theorem 1.5 below) then implies that the fusion systems are isotypically equivalent.

Since p -completion of spaces plays a central role in our proofs, we give a very brief outline here of what it means, and refer to the book of Bousfield and Kan [BK] for more details. They define p -completion as a functor from spaces to spaces, which we denote $(-)_p^\wedge$ here, and this functor comes with a map $X \xrightarrow{\kappa_p(X)} X_p^\wedge$ which is natural in X . For any map $f: X \longrightarrow Y$, f_p^\wedge is a homotopy equivalence if and only if f is a mod p equivalence; i.e., $H^*(f; \mathbb{F}_p)$ is an isomorphism from $H^*(Y; \mathbb{F}_p)$ to $H^*(X; \mathbb{F}_p)$. A space X is called “ p -good” if $\kappa_p(X_p^\wedge)$ is a homotopy equivalence (equivalently, $\kappa_p(X)$ is a mod p equivalence). In particular, all spaces with finite fundamental group are p -good. If X is p -good, then $\kappa_p(X): X \longrightarrow X_p^\wedge$ is universal among all mod p equivalences $X \longrightarrow Y$. If X and Y are both p -good, then $X_p^\wedge \simeq Y_p^\wedge$ (the p -completions are homotopy equivalent) if and only if there is a third space Z , and mod p equivalences $X \longrightarrow Z \longleftarrow Y$.

By a theorem of Friedlander (stated as Theorem 3.1 below), $B(\tau\mathbb{G}(q))_p^\wedge$ is the homotopy fixed space (Definition 2.1) of the action of $\tau\psi^q$ on $B\mathbb{G}(\mathbb{C})_p^\wedge$. Theorem A follows from this together with a general result about homotopy fixed spaces (Theorem 2.4), which says that under certain conditions on a space X , two self homotopy equivalences have equivalent homotopy fixed sets if they generate the same closed subgroup of the group of all self equivalences.

Corresponding results for the Suzuki and Ree groups can also be shown using this method of proof. But since there are much more elementary proofs of these results (all equivalences are induced by inclusions of groups), and since it seemed difficult to find a nice formulation of the theorem which included everything, we decided to leave them out of the statement.

As another application of these results, we prove that for any prime p and any prime power $q \equiv 1 \pmod{p}$, the fusion systems $\mathcal{F}_p(G_2(q))$ and $\mathcal{F}_p({}^3D_4(q))$ are isotypically equivalent if $p \neq 3$, and the fusion systems $\mathcal{F}_p(F_4(q))$ and $\mathcal{F}_p({}^2E_6(q))$ are isotypically equivalent if $p \neq 2$ (Example 4.5). However, while this provides another example of how our methods can be applied, the first equivalence (at least) can also be shown by much simpler methods.

Theorem A is certainly not surprising to the experts, who are familiar with it by observation. It seems likely that it can also be shown directly using a purely algebraic proof, but the people we have asked do not know of one, and there does not seem to be any in the literature. There is a very closely related result by Michael Larsen [GR, Theorem A.12], restated below as Theorem 3.4. It implies that two Chevalley groups $\mathbb{G}(K)$ and $\mathbb{G}(K')$ over algebraically closed fields of characteristic prime to p have equivalent p -fusion systems when defined appropriately for these infinite groups. There are standard methods for comparing the finite subgroups of $\mathbb{G}(\overline{\mathbb{F}}_q)$ (of order prime to q) with those in its finite Chevalley subgroups (see, e.g.,

Proposition 3.5), but we have been unable to get enough control over them to prove Theorem A using Larsen's theorem.

The paper is organized as follows. In Section 1, we give a general survey of fusion categories of finite groups and their relationship to p -completed classifying spaces. Then, in Section 2, we prove a general theorem (Theorem 2.4) comparing homotopy fixed points of different actions on the same space, and apply it in Section 3 to prove Theorem A. In Section 4, we show a second result about homotopy fixed points, which is used to prove the result comparing fusion systems of $G_2(q)$ and ${}^3D_4(q)$, and $F_4(q)$ and ${}^2E_6(q)$. We finish with a brief sketch in Section 5 of some elementary techniques for proving special cases of Theorem A for some classical groups, and more generally a comparison of fusion systems of classical groups at odd primes.

1. FUSION CATEGORIES

We begin with a quick summary of what is needed here about fusion systems of finite groups.

Definition 1.1. *For any finite group G and any prime p , $\mathcal{F}_p(G)$ denotes the category whose objects are the p -subgroups of G , and where*

$$\text{Mor}_{\mathcal{F}_p(G)}(P, Q) = \{\varphi \in \text{Hom}(P, Q) \mid \varphi = c_x \text{ for some } x \in G\}.$$

Here, c_x denotes the conjugation homomorphism: $c_x(g) = xgx^{-1}$. If $S \in \text{Syl}_p(G)$ is a Sylow p -subgroup, then $\mathcal{F}_S(G) \subseteq \mathcal{F}_p(G)$ denotes the full subcategory with objects the subgroups of S .

A functor $F: \mathcal{C} \longrightarrow \mathcal{C}'$ is an *equivalence* of categories if it induces bijections on isomorphism classes of objects and on all morphism sets. This is equivalent to the condition that there be a functor from \mathcal{C}' to \mathcal{C} such that both composites are naturally isomorphic to the identity. An inclusion of a full subcategory is an equivalence if and only if every object in the larger category is isomorphic to some object in the smaller one. Thus when G is finite and $S \in \text{Syl}_p(G)$, the inclusion $\mathcal{F}_S(G) \subseteq \mathcal{F}_p(G)$ is an equivalence of categories by the Sylow theorems.

In general, we write $\Psi: \mathcal{C}_1 \xrightarrow{\cong} \mathcal{C}_2$ to mean that Ψ is an isomorphism of categories (bijective on objects and on morphisms); and $\Psi: \mathcal{C}_1 \xrightarrow{\cong} \mathcal{C}_2$ to mean that Ψ is an equivalence of categories.

In the following definition, for any finite G , λ_G denotes the forgetful functor from $\mathcal{F}_p(G)$ to the category of groups.

Definition 1.2. *Fix a prime p , a pair of finite groups G and G^* , and Sylow p -subgroups $S \in \text{Syl}_p(G)$ and $S^* \in \text{Syl}_p(G^*)$.*

- (a) *An isomorphism $\varphi: S \xrightarrow{\cong} S^*$ is fusion preserving if for all $P, Q \leq S$ and $\alpha \in \text{Hom}(P, Q)$,*

$$\alpha \in \text{Mor}_{\mathcal{F}_p(G)}(P, Q) \iff \varphi\alpha\varphi^{-1} \in \text{Mor}_{\mathcal{F}_p(G^*)}(\varphi(P), \varphi(Q)).$$

- (b) *An equivalence of categories $T: \mathcal{F}_p(G) \longrightarrow \mathcal{F}_p(G^*)$ is isotypical if there is a natural isomorphism of functors $\omega: \lambda_G \longrightarrow \lambda_{G^*} \circ T$; i.e., if there are isomorphisms $\omega_P: P \xrightarrow{\cong} T(P)$ such that $\omega_Q \circ \varphi = T(\varphi) \circ \omega_P$ for each $\varphi \in \text{Hom}_G(P, Q)$.*

In other words, in the above situation, an isomorphism $\varphi: S \longrightarrow S^*$ is fusion preserving if and only if it induces an isomorphism from $\mathcal{F}_S(G) \xrightarrow{\cong} \mathcal{F}_{S^*}(G^*)$ by

sending P to $\varphi(P)$ and α to $\varphi\alpha\varphi^{-1}$. Any such isomorphism of categories extends to an equivalence $\mathcal{F}_p(G) \xrightarrow{\simeq} \mathcal{F}_p(G^*)$, which is easily seen to be isotypical. In fact, two fusion categories $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G^*)$ are isotypically equivalent if and only if there is a fusion preserving isomorphism between Sylow p -subgroups, as is shown in the following proposition.

Proposition 1.3. *Fix a pair of finite groups G and G^* , a prime p and Sylow p -subgroups $S \leq G$ and $S^* \leq G^*$. Then the following are equivalent:*

- (a) *There is a fusion preserving isomorphism $\varphi: S \xrightarrow{\cong} S^*$.*
- (b) *$\mathcal{F}_p(G)$ and $\mathcal{F}_p(G^*)$ are isotypically equivalent.*
- (c) *There are bijections $\text{Rep}(P, G) \xrightarrow{\cong} \text{Rep}(P, G^*)$, for all finite p -groups P , which are natural in P .*

Proof. This was essentially shown by Martino and Priddy [MP], but not completely explicitly. By the above remarks, (a) implies (b).

(b \implies c) : Fix an isotypical equivalence $T: \mathcal{F}_p(G) \longrightarrow \mathcal{F}_p(G^*)$, and let ω be an associated natural isomorphism. Thus $\omega_P \in \text{Iso}(P, T(P))$ for each p -subgroup $P \leq G$, and $\omega_Q \circ \alpha = T(\alpha) \circ \omega_P$ for all $\alpha \in \text{Hom}_G(P, Q)$. For each p -group Q , ω defines a bijection from $\text{Hom}(Q, G)$ to $\text{Hom}(Q, G^*)$ by sending ρ to $\omega_{\rho(Q)} \circ \rho$. For $\alpha, \beta \in \text{Hom}(Q, G)$, the diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\alpha} & \alpha(Q) & \xrightarrow{\omega_{\alpha(Q)}} & T(\alpha(Q)) \\ \parallel & & \downarrow \gamma & & \downarrow T(\gamma) \\ Q & \xrightarrow{\beta} & \beta(Q) & \xrightarrow{\omega_{\beta(Q)}} & T(\beta(Q)) \end{array},$$

together with the fact that T is an equivalence, proves that α and β are G -conjugate (there exists γ which makes the left hand square commute) if and only if $\omega_{\alpha(Q)} \circ \alpha$ and $\omega_{\beta(Q)} \circ \beta$ are G^* -conjugate (there exists $T(\gamma)$). Thus T induces bijections $\Phi: \text{Rep}(Q, G) \xrightarrow{\cong} \text{Rep}(Q, G^*)$; and similar arguments show that Φ is natural in Q .

(c \implies a) : Fix a natural bijection $\Phi: \text{Rep}(-, G) \xrightarrow{\cong} \text{Rep}(-, G^*)$ of functors on finite p -groups. By naturality, Φ preserves kernels, and hence restricts to a bijection between classes of injections. In particular, there are injections of S into G^* and S^* into G , and thus $S \cong S^*$. Since conjugation defines a fusion preserving isomorphism between any two Sylow p -subgroups of G^* , we can assume $S^* = \text{Im}(\Phi_S(\text{incl}_S^G))$. Using the naturality of Φ , it is straightforward to check that $\Phi_S(\text{incl}_S^G)$ is fusion preserving as an isomorphism from S to S^* . \square

We also note the following, very elementary result about comparing fusion systems.

Proposition 1.4. *Fix a prime p , and a pair of groups G_1 and G_2 such that $\mathcal{F}_p(G_1)$ is isotypically equivalent to $\mathcal{F}_p(G_2)$. Then the following hold, where “ \simeq ” always means isotypically equivalent.*

- (a) *If $Z_i \leq Z(G_i)$ is central of order prime to p , then $\mathcal{F}_p(G_i/Z_i) \simeq \mathcal{F}_p(G_i)$.*
- (b) *If $Z_1 \leq Z(G_1)$ is a central p -subgroup, and $Z_2 \leq G_2$ is its image under some fusion preserving isomorphism between Sylow p -subgroups of the G_i , then $\mathcal{F}_p(G_1) \simeq \mathcal{F}_p(C_{G_2}(Z_2))$ and $\mathcal{F}_p(G_1/Z_1) \simeq \mathcal{F}_p(C_{G_2}(Z_2)/Z_2)$.*

(c) $\mathcal{F}_p([G_1, G_1]) \simeq \mathcal{F}_p([G_2, G_2])$.

Proof. Points (a) and (b) are elementary. To prove (c), first fix Sylow subgroups $S_i \in \text{Syl}_p(G_i)$ and a fusion preserving isomorphism $\varphi: S_1 \xrightarrow{\cong} S_2$. By the focal subgroup theorem (cf. [Go, Theorem 7.3.4]), $\varphi(S_1 \cap [G_1, G_1]) = S_2 \cap [G_2, G_2]$. By [BCGLO2, Theorem 4.4], for each $i = 1, 2$, there is a unique fusion subsystem “of p -power index” in $\mathcal{F}_{S_i}(G_i)$ over the focal subgroup $S_i \cap [G_i, G_i]$, which must be the fusion system of $[G_i, G_i]$. Hence φ restricts to an isomorphism which is fusion preserving with respect to the commutator subgroups. \square

Proposition 1.4 implies, for example, that whenever $\mathcal{F}_p(GL_n(q)) \simeq \mathcal{F}_p(GL_n(q'))$ for q and q' prime to p , then there are also equivalences $\mathcal{F}_p(SL_n(q)) \simeq \mathcal{F}_p(SL_n(q'))$, $\mathcal{F}_p(PSL_n(q)) \simeq \mathcal{F}_p(PSL_n(q'))$, etc.

The following theorem of Martino and Priddy shows that the p -fusion in a finite group is determined by the homotopy type of its p -completed classifying space. The converse (the Martino-Priddy conjecture) is also true, but the only known proof uses the classification of finite simple groups [O1, O2].

Theorem 1.5. *Assume p is a prime, and G and G' are finite groups, such that $BG_p^\wedge \simeq BG'_p^\wedge$. Then $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G')$ are isotypically equivalent.*

Proof. This was shown by Martino and Priddy in [MP]. The key ingredient in the proof is a theorem of Mislin [Ms, pp.457–458], which says that for any finite p -group Q and any finite group G , there is a bijection

$$\text{Rep}(Q, G) \xrightarrow[\cong]{B_p^\wedge} [BQ, BG_p^\wedge],$$

where B_p^\wedge sends the class of a homomorphism $\rho: Q \longrightarrow G$ to the p -completion of $B\rho: BQ \longrightarrow BG$. Thus any homotopy equivalence $BG_p^\wedge \xrightarrow{\cong} BG'_p^\wedge$ induces bijections $\text{Rep}(Q, G) \cong \text{Rep}(Q, G')$, for all p -groups Q , which are natural in Q . The theorem now follows from Proposition 1.3. \square

The following proposition will also be useful. When $H \leq G$ is a pair of groups, we regard $\mathcal{F}_p(H)$ as a subcategory of $\mathcal{F}_p(G)$.

Proposition 1.6. *If $H \leq G$ is a pair of groups, then $\mathcal{F}_p(H)$ is a full subcategory of $\mathcal{F}_p(G)$ if and only if the induced map*

$$\text{Rep}(P, H) \longrightarrow \text{Rep}(P, G)$$

is injective for all finite p -groups P .

Proof. Assume $\text{Rep}(P, H)$ injects into $\text{Rep}(P, G)$ for all P . For each pair of p -subgroups $P, Q \leq H$ and each $\varphi \in \text{Hom}_G(P, Q)$, $[\text{incl}_P^G] = [\text{incl}_Q^G \circ \varphi]$ in $\text{Rep}(P, G)$, so $[\text{incl}_P^H] = [\text{incl}_Q^H \circ \varphi]$ in $\text{Rep}(P, H)$, and thus $\varphi \in \text{Hom}_H(P, Q)$. This proves that $\mathcal{F}_p(H)$ is a full subcategory of $\mathcal{F}_p(G)$.

Conversely, assume $\mathcal{F}_p(H)$ is a full subcategory. Fix a finite p -group P , and $\alpha, \beta \in \text{Hom}(P, H)$ such that $[\alpha] = [\beta]$ in $\text{Rep}(P, G)$. Let $\varphi \in \text{Hom}_G(\alpha(P), \beta(P))$ be such that $\varphi \circ \alpha = \beta$. Then $\varphi \in \text{Hom}_H(\alpha(P), \beta(P))$ since $\mathcal{F}_p(H)$ is a full subcategory, and so $[\alpha] = [\beta]$ in $\text{Rep}(P, H)$. This proves injectivity. \square

Our goal in the next three sections is to construct isotypical equivalences between fusion systems of finite groups at a prime p by constructing homotopy equivalences between their p -completed classifying spaces.

2. HOMOTOPY FIXED POINTS OF SELF HOMOTOPY EQUIVALENCES

We start by defining homotopy orbit spaces and homotopy fixed spaces for a self homotopy equivalence of a space; i.e., for a homotopy action of the group \mathbb{Z} . As usual, I denotes the unit interval $[0, 1]$.

Definition 2.1. Fix a space X , and a map $\alpha: X \longrightarrow X$.

- (a) When α is a homeomorphism, the homotopy orbit space $X_{h\alpha}$ and homotopy fixed space $X^{h\alpha}$ of α are defined as follows:
- $X_{h\alpha} = (X \times I)/\sim$, where $(x, 1) \sim (\alpha(x), 0)$ for all $x \in X$.
 - $X^{h\alpha}$ is the space of all continuous maps $\gamma: I \longrightarrow X$ such that $\gamma(1) = \alpha(\gamma(0))$.
- (b) When α is a homotopy equivalence but not a homeomorphism, define the double mapping telescope of α by setting

$$\text{Tel}(\alpha) = (X \times I \times \mathbb{Z})/\sim \quad \text{where} \quad (x, 1, n) \sim (\alpha(x), 0, n+1) \quad \forall x \in X, n \in \mathbb{Z}.$$

Let $\hat{\alpha}: \text{Tel}(\alpha) \longrightarrow \text{Tel}(\alpha)$ be the homeomorphism $\hat{\alpha}([x, t, n]) = [x, t, n-1]$. Then set

$$X_{h\alpha} = \text{Tel}(\alpha)_{h\hat{\alpha}} \quad \text{and} \quad X^{h\alpha} = \text{Tel}(\alpha)^{h\hat{\alpha}},$$

where $\text{Tel}(\alpha)_{h\hat{\alpha}}$ and $\text{Tel}(\alpha)^{h\hat{\alpha}}$ are defined as in (a).

The space $X_{h\alpha}$, when defined as in (a), is also known as the *mapping torus* of α . In this situation, $X^{h\alpha}$ is clearly the space of sections of the bundle $X_{h\alpha} \xrightarrow{p_\alpha} S^1$, defined by identifying S^1 with $I/(0 \sim 1)$.

When α is a homotopy equivalence, the double mapping telescope $\text{Tel}(\alpha)$ is homotopy equivalent to X . Thus the idea in part (b) of the above definition is to replace (X, α) by a pair $(\hat{X}, \hat{\alpha})$ with the same homotopy type, but such that $\hat{\alpha}$ is a homeomorphism. The following lemma helps to motivate this approach.

Lemma 2.2. Fix spaces X and Y , a homotopy equivalence $f: X \longrightarrow Y$, and homeomorphisms $\alpha: X \longrightarrow X$ and $\beta: Y \longrightarrow Y$ such that $\beta \circ f \simeq f \circ \alpha$. Then there are homotopy equivalences

$$X_{h\alpha} \simeq Y_{h\beta} \quad \text{and} \quad X^{h\alpha} \simeq Y^{h\beta},$$

where these spaces are defined as in Definition 2.1(a).

Proof. Fix a homotopy $F: X \times I \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = \beta^{-1} \circ f \circ \alpha$. Define

$$h: \begin{array}{ccc} X_{h\alpha} & \longrightarrow & Y_{h\beta} \\ = (X \times I)/\sim & & = (Y \times I)/\sim \end{array}$$

by setting $h(x, t) = (F(x, t), t)$. If g is a homotopy inverse to f and G is a homotopy from g to $\alpha^{-1} \circ g \circ \beta$, then these define a map from $Y_{h\beta}$ to $X_{h\alpha}$ which is easily seen to be a homotopy inverse to h .

We thus have a homotopy equivalence between $X_{h\alpha}$ and $X_{h\beta}$ which commutes with the projections to S^1 . Hence the spaces $X^{h\alpha}$ and $Y^{h\beta}$ of sections of these bundles are homotopy equivalent. \square

Lemma 2.2 also shows that when α is a homeomorphism, the two constructions of $X_{h\alpha}$ and $X^{h\alpha}$ given in parts (a) and (b) of Definition 2.1 are homotopy equivalent.

Remark 2.3. *The homotopy fixed point space $X^{h\alpha}$ of a homeomorphism α can also be described as the homotopy pullback of the maps*

$$X \xrightarrow{\Delta} X \times X \xleftarrow{(\text{Id}, \alpha)} X,$$

where Δ is the diagonal map $\Delta(x) = (x, x)$. In other words, $X^{h\alpha}$ is the space of triples (x_1, x_2, ϕ) , where $x_1, x_2 \in X$, and ϕ is a path in $X \times X$ from $\Delta(x_1) = (x_1, x_1)$ to $(x_2, \alpha(x_2))$. Thus ϕ is a pair of paths in X , one from x_1 to x_2 and the other from x_1 to $\alpha(x_2)$, and these two paths can be composed to give a single path from x_2 to $\alpha(x_2)$ which passes through the (arbitrary) point x_1 . Hence this definition is equivalent to the one given above. It helps to explain the name ‘‘homotopy fixed point set’’, since the ordinary pullback of the above maps can be identified with the space of all $x \in X$ such that $\alpha(x) = x$.

For any space X , we set $\widehat{H}^i(X; \mathbb{Z}_p) = \varprojlim H^i(X; \mathbb{Z}/p^k)$ for each i , and let $\widehat{H}^*(X; \mathbb{Z}_p)$ be the sum of the $\widehat{H}^i(X; \mathbb{Z}_p)$. If $H^*(X; \mathbb{F}_p)$ is finite in each degree, then $\widehat{H}^*(X; \mathbb{Z}_p)$ is isomorphic to the usual cohomology ring $H^*(X; \mathbb{Z}_p)$ with coefficients in the p -adics.

By the p -adic topology on $\text{Out}(X)$, we mean the topology for which $\{U_k\}$ is a basis of open neighborhoods of the identity, where $U_k \leq \text{Out}(X)$ is the group of automorphisms which induce the identity on $H^*(X; \mathbb{Z}/p^k)$. Thus this topology is Hausdorff if and only if $\text{Out}(X)$ is detected on $\widehat{H}^*(X; \mathbb{Z}_p)$.

Theorem 2.4. *Fix a prime p . Let X be a connected, p -complete space such that*

- $H^*(X; \mathbb{F}_p)$ is noetherian, and
- $\text{Out}(X)$ is detected on $\widehat{H}^*(X; \mathbb{Z}_p)$.

Let α and β be self homotopy equivalences of X which generate the same closed subgroup of $\text{Out}(X)$ under the p -adic topology. Then $X^{h\alpha} \simeq X^{h\beta}$.

Proof. Upon replacing X by the double mapping telescope of α , we can assume that α is a homeomorphism. By Lemma 2.2, this does not change the homotopy type of $X^{h\alpha}$ or $X^{h\beta}$.

Let $r \geq 1$ be the smallest integer prime to p such that the action of α^r on $H^*(X; \mathbb{F}_p)$ has p -power order. (The action of α on the noetherian ring $H^*(X; \mathbb{F}_p)$ has finite order.) Since $\langle \alpha \rangle = \langle \beta \rangle$, $H^*(\alpha; \mathbb{F}_p)$ and $H^*(\beta; \mathbb{F}_p)$ generate the same subgroup in $\text{Aut}(H^*(X; \mathbb{F}_p))$, hence have the same order, and so r is also the smallest integer prime to p such that the action of β^r on $H^*(X; \mathbb{F}_p)$ has p -power order.

Let

$$p_\alpha: X_{h\alpha} \longrightarrow S^1 \quad \text{and} \quad p_\beta: X_{h\beta} \longrightarrow S^1$$

be the canonical fibrations. Let

$$\tilde{p}_\alpha: \tilde{X}_{h\alpha} \longrightarrow S^1 \quad \text{and} \quad \tilde{p}_\beta: \tilde{X}_{h\beta} \longrightarrow S^1$$

be their r -fold cyclic covers, considered as equivariant maps between spaces with \mathbb{Z}/r -action. Thus $\tilde{X}_\alpha \simeq X_{h\alpha^r}$ (and \tilde{p}_α is its canonical fibration), and the same for \tilde{X}_β . Also, since each section of p_α lifts to a unique equivariant section of \tilde{p}_α , and each equivariant section factors through as section of p_α by taking the orbit map, $X^{h\alpha}$ is the space of all equivariant sections of \tilde{p}_α .

Since α^r acts on $H^*(X; \mathbb{F}_p)$ with order a power of p , this action is nilpotent; and by [BK, II.5.1], the homotopy fiber of $\tilde{p}_\alpha \wedge_p$ has the homotopy type of $X_p^\wedge \simeq X$.

Thus the rows in the following diagram are (homotopy) fibration sequences:

$$\begin{array}{ccccc} X & \longrightarrow & \tilde{X}_{h\alpha} & \xrightarrow{\tilde{p}_\alpha} & S^1 \\ \parallel & & \downarrow \kappa_p & & \downarrow \kappa_p \\ X & \longrightarrow & (\tilde{X}_{h\alpha})_p^\wedge & \xrightarrow{\tilde{p}_{\alpha_p}^\wedge} & S_p^{1\wedge} \end{array},$$

and so the right hand square is a homotopy pullback. By the definition of p -completion in [BK], the induced actions of \mathbb{Z}/r on $S_p^{1\wedge}$ and on $(\tilde{X}_{h\alpha})_p^\wedge$ are free, since the actions on the uncompleted spaces are free. Hence $X^{h\alpha}$ can be described (up to homotopy), not only as the space of \mathbb{Z}/r -equivariant sections of \tilde{p}_α , but also as the space of \mathbb{Z}/r -equivariant liftings of $\kappa_p(S^1): S^1 \longrightarrow S_p^{1\wedge}$ along $\tilde{p}_{\alpha_p}^\wedge$. In other words,

$$X^{h\alpha} \simeq \text{fiber}(\text{map}_{\mathbb{Z}/r}(S^1, (\tilde{X}_{h\alpha})_p^\wedge) \xrightarrow{\tilde{p}_{\alpha_p}^\wedge \circ -} \text{map}_{\mathbb{Z}/r}(S^1, S_p^{1\wedge})) \quad (1)$$

(the fiber over $\kappa_p(S^1)$).

Consider again the p -completed fibration sequence

$$X \longrightarrow (\tilde{X}_{h\alpha})_p^\wedge \xrightarrow{\tilde{p}_\alpha} BS_p^{1\wedge},$$

and its orbit fibration

$$X \longrightarrow (\tilde{X}_{h\alpha})_p^\wedge / (\mathbb{Z}/r) \xrightarrow{\hat{p}_\alpha} BS_p^{1\wedge} / (\mathbb{Z}/r).$$

Here, $\pi_1(BS_p^{1\wedge}) \cong \mathbb{Z}_p$, and $\pi_1(BS_p^{1\wedge} / (\mathbb{Z}/r)) \cong \mathbb{Z}_p \times \mathbb{Z}/r$ (the completion of \mathbb{Z} with respect to the ideals $rp^i\mathbb{Z}$). Since α^r acts on $H^*(X; \mathbb{Z}/p)$ with order a power of p , it also acts on each $H^*(X; \mathbb{Z}/p^k)$ with order a power of p , and hence the homotopy action of $\pi_1(BS_p^{1\wedge})$ on X has as image the p -adic closure of $\langle \alpha^r \rangle$. Thus the homotopy action of $\pi_1(BS_p^{1\wedge} / (\mathbb{Z}/r))$ on X , defined by the fibration \hat{p}_α , has as image the p -adic closure of $\langle \alpha \rangle$.

Since $\beta \in \overline{\langle \alpha \rangle}$ by assumption, we can represent it by a map $S^1 \xrightarrow{b} S_p^{1\wedge} / (\mathbb{Z}/r)$. Define Y to be the homotopy pullback defined by the following diagram:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \xrightarrow{p'} & S^1 \\ \parallel & & \downarrow f & & \downarrow b \\ X & \longrightarrow & (\tilde{X}_{h\alpha})_p^\wedge / (\mathbb{Z}/r) & \xrightarrow{\hat{p}_\alpha} & S_p^{1\wedge} / (\mathbb{Z}/r) \end{array}.$$

Thus the canonical generator of $\pi_1(S^1)$ induces $\beta \in \text{Out}(X)$, and so $(Y, p') \simeq (X_{h\beta}, p_\beta)$. Upon taking r -fold covers and then completing the first row, this induces a map of fibrations

$$\begin{array}{ccccc} X & \longrightarrow & (\tilde{X}_{h\beta})_p^\wedge & \xrightarrow{\tilde{p}_\beta} & S_p^{1\wedge} \\ \parallel & & \simeq \downarrow f & & \downarrow \tilde{b} \\ X & \longrightarrow & (\tilde{X}_{h\alpha})_p^\wedge & \xrightarrow{\tilde{p}_\alpha} & S_p^{1\wedge} \end{array},$$

which is an equivalence since \tilde{b} is an equivalence (since $\langle \beta \rangle$ is dense in $\overline{\langle \alpha \rangle}$). Also, f and \tilde{b} are equivariant with respect to some automorphism of \mathbb{Z}/r .

The maps $S^1 \longrightarrow S_p^{1\wedge} \longleftarrow (X)_{h\mathbb{Z}_p}$ determine a commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathbb{Z}/r}(S_p^{1\wedge}, (\tilde{X}_{h\alpha})_p^\wedge) & \longrightarrow & \mathrm{map}_{\mathbb{Z}/r}(S^1, (\tilde{X}_{h\alpha})_p^\wedge) \\ \downarrow t & & \downarrow u \\ \mathrm{map}_{\mathbb{Z}/r}(S_p^{1\wedge}, S_p^{1\wedge}) & \longrightarrow & \mathrm{map}_{\mathbb{Z}/r}(S^1, S_p^{1\wedge}) \end{array} \quad (2)$$

which in turn induces a map between the respective fibres. The horizontal arrows in (2) are homotopy equivalences because the target spaces in the respective mapping spaces are p -complete and \mathbb{Z}/r acts freely on the source spaces. Hence the fibers of the vertical maps in (2) are homotopy equivalent. Since u has fiber $X^{h\alpha}$ by (1), this proves that $X^{h\alpha}$ has the homotopy type of the space of equivariant sections of the bundle $(X_{h\alpha})_p^\wedge \xrightarrow{\tilde{p}_{\alpha p}^\wedge} S_p^{1\wedge}$. Since this bundle is equivariantly equivalent to the one with total space $(\tilde{X}_{h\beta})_p^\wedge$, the same argument applied to β proves that $X^{h\alpha} \simeq X^{h\beta}$. \square

Our main application of Theorem 2.4 is to the case where $X = BG_p^\wedge$ for a compact connected Lie group G .

Corollary 2.5. *Let G be a compact connected Lie group, and let $\alpha, \beta \in \mathrm{Out}(BG_p^\wedge)$ be two self equivalences of the p -completed classifying space. If α and β generate the same closed subgroup of $\mathrm{Out}(BG_p^\wedge)$, then $(BG_p^\wedge)^{h\alpha} \simeq (BG_p^\wedge)^{h\beta}$.*

Proof. From the spectral sequence for the fibration $U(n)/G \longrightarrow BG \longrightarrow BU(n)$ for any embedding $G \leq U(n)$, we see that $H^*(BG; \mathbb{F}_p)$ is noetherian. By [JMO, Theorem 2.5], $\mathrm{Out}(BG_p^\wedge)$ is detected by its restriction to BT_p^\wedge for a maximal torus T , and hence by invariant theory is detected by $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{H}^*(BG; \mathbb{Z}_p)$. So the hypotheses of Theorem 2.4 hold when $X = BG_p^\wedge$. \square

The hypotheses on X in Theorem 2.4 also apply whenever X is the classifying space of a connected p -compact group. The condition on cohomology holds by [DW, Theorem 2.3]. Automorphisms are detected by restriction to the maximal torus by [AGMV, Theorem 1.1] (when p is odd) and [ML, Theorem 1.1] (when $p = 2$).

3. FINITE GROUPS OF LIE TYPE

We first fix our terminology. Let \mathbb{G} be a connected reductive integral group scheme. Thus for each algebraically closed field K , $\mathbb{G}(K)$ is a complex connected algebraic group such that for some finite central subgroup $Z \leq Z(\mathbb{G}(K))$, $\mathbb{G}(K)/Z$ is the product of a K -torus and a semisimple group. For any prime power q , we let $\mathbb{G}(q)$ be the fixed subgroup of the field automorphism ψ^q . Also, if τ is any automorphism of \mathbb{G} of finite order, then ${}^\tau\mathbb{G}(q)$ will denote the fixed subgroup of the composite $\tau\psi^q$ on $\mathbb{G}(\bar{\mathbb{F}}_q)$.

Note that with this definition, when $\mathbb{G} = PSL_n$, $\mathbb{G}(q)$ does not mean $PSL_n(q)$ in the usual sense, but rather its extension by diagonal automorphisms (i.e., $PGL_n(q)$). By Proposition 1.4, however, any equivalence between fusion systems over groups $SL_n(-)$ will also induce an equivalence between fusion systems over $PSL_n(-)$. Also, we are not including the case $\mathbb{G} = SO_n$ for n even, since $SO_n(\bar{\mathbb{F}}_2)$ is not connected. Instead, when working with orthogonal groups in even dimension, we take $\mathbb{G} = \Omega_n$ (and $\Omega_n(K) = SO_n(K)$ when K is algebraically closed of characteristic different from two).

The results in this section are based on Corollary 2.5, together with the following theorem of Friedlander. Following the terminology of [GLS3], we define a *Steinberg endomorphism* of an algebraic group \bar{G} over an algebraically closed field to be an algebraic endomorphism $\psi: \bar{G} \rightarrow \bar{G}$ which is bijective, and whose fixed subgroup is finite. For any connected complex Lie group $\mathbb{G}(\mathbb{C})$ with maximal torus $\mathbb{T}(\mathbb{C})$, any prime p , and any $m \in \mathbb{Z}$ prime to p , $\Psi^m: B\mathbb{G}(\mathbb{C})_p^\wedge \rightarrow B\mathbb{G}(\mathbb{C})_p^\wedge$ denotes a self equivalence whose restriction to $B\mathbb{T}(\mathbb{C})_p^\wedge$ is induced by $(x \mapsto x^m)$ (an “unstable Adams operation”). Such a map is unique up to homotopy by [JMO, Theorem 2.5] (applied to $BG \simeq B\mathbb{G}(\mathbb{C})$, where G is a maximal compact subgroup of $\mathbb{G}(\mathbb{C})$).

Theorem 3.1. *Fix a connected reductive group scheme \mathbb{G} , a prime power q , and a prime p which does not divide q . Then for any Steinberg endomorphism ψ of $\mathbb{G}(\bar{\mathbb{F}}_q)$ with fixed subgroup H ,*

$$BH_p^\wedge \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h\Psi}$$

for some $\Psi: B\mathbb{G}(\mathbb{C})_p^\wedge \xrightarrow{\simeq} B\mathbb{G}(\mathbb{C})_p^\wedge$. If $\psi = \tau(\bar{\mathbb{F}}_q) \circ \mathbb{G}(\psi^q)$, where $\tau \in \text{Aut}(\mathbb{G})$ and $\psi^q \in \text{Aut}(\bar{\mathbb{F}}_q)$ is the automorphism $(x \mapsto x^q)$, then $\Psi \simeq B\tau(\mathbb{C}) \circ \Psi^q$ where Ψ^q is as described above.

Proof. By [Fr, Theorem 12.2], BH_p^\wedge is homotopy equivalent to a homotopy pullback of maps

$$B\mathbb{G}(\mathbb{C})_p^\wedge \xrightarrow{(\text{Id}, \Psi)} B\mathbb{G}(\mathbb{C})_p^\wedge \times B\mathbb{G}(\bar{\mathbb{F}}_q)_p^\wedge \xleftarrow{(\text{Id}, \text{Id})} B\mathbb{G}(\mathbb{C})_p^\wedge$$

for some Ψ ; and hence is homotopy equivalent to $(B\mathbb{G}(\mathbb{C})_p^\wedge)^{h\Psi}$ by Remark 2.3. From the proof of Friedlander’s theorem, one sees that Ψ is induced by $B\psi$, together with the homotopy equivalence $B\mathbb{G}(\mathbb{C})_p^\wedge \simeq \text{holim}((B\mathbb{G}(\bar{\mathbb{F}}_q)_{\text{et}})_p^\wedge)$ of [Fr, Proposition 8.8]. This equivalence is natural with respect to the inclusion of a maximal torus \mathbb{T} in \mathbb{G} . Hence when $\psi = \tau(\bar{\mathbb{F}}_q) \circ \mathbb{G}(\psi^q)$, Ψ restricts to the action on $B\mathbb{T}(\mathbb{C})_p^\wedge$ induced by τ and $(x \mapsto x^q)$. \square

Theorem 3.1 can now be combined with Corollary 2.5 to prove Theorem A; i.e., to compare fusion systems over different Chevalley groups associated to the same connected group scheme \mathbb{G} . This will be done in the next two propositions.

Proposition 3.2. *Fix a prime p , a connected reductive integral group scheme \mathbb{G} , and an automorphism τ of \mathbb{G} of finite order k . Assume, for each m prime to k , that τ^m is conjugate to τ in the group of all automorphisms of \mathbb{G} . Let q and q' be prime powers prime to p . Assume either*

- (a) $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$ as subgroups of \mathbb{Z}_p^\times ; or
- (b) there is some ψ^{-1} in the Weyl group of \mathbb{G} which inverts all elements of the maximal torus, and $\overline{\langle -1, q \rangle} = \overline{\langle -1, q' \rangle}$ as subgroups of \mathbb{Z}_p^\times .

Then there is an isotypical equivalence $\mathcal{F}_p(\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\tau\mathbb{G}(q'))$.

Proof. By [JMO, Theorem 2.5], the group $\text{Out}(B\mathbb{G}(\mathbb{C})_p^\wedge)$ is detected by restricting maps to a maximal torus. (Every class in this group is represented by some map which sends BT_p^\wedge to itself for some maximal torus T .) Hence $\Psi^q, \Psi^{q'} \in \text{Out}(B\mathbb{G}(\mathbb{C})_p^\wedge)$ (the maps whose restrictions to the maximal torus are induced by $(x \mapsto x^q)$ and $(x \mapsto x^{q'})$, respectively) generate the same closed subgroup of $\text{Out}(B\mathbb{G}(\mathbb{C})_p^\wedge)$ whenever $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$.

Since τ is an automorphism of \mathbb{G} , its actions on $\mathbb{G}(\overline{\mathbb{F}}_q)$ and $\mathbb{G}(\overline{\mathbb{F}}_{q'})$ commute with the field automorphisms ψ^q and $\psi^{q'}$. Thus $(\tau\psi^q)^k = \psi^{q^k}$ has finite fixed subgroup, so $\tau\psi^q$ also has finite fixed subgroup, and similarly for $\tau\psi^{q'}$. So by Theorem 3.1,

$$B(\tau\mathbb{G}(q))_p^\wedge \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau\circ\Psi^q)} \quad \text{and} \quad B(\tau\mathbb{G}(q'))_p^\wedge \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau\circ\Psi^{q'})}.$$

Assume $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$. Then for some m prime to $k = |\tau|$, $q \equiv (q')^m$ modulo $\langle q^k \rangle = \langle q'^k \rangle$. Hence $B\tau^m \circ \Psi^q$ and $B\tau \circ \Psi^{q'}$ generate the same closed subgroup of $\text{Out}(\mathbb{G}(\mathbb{C})_p^\wedge)$ under the p -adic topology, since they generate the same subgroup modulo $\langle \Psi^{q^k} \rangle$. Thus

$$(B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau\circ\Psi^{q'})} \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau^m\circ\Psi^q)} \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau\circ\Psi^q)},$$

where the first equivalence holds by Corollary 2.5, and the second since τ and τ^m are conjugate in the group of all automorphisms of \mathbb{G} . So $B(\tau\mathbb{G}(q))_p^\wedge \simeq B(\tau\mathbb{G}(q'))_p^\wedge$, and there is an isotypical equivalence between the fusion systems of these groups by Theorem 1.5.

If $-\text{Id}$ is in the Weyl group, then we can regard this as an inner automorphism of $\mathbb{G}(\mathbb{C})$ which inverts all elements in a maximal torus. Thus by [JMO] again, $\Psi^{-1} \simeq \text{Id}$ in this case. So by the same argument as that just given, $\overline{\langle -1, q \rangle} = \overline{\langle -1, q' \rangle}$ implies $(B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau\circ\Psi^q)} \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h(B\tau\circ\Psi^{q'})}$, and hence $\mathcal{F}_p(\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\tau\mathbb{G}(q'))$. \square

To make the condition $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$ more concrete, note that for any prime p , and any q, q' prime to p of order s and s' in \mathbb{F}_p^\times , respectively,

$$\overline{\langle q \rangle} = \overline{\langle q' \rangle} \iff \begin{cases} s = s' \text{ and } v_p(q^s - 1) = v_p(q'^s - 1) & \text{if } p \text{ is odd} \\ q \equiv q' \pmod{8} \text{ and } v_p(q^2 - 1) = v_p(q'^2 - 1) & \text{if } p = 2. \end{cases}$$

Corollary 2.5 can also be applied to compare fusion systems of Steinberg groups with those of related Chevalley groups.

Proposition 3.3. *Fix a prime p , and a pair q, q' of prime powers prime to p such that $\overline{\langle -q \rangle} = \overline{\langle q' \rangle}$ as subgroups of \mathbb{Z}_p^\times . Then there are isotypical equivalences:*

- (a) $\mathcal{F}_p(SU_n(q)) \simeq \mathcal{F}_p(SL_n(q'))$ for all n .
- (b) $\mathcal{F}_p(\text{Spin}_{2n}^-(q)) \simeq \mathcal{F}_p(\text{Spin}_{2n}^+(q'))$ for all odd n .
- (c) $\mathcal{F}_p({}^2E_6(q)) \simeq \mathcal{F}_p(E_6(q'))$.

Proof. Set $\mathbb{G} = SL_n, \text{Spin}_{2n}$ for n odd, or the simply connected E_6 ; and let τ be the graph automorphism of order two. In all of these cases, τ acts by inverting the elements of some maximal torus. Hence by Theorem 3.1, $B(\tau\mathbb{G}(q))_p^\wedge \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h\Psi^{-q}}$ and $B\mathbb{G}(q')_p^\wedge \simeq (B\mathbb{G}(\mathbb{C})_p^\wedge)^{h\Psi^{q'}}$. So $B\mathbb{G}(q)_p^\wedge \simeq B\mathbb{G}(q')_p^\wedge$ by Corollary 2.5, [JMO, Theorem 2.5], and the assumption $\overline{\langle -q \rangle} = \overline{\langle q' \rangle}$; and there is an isotypical equivalence between the fusion systems of these groups by Theorem 1.5. \square

Upon combining this with Proposition 1.4, one gets similar results for $PSU_n(q), \Omega_{2n}^-(q), P\Omega_{2n}^-(q)$, etc.

This finishes the proof of Theorem A. We now finish the section with some remarks about a possible algebraic proof of this result. The following theorem of Michael Larsen in his appendix to [GR] implies roughly that two Chevalley groups $\mathbb{G}(K)$ and $\mathbb{G}(K')$ over algebraically closed fields K and K' have equivalent p -fusion

systems (appropriately defined) for p different from the characteristics of K and K' .

Theorem 3.4. *Fix a connected group scheme \mathbb{G} , and let K and K' be two algebraically closed fields. Then there are bijections*

$$\nu_P: \operatorname{Rep}(P, \mathbb{G}(K)) \xrightarrow{\cong} \operatorname{Rep}(P, \mathbb{G}(K')),$$

for all finite groups P of order prime to $\operatorname{char}(K)$ and $\operatorname{char}(K')$, and which are natural with respect to P , and also with respect to automorphisms of \mathbb{G} .

Proof. Except for the statement of naturality, this is [GR, Theorem A.12]. For fields of the same characteristic, the bijection is induced by the inclusions of the algebraic closures of their prime subfields ([GR, Lemma A.11]). When $K = \bar{\mathbb{F}}_q$ for a prime q , $W(K)$ is its ring of Witt vectors (the extension of \mathbb{Z}_q by all roots of unity of order prime to q), and K' is the algebraic closure of $W(K)$, then the bijections $\operatorname{Rep}(P, \mathbb{G}(K)) \cong \operatorname{Rep}(P, \mathbb{G}(W(K))) \cong \operatorname{Rep}(P, \mathbb{G}(K'))$ are induced by the projection $W(K) \longrightarrow K$ and inclusion $W(K) \subseteq K'$ in the obvious way. All of these are natural in P , and also commute with automorphisms of \mathbb{G} . \square

The next proposition describes how to compare $\operatorname{Rep}(P, \mathbb{G}(\bar{\mathbb{F}}_q))$ to the Chevalley and Steinberg groups over \mathbb{F}_q .

Proposition 3.5. *Fix a connected algebraic group \bar{G} over $\bar{\mathbb{F}}_q$ for some q , and σ be a Steinberg endomorphism of \bar{G} , and set $G = C_{\bar{G}}(\sigma)$. Let P be a finite group, and consider the map of sets*

$$\rho: \operatorname{Rep}(P, G) \longrightarrow \operatorname{Rep}(P, \bar{G}).$$

Fix $\varphi \in \operatorname{Hom}(P, \bar{G})$, and let $[\varphi]$ be its class in $\operatorname{Rep}(P, \bar{G})$. Then $[\varphi] \in \operatorname{Im}(\rho)$ if and only if $[\varphi]$ is fixed under the action of σ on $\operatorname{Rep}(P, \bar{G})$. When $\varphi(P) \leq G$, set

$$C = C_{\bar{G}}(P), \quad N = N_{\bar{G}}(P), \quad \hat{C} = \pi_0(C),$$

and let $g \in N$ act on \bar{C} by sending x to $gx\sigma(g)^{-1}$. Then there is a bijection

$$B: \rho^{-1}([\varphi]) \xrightarrow{\cong} \hat{C}/C,$$

where $B([c_y \circ \varphi]) = y^{-1}\sigma(y)$ for any $y \in \bar{G}$ such that $y\varphi(P)y^{-1} \leq G$. Also, $\operatorname{Aut}_{\bar{G}}(\varphi(P))$ is the stabilizer, under the action of $\operatorname{Aut}_{\bar{G}}(\varphi(P)) \cong N/C$, of the class of the identity element in \hat{C} .

Proof. This is a special case of [GLS3, Theorem 2.1.5], when applied to the set Ω of all homomorphisms $\varphi': P \longrightarrow \bar{G}$ which are \bar{G} -conjugate to φ . \square

We assumed at first that an algebraic proof of Theorem A could easily be constructed by applying Theorem 3.4 and Proposition 3.5. But so far, we have been unable to do so, nor do we know of any other algebraic proof of this result.

4. HOMOTOPY FIXED POINTS OF PROXY ACTIONS

To get more results of this type, we need to look at more general types of actions and their homotopy fixed points. The concept of “proxy actions” is due to Dwyer and Wilkerson.

Definition 4.1. For any discrete group G and any space X , a proxy action of G on X is a fibration $f: X_{hG} \longrightarrow BG$ with fiber X . An equivalence of proxy actions $f: X_{hG} \longrightarrow BG$ on X and $f': Y_{hG} \longrightarrow BG$ on Y is a homotopy equivalence $\alpha: X_{hG} \longrightarrow Y_{hG}$ such that $f' \circ \alpha = f$. The homotopy fixed space X^{hG} of a proxy action $f: X_{hG} \longrightarrow BG$ on X is the space of sections $s: BG \longrightarrow X_{hG}$ of the fibration f .

Any (genuine) action of G on X can be regarded as a proxy action via the Borel construction: $X_{hG} = EG \times_G X$ is the orbit space of the diagonal G -action on $EG \times X$. In this case, we can identify

$$X^{hG} = \text{map}_G(EG, X)$$

via covering space theory.

If $\alpha: X \longrightarrow X$ is a homeomorphism, regarded as a \mathbb{Z} -action, then the mapping torus

$$X_{h\alpha} = (X \times I) / ((x, 1) \sim (\alpha(x), 0)) ,$$

as defined in Definition 2.1, is homeomorphic to the Borel construction $E\mathbb{Z} \times_{\mathbb{Z}} X$. So in this case, the homotopy fixed set $X^{h\mathbb{Z}}$ of Definition 4.1 is the same as the space $X^{h\alpha}$ of Definition 2.1. If α is a self homotopy equivalence of X , then the map from the mapping torus to $S^1 = B\mathbb{Z}$ need not be a fibration, which is why we need to first replace X by the double telescope before defining the homotopy fixed set.

Now assume, furthermore, that X is p -complete, and that the action of α on $H^n(X; \mathbb{F}_p)$ is nilpotent for each n . Then by [BK, Lemma II.5.1], the homotopy fiber of the p -completion $(X_{h\alpha})_p^\wedge \longrightarrow S^1_p^\wedge$ has the homotopy type of $X_p^\wedge \simeq X$. Also, $S^1_p^\wedge \simeq B\mathbb{Z}_p$, and so this defines a proxy action of the p -adics on X . By the arguments used in the proof of Theorem 2.4, the homotopy fixed space of this action has the homotopy type of $X^{h\alpha}$.

Some of the basic properties of proxy actions and their homotopy fixed spaces are listed in the following proposition.

Proposition 4.2. Fix a proxy action $X_{hG} \xrightarrow{f} BG$ of a discrete group G on a space X .

- (a) If $Y_{hG} \xrightarrow{f'} BG$ is a proxy action of the same group G on another space Y , and $\varphi: X_{hG} \longrightarrow Y_{hG}$ is a homotopy equivalence such that $f' \circ \varphi = f$, then φ induces a homotopy equivalence $X^{hG} \simeq Y^{hG}$.
- (b) Let \tilde{X} be the pullback of X_{hG} and EG over BG . Then G has a genuine action on \tilde{X} , which as a proxy action, is equivalent to that on X .

Proof. Point (a) follows easily from the definition.

In the situation of (b), the action of G on EG induces a free action on \tilde{X} . Consider the two G -maps

$$\text{pr}_1, \tilde{f} \circ \text{pr}_2: EG \times_G \tilde{X} \longrightarrow EG ,$$

where $\tilde{f}: \tilde{X} \longrightarrow EG$ is the map coming from the pullback square used to define \tilde{X} . By the universality of EG , all maps from $EG \times_G \tilde{X}$ to EG are equivariantly homotopic (cf. [Hu, Theorem 4.12.4], and recall that these spaces are the total spaces of principal G -bundles). So upon passing to the orbit space, the composite

$$EG \times_G \tilde{X} \xrightarrow{\text{pr}_2/G} X_{hG} \xrightarrow[=f]{\tilde{f}/G} BG$$

is homotopic to pr_1/G , the map which defines the proxy action of G induced by the action on \tilde{X} . By the homotopy lifting property for the fibration f , pr_2/G is homotopic to a map α such that $f \circ \alpha = \text{pr}_1/G$, and this is an equivalence between the proxy actions. \square

If $X_{hG} \xrightarrow{f} BG$ is a proxy action of G on X , and $H \leq G$ is a subgroup, then the pullback of BH and X_{hG} over BG defines a proxy action $X_{hH} \xrightarrow{f|_H} BH$ of H on X . The first statement in the following proposition is due to Dwyer and Wilkerson [DW, 10.5].

Proposition 4.3. *Let $f: X_{hG} \longrightarrow BG$ be a proxy action of G on X , and let H be a normal subgroup of G with quotient group $\pi = G/H$. Then the following hold.*

- (a) *There is a proxy action of π on X^{hH} with $X^{hG} \simeq (X^{hH})^{h\pi}$.*
- (b) *Assume $G_0 \leq G$ and $H_0 = H \cap G_0$ are such that the inclusion induces an isomorphism $G_0/H_0 \xrightarrow{\cong} G/H = \pi$. Assume also that the natural map $X^{hH} \longrightarrow X^{hH_0}$ induced by restricting the action of G is a homotopy equivalence. Then $X^{hG} \longrightarrow X^{hG_0}$ is also a homotopy equivalence.*

Proof. We give an argument for (a) that will be useful in the proof of (b). Write $\widehat{BH} = EG/H \simeq BH$ and $\widehat{BG} = EG \times_G E\pi \simeq BG$, where $EG \times_G E\pi$ is the orbit space of the diagonal action of G on $EG \times E\pi$. Let

$$\gamma: BG \xrightarrow{\cong} \widehat{BG} \quad \text{and} \quad \widehat{B}l: \widehat{BH} \longrightarrow \widehat{BG}$$

be induced by the diagonal map $EG \longrightarrow EG \times E\pi$ and its composite with the inclusion $E_l: EH \longrightarrow EG$. The adjoint of the projection

$$\widehat{BH} \times E\pi = EG \times_H E\pi \longrightarrow EG \times_G E\pi = \widehat{BG}$$

is a π -equivariant map $E\pi \longrightarrow \text{map}(\widehat{BH}, \widehat{BG})_{\widehat{B}l}$, where $\text{map}(\widehat{BH}, \widehat{BG})_{\widehat{B}l}$ is the space of all maps homotopic to $\widehat{B}l$. Consider the following homotopy pullback diagram of spaces with π -action:

$$\begin{array}{ccc} Y & \longrightarrow & \text{map}(EG/H, X_{hG})_{[\widehat{B}l]} \\ \downarrow & & \downarrow f' \circ - \\ E\pi & \longrightarrow & \text{map}(EG/H, \widehat{BG})_{\widehat{B}l} . \end{array}$$

Here, $f' = \gamma \circ f: X_{hG} \longrightarrow \widehat{BG}$, and $\text{map}(-, -)_{[\widehat{B}l]}$ means the space of all maps whose composite with f' is homotopic to $\widehat{B}l$. Since $E\pi$ is contractible, Y is the homotopy fiber of the map on the right, and hence homotopy equivalent to X^{hH} . After taking homotopy fixed spaces $(-)^{h\pi}$, and since $(EG/H \times E\pi)/G \cong \widehat{BG}$, we get a new homotopy pullback square

$$\begin{array}{ccc} (X^{hH})^{h\pi} \simeq Y^{h\pi} & \longrightarrow & \text{map}(\widehat{BG}, X_{hG})_{\widehat{B}l} \\ \downarrow & & \downarrow \\ * \simeq E\pi^{h\pi} & \longrightarrow & \text{map}(\widehat{BG}, \widehat{BG})_{\widehat{B}l} . \end{array}$$

Thus $(X^{hH})^{h\pi}$ is the homotopy fiber of the right hand map, hence equivalent to the space of sections of the fibration $X_{hG} \longrightarrow BG$, which is X^{hG} .

Now assume that we are in the situation of (b). In this situation, if G acts on X , the restriction map $X^{hH} \xrightarrow{\cong} X^{hH_0}$ is π -equivariant, provided we use the models for X^{hH} and X^{hH_0} constructed above. Taking homotopy fixed points on both

sides for the action of π , we see that the inclusion $X^{hG} \longrightarrow X^{hG_0}$ is a homotopy equivalence.

Alternatively, by Proposition 4.2(a,b), it suffices to prove this for a genuine action of G on X . In this case,

$$X^{hG} = \text{map}_G(EG, X) \simeq \text{map}_G(EG \times E\pi, X) \cong \text{map}_\pi(E\pi, \text{map}_H(EG, X)),$$

where the last equivalence follows by adjunction. We can identify $\text{map}_H(EG, X)$ with X^{hH} , and thus $X^{hG} \simeq (X^{hH})^{h\pi}$. In the situation of (b), we get a commutative square

$$\begin{array}{ccc} X^{hG} \simeq \text{map}_G(EG \times E\pi, X) & \longrightarrow & \text{map}_\pi(E\pi, \text{map}_H(EG, X)) \simeq (X^{hH})^{h\pi} \\ \downarrow r_1 & & \downarrow r_2 \\ X^{hG_0} \simeq \text{map}_{G_0}(EG \times E\pi, X) & \longrightarrow & \text{map}_\pi(E\pi, \text{map}_{H_0}(EG, X)) \simeq (X^{hH_0})^{h\pi}. \end{array}$$

where r_1 and r_2 are induced by restriction to G_0 or H_0 . Since r_2 is a homotopy equivalence by assumption (and by Proposition 4.2(a)), r_1 is also a homotopy equivalence. \square

The following theorem deals with homotopy fixed points of actions of $K \times \mathbb{Z}$, where K is a finite cyclic group of order prime to p . This can be applied when K is a group of graph automorphisms of BG_p^\wedge (and G is a compact connected Lie group), or when K is a group of elements of finite order in \mathbb{Z}_p^\times (for odd p).

Theorem 4.4. *Fix a prime p . Let X be a connected, p -complete space such that*

- $H^*(X; \mathbb{F}_p)$ is noetherian, and
- $\text{Out}(X)$ is detected on $\widehat{H}^*(X; \mathbb{Z}_p)$.

Fix a finite cyclic group $K = \langle g \rangle$ of order r prime to p , together with a proxy action $f: X_{hK} \longrightarrow BK$ of K on X . Let $\beta: X_{hK} \longrightarrow X_{hK}$ be a self homotopy equivalence such that $f \circ \beta = f$; and set $\alpha = \beta|_X$, a self homotopy equivalence of X . Assume $H_(\alpha; \mathbb{F}_p)$ is an automorphism of $H_*(X; \mathbb{F}_p)$ of p -power order (equivalently, the action is nilpotent). Let $\kappa: X \longrightarrow X$ be the self homotopy equivalence induced by lifting a loop representing $g \in \pi_1(BK)$ to a homotopy of the inclusion $X \longrightarrow X_{hK}$. Then*

$$X^{h(\kappa\alpha)} \simeq (X^{hK})^{h\alpha}.$$

Proof. Upon first replacing X by the pullback of X_{hK} and EK over BK , and then by taking the double mapping telescope of the map from that space to itself induced by β , we can assume that X has a genuine free action of the group $K \times \mathbb{Z}$, and that κ and α are (commuting) homeomorphisms of X which are the actions of generators of K and of \mathbb{Z} . In particular, $(\kappa\alpha)^r = \alpha^r$.

For each $k \geq 1$, α acts on $H_*(X; \mathbb{Z}/p^k)$ as an automorphism of p -power order. Since r is prime to p , this implies that α and α^r generate the same closed subgroup of $\text{Out}(X)$. So by Theorem 2.4, the inclusion of $X^{h\alpha}$ into $X^{h\alpha^r}$ is a homotopy equivalence.

By Proposition 4.3(b), applied with $G = \langle \kappa \rangle \times \langle \alpha \rangle$, $G_0 = \langle \kappa\alpha \rangle$, and $H = \langle \alpha \rangle$, the inclusion of $X^{h(K \times \langle \alpha \rangle)}$ into $X^{h(\kappa\alpha)}$ is a homotopy equivalence. By Proposition 4.3(a), $X^{h(K \times \langle \alpha \rangle)} \simeq (X^{hK})^{h\alpha}$, and this proves the theorem. \square

The following result comparing fusion systems of $G_2(q)$ and ${}^3D_4(q)$, and those of $F_4(q)$ and ${}^2E_6(q)$, is well known. For example, the first part follows easily from the lists of maximal subgroups of these groups in [Kl1] and [Kl2], and also follows

from the cohomology calculations in [FM] and [Mi] (together with Theorem 1.5). We present it here as one example of how Theorem 4.4 can be applied.

Example 4.5. *Fix a prime p , and a prime power $q \equiv 1 \pmod{p}$. Then the following hold.*

- (a) *If $p \neq 3$, the fusion systems $\mathcal{F}_p(G_2(q))$ and $\mathcal{F}_p({}^3D_4(q))$ are isotypically equivalent.*
- (b) *If $p \neq 2$, the fusion systems $\mathcal{F}_p(F_4(q))$ and $\mathcal{F}_p({}^2E_6(q))$ are isotypically equivalent.*

Proof. To prove (a), we apply Theorem 4.4, with $X = BSpin_8(\mathbb{C})_p^\wedge \simeq BSpin(8)_p^\wedge$, with $K \cong C_3$ having the action on X induced by the triality automorphism, and with $\alpha = \Psi^q$ the unstable Adams operation on X .

We first show that the inclusion of G_2 into $Spin(8)$ induces a homotopy equivalence $(BG_2)_p^\wedge \simeq X^{hK}$. Since there is always a map from the fixed point set of an action to its homotopy fixed point set, the inclusion of $G_2(\mathbb{C}) \cong Spin_8(\mathbb{C})^K$ (cf. [GLS3, Theorem 1.15.2]) into $Spin_8(\mathbb{C})$ induces maps $(BG_2)_p^\wedge \longrightarrow X^{hK} \longrightarrow X$. The first map is a monomorphism in the sense of Dwyer and Wilkerson [DW, §3.2], since the composite is a monomorphism.

By [BM, Theorem B(2)], X^{hK} is the classifying space of a connected 2-compact group. Hence by [BM, Theorem B(2)], $H^*(X^{hK}; \mathbb{Q}_p)$ is the polynomial algebra generated by the coinvariants $QH^*(X; \mathbb{Q}_p)_K$; i.e. the coinvariants of the K -action on the polynomial generators of $H^*(Spin_8(\mathbb{C}); \mathbb{Q}_p)$. For any compact connected Lie group G with maximal torus T , $H^*(BG; \mathbb{Q})$ is the ring of invariants of the action of the Weyl group on $H^*(BT; \mathbb{Q})$ [Bor, Proposition 27.1], and is a polynomial algebra with degrees listed in [ST, Table VII]. In particular, $H^*(X; \mathbb{Q}_p)$ has polynomial generators in degrees 4, 8, 12, 8, and an explicit computation shows that K fixes generators in degrees 4 and 12. Thus $H^*(X^{hK}; \mathbb{Q}_p) \cong H^*(BG_2(\mathbb{C}); \mathbb{Q}_p)$ (as graded \mathbb{Q}_p -algebras). It follows from [MN, Proposition 3.7] that $(BG_2)_p^\wedge \longrightarrow X^{hK}$ is an isomorphism of connected 2-compact groups because it is a monomorphism and a rational isomorphism.

Now let $\kappa \in \text{Aut}(X)$ generate the action of K . By Theorem 3.1,

$$X^{h(\kappa\alpha)} \simeq B({}^3D_4(q))_p^\wedge \quad \text{and} \quad (X^{hK})^{h\alpha} \simeq (BG_2)_p^\wedge \simeq BG_2(q)_p^\wedge.$$

Since $q \equiv 1 \pmod{p}$, the action of $\alpha = \Psi^q$ on $H^*(X; \mathbb{F}_p)$ has p -power order. Hence $B({}^3D_4(q))_p^\wedge \simeq BG_2(q)_p^\wedge$ by Theorem 4.4, and so these groups have isotypically equivalent p -fusion systems by Theorem 1.5. This proves (a).

Now set $X = BE_6(\mathbb{C})_p^\wedge$ and $K = \langle \tau \rangle$, where τ is an outer automorphism of order two. For each $k \geq 0$, τ acts on $H^{2k}(X; \mathbb{Q}_p)$ via $(-1)^k$: this follows since $H^*(X; \mathbb{Q}_p)$ injects into the cohomology of any maximal torus and τ acts on an appropriate choice of maximal torus by via $(g \mapsto g^{-1})$. Since $H^*(X; \mathbb{Q}_p)$ is polynomial with generators in degrees 4, 10, 12, 16, 18, 24, [BM, Theorem B(2)] implies that $H^*(X^{hK}; \mathbb{Q}_p)$ is polynomial with generators in degrees 4, 12, 16, 24, and hence is isomorphic to $H^*(BF_4(\mathbb{C}); \mathbb{Q}_p)$. The rest of the proof of (b) is identical to that of (a). \square

5. CLASSICAL GROUPS

In the case of many of the classical groups, there is a much more elementary approach to Theorem A. Recall that the modular character χ_V of an $\mathbb{F}_q[G]$ -module

V is defined by identifying \mathbb{F}_q^\times with a subgroup of \mathbb{C}^\times , and then letting $\chi_V(g) \in \mathbb{C}$ (when $(|g|, q) = 1$) be the sum of the eigenvalues of $V \xrightarrow{g} V$ lifted to \mathbb{C} . We always consider this in the case where G has order prime to q , and hence when two representations with the same character are isomorphic. See [Se, §18] for more details.

For any finite group G , let $\text{Rep}_n(G)$ be the set of isomorphism classes of n -dimensional irreducible complex representations (i.e., $\text{Rep}_n(G) = \text{Rep}(G, GL_n(\mathbb{C}))$ in the notation used elsewhere). For any prime p and any q prime to p , $\overline{\langle q \rangle} \subseteq (\widehat{\mathbb{Z}}_p)^\times$ denotes the closure of the subgroup generated by q .

In the following theorem, we set $GL_n^+(q) = GL_n(q)$ and $GL_n^-(q) = GU_n(q)$ for convenience.

Proposition 5.1. *Fix a prime p , and let q be a prime power which is prime to p .*

- (a) *Fix $n \geq 2$ and $\epsilon = \pm 1$. For any finite p -group P , $\text{Rep}(P, GL_n^\epsilon(q))$ can be identified with the set of those $V \in \text{Rep}_n(P)$ such that $\chi_V(g^{\epsilon q}) = \chi_V(g)$ for all $g \in P$.*
- (b) *Assume p is odd and fix $n \geq 1$. $G = Sp_{2n}(q)$ and $G_1 = GO_{2n+1}(q)$. Then for any finite p -group P , $\text{Rep}(P, Sp_{2n}(q))$ and $\text{Rep}(P, GO_{2n+1}(q))$ can be identified with the set of those $V \in \text{Rep}_{2n}(P) \cong \text{Rep}_{2n+1}(P)$ such that $\chi_V(g^q) = \chi_V(g) = \chi_V(g^{-1})$ for all $g \in P$. In particular, the fusion systems $\mathcal{F}_p(Sp_{2n}(q))$ and $\mathcal{F}_p(GO_{2n+1}(q))$ are isotypically equivalent.*

Proof. Let $K \subseteq \mathbb{C}$ be the subfield generated by all p -th power roots of unity. For each $r \in \mathbb{Z}_p^\times$, let $\psi^r \in \text{Aut}(K)$ be the field automorphism $\psi^r(\zeta) = \zeta^r$ for each root of unity ζ .

(a) Let \widehat{K} be the extension of \mathbb{Q}_q by all roots of unity prime to q , let $A \subseteq \mathbb{Q}_q$ be the ring of integers, and let $\mathfrak{p} \subseteq A$ be the maximal ideal. Thus $A/\mathfrak{p} \cong \overline{\mathbb{F}}_q$. By modular representation theory (cf. [Se, Theorems 33 & 42]), for each finite p -group P , there is an isomorphism of representation rings $R_{\widehat{K}}(P) \xrightarrow{\cong} R_{\overline{\mathbb{F}}_q}(P)$, which sends the class of a $\widehat{K}[P]$ -module V to $M/\mathfrak{p}M$ for any P -invariant A -lattice $M \subseteq V$. This clearly sends an actual representation to an actual representation. If $M_1 \subseteq V_1$ and $M_2 \subseteq V_2$ are such that $M_1/\mathfrak{p}M_1$ and $M_2/\mathfrak{p}M_2$ have an irreducible factor in common, then since $|P|$ is invertible in $\overline{\mathbb{F}}_q$ and in A , any nonzero homomorphism $\varphi \in \text{Hom}_P(M_1/\mathfrak{p}M_1, M_2/\mathfrak{p}M_2)$ can be lifted (by averaging over the elements of P) to a homomorphism $\widehat{\varphi} \in \text{Hom}_P(V_1, V_2)$. From this we see that the isomorphism $R_{\widehat{K}}(P) \cong R_{\overline{\mathbb{F}}_q}(P)$ restricts to a bijection between irreducible representations, and also between n -dimensional representations for any given n . So

$$\text{Rep}(P, GL_n(\overline{\mathbb{F}}_q)) \cong \text{Rep}(P, GL_n(\widehat{K})) \cong \text{Rep}(P, GL_n(K)) \cong \text{Rep}(P, GL_n(\mathbb{C})),$$

where the last two bijections follow from [Se, Theorem 24].

The centralizer of any finite p -subgroup of $GL_n(\overline{\mathbb{F}}_q)$ is a product of general linear groups, and hence connected. Thus by Proposition 3.5, $\text{Rep}(P, GL_n^\epsilon(q))$ injects into $\text{Rep}(P, GL_n(\overline{\mathbb{F}}_q))$, and its image is the set of representations which are fixed by the Steinberg endomorphism $\psi^{\epsilon q}$ on $GL_n(\overline{\mathbb{F}}_q)$.

Fix $V \in \text{Rep}(P, GL_n(\overline{\mathbb{F}}_q))$ and $g \in P$, let $\xi_1, \dots, \xi_n \in \overline{\mathbb{F}}_q$ be the eigenvalues of the action of g on V , and let $\zeta_1, \dots, \zeta_n \in K$ be the corresponding p -th power roots of unity in \mathbb{C} . Then $\psi^q(V)$ has eigenvalues ξ_1^q, \dots, ξ_n^q , $\psi^{-1}(V) = V^*$ has eigenvalues

$\xi_1^{-1}, \dots, \xi_n^{-1}$, and so $\psi^{\epsilon q}(V)$ has eigenvalues $\xi_1^{\epsilon q}, \dots, \xi_n^{\epsilon q}$. This proves that

$$\chi_{\psi^{\epsilon q}(V)}(g) = \chi_1^{\epsilon q} + \dots + \chi_n^{\epsilon q} = \chi_V(g^{\epsilon q})$$

for all $V \in \text{Rep}_n(P)$ and all $g \in P$. Thus $V \in \text{Rep}_n(P)$ is in the image of $\text{Rep}(P, GL_n^\epsilon(q))$ if and only if $\chi_V(g^{\epsilon q}) = \chi_V(g)$ for all $g \in P$.

(b) Now assume p is odd, and let P be a finite p -group. For any irreducible $\bar{\mathbb{F}}_q[P]$ -representation W which is self dual, $\sum_{g \in P} \chi_W(g^2) \neq 0$: this is shown in [BtD, Proposition II.6.8] for complex representations, and the same proof applies in our situation. Since $|P|$ is odd, this implies that $\sum_{g \in P} \chi_W(g) \neq 0$, and hence that W is the trivial representation. In other words, the only self dual irreducible representation is the trivial one. Hence every self dual $\bar{\mathbb{F}}_q[P]$ -representation V has the form $V = V_0 \oplus W \oplus W'$, where P acts trivially on V_0 and has no fixed component on W and W' , $W' \cong W^*$, and $\text{Hom}_{\bar{\mathbb{F}}_q[P]}(W, W') = 0$.

Fix a self dual $\bar{\mathbb{F}}_q[P]$ -representation V , and write $V = V_0 \oplus W \oplus W'$ as above. Fix $\epsilon = \pm 1$, where $\epsilon = +1$ if $\dim_{\bar{\mathbb{F}}_q}(V)$ is odd, and write “ ϵ -symmetric” to mean symmetric ($\epsilon = +1$) or symplectic ($\epsilon = -1$). For any nondegenerate ϵ -symmetric form \mathfrak{b}_0 on V_0 and any $\bar{\mathbb{F}}_q[P]$ -linear isomorphism $f: W' \xrightarrow{\cong} W^*$, there is a nondegenerate ϵ -symmetric form \mathfrak{b} on V defined by

$$\mathfrak{b}((v_1, w_1, w'_1), (v_2, w_2, w'_2)) = \mathfrak{b}_0(v_1, v_2) + f(w'_1)(w_2) + \epsilon f(w'_2)(w_1) \quad (3)$$

for $v_i \in V_0$, $w_i \in W$, and $w'_i \in W'$. Conversely, if \mathfrak{b} is any nonsingular ϵ -symmetric form on V , then \mathfrak{b} must be nonsingular on V , and zero on W and on W' , and hence has the form (3) for some \mathfrak{b}_0 and f . Since all such forms are isomorphic, this proves that $\text{Rep}(P, Sp_{2n}(\bar{\mathbb{F}}_q))$ can be identified with the set of self dual elements of $\text{Rep}_{2n}(P)$, and $\text{Rep}(P, GO_{2n+1}(\bar{\mathbb{F}}_q))$ with the set of self dual elements of $\text{Rep}_{2n+1}(P)$.

Thus $\text{Rep}(P, Sp_{2n}(\bar{\mathbb{F}}_q)) \subseteq \text{Rep}_{2n}(P)$ and $\text{Rep}(P, GO_{2n+1}(q)) \subseteq \text{Rep}_{2n+1}(P)$ are both the sets of self dual elements. Since $GO_{2n+1}(\bar{\mathbb{F}}_q) = SO_{2n+1}(\bar{\mathbb{F}}_q) \times \langle \pm \text{Id} \rangle$, $\text{Rep}(P, SO_{2n+1}(\bar{\mathbb{F}}_q)) = \text{Rep}(P, GO_{2n+1}(\bar{\mathbb{F}}_q))$. Also, as we just saw, each odd dimensional self dual P -representation has odd dimensional fixed component, and thus $\text{Rep}(P, SO_{2n+1}(\bar{\mathbb{F}}_q)) = \text{Rep}(P, Sp_{2n}(\bar{\mathbb{F}}_q))$ as subsets of $\text{Rep}_{2n+1}(P)$.

We claim that the centralizer of any finite p -subgroup of $Sp_{2n}(\bar{\mathbb{F}}_q)$ or $SO_{2n+1}(\bar{\mathbb{F}}_q)$ is connected. To see this, fix such a subgroup P , let V be the corresponding representation with symmetric or symplectic form \mathfrak{b} , and let $V = V_0 \oplus W \oplus W'$ be a decomposition such that \mathfrak{b} is as in (3). Then the centralizer of P in $\text{Aut}(V, \mathfrak{b})$ is the product of $\text{Aut}(V_0, \mathfrak{b}_0)$ with $\text{Aut}(W)$, and hence its centralizer in $Sp_{2n}(\bar{\mathbb{F}}_q)$ or $SO_{2n+1}(\bar{\mathbb{F}}_q)$ is connected.

We now apply Proposition 3.5, exactly as in the proof of (a), to show that for a p -group P , $\text{Rep}(P, Sp_{2n}(q))$ injects into $\text{Rep}_{2n}(P)$ with image the set of those V with $\chi_V(g) = \chi_V(g^q) = \chi_V(g^{-1})$ for all $g \in P$; and similarly for $\text{Rep}(P, SO_{2n+1}(q))$. \square

For the linear and unitary groups, Theorem A follows immediately from Proposition 5.1(a). Also, Proposition 5.1(b) implies that when p is odd, Theorem A holds for the symplectic and odd orthogonal groups; and also that

$$\mathcal{F}_p(Sp_{2n}(q)) \simeq \mathcal{F}_p(GO_{2n+1}(q)) (\simeq \mathcal{F}_p(SO_{2n+1}(q)))$$

for each odd p , each $n \geq 1$, and each q prime to p . Theorem A for the even orthogonal groups then follows from the following observation.

Proposition 5.2. *For each odd prime p , each prime power q prime to p , and each $n \geq 1$,*

$$\begin{aligned} \mathcal{F}_p(GO_{2n}^\pm(q)) &\simeq \mathcal{F}_p(SO_{2n+1}(q)) \simeq \mathcal{F}_p(Sp_{2n}(q)) && \text{if } q^n \not\equiv \mp 1 \pmod{p} \\ \mathcal{F}_p(GO_{2n}^\pm(q)) &\simeq \mathcal{F}_p(SO_{2n-1}(q)) \simeq \mathcal{F}_p(Sp_{2n-2}(q)) && \text{if } q^n \not\equiv \pm 1 \pmod{p}. \end{aligned}$$

Proof. Any inclusion $GO_k^\pm(q) \leq GO_{k+1}^\pm(q)$ induces an injection of $\text{Rep}(P, GO_k^\pm(q))$ into $\text{Rep}(P, GO_{k+1}^\pm(q))$ for each p -group P . Thus $\mathcal{F}_p(GO_k^\pm(q))$ is a full subcategory of $\mathcal{F}_p(GO_{k+1}^\pm(q))$ by Proposition 1.6. Hence we get an equivalence of p -fusion categories whenever $GO_k^\pm(q)$ has index prime to p in $GO_{k+1}^\pm(q)$. By the standard formulas for the orders of these groups,

$$\begin{aligned} [GO_{2n+1}(q) : GO_{2n}^\pm(q)] &= q^n(q^n \pm 1) \\ [GO_{2n}^\pm(q) : GO_{2n-1}(q)] &= q^{n-1}(q^n \mp 1), \end{aligned}$$

and the proposition follows. \square

Another consequence of Proposition 5.1 is the following:

Proposition 5.3. *Fix an odd prime p , and a prime power q prime to p . Set $s = \text{order}(q) \pmod{p}$.*

- (a) *If s is even, then for each $n \geq 1$, the inclusion $Sp_{2n}(q) \leq GL_{2n}(q)$ induces an equivalence $\mathcal{F}_p(Sp_{2n}(q)) \simeq \mathcal{F}_p(GL_{2n}(q))$ of fusion systems.*
- (b) *If $s \equiv 2 \pmod{4}$, then for each $n \geq 1$, $Sp_{2n}(q) \leq Sp_{2n}(q^2)$ induces an equivalence of p -fusion systems.*

Proof. If s is even, then -1 is a power of q modulo p , and also modulo p^n for all $n \geq 2$. Hence if P is a p -group, and $V \in \text{Rep}_{2n}(P)$ is such that $\chi_V(g) = \chi_V(g^q)$ for all $g \in P$, then also $\chi_V(g) = \chi_V(g^{-1})$ for all g . So by Proposition 5.1, $\text{Rep}(P, Sp_{2n}(q)) \cong \text{Rep}(P, GL_{2n}(q))$, and so (a) follows from Proposition 1.3(a,b).

If $s \equiv 2 \pmod{4}$, then q^2 has odd order in $(\mathbb{Z}/p)^\times$, and hence in $(\mathbb{Z}/p^n)^\times$ for all n . So $\langle q \rangle = \langle q^2, -1 \rangle$ in $(\mathbb{Z}/p^n)^\times$ for all n . Thus for a p -group P and $V \in \text{Rep}_k(P)$, $\chi_V(g) = \chi_V(g^q) = \chi_V(g^{-1})$ for all $g \in P$ if and only if $\chi_V(g) = \chi_V(g^{q^2}) = \chi_V(g^{-1})$ for all g . So (c) follows from Proposition 1.3(b). \square

Upon combining Propositions 5.2 and 5.3 with Theorem A, we see that for p odd and q prime to p , each of the fusion systems $\mathcal{F}_p(Sp_{2n}(q)) \simeq \mathcal{F}_p(SO_{2n+1}(q))$ and $\mathcal{F}_p(GO_{2n}^\pm(q))$ is isotypically equivalent to the p -fusion system of some general linear group. Note, for example, that when q has odd order in $(\mathbb{Z}/p)^\times$, there is some q' such that $\langle q \rangle = \langle q'^2 \rangle$ in \mathbb{Z}_p^\times , and so

$$\mathcal{F}_p(Sp_{2n}(q)) \simeq \mathcal{F}_p(Sp_{2n}(q'^2)) \simeq \mathcal{F}_p(Sp_{2n}(q')) \simeq \mathcal{F}_p(GL_{2n}(q')).$$

Since $\mathcal{F}_p(SO_{2n}^\pm(q))$ is always normal of index at most two in $\mathcal{F}_p(SO_{2n}^\pm(q))$, this also gives a description of those fusion systems in terms of fusion systems of general linear groups.

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