Motivic double shuffle

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MOTIVIC DOUBLE SHUFFLE

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Introduction

For a p-tuple $\mathbf{k} = (k_1, \ldots, k_p)$ of positive integers and $k_1 \geqslant 2$, the multiple zeta value $\zeta(\mathbf{k})$ is defined as

$$\zeta(\mathbf{k}) = \sum_{n_1 > \ldots > n_p > 0} \frac{1}{n_1^{k_1} \cdots n_p^{k_p}}.$$

These values satisfy two families of algebraic (quadratic) relations known as double shuffle relations, or shuffle and stuffle described below.

In [GM04] A.B. Goncharov and Y. Manin defined a motivic version of multiple zeta values using certain framed mixed Tate motives attached to moduli spaces of genus 0 curves. In this context, the real multiple zeta values appear naturally as periods of those motives attached to the moduli spaces of curves. They do not prove the double shuffle relations directly, referring instead to previous work by A.B. Goncharov in which, using a different definition of motivic multiple polylogarithms based on $(\mathbb{P}^1)^n$ rather than moduli spaces, the motivic double shuffle relations are shown via results on variations of mixed Hodge structure.

The goal of this article is to give an elementary proof of the double shuffle relations directly for the Goncharov and Manin motivic multiple zeta values. The shuffle relation (Proposition 3.8) is straightforward, but for the stuffle (Proposition

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4.23) we use a modification of a method first introduced by P. Cartier for the purpose of proving stuffle for the real multiple zeta values via integrals and blow-up sequences. In this article, we will work over the base field $\mathbb{Q}$.

1. Integral representation of the double shuffle relations

1.1. **Series representation of the stuffle relations.** The stuffle product of a p-tuple $\mathbf{k} = (k_1, \dots, k_p)$ and a q-tuple $\mathbf{l} = (l_1, \dots, l_q)$ is defined recursively by the formula:

$$\mathbf{k} * \mathbf{l} = (\mathbf{k} * (l_1, \dots, l_{q-1})) \cdot l_q + ((k_1, \dots, k_{p-1}) * \mathbf{l}) \cdot k_p + ((k_1, \dots, k_{p-1}) * (l_1, \dots, l_{q-1})) \cdot (k_p + l_q) \tag{1}$$

and $\mathbf{k} * () = () * \mathbf{k} = \mathbf{k}$. Here the $+$ is a formal sum, $A \cdot a$ means that we concatenate a at the end of the tuple A and $\cdot$ is linear in A.

Let $\mathbf{k}$ and $\mathbf{l}$ be two such tuples of integers. We will write $\mathrm{st}(\mathbf{k}, \mathbf{l})$ for the set of the individual terms of the formal sum $\mathbf{k} * \mathbf{l}$ whose coefficients are all equal to 1, such a generic term is then denoted by $\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})$.

In order to multiply two multiple zeta values $\zeta(\mathbf{k})$ and $\zeta(\mathbf{l})$, we split the summation domain of the product $\zeta(\mathbf{k})\zeta(\mathbf{l})$

$$\{0 < n_1 < \ldots < n_p\} \times \{0 < m_1 < \ldots < m_q\}$$

into all the domains that preserve the order of the n_i as well as the order of the m_j and into the boundary domains where some n_i are equal to some m_j. We obtain the following well-known proposition, giving the quadratic relations (2) between multiple zeta values known as the *stuffle relations*:

Proposition 1.1. *Let $\mathbf{k} = (k_1, \dots, k_p)$ and $\mathbf{l} = (l_1, \dots, l_q)$ as above with k_1, $l_1 \geqslant 2$. Then we have:*

(2)

$$\zeta(\mathbf{k})\zeta(\mathbf{l}) = \left(\sum_{n_1 > \ldots > n_p > 0} \frac{1}{n_1^{k_1} \cdots n_p^{k_p}} \right) \left(\sum_{m_1 > \ldots > m_q > 0} \frac{1}{m_1^{l_1} \cdots m_q^{l_q}} \right) = \sum_{\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})} \zeta(\sigma).$$

1.2. **Integral representation of the shuffle relations.** To the tuple $\mathbf{k}$, with $n = k_1 + \cdots + k_p$, we associate the n-tuple:

$$\overline{k} = (\underbrace{0, \dots, 0}_{k_1 - 1 \text{ times}}, 1, \dots, \underbrace{0, \dots, 0}_{k_p - 1 \text{ times}}, 1) = (\varepsilon_n, \dots, \varepsilon_1)$$

and the differential form, introduced by Kontsevich

$$\omega_{\mathbf{k}} = \omega_{\overline{k}} = (-1)^p \frac{dt_1}{t_1 - \varepsilon_1} \wedge \cdots \wedge \frac{dt_n}{t_n - \varepsilon_n}. \tag{3}$$

Then, setting $\Delta_n = \{0 < t_1 < \ldots < t_n < 1\}$, direct integration yields:

$$\zeta(\mathbf{k}) = \int_{\Delta_n} \omega_{\mathbf{k}}.$$

The shuffle product of an n-tuple $(e_1, \dots, e_n) = e_1 \cdot \overline{e}$ and an m-tuple $(f_1, \dots, f_m) = f_1 \cdot \overline{f}$ is defined recursively by:

$$(e_1, \dots, e_n) \text{ ш } (f_1, \dots, f_m) = e_1 \cdot (\overline{e} \text{ ш } (f_1 \cdot \overline{f})) + f_1 \cdot ((e_1 \cdot \overline{e}) \text{ ш } \overline{f}) \tag{4}$$

and $\overline{e} \text{ ш } () = () \text{ ш } \overline{e} = \overline{e}$. Here, as above, the $+$ is a formal sum, $b \cdot B$ means that we concatenate b at the beginning of the tuple B and $\cdot$ is linear in B.

Let $\mathbf{k}$ and $\mathbf{l}$ be two tuples of integers as above. We will write $\mathrm{sh}(\overline{k}, \overline{l})$ for the set of the individual terms of the formal sum $\overline{k} \text{ ш } \overline{l}$ whose coefficients are all equal to 1. Such a generic term is then denoted by $\sigma \in \mathrm{sh}(\overline{k}, \overline{l})$ and can be identify

with a unique permutation $\tilde{\sigma}$ of $\{1,\dots n+m\}$ such that $\tilde{\sigma}(1)<\ldots<\tilde{\sigma}(n)$ and $\tilde{\sigma}(n+1)<\ldots<\tilde{\sigma}(n+m)$. The permutation $\tilde{\sigma}$ will simply be denoted by σ when the context will be clear enough.

We will put an index σ on any object which naturally depends on a shuffle. The following proposition yields the quadratic relations (5) known as the *shuffle relations*.

Proposition 1.2. *Let* $\mathbf{k}=(k_1,\dots,k_p)$ *and* $\mathbf{l}=(l_1,\dots,l_q)$ *with* $k_1,\, l_1\geqslant 2$. *Then:*

$$\int_{\Delta_n}\omega_{\overline{k}}\int_{\Delta_m}\omega_{\overline{l}}=\sum_{\sigma\in \mathrm{sh}(\overline{k},\overline{l})}\int_{\Delta_{n+m}}\omega_\sigma. \tag{5}$$

Proof. Let $n=k_1+\ldots+k_p$ and $m=l_1+\ldots+l_q$. Then we have:

$$\begin{aligned}\int_{\Delta_n}\omega_{\overline{k}}\int_{\Delta_m}\omega_{\overline{l}}&=\left(\int_{\Delta_n}\frac{dt_1}{1-t_1}\cdots\frac{dt_n}{t_n}\right)\left(\int_{\Delta_m}\frac{dt_{n+1}}{1-t_{n+1}}\cdots\frac{dt_{n+m}}{t_{n+m}}\right)\\&=\int_{\Delta}\frac{dt_1}{1-t_1}\cdots\frac{dt_n}{t_n}\frac{dt_{n+1}}{1-t_{n+1}}\cdots\frac{dt_{n+m}}{t_{n+m}}.\end{aligned}$$

The set $\Delta=\{0<t_1<\ldots<t_n<1\}\times\{0<t_{n+1}<\ldots<t_{n+m}<1\}$ can be, up to codimension 1 sets, split into a union of simplices

$$\coprod_{\sigma\in\mathrm{sh}(\llbracket 1,n\rrbracket,\llbracket n+1,m\rrbracket)}\Delta_\sigma\qquad\text{with }\Delta_\sigma=\{0<t_{\sigma(1)}<t_{\sigma(2)}<\ldots<t_{\sigma(n+m)}<1\},$$

where $\llbracket a,b\rrbracket$ denotes the ordered sequence of integers from a to b.

The integral over Δ is the sum of the integrals over the individual simplices. But the integral over one of these simplices is, up to the numbering of the variables, exactly one term of the sum $\displaystyle\sum_{\sigma\in\mathrm{sh}(\overline{k},\overline{l})}\int_{\Delta_{n+m}}\omega_\sigma$. □

1.3. **The stuffle relations in terms of integrals.** We explain here ideas already written in articles of Goncharov [Gon02] and in Francis Brown's Ph.D. thesis [Bro06], showing how to express the stuffle relations (2) in terms of integrals.
Example. We have $\zeta(2)=\int_{\Delta_2}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1}$. The change of variables $t_2=x_1$ and $t_1=x_1x_2$ gives:

$$\zeta(2)=\int_{[0,1]^2}\frac{dx_1}{x_1}\frac{x_1dx_2}{1-x_1x_2}=\int_{[0,1]^2}\frac{dx_1dx_2}{1-x_1x_2}.$$

This change of variables is nothing but the blow-up of the point $(0,0)$ in the projective plane, given in n dimensions by a sequence of blow-ups:

$$t_n=x_1,\ t_{n-1}=x_1x_2,\ \ldots,\ t_1=x_1\ldots x_n. \tag{6}$$

We will write d^nx for $dx_1\cdots dx_n$ where n is the number of variables under the integral. Using the change of variables (6) for $n=4$ we write the Kontsevich forms as follows:

$$\zeta(4)=\int_{[0,1]^4}\frac{d^4x}{1-x_1x_2x_3x_4},\quad \zeta(2,2)=\int_{[0,1]^4}\frac{x_1x_2d^4x}{(1-x_1x_2)(1-x_1x_2x_3x_4)}$$

and

$$\zeta(2)\zeta(2)=\int_{[0,1]^4}\frac{1}{(1-x_1x_2)}\frac{1}{(1-x_3x_4)}d^4x.$$

For any variables α and β we have the equality:

$$\frac{1}{(1-\alpha)(1-\beta)}=\frac{\alpha}{(1-\alpha)(1-\alpha\beta)}+\frac{\beta}{(1-\beta)(1-\beta\alpha)}+\frac{1}{1-\alpha\beta}. \tag{7}$$

This identity will be the key of this section.

Setting $\alpha = x_1x_2$ and $\beta = x_3x_4$ and applying (7), we recover the stuffle relation:

$$\zeta(2)\zeta(2) = \int_{[0,1]^4} \left(\frac{x_1x_2}{(1-x_1x_2)(1-x_1x_2x_3x_4)} + \frac{x_3x_4}{(1-x_3x_4)(1-x_3x_4x_1x_2)} + \frac{1}{1-x_1x_2x_3x_4} \right) d^4x$$

$$\zeta(2)\zeta(2) = \zeta(2,2) + \zeta(2,2) + \zeta(4).$$

General case. We will show that the Cartier decomposition (9) below makes it possible to express all the stuffle relations in terms of integrals as in the example above.

Let $\mathbf{k} = (k_1, \dots, k_p)$ and $\mathbf{l} = (l_1, \dots, l_q)$ two tuples of integers with k_1, $l_1 \geqslant 2$. As above, if σ is a term of the formal sum $\mathbf{k} * \mathbf{l}$, we will write $\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})$. We will put an index σ on any object which naturally depends on a stuffle.

Let $\mathbf{k} = (k_1, \dots, k_p)$ be as above and $n = k_1 + \cdots + k_p$. We define $f_{k_1,\dots,k_p}$ to be the function of n variables defined on $[0,1]^n$ given by:

$$f_{k_1,\dots,k_p}(x_1, \dots, x_n) = \frac{1}{1-x_1\cdots x_{k_1}} \frac{x_1\cdots x_{k_1}}{1-x_1\cdots x_{k_1}x_{k_1+1}\cdots x_{k_1+k_2}} \frac{x_1\cdots x_{k_1+k_2}}{1-x_1\cdots x_{k_1+k_2+k_3}} \cdots \frac{x_1\cdots x_{k_1+\dots+k_{p-1}}}{1-x_1\cdots x_{k_1+\cdots+k_p}}.$$

Proposition 1.3. *For all p-tuples of integers $(k_1, \dots, k_p)$ with $k_1 \geqslant 2$, we have (with $n = k_1 + \cdots + k_p$):*

$$\zeta(k_1, \dots, k_p) = \int_{[0,1]^n} f_{k_1,\dots,k_p}(x_1, \dots, x_n) d^n x. \tag{8}$$

Proof. Let $\omega_{\mathbf{k}}$ be the Kontsevich form associated to a p-tuple $(k_1, \dots, k_p)$ with $n = k_1 + \cdots + k_p$, so that $\zeta(k_1, \dots, k_p) = \int_{\Delta_n} \omega_{\mathbf{k}}$.

Applying the variable change (6) to $\omega_{\mathbf{k}}$, we see that for each term $\frac{dt_i}{t_i}$, there arises from the $\frac{1}{t_i}$ a term $\frac{1}{x_1\cdots x_{n-i+1}}$ which cancels with $\frac{dt_{i-1}}{\cdots} = \frac{x_1\cdots x_{n-i+1}dx_{n-i+2}}{\cdots}$. This gives the result. □

To derive the stuffle relations in general using integrals and the functions $f_{k_1,\dots,k_p}$, we will use the following notation.

Notation. Let $\mathbf{k}$ be a sequence $(k_1, \dots, k_p)$, $n = k_1 + \cdots + k_p$. We have n variables $x_1, \dots, x_n$.

- For any sequence $\mathbf{a} = (a_1, \dots, a_r)$, we will write $\prod \mathbf{a} = a_1 \cdots a_r$.
- The sequence $(x_1, \dots, x_n)$ will be written $\mathbf{x}$. We set $\mathbf{x}(\mathbf{k}, 1) = (x_1, \dots, x_{k_1})$ and
$$\mathbf{x}(\mathbf{k}, i) = (x_{k_1+\cdots+k_{i-1}+1}, \dots, x_{k_1+\cdots+k_i}),$$
so the $\mathbf{x}$ is the concatenation of sequences $\mathbf{x}(\mathbf{k}, 1) \cdots \mathbf{x}(\mathbf{k}, p)$.
- The sequence $(x_1, \dots, x_{k_1+\cdots+k_i}) = \mathbf{x}(\mathbf{k}, 1) \cdots \mathbf{x}(\mathbf{k}, i)$ will be denoted by $\mathbf{x}(\mathbf{k}, \leqslant i)$. If $\mathbf{k} = (\mathbf{k_0}, k_p)$, $\mathbf{x}_0 = \mathbf{x}(\mathbf{k}, \leqslant p-1)$ will be the sequence
$$(x_1, \dots, x_{k_1+\cdots+k_{p-1}}).$$
- If $\mathbf{l}$ is a q-tuple with $l_1 + \cdots l_q = m$ and $\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})$, y_σ will be the sequence in the variables $x_1, \dots, x_n, x'_1, \dots, x'_m$ in which each group of variables
$$\mathbf{x}(\mathbf{k}, i) = (x_{k_1+\cdots+k_{i-1}+1}, \dots, x_{k_1+\cdots+k_i})$$
$$(\text{resp. } \mathbf{x}'(\mathbf{l}, j) = (x'_{l_1+\cdots+l_{j-1}+1}, \dots, x'_{l_1+\cdots+l_j}))$$

is in the position of k_i (resp. l_j) in σ. Components of σ of the form $k_i + l_j$ give rise to subsequences like

$$(x_{k_1+\cdots+k_{i-1}+1}, \ldots, x_{k_1+\cdots+k_i}, x'_{l_1+\cdots+l_{j-1}+1}, \ldots, x'_{l_1+\cdots+l_j}) = (\mathbf{x}(\mathbf{k}, i), \mathbf{x}'(\mathbf{l}, j)).$$

- Following these notations, products $x_1 \cdots x_{k_1}$, $x_{k_1+\cdots+k_{i-1}+1} \cdots x_{k_1+\cdots+k_i}$, $x_1 \cdots x_{k_1+\cdots+k_i}$ will be written respectively $\prod \mathbf{x}(\mathbf{k}, 1)$, $\prod \mathbf{x}(\mathbf{k}, i)$, $\prod \mathbf{x}(\mathbf{k}, \leqslant i)$. As $\mathbf{x}(\mathbf{k}, \leqslant p-1) = \mathbf{x}_0$ and $\mathbf{x}(\mathbf{k}, \leqslant p) = \mathbf{x}$, products $\prod \mathbf{x}(\mathbf{k}, \leqslant p-1)$ and $\prod \mathbf{x}(\mathbf{k}, \leqslant p)$ will be written $\prod \mathbf{x}_0$ and $\prod \mathbf{x}$.
  We remark that for each $\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})$, $\prod \sigma = \prod \mathbf{x} \prod \mathbf{x}'$.

Remark 1.4. Let $(k_1, \ldots, k_p) = (\mathbf{k_0}, k_p)$ be a sequence of integers. Then:

$$f_{k_1,\ldots,k_p}(\mathbf{x}) = f_{k_1,\ldots,k_{p-1}}(\mathbf{x}(\mathbf{k}, \leqslant p-1)) \frac{\prod \mathbf{x}(\mathbf{k}, \leqslant p-1)}{1 - \prod \mathbf{x}(\mathbf{k}, \leqslant p)} = f_{k_1,\ldots,k_{p-1}}(\mathbf{x}_0) \frac{\prod \mathbf{x}_0}{1 - \prod \mathbf{x}}.$$

Proposition 1.5. *Let* $\mathbf{k} = (k_1, \ldots, k_p)$ *and* $\mathbf{l} = (l_1, \ldots, l_q)$ *be two sequences of weight* n *and* m. *Then:*

$$f_{k_1,\ldots,k_p}(\mathbf{x}(\mathbf{k}, 1), \ldots, \mathbf{x}(\mathbf{k}, p)) \cdot f_{l_1,\ldots,l_q}(\mathbf{x}'(\mathbf{l}, 1), \ldots, \mathbf{x}'(\mathbf{l}, q)) = \sum_{\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})} f_\sigma(y_\sigma). \tag{9}$$

Proof. We proceed by induction on the depth of the sequence. The recursion formula for the stuffle is given in (1).

If $p = q = 1$**:** As we have

$$f_n(\mathbf{x}(\mathbf{k}, 1)) f_m(\mathbf{x}'(\mathbf{l}, 1)) = \frac{1}{1 - \prod \mathbf{x}(\mathbf{k}, \leqslant 1)} \cdot \frac{1}{1 - \prod \mathbf{x}'(\mathbf{l}, \leqslant 1)} = \frac{1}{1 - \prod \mathbf{x}} \cdot \frac{1}{1 - \prod \mathbf{x}'},$$

using the formula (7) with $\alpha = \prod \mathbf{x}$ and $\beta = \prod \mathbf{x}'$ leads to

$$f_n(\mathbf{x}(\mathbf{k}, 1)) f_m(\mathbf{x}'(\mathbf{l}, 1)) = \frac{\prod \mathbf{x}}{(1 - \prod \mathbf{x})(1 - \prod \mathbf{x} \prod \mathbf{x}')} + \frac{\prod \mathbf{x}'}{(1 - \prod \mathbf{x}')(1 - \prod \mathbf{x}' \prod \mathbf{x})} + \frac{1}{1 - \prod \mathbf{x} \prod \mathbf{x}'}. \tag{10}$$

Inductive step: Let $(k_1, \ldots, k_p) = (\mathbf{k_0}, k_p)$ and $(l_1, \ldots, l_q) = (\mathbf{l_0}, l_q)$ be two sequences. By Remark 1.4, the following equality holds

$$f_{\mathbf{k_0}, k_p}(\mathbf{x}_0, \mathbf{x}(\mathbf{k}, p)) f_{\mathbf{l_0}, l_q}(\mathbf{x_0}', \mathbf{x}'(\mathbf{l}, q)) = f_{\mathbf{k_0}}(\mathbf{x}_0) \frac{\prod \mathbf{x}_0}{1 - \prod \mathbf{x}} f_{\mathbf{l_0}}(\mathbf{x_0}') \frac{\prod \mathbf{x}'_0}{1 - \prod \mathbf{x}'}.$$

Applying the formula (7) with $\alpha = \prod \mathbf{x}$ and $\beta = \prod \mathbf{x}'$, one sees that the RHS of the previous equation is equal to

$$f_{\mathbf{k_0}}(\mathbf{x}_0) f_{\mathbf{l_0}}(\mathbf{x_0}') \cdot (\prod \mathbf{x}_0 \cdot \prod \mathbf{x}'_0) \left(\frac{\prod \mathbf{x}}{(1 - \prod \mathbf{x})(1 - \prod \mathbf{x} \prod \mathbf{x}')} + \frac{\prod \mathbf{x}'}{(1 - \prod \mathbf{x}')(1 - \prod \mathbf{x}' \prod \mathbf{x})} + \frac{1}{(1 - \prod \mathbf{x} \prod \mathbf{x}')} \right).$$

Expanding and using the Remark 1.4 we obtain:

$$f_{\mathbf{k_o},k_p}(\mathbf{x}_0,\mathbf{x}(\mathbf{k},p))f_{\mathbf{l_o},l_q}(\mathbf{x_0}',\mathbf{x}'(\mathbf{l},q)) = \tag{11}$$

$$\left(f_{\mathbf{k_o},k_p}(\mathbf{x})f_{\mathbf{l_o}}(\mathbf{x_0}')\right)\cdot\frac{\prod\mathbf{x}\prod\mathbf{x}'_0}{1-\prod\mathbf{x}\prod\mathbf{x}'}+\left(f_{\mathbf{k_o}}(\mathbf{x}_0)f_{\mathbf{l_o},l_q}(\mathbf{x}')\right)\cdot\frac{\prod\mathbf{x}'\prod\mathbf{x}_0}{1-\prod\mathbf{x}'\prod\mathbf{x}}$$

$$+\left(f_{\mathbf{k_o}}(\mathbf{x}_0)f_{\mathbf{l_o}}(\mathbf{x_0}')\right)\cdot\frac{\prod\mathbf{x}_0\prod\mathbf{x}'_0}{1-\prod\mathbf{x}\prod\mathbf{x}'}.$$

Hence, the product of functions $f_{k_1,\dots,k_p}$ and $f_{l_1,\dots,l_q}$ satisfies a recursion formula identical to the formula (1) that defines the stuffle product. Using induction, the proposition follows. □

Corollary 1.6 (integral representation of the stuffle)**.** *Integrating the statement of the previous proposition over the cube and permuting the variables in each term of the RHS, we obtain:*

$$\zeta(\mathbf{k})\zeta(\mathbf{l}) = \int_{[0,1]^n} f_{\mathbf{k}}d^nx\int_{[0,1]^m} f_{\mathbf{l}}d^mx = \int_{[0,1]^{n+m}}\sum_{\sigma\in\mathrm{st}(\mathbf{k},\mathbf{l})} f_\sigma\ d^{n+m}x = \sum_{\sigma\in\mathrm{st}(\mathbf{k},\mathbf{l})}\zeta(\sigma).$$

Proof. We only need to check that all integrals are convergent. As all the functions are positive on the integration domain, all changes of variable are allowed and we can deduce the convergence of each term from the convergence of the iterated integral representation for the multiple zeta values.

Another argument is to remark that the orders of the poles of our functions along a codimension k subvariety is at most k. Then, for each integral, a succession of blow-up ensures that the integral converge. □

2. Moduli spaces of curves; double shuffle and forgetful maps

2.1. **Shuffle and moduli spaces of curves.** Let $\mathbf{k}$ and $\mathbf{l}$ be as in the previous section, let $n = k_1+\cdots+k_p$ and $m = l_1+\cdots+l_q$. Following the article of Goncharov and Manin [GM04], we will identify a point of $\mathcal{M}_{0,j+3}$, the moduli space of curves of genus 0 with $j+3$ marked points, with a sequence $(0, z_1, \dots, z_j, 1, \infty)$, the z_i being pairwise distinct and distinct from 0, 1 and ∞, and write Φ_j for the open cell in $\mathcal{M}_{0,j+3}(\mathbb{R})$ which is mapped onto Δ_j, the standard simplex, by the map: $\mathcal{M}_{0,j+3}\to(\mathbb{P}^1)^j$, $(0,z_1,\dots,z_j,1,\infty)\mapsto(z_1,\dots,z_j)$. Then we have:

$$\zeta(k_1,\dots,k_p) = \int_{\Phi_n}\omega_{\mathbf{k}}.$$

Proposition 2.1. *Let β be the map defined by*

$$\begin{array}{ccc}\mathcal{M}_{0,n+m+3} & \xrightarrow{\ \beta\ } & \mathcal{M}_{0,n+3}\times\mathcal{M}_{0,m+3}\\ (0,z_1,\dots,z_{n+m},1,\infty) & \longmapsto & (0,z_1,\dots,z_n,1,\infty)\times(0,z_{n+1},\dots,z_{n+m},1,\infty).\end{array}$$

Then, letting t_i be the coordinate such that $t_i(0,z_1,\dots,z_{n+m},1,\infty) = z_i$, we have

$$\beta^*(\omega_{\mathbf{k}}\wedge\omega_{\mathbf{l}}) = \frac{dt_1}{1-t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}\wedge\frac{dt_{n+1}}{1-t_{n+1}}\wedge\cdots\wedge\frac{dt_{n+m}}{t_{n+m}}.$$

Furthermore, if for $\sigma\in\mathrm{sh}([\![1,n]\!],[\![n+1,n+m]\!])$ we write Φ^σ_{n+m} for the open cell of $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ in which the points are in the same order as their indices are in σ, we have

$$\beta^{-1}(\Phi_n\times\Phi_m) = \coprod_{\sigma\in\mathrm{sh}([\![1,n]\!],[\![n+1,n+m]\!])}\Phi^\sigma_{n+m}.$$

Proof. The first part is obvious.

In order to show that $\beta^{-1}(\Phi_n \times \Phi_m) = \coprod \Phi^\sigma_{n+m}$ we have to remember that a cell in $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ is given by a cyclic order on the marked points. Let $X = (0, z_1, \ldots, z_{n+m}, 1, \infty)$ be a point in $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ such that $\beta(X) \in \Phi_n \times \Phi_m$. The values of the z_i have to be such that

$$0 < z_1 < \ldots < z_n < 1 \ (< \infty) \quad \text{and} \quad 0 < z_{n+1} < \ldots < z_{n+m} < 1 \ (< \infty). \tag{12}$$

However there is no order condition relating, say z_1 to z_{n+1}.

So, points on $\mathcal{M}_{0,n+m+3}(\mathbb{R})$ which are in $\beta^{-1}(\Phi_n \times \Phi_m)$ are such that the z_i are compatible with (12). That is there are in $\coprod_{\sigma \in \mathrm{sh}([\![1,n]\!],[\![n+1,n+m]\!])} \Phi^\sigma_{n+m}$. □

The open embedding β being such that $\Phi_n \times \Phi_m \setminus \big(\beta(\beta^{-1}(\Phi_n \times \Phi_m))\big)$ is of codimension 1, we have the following proposition

Proposition 2.2. *The shuffle relation $\zeta(\mathbf{k})\zeta(\mathbf{l}) = \sum_{\sigma \in \mathrm{sh}(\mathbf{k},\mathbf{l})} \zeta(\sigma)$ is a consequence of the following change of variables:*

$$\int_{\Phi_n \times \Phi_m} \omega_\mathbf{k} \wedge \omega_\mathbf{l} = \int_{\beta^{-1}(\Phi_n \times \Phi_m)} \beta^*(\omega_\mathbf{k} \wedge \omega_\mathbf{l}).$$

Proof. Using the previous proposition, the right hand side of this equality is equal to

$$\sum_{\sigma \in \mathrm{sh}([\![1,n]\!],[\![n+1,n+m]\!])} \int_{\Phi^\sigma_{n+m}} \frac{dt_1}{1-t_1} \wedge \cdots \wedge \frac{dt_{n+m}}{t_{n+m}}.$$

Then we permute the variables and change their names in order to have an integral over Φ_{n+m} for each term. This is the same computation we did for the integral over $\mathbb{R}^{n+m}$ in proposition 1.2.

As the form $\frac{dt_1}{1-t_1} \wedge \cdots \wedge \frac{dt_{n+m}}{t_{n+m}}$ (resp. $\frac{dt_{\sigma(1)}}{1-t_{\sigma(1)}} \wedge \cdots \wedge \frac{dt_{\sigma(n+m)}}{t_{\sigma(n+m)}}$) does not have any pole on the boundary of Φ^σ_{n+m} (resp. Φ_{n+m}), all the integrals are convergent. □

2.2. **Stuffle and moduli spaces of curves.** In Section 1.3, in order to have an integral representation of the stuffle product, we introduced, using the integral over a simplex and a change of variables, a cubical representation of the MZVs (integral over a cube). We use here a similar change of variable to introduce an other system of local coordinates on $\overline{\mathcal{M}_{0,r+3}}$, the Deligne-Mumford compactification of the moduli space of curves. We will, following [Bro06], speak of cubical coordinates. Those cubical coordinates, u_i, are defined on an open of $\overline{\mathcal{M}_{0,r+3}}$ by $u_1 = t_r$ and $u_i = t_{r-i+1}/t_{r-i+2}$ for $i < r$ where the t_i are the usual (simplicial) coordinates on $\overline{\mathcal{M}_{0,r+3}}$. This cubical system is well adapted to express the stuffle relations on the moduli spaces of curves.

Proposition 2.3. *Let δ be the map defined by*

$$\begin{aligned} \mathcal{M}_{0,n+m+3} &\xrightarrow{\quad\delta\quad} \mathcal{M}_{0,n+3} \times \mathcal{M}_{0,m+3} \\ (0, z_1, \ldots, z_{n+m}, 1, \infty) &\longmapsto (0, z_{m+1}, \ldots, z_{m+n}, 1, \infty) \times (0, z_1, \ldots, z_m, z_{m+1}, \infty). \end{aligned}$$

Writing the expression of $\omega_\mathbf{k}$ and $\omega_\mathbf{l}$ in the cubical coordinates, one finds $\omega_\mathbf{k} = f_\mathbf{k}(u_1, \ldots, u_n)d^n u$ and $\omega_\mathbf{l} = f_\mathbf{l}(u_{n+1}, \ldots, u_{n+m})d^m u$ where the $f_\mathbf{k}$ are as in section 1.3. Then, using those coordinates we have

$$\delta^*(\omega_\mathbf{k} \wedge \omega_\mathbf{l}) = f_{k_1,\ldots,k_p}(u_1, \ldots, u_n) f_{l_1,\ldots,l_q}(u_{n+1}, \ldots, u_{n+m}) d^{n+m} u$$

and

$$\delta^{-1}(\Phi_n \times \Phi_m) = \Phi_{n+m}.$$

Proof. To prove the second statement, let $X = (0, z_1, \dots, z_{n+m}, 1, \infty)$ such that $\delta(X) \in \Phi_n \times \Phi_m$. Then the values of the z_i's have to verify
(13)
$$0 < z_1 < \ldots < z_m < z_{m+1} \ (< \infty) \quad \text{and} \quad 0 < z_{m+1} < \ldots < z_{n+m} < 1 \ (< \infty).$$
These conditions show that $0 < z_1 < \ldots < z_m < z_{m+1} < \ldots < 1 < \infty$, so $X \in \Phi_{n+m}$.

To prove the first statement, we claim that δ is expressed in cubical coordinates by
$$(u_1, \dots, u_{n+m}) \longmapsto (u_1, \dots, u_n) \times (u_{n+1}, \dots, u_{n+m}).$$
It is obvious to see that for the left hand factor the coordinates are not changed. For the right hand factor we have to rewrite the expression of the right side in terms of the standard representatives on $\mathcal{M}_{0,m+3}$. We have
$$(0, z_1, \dots, z_m, z_{m+1}, \infty) = (0, z_1/z_{m+1}, \dots, z_m/z_{m+1}, 1, \infty) = (0, t_1, \dots, t_m, 1, \infty)$$
in simplicial coordinates. This point is given in cubical coordinates on $\mathcal{M}_{0,m+3}$ by
$$(t_m, t_{m-1}/t_m, \dots, t_1/t_2) = (z_m/z_{m+1}, \dots, z_1/z_2) = (u_{n+1}, \dots, u_{n+m}).$$
□

As a consequence of this discussion and the results of Section 1.3, we have the following proposition.

Proposition 2.4. *Using the Cartier decomposition (9), the stuffle product can be viewed as the change of variables:*
$$\int_{\Phi_n \times \Phi_m} \omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}} = \int_{\delta^{-1}(\Phi_n \times \Phi_m)} \delta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}).$$

Remark 2.5. We should point out here the fact that the Cartier decomposition "does not lie in the moduli spaces of curves", in the sense that forms appear in the decomposition which are not holomorphic on the moduli space. For example, in the Cartier decomposition of $f_{2,1}(u_1, u_2, u_3) f_{2,1}(u_4, u_5, u_6)$, we see the term
$$\frac{u_1 u_2 u_4 u_5 du_1 du_2 du_3 du_4 du_5 du_6}{(1 - u_1 u_2 u_4 u_5)(1 - u_1 u_2 u_3 u_4 u_5 u_6)}$$
which is not a holomorphic differential form on $\mathcal{M}_{0,6}$. However, it is a well-defined convergent form on the standard cell where it is integrated. Changing the numbering of the variables (which stabilises the standard cell) gives the equality with $\zeta(4, 2)$. This example represents the situation in the general case: when simply dealing with integrals, the non-holomorphic forms are not a problem. However, in the context of framed motives they are.

3. Motivic shuffle for the "convergent" words

3.1. **Framed mixed Tate motives and motivic multiple zeta values.** This section is a short introduction to the motivic tools we will use to prove the motivic double shuffle. The motivic context is a cohomological version of Voevodsky's category $\mathcal{DM}_{\mathbb{Q}}$ [Voe00]. Goncharov developed in [Gon99], [Gon05] and [Gon01] an additional structure on mixed Tate motives, introduced in [BGSV90], in order to select a specific period of a mixed Tate motive.

An n-framed mixed Tate motive is a mixed Tate motive M equipped with two non-zero morphisms:
$$v : \mathbb{Q}(-n) \to \mathrm{Gr}^W_{2n} M \qquad f : \mathbb{Q}(0) \to \left(\mathrm{Gr}^W_0 M\right)^\vee = \mathrm{Gr}^W_0 M^\vee.$$
On the set of all n-framed mixed motives, we consider the coarsest equivalence relation for which $(M, v, f) \sim (M', v', f')$ if there is a linear map $M \to M'$ respecting

the frames. Let $\mathcal{A}_n$ be the set of equivalence classes and $\mathcal{A}_\bullet$ be the direct sum of the $\mathcal{A}_n$. We write $[M; v; f]$ for an equivalence class

Theorem 3.1 ([Gon05]). *$\mathcal{A}_\bullet$ has a natural structure of graded commutative Hopf algebra over $\mathbb{Q}$.*

$\mathcal{A}_\bullet$ is canonically isomorphic to the dual of Hopf algebra of all endomorphisms of the fibre functor of the Tannakian category of mixed Tate motives.

In our context, the morphism v of a frame should be linked with some differential form and the morphism f is a homological counterpart of v, that is a real simplex.

We give here two technical lemmas that will be used in the next sections. We write $[M, v, f]$ for the equivalence class of (M, v, f) in $\mathcal{A}_\bullet$. We will speak of framed mixed Tate motives in both cases.

We recall that the adition of two framed mixed Tate motives $[M, v, f]$ and $[M', v', f']$ is

$$[M, v, f] \oplus [M', v', f'] \coloneqq [M \oplus M', (v, v'), f + f'].$$

Lemma 3.2. *Let M be a mixed Tate motive. v, v_1, $v_2 : \mathbb{Q}(-n) \to \mathrm{Gr}^W_{2n} M$ and f, f_1, $f_2 : \mathbb{Q}(0) \to \mathrm{Gr}^W_0 M^\vee$. We have:*

$$[M; v; f_1 + f_2] = [M; v; f_1] + [M; v; f_2]$$

and

$$[M; v_1 + v_2; f] = [M; v_1; f] + [M; v_2; f]$$

Proof. It follow directly from the definition in [Gon05]. For the first case, it is straightforward to check that the diagonal map $\varphi : M \to M \oplus M$ is compatible with the frames. For the second equality, the map from $M \oplus M$ to M which sends (m_1, m_2) to $m_1 + m_2$ gives the map between the underlying vector space and respects the frames. □

Lemma 3.3. *Let M and M' be two mixed Tate motives. Let M be framed by $v : \mathbb{Q}(-n) \to \mathrm{Gr}^W_{2n}$ and $f : \mathbb{Q}(0) \to \mathrm{Gr}^W_0 M^\vee$. Suppose there exists $v' : \mathbb{Q}(-n) \to \mathrm{Gr}^W_{2n} M'$ and $\varphi : M' \to M$ compatible with v and v'. Then f induces a map $f' : \mathbb{Q}(0) \to \mathrm{Gr}^W_0 M'^\vee$ and if f' is non zero, then φ gives an equality of framed mixed Tate motives $[M; v; f] = [M; v'; f']$*

We recall a classical result, used in [GM04] and described more explicitly in [Gon02] that allows us to build mixed Tate motives from natural geometric situations. In [Gon02], A.B. Goncharov defined a *Tate variety* as a smooth projective variety $\mathcal{M}$ such that the motive of $\mathcal{M}$ is a direct sum of copies of the Tate motive $\mathbb{Q}(m)$ (for certain m). We say that a divisor D on $\mathcal{M}$ provides a Tate stratification on $\mathcal{M}$ if all strata of D, including $D_\emptyset = \mathcal{M}$, are Tate varieties.

Let $\mathcal{M}$ be a smooth variety and X and Y be two normal crossing divisors on $\mathcal{M}$. Let Y^X be $Y \setminus (Y \cap X)$, which is a normal crossing divisor on $\mathcal{M} \setminus X$.

Lemma 3.4. *Let $\mathcal{M}$ be a smooth variety of dimension n over $\mathbb{Q}$ and $X \cup Y$ be a normal crossing divisor on $\mathcal{M}$ providing a Tate stratification of $\mathcal{M}$. If X and Y share no common irreducible components then there exists a mixed Tate motive:*

$$\mathrm{H}^n(\mathcal{M} \setminus X; Y^X)$$

such that its different realisations are given by the respective relative cohomology groups.

We have the following version given in [GM04].

Corollary 3.5. *Let X and Y be two normal crossing divisors on $\partial\overline{\mathcal{M}_{0,n+3}}$ and suppose they do not share any irreducible components. Then, any choice of non-zero elements*

$$[\omega_X] \in \mathrm{Gr}^W_{2n}(\mathrm{H}^n(\overline{\mathcal{M}_{0,n+3}} \setminus X)); \qquad [\Phi_Y] \in \mathrm{Gr}^W_0(\mathrm{H}^n(\overline{\mathcal{M}_{0,n+3}}; Y))^\vee$$

defines a framed *mixed Tate motive given by*

$$\left[\mathrm{H}^n(\overline{\mathcal{M}_{0,n+3}} \setminus X; Y^X); [\omega_X]; [\Phi_Y]\right].$$

The following lemma shows that we have some flexibility in choosing X and Y for the framed mixed Tate motive $\left[\mathrm{H}^n(\mathcal{M} \setminus X; Y^X); [\omega_X]; [\Phi_Y]\right]$.

Lemma 3.6. *With the notations of Lemma 3.4, let X' be a normal crossing divisor containing X which still does not share any irreducible component with Y, $X' \cup Y$ being a normal crossing divisor. Then:*

$$\left[\mathrm{H}^n(\mathcal{M} \setminus X; Y^X); [\omega_X]; [\Phi_Y]\right] = \left[\mathrm{H}^n(\mathcal{M} \setminus X'; Y^{X'}); [\omega_X]; [\Phi_Y]\right].$$

Suppose now that Y' is a normal crossing divisor containing Y which does not share any irreducible component with X', $X' \cup Y'$ being a normal crossing divisor. Then:

$$\left[\mathrm{H}^n(\mathcal{M} \setminus X'; Y^{X'}); [\omega_X]; [\Phi_Y]\right] = \left[\mathrm{H}^n(\mathcal{M} \setminus X'; Y'^{X'}); [\omega_X]; [\Phi_Y]\right].$$

We are now in a position to introduce Goncharov's and Manin's definition of motivic multiple zeta values.

Definition 3.7. In particular, let $\mathbf{k}$ be a p-tuple with $k_1 \geqslant 2$ and let $A_{\mathbf{k}}$ be the divisor of singularities of $\omega_{\mathbf{k}}$. Let B_n be the Zariski closure of the boundary of Φ_n. The motivic multiple zeta value is defined in [GM04] by:

$$\left[\mathrm{H}^n(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B_n^{A_{\mathbf{k}}}); [\omega_{\mathbf{k}}]; [\Phi_n]\right]$$

3.2. **Motivic Shuffle.** The map β defined in Proposition 2.1 will be the key to check that the motivic multiple zeta values satisfy the shuffle relations.This map extends continuously to the Deligne-Mumford compactification of the moduli spaces of curves:

$$\overline{\mathcal{M}_{0,n+m+3}} \quad \xrightarrow{\beta} \quad \overline{\mathcal{M}_{0,n+3}} \times \overline{\mathcal{M}_{0,m+3}}.$$

Let $\omega_{\mathbf{k}}$ and $\omega_{\mathbf{l}}$ be as in section 2.1, and write $A_{\mathbf{k}}$ and $A_{\mathbf{l}}$ for their respective singularity divisors. Let B_n and B_m denote the Zariski closures of the boundary of Φ_n and Φ_m respectively. For $\sigma \in \mathrm{sh}([\![1,n]\!],[\![n+1,n+m]\!])$, let ω_σ denote the differential form which corresponds to the shuffled MZV and let A_σ denote its divisor of singularities. Let B_{n+m} denote the Zariski closure of the boundary of Φ_{n+m} and B_σ that of Φ^σ_{n+m}. The shuffle relations between motivic multiple zeta values are given in the following proposition.

Proposition 3.8. *We have an equality of framed motives:*

$$\left[\mathrm{H}^n\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B_n^{A_{\mathbf{k}}}\right); [\omega_{\mathbf{k}}]; [\Phi_n]\right] \cdot \left[\mathrm{H}^m\left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}; B_m^{A_{\mathbf{l}}}\right); [\omega_{\mathbf{l}}]; [\Phi_m]\right] =$$

$$\sum_{\sigma \in \mathrm{sh}([\![1,n]\!],[\![n+1,n+m]\!])} \left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_\sigma; B^{A_\sigma}_{n+m}\right); [\omega_\sigma]; [\Phi_{n+m}]\right].$$

Proof. To prove this equality, we need to display a map between the underlying vector spaces which respects the frames.

We set A' the boundary of $(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}})$, it is equal to the divisor of singularities of $\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}$ on $\overline{\mathcal{M}_{0,n+3}} \times \overline{\mathcal{M}_{0,m+3}}$.

Let $A_0 = \beta^{-1}(A')$ and let B_0 be the Zariski closure of the boundary of $\Phi_0 = \beta^{-1}(\Phi_n \times \Phi_m)$. Let $B_{n,m}$ be the Zariski closure of the boundary of $\Phi_n \times \Phi_m$. The map β gives a map:

$$(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_0^{A_0}) \xrightarrow{\beta} \left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); \beta(B_0)^{A'}\right)$$
$$\uparrow \alpha$$
$$\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); B_{n,m}^{A'}\right).$$

We introduce the the right-hand inclusion α because B_0 does not map onto $B_{n,m}$ via β. The map α induces a map on the mixed Tate motives:

$$\begin{aligned} &\mathrm{H}^{n+m}\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); \beta(B_0)^{A'}\right) \xrightarrow{\alpha^*} \\ &\qquad \mathrm{H}^{n+m}\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); B_{n,m}^{A'}\right) \end{aligned} \tag{14}$$

The frames on the RHS of (14) is given by $[\Phi_n \times \Phi_m]$ and $[\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}]$. Applying lemma 3.3 to (14), $[\Phi_n \times \Phi_m]$ induces a map $\tilde{\Phi}$ from $\mathbb{Q}(0)$ to the $-2(n+m)$ graded part of the LHS of (14). In fact, since α is the identity map, we have $[\tilde{\Phi}] = [\Phi_n \times \Phi_m]$, so $[\Phi_n \times \Phi_m]$ and $[\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}]$ give a frames on the LHS of (14) which is compatible with the map α^*.

The map β induces a map on the mixed Tate motives:

$$\begin{aligned} &\mathrm{H}^{n+m}\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); \beta(B_0)^{A'}\right) \xrightarrow{\beta^*} \\ &\qquad \mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_0^{A_0}) \end{aligned} \tag{15}$$

On the RHS of (15) the frames given by $[\omega_0]$ where ω_0 is $\beta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}})$ and $[\Phi_0] = [\beta^{-1}(\Phi_n \times \Phi_m)]$ which is compatible with the map β^*.

Now we can prove the proposition. The Künneth formula gives us a map:

$$\begin{aligned} &\mathrm{H}^n\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B_n^{A_{\mathbf{k}}}\right) \otimes \mathrm{H}^m\left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}; B_m^{A_{\mathbf{l}}}\right) \longrightarrow \\ &\qquad \mathrm{H}^{n+m}\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); B_{n,m}^{A'}.\right) \end{aligned}$$

By theorem 3.1, this map also respects the frames, so the associated framed mixed Tate motives are equal. By (14),

$$\left[\mathrm{H}^{n+m}\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); B_{n,m}^{A'}\right); [\omega_{\mathbf{k}} \otimes \omega_{\mathbf{l}}]; [\Phi_n \times \Phi_m]\right]$$

is equal to

$$\left[\mathrm{H}^{n+m}\left((\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}) \times (\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}); \beta(B_0)^{A'}\right); [\omega_{\mathbf{k}} \otimes \omega_{\mathbf{l}}]; [\Phi_n \times \Phi_m]\right],$$

which, using (15), is equal to

$$\left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_0^{A_0}); [\omega_0]; [\Phi_0]\right].$$

It remains to show that:

$$\begin{aligned} &\left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_0^{A_0}); [\omega_0]; [\Phi_0]\right] = \\ &\qquad \sum_\sigma \left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_\sigma; B_{n+m}^{A_\sigma}); [\omega_\sigma]; [\Phi_{n+m}]\right]. \end{aligned} \tag{16}$$

In the LHS of (16), B_0 being included in $B_{\mathrm{sh}} = \bigcup_\sigma B_\sigma$, we can replace B_0 by B_{sh} using lemma 3.6.

As $[\Phi_0] = \sum_\sigma [\Phi^\sigma_{n+m}]$, lemma 3.2 shows that the LHS of 16 is equal to

$$\sum_\sigma \left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B^{A_0}_{\mathrm{sh}}); [\omega_0]; [\Phi^\sigma_{n+m}]\right].$$

Using the fact that $B_\sigma \subset B_{\mathrm{sh}}$ and the identity map, lemma 3.6 shows that this framed motive is equal to

$$\sum_\sigma \left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B^{A_0}_{\sigma}); [\omega_0]; [\Phi^\sigma_{n+m}]\right].$$

As the divisor of singularities A of ω_0 is included in A_0, using lemma 3.6 we can replace A_0 by A in this framed motive. Then permuting the points gives an equality of framed motives on each term of the sum,

$$\left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B^{A_0}_{\sigma}); [\omega_0]; [\Phi^\sigma_{n+m}]\right],$$

with

$$\left[\mathrm{H}^{n+m}(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_\sigma; B^{A_\sigma}_{n+m}); [\omega_\sigma]; [\Phi_{n+m}]\right].$$

Thus, we obtain the desired formula:

$$\left[\mathrm{H}^{n}\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B^{A_{\mathbf{k}}}_{n}\right); [\omega_{\mathbf{k}}]; [\Phi_n]\right] \cdot \left[\mathrm{H}^{m}\left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}; B^{A_{\mathbf{l}}}_{m}\right); [\omega_{\mathbf{l}}]; [\Phi_m]\right] =$$
$$\sum_{\sigma \in \mathrm{sh}((1,\dots,n),(n+1,\dots,n+m))} \left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_\sigma; B^{A_\sigma}_{n+m}\right); [\omega_\sigma]; [\Phi_{n+m}]\right].$$

□

4. The stuffle case

The goal of this section is to be able to translate all the calculations done in Section 1.3 into a motivic context. In order to achieve this goal, we need to define, for all n greater than 2, a variety $X_n \to \mathbb{A}^n$ resulting of successive blow-ups of $\mathbb{A}^n$ together with a differential form $\Omega^s_{k_1,\dots,k_p}$ for any tuple of integer $(k_1,\dots,k_p)$ (with $k_1 + \cdots k_p = n$) and any permutation s of $[\![1,n]\!]$. After defining another but equivalent motivic counterpart of the multiple zeta values, we will show, using a natural map from X_{n+m} to an open subset of $\mathcal{M}_{0,n+m+3}$, that the stuffle product is defined at a motivic level.

4.1. **Blow up preliminaries.**

Lemma 4.1 (Flag Blowup Lemma; [Uly02].)**.** *Let $V^1_0 \subset V^2_0 \subset \cdots V^r_0 \subset W_0$ be a flag of smooth subvarieties in a smooth algebraic variety W_0. For $k = 1,\dots,r$, define inductively W_k as the blow-up of W_{k-1} along V^k_{k-1}, then V^k_k as the exceptional divisor in W_k and V^i_k, $k \leqslant i$, as the proper transform of V^i_{k-1} in W_k. Then the preimage of V^r_0 in the resulting variety W_r is a normal crossing divisor $V^1_r \cup \cdots \cup V^r_r$*
.

If $\mathscr{F}$ is a flag of subvarieties V^i_0 of a smooth algebraic variety W_0 as in the previous lemma, the resulting space W_s will be denoted by $\mathrm{Bl}_{\mathscr{F}} W_0$.

Theorem 4.2 ([Hu03])**.** *Let X_0 be an open subset of a nonsingular algebraic variety X. Assume that $X \setminus X_0$ can be decomposed as a finite union $\cup_{i\in I} D_i$ of closed irreducible subvarieties such that*

1. *For all $i \in I$, D_i is smooth;*
2. *for all $i, j \in I$, D_i and D_j meet cleanly, that is the scheme-theoretic intersection is smooth and the intersection of the tangeant space $T_X(D_i) \cap T_X(D_j)$ is the tangeant space of the intersection $T_X(D_i \cap D_j)$;*
3. *for all $i, j \in I$, $D_i \cap D_j = \emptyset$; or a disjoint union of D_l.*

The set $\mathcal{D} = \{D_i\}_{i\in I}$ is then a poset. Let k be the rank of $\mathcal{D}$. Then there is a sequence of well-defined blow-ups

$$\mathrm{Bl}_{\mathcal{D}} X \to \mathrm{Bl}_{\mathcal{D}\leqslant k-1} X \to \cdots \to \mathrm{Bl}_{\mathcal{D}\leqslant 0} X \to X$$

where $\mathrm{Bl}_{\mathcal{D}\leqslant 0} X \to X$ is the blowup of X along D_i of rank 0, and, inductively, $\mathrm{Bl}_{\mathcal{D}\leqslant r} X \to \mathrm{Bl}_{\mathcal{D}\leqslant r-1} X$ is the blowup of $\mathrm{Bl}_{\mathcal{D}\leqslant r-1} X$ along the proper transforms of D_j of rank r, such that

1. *$\mathrm{Bl}_{\mathcal{D}} X$ is smooth;*
2. *$\mathrm{Bl}_{\mathcal{D}} X \setminus X_0 = \bigcup_{i\in I} \widetilde{D_i}$ is a divisor with normal crossings;*
3. *For any integer k, $\widetilde{D_{i_1}} \cap \cdots \cap \widetilde{D_{i_k}}$ is non-empty if and only if, up to numbering, $D_{i_1} \subset \cdots \subset D_{i_k}$ form a chain in the poset $\mathcal{D}$. Consequently, $\widetilde{D_i}$ and $\widetilde{D_j}$ meet if and only if D_i and D_j are comparable.*

The fact that blow-ups are local constructions yields directly to the following

Corollary 4.3 (Flags blow-up sequence)**.** *Let X and $\mathcal{D}$ be as in the previous theorem. Let $\mathscr{F}_1, \ldots, \mathscr{F}_k$ be flags of subvarieties of $\mathcal{D}$ such that*

1. *$\mathscr{F}_1, \ldots, \mathscr{F}_k$ is a partition of $\mathcal{D}$,*
2. *If D is in some $\mathscr{F}_i$, then for all $D' \in \mathcal{D}$ with $D' < D$ there exists some $j \leqslant i$ such that $D' \in \mathscr{F}_j$.*

If $\mathscr{F}_j^i$ denotes the flag of the proper transform of elements of $\mathscr{F}_j^{i-1}$ in

$$\mathrm{Bl}_{\mathscr{F}_i^{i-1}} (\cdots (\mathrm{Bl}_{\mathscr{F}_1} X) \cdots),$$

then

$$\mathrm{Bl}_{\mathcal{D}} X = \mathrm{Bl}_{\mathscr{F}_k^{k-1}} (\cdots (\mathrm{Bl}_{\mathscr{F}_1} X) \cdots)$$

We will denote such sequence of blow-up by

$$\mathrm{Bl}_{\mathcal{F}_k,\ldots,\mathcal{F}_1} X$$

As we want to apply these results in order to have a motivic description of the stuffle product in terms of blow-ups, we need some precisions about what sort of motives arise from the construction of Theorem 4.2. Following the notation of the article [Hu03], in particular using the proof of theorems 1.4, 1.7 and Corollary 1.6, we have the following proposition:

Proposition 4.4. *Suppose that X and $\mathcal{D} = \cup D_i$ as in proposition 4.2 are such that X and all the D_i are Tate varieties. Let $\mathcal{E}^{r+1}$ be the set of exceptional divisors of $\mathrm{Bl}_{\mathcal{D}\leqslant r} X \to X$. Then all possible intersection of strata of $\mathcal{D}^{r+1} \cup \mathcal{E}^{r+1}$ are Tate Varieties and so is $\mathrm{Bl}_{\mathcal{D}\leqslant r} X$*

Proof. Mainly following the proof of theorem 1.7 in [Hu03], we use an induction on r.

If $r = 0$ then $\mathrm{Bl}_{\mathcal{D}\leqslant 0} X \to X$ is the blow up along the disjoint subvarities D_i of rank 0.

All the exceptional divisors in $\mathcal{E}^1$ are of the form $\mathbb{P}(N_X D_i)$ (with D_i of rank 0) and as the D_i are Tate, so are the exceptional divisors.

The Blow-up formula

$$h(X_Z) = \mathfrak{H}(X) \bigoplus_{i=0}^{d-1} h(Z)(-i)[-2i] \tag{17}$$

tell us that the blow-up of a Tate variety X along some Tate variety Z o f codimesion d is a Tate variety. Then $\mathrm{Bl}_{\mathcal{D}\leqslant 0} X$ is Tate. More over let D_i^1 be an element of $\mathcal{D}^1$, it is the proper transform of an element D_i in $\mathcal{D}$ of rank bigger than 1. And theorem 1.4 in [Hu03] tells us that $D_i^1 = \mathrm{Bl}_{D_j \subset D_i; rank(D_j)=0} D_j$ and therefore is a Tate variety

We now need to show that all intersection of strata of $\mathcal{D}^1 \cap \mathcal{E}^1$ is Tate. As the centre of the Blow up were disjoint, elements in $\mathcal{E}^1$ do not intersect.

Let D_i^1 and D_j^1 be two elements of $\mathcal{D}^1$ being the proper transform of D_i and D_j in $\mathcal{D}$. If $D_i \cap D_j = \emptyset$ then the same hold for their proper transform and there is nothing to prove, else $D_i \cap D_j = \cup D_l$. If the maximal rank of the D_l is 0 then the lemma 2.1 in [Hu03] ensures that the proper transform have an empty intersection. If the maximal rank of the D_l is bigger than 1 the fact that D_i and D_j meet cleanly ensures that the proper transform of the intersection is the intersection of the proper transform, that is

$$D_i^1 \cap D_j^1 = \mathrm{Bl}_{D_l \subset D_i \cap D_j ; rank(D_l)=0} D_i \cap D_j$$

And the intersection is Tate. Moreover from theorem 1.4 ([Hu03]) we have $D_i^1 \cap D_j^1 = \cup D_l^1$. This allow to consider only intersection of the form $E^1 \cap D_i^1$ with E^1 in $\mathcal{E}^1$ and D_i^1 in $\mathcal{D}^1$. Such an intersection is non empty if and only if E^1 comes from an element D_j of rank 0 in $\mathcal{D}$ with $D_j \subset D_i$. Then $E^1 \cap D_i^1$ is $\mathbb{P}(N_{D_i} D_j)$ and is a Tate variety.

Assume the statement is true for $\mathrm{Bl}_{\mathcal{D}^{\leqslant r-1}} X$, $\mathcal{E}^r$ and $\mathcal{D}^r$. By corollary 1.6 in [Hu03], the blow-up $\mathrm{Bl}_{\mathcal{D}^{\leqslant r}} X \to \mathrm{Bl}_{\mathcal{D}^{\leqslant r-1}} X$ is

$$\mathrm{Bl}_{\mathcal{D}^r_{\leqslant 0}} (\mathrm{Bl}_{\mathcal{D}^{\leqslant r}} X) \longrightarrow \mathrm{Bl}_{\mathcal{D}^{\leqslant r-1}} X.$$

The centre of the blow-up are the element in $\mathcal{D}^r$ of rank r which by assumption are Tate, as $\mathrm{Bl}_{\mathcal{D}^{\leqslant r-1}} X$, then $\mathrm{Bl}_{\mathcal{D}^{\leqslant r}} X$ and the new exceptional divisors are Tate. The other exceptional divisor are proper transform of element in $\mathcal{E}^r$ and are of the form

$$E_i^{r+1} = \mathrm{Bl}_{E_i^r \cap D_l^r ; rank(D_l)=r} E_i^r$$

with E_i^r in $\mathcal{E}^r$ and D_l^r in $\mathcal{D}^r$ coming from some D_l in $\mathcal{D}$. As by induction hypothesis both E_i^r and $E_i^r \cap D_l^r$ are Tate, E_i^{r+1} is a Tate variety. The same argument prove that all element in $\mathcal{D}^{r+1}$ are Tate. As previously the intersection of two element in $\mathcal{D}^{r+1}$ is either empty or the proper transform of the intersection of two element in $\mathcal{D}^r$; again this proper transform is Tate.

Theorem 1.4 tells us that the intersection $D_i^{r+1} \cap D_j^{r+1}$ of two elements of $\mathcal{D}^{r+1}$ is either empty either the union of some elements D_l^{r+1} in $\mathcal{D}^{r+1}$. Then, to prove that all possible intersections of strata of $\mathcal{E}^{r+1} \cup \mathcal{D}^{r+1}$ is Tate it is enough to prove that the intersection of some D_i^{r+1} with any intersection $E_1^{r+1} \cap \cdots E_k^{r+1}$is Tate.

If two of the E_i^{r+1} are exceptional divisor of $\mathrm{Bl}_{\mathcal{D}^r_{\leqslant 0}} (\mathrm{Bl}_{\mathcal{D}^{\leqslant r}} X) \to \mathrm{Bl}_{\mathcal{D}^{\leqslant r-1}} X$ then the intersection is empty because the corresponding strata D_i^r and D_j^r have an empty intersection (they have been separated at a previous stage).

Hence at most one of E_i^{r+1} is an exceptional divisor coming from the last blow-up and we can suppose that the strata $D_i^{r+1}, E_1^{r+1}, \ldots, E_{k-1}^{r+1}$ are coming from strata at the previous stage $D_i^r, E_1^r, \ldots, E_{k-1}^r$.

- Suppose that E_k^{r+1} is the proper transform of an exceptional divisor E_k^r in $\mathcal{E}^r$. The subvariety $Y = D_i^r \cap E_1^r \cap \cdots E_k^r$ is Tate by induction hypothesis and its proper transform is

  $$\mathrm{Bl}_{D_j^r \cap Y ; rank(D_j)=r} Y$$

  which is a Tate variety ($D_j^r \cap Y$ is either empty or Tate and Y is Tate). On the other side the proper transform of Y is the intersection $D_i^{r+1} \cap E_1^{r+1} \cap \cdots \cap E_k^{r+1}$ which is therefore Tate.

- Suppose that E_k^r is the exceptional divisor coming from the blow-up of $\mathrm{Bl}_{\mathcal{D}^{\leqslant r-1}} X$ along D_j^r. Let Y be the intersection $D_i^r \cap E_1^r \cap \cdots E_{k-1}^r$. Then $D_j^r \cap Y$ is either empty or a Tate variety In the first case the intersection $D_i^{r+1} \cap E_1^{r+1} \cap \cdots \cap E_k^{r+1}$ is empty. In the later case we have
$$D_i^{r+1} \cap E_1^{r+1} \cap \cdots \cap E_k^{r+1} = \mathbb{P}(N_Y Y \cap D_j^r)$$
which is Tate.

□

4.2. **The space X_n and some of its properties.** Let n be an integer greater than 2 and for $i \in [\![1, n]\!]$ and let $x_1, \ldots, x_n$ be the natural coordinates on $\mathbb{A}^n$. We define the divisors A_I, B_i^0, B_i^1, A_n, B_n, D_n^1, D_n^0 and D_n as follow:

- For all subset non empty I of $[\![1, n]\!]$, A_I is the divisor defined by
$$1 - \prod_{i \in I} x_i = 0;$$
- for all $i \in [\![1, n]\!]$, B_i^0 is the divisor defined by $x_i = 0$;
- for all $i \in [\![1, n]\!]$, $B_i^1 = A_{\{i\}}$ is the divisor defined by $1 - x_i = 0$;
- B_n is the union $(\bigcup_i B_i^0) \bigcup (\bigcup_i B_i^1)$;
- A_n is the union $\bigcup_{I \subset [\![1,n]\!]; |I| \geqslant 2} A_I$;
- D_n^1 is the union $\bigcup_{I \subset [\![1,n]\!]; I \neq \emptyset} A_I$;
- D_n^0 is the union $\bigcup_i B_i^0$;
- D_n is the union $D_n^0 \bigcup D_n^1$.

Remark 4.5. The divisor B_n is the Zariski closure of the boundary of the real cube $C_n = [0, 1]^n$ in $\mathbb{A}^n(\mathbb{R})$.

As the divisor D_n is not normal crossing, we would like to find a suitable succession of blow-up that will allow us to have a normal crossing divisor $\widehat{D}_n$ over D_n. In order to achieve that we first need the following remark and lemma.

Remark 4.6. Let I be a non-empty subset of $[\![1, n]\!]$ and $x = (x_1, \ldots, x_n)$ a point in A_I, then the normal vector of A_I at the point x is
$$n_{|x}^{A_I} = \sum_{i \in I} \frac{1}{x_i} \mathrm{d}x_i. \tag{18}$$

Therefore, if I and J are two distinct non-empty subsets of $[\![1, n]\!]$, the intersection of A_I and A_J is transverse.

Lemma 4.7. *Let $I_1, \ldots, I_k$ ($k \geqslant 3$) be distinct non-empty subsets of $[\![1, n]\!]$ and let x be a point in*
$$A_{I_1} \cap \cdots \cap A_{I_k}.$$
Suppose that $n_{|x}^{A_{I_k}}$ is in $\mathrm{Vect}(n_{|x}^{A_{I_1}}, \ldots, n_{|x}^{A_{I_{k-1}}})$ then
$$A_{I_1} \cap \cdots \cap A_{I_{k-1}} = A_{I_1} \cap \cdots \cap A_{I_k}.$$

Proof. By assumption, there exists rational numbers $\alpha_1, \ldots, \alpha_k$ such that
$$n_{|x}^{A_{I_k}} = \alpha_1 n_{|x}^{A_{I_1}} + \cdots + \alpha_{k-1} n_{|x}^{A_{I_{k-1}}}.$$
Considering the expression (18), if δ_I is the characteristic function of I, we find that for all i in $[\![1, n]\!]$
$$\delta_{I_k}(i) = \alpha_1 \delta_{I_1}(i) + \cdots + \alpha_{k-1} \delta_{I_{k-1}}(i).$$

Now let $y = (y_1, \dots, y_n)$ be a point in $A_{I_1} \cap \cdots \cap A_{I_{k-1}}$ we have

$$\prod_{i \in I_k} y_i = \prod_{i \in [\![1,n]\!]} y_i^{\delta_{I_k}(i)} = \prod_{i \in [\![1,n]\!]} (y_i^{\delta_{I_1}(i)})^{\alpha_1} \cdots \prod_{i \in [\![1,n]\!]} (y_i^{\delta_{I_{k-1}}(i)})^{\alpha_{k-1}} = 1$$

□

Lemma 4.8. *Let $\mathcal{D}_n^1$ be the poset (for the inclusion) formed by all the irreducible components of all possible intersections of divisors A_I. Then the poset $\mathcal{D}_n^1$ satisfy the condition (1), (2) and (3) of theorem 4.2.*

Proof. The intersection condition (3) follows from the definition of $\mathcal{D}_n^1$. From the former lemma, we deduce that the dimension of the normal space at a point of an intersection is the codimension of this intersection. Hence the irreducible components of the intersections are smooth.

Let S_1 and S_2 be two elements of $\mathcal{D}_n^1$. To show that S_1 and S_2 meet cleanly, it is enough to show that the normal bundle of the intersection is spanned by the normal bundles of S_1 and S_2, that is

$$N_{\mathbb{A}^n}(S_1 \cap S_2) = N_{\mathbb{A}^n}(S_1) + N_{\mathbb{A}^n}(S_2).$$

As S_1 and S_2 are intersection of some A_I, it is enough show that the normal bundle of $A_{I_1} \cap \cdots \cap A_{I_k}$ is spanned by the normal vector of the A_{I_j} and that is ensured by lemma 4.7 and remark 4.6. □

Applying the construction of theorem 4.2 with $\mathcal{D} = \mathcal{D}_n^1$ and $X = \mathbb{A}^n$ leads to a variety $X_n \xrightarrow{p_n} \mathbb{A}^n$, which result from successive blow-up of all the strata of $\mathcal{D}_n^1$ such that the preimage $\widehat{\mathcal{D}_n^1}$ of $\mathcal{D}_n^1$ is a normal crossing divisor. We will write $\widehat{D_n^1}$ to mean the preimage of D_n^1.

Lemma 4.9. *Let $\widehat{D_n^0}$ be the proper transform in X_n of the divisor D_n^0. Then $\widehat{D}_n = \widehat{D_n^1} \bigcup \widehat{D_n^0}$ is a normal crossing divisor.*

Proof. Let I be a non-empty subset of $[\![1, n]\!]$, $\widehat{B_I^0}$ (resp. B_I^0) be the intersection in X_n (resp. $\mathbb{A}^n$) of divisors $\{x_i = 0\}$ for i in I. And let $\widehat{S_1}, \dots, \widehat{S_k}$ be strata of $\widehat{\mathcal{D}_1^n}$ such that the intersection of the $\widehat{S_i}$ is non-empty. We want to show that there is a neighbourhood V of $\widehat{B_I^0} \bigcap \widehat{S_1} \bigcap \cdots \bigcap \widehat{S_k}$ such that $V \cap \widehat{D}_n$ is normal crossing. By theorem 4.2, the $\widehat{S_i}$ are coming from strata of $\mathcal{D}_n^1$, $S_1 \subset \cdots \subset S_k$. As the intersection of the $\widehat{S_i}$'s with $\widehat{B_I^0}$ is non-empty, the intersection of B_I^0 with S_1 is non-empty. There exists $I_1, \dots, I_l$ non-empty subsets of $[\![1, n]\!]$ such that $S_1 = A_{I_1} \cap \cdots \cap A_{I_l}$.

As $B_I^0 \bigcap S_1$ is non-empty, we have

$$I \bigcap (I_1 \bigcup \cdots \bigcup I_l) = \emptyset.$$

Then, in $\mathbb{A}^n$, we have a neighbourhood V_0 of $B_I^0 \bigcap S_1$ isomorphic to a product $\mathbb{A}^d \times \mathbb{A}^{|I|}$ with $d = n - |I|$:

$$\begin{array}{ccc} \mathbb{A}^d & \times & \mathbb{A}^{|I|} \\ \cup & & \cup \\ \tilde{D}_d^1 & & \bigcup_{i \in I} \tilde{B}_i^0, \end{array}$$

where $\tilde{B}_i^0$ is the hyperplane corresponding to $\{x_i = 0\}$ inside $\mathbb{A}^{|I|}$.

Lifting this neighbourhood to $\widehat{V_0}$ in X_n, it becomes isomorphic to $X_d \times \mathbb{A}^{|I|}$ with $\widehat{D}_d^1 \subset X_d$. Then, for any $\widehat{S_i}$ there is a stratum $\widehat{S_i^d}$ of $\widehat{D}_d^1$ such that $\widehat{V_0} \cap \widehat{S_i} \simeq \widehat{S_i^d} \times \mathbb{A}^{|I|}$. As the $\widehat{S_i^d}$'s give a normal crossing divisor in X_d by Theorem 4.2, $\widehat{V_0}$ gives the neighbourhood of $\widehat{B_I^0} \bigcap \widehat{S_1} \bigcap \cdots \bigcap \widehat{S_k}$ such that $V \cap \widehat{D}_n$ is a normal crossing divisor in X_n.

□

Definition 4.10. Let $\widehat{B}_n$ denote the preimage of B_n and $\widehat{A}_n$ be the divisor $\widehat{D}_n \setminus \widehat{B}_n$.

Remark 4.11. The divisors $\widehat{A}_n$ and $\widehat{B}_n$ do not share any irreducible components and are both normal crossing divisors.

Let $\widehat{C}_n$ be the preimage of $C_n = [0,1]^n$ in X_n and $\overline{\widehat{C}_n}$ its closure. Then $\widehat{B}_n$ is the Zariski closure of the boundary of $\widehat{C}_n$ and there is a non-zero class

$$[\widehat{C}_n] \in \mathrm{Gr}_0^W \mathrm{H}^n(X_n, \widehat{B}_n). \tag{19}$$

If I is a subset of $[\![1,n]\!]$, we define F_I and G_I to be the functions

$$\begin{array}{l} G_I : (x_1, \ldots, x_n) \longmapsto \prod_{i\in I} x_i \\ F_I : (x_1, \ldots, x_n) \longmapsto 1 - \prod_{i\in I} x_i. \end{array}$$

Definition 4.12. A flag $\mathcal{F}$ of $[\![1,n]\!]$ is a collection of non-empty distinct subsets I_j of $[\![1,n]\!]$ such that $I_1 \subsetneq \ldots \subsetneq I_r$. The length of the flag $\mathcal{F}$ is the integer r and we may say that $\mathcal{F}$ is an r-flag of $[\![1,n]\!]$. A flag on length n will be a maximal flag. A distinguished r-flag $(\mathcal{F}, i_1, \ldots i_p)$ will be a flag $\mathcal{F}$ of length r together with element $i_1 < \ldots < i_p$ of $[\![1,r]\!]$.

Definition 4.13. Let $(\mathcal{F}, i_1, \ldots i_p)$ be a distinguished r-flag of $[\![1,n]\!]$. Let $\Omega^{\mathcal{F}}_{i_1,\ldots,i_p}$ denote the differential form of $\Omega^\bullet_{log}(\mathbb{A}^n \setminus D_n)$ defined by

$$\Omega^{\mathcal{F}}_{i_1,\ldots i_p} = \bigwedge_{j=1}^{r} \mathrm{d}\log(g_j)$$

where

$$g_j = \begin{cases} F_{I_j} \text{if } j \in \{i_1, \ldots, i_p\} \\ G_{I_j} \text{otherwise} \end{cases}$$

Let $\mathbf{k} = (k_1, \ldots, k_p)$ be a tuple of positive integers with $k_1 \geqslant 2$ such that $k_1 + \cdots + k_p = n$ and s be a permutation of $[\![1,n]\!]$. We define a differential form $\Omega_{\mathbf{k},s} \in \Omega^n_{log}(\mathbb{A}^n \setminus D_n)$ by

$$\Omega_{\mathbf{k},s} = f_{k_1,\ldots,k_n}(x_{s(1)}, \ldots, x_{s(n)})\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Remark 4.14. Let $\mathbf{k}$ and s be as in the previous definition. We associate to the pair $(\mathbf{k}, s)$ the maximal distinguished flag $(F_k, i_1, \ldots, i_p)$ defined by $I_i = \{s(1), \ldots, s(i)\}$ and $i_j = k_1 + \cdots + k_j$ for j running from 1 to p. Then we can see that there exists an integer r_s such that

$$\Omega_{\mathbf{k},s} = (-1)^{r_s} \Omega^{\mathcal{F}_k}_{i_1,\ldots i_p}.$$

Definition 4.15. We shall write $\omega^{\mathcal{F}}_{i_1,\ldots,i_p}$ and $\omega_{\mathbf{k},s}$ for the pull back on $X_n \setminus \widehat{D}_n$ of, respectively, the forms $\Omega_{I_1,\ldots,I_p}$ and $\Omega_{\mathbf{k},s}$.

Proposition 4.16. *If $(\mathcal{F}, i_1, \ldots, i_p)$ is a maximal flag of $[\![1,n]\!]$ such that $i_1 \geqslant 2$ and $i_p = n$ then:*

- *The divisor of singularities $A^{\mathcal{F}}_{i_1,\ldots,i_p}$ of $\Omega^{\mathcal{F}}_{i_1,\ldots,i_p}$ is $A_{I_{i_1}} \cup \cdots \cup A_{I_{i_p}}$.*
- *The divisor of singularities $\widehat{A}^{\mathcal{F}}_{i_1,\ldots,i_p}$ of $\omega^{\mathcal{F}}_{i_1,\ldots,i_p}$ lies in $\widehat{A}_n$. Thus, the divisor of singularities of $\omega_{\mathbf{k},s}$ lies in $\widehat{A}_n$.*

Moreover, if $(\mathcal{F}, i_1, \ldots, i_p)$ and $(\mathcal{F}', i'_1, \ldots, i'_q)$ are two distinguished flags of length i_p and i'_q with $|i_1| \geqslant 2$, $|i'_1| \geqslant 2$ and I_{i_p}, I'_{i_q} being a partition of $[\![1,n]\!]$, then the divisor of singularities of $\omega^{\mathcal{F}}_{i_1,\ldots,i_p} \wedge \omega^{\mathcal{F}'}_{i'_1,\ldots,i'_q}$ lies in $\widehat{A}_n$.

Let $(\mathcal{F}, i_1 \subsetneq \ldots \subsetneq i_p)$ be a flag as in the previous proposition.

It is straightforward to see that $A^{\mathcal{F}}_{I_{i_1},\ldots I_{i_p}}$ is $A_{I_{i_1}} \cup \cdots \cup A_{I_{i_p}}$. The following lemma from Goncharov can easily be modify to fit into our situation.

Lemma 4.17 ([Gon02][lemma 3.8])**.** *Let Y be a normal crossing divisor in a smooth variety X and $\omega \in \Omega^n_{log}(X \setminus Y)$. Let $p : \widehat{X} \longrightarrow X$ be the blow-up of an irreducible variety Z. Suppose that the generic point of Z is different from the generic points of strata of Y. Then $p^*\omega$ does not have a singularity at the special divisor of $\widehat{X}$.*

That is:

Lemma 4.18. *Let Y be a normal crossing divisor in $\mathbb{A}^n$ and $\omega \in \Omega^n_{log}(\mathbb{A}^n \setminus Y)$. Let $p_n : X_n \to \mathbb{A}^n$ be the map of our previous construction. Suppose that the generic points of the strata of B_n that are blow-up in the construction of X_n are different from the generic points of strata of Y. Then $p_n^*\omega$ does not have singularities at the corresponding exceptional divisors in $\widehat{B}_n$.*

It is enough to check that the divisor of singularities of $\Omega^{\mathcal{F}}_{i_1,\ldots,i_P}$ is a a normal crossing divisor and that none of its strata is a blown up strata of B_n.

The divisor of singularities of $\Omega^{\mathcal{F}}_{i_1,\ldots,i_p}$ is $A_{I_{i_1}} \cup \cdots \cup A_{I_{i_p}}$ and to show it is a normal crossing divisor it is enough to show that the normal vectors of the $A_{I_{i_j}}$ at any intersection of some of them are linearly independent. The normal vector of $A_{I_{i_j}}$ is $\sum_{i \in I_{i_j}} 1/x_i \, \mathrm{d}x_i$ and as we have $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_p$, they are linearly independent.

We now have to show that none of the strata of B_n that are blown up in the construction of X_n are exactly some strata of $A_{I_1} \cup \cdots A_{I_p}$. Let S be such a strata of B_n of codimension k. The strata S is defined by the equations $x_{r_1} = 1, \ldots, x_{r_k} = 1$. If I_S denotes the set $\{r_1, \ldots, r_k\}$ then, for any subset I of $[\![1, n]\!]$, S is included in A_I if and only I is included in I_S. As $I_i \subset I_{i'}$ for $i < i'$, if S is included in a strata S_A of $A_{I_1,\ldots,I_p}$, that strata is of the form $A_{I_{i_1}} \cap \cdots \cap A_{A_{i_j}}$ with $j < k$ because $|I_1| \leqslant 2$. As a consequence, S_A is of codimension at most $k-1$ and S can not be a strata of $A^{\mathcal{F}}_{i_1,\ldots,i_p}$.

We use the same argument in the case of two distinguished flags as in the lemma and the proposition 4.16 is proved.

Proposition 4.19. *The divisor $\widehat{A}_n$ does not intersect the boundary of $\widehat{C}_n$ in $X_n(R)$.*

Proof. Let S be an irreducible codimension 1 stratum of $\widehat{B}_n$ containing an intersection of some $\widehat{A}_n$ strata with the boundary of $\overline{\widehat{C}_n}$. As, the divisor A_n intersect the boundary of the real cube C_n only on strata of B_n that are of codimension at least 2, S have to be such that $p_n(S)$ is a stratum of B_n of codimension at least 2.

Using the symmetry, with respect to the standard coordinates on $\mathbb{A}^n$, we can suppose that $p_n(S)$ is defined in those coordinates by $x_k = x_{k+1} = \ldots = x_n$.

Starting from $\mathbb{A}^n$ and blowing up first the point $x_1 = x_2 = \ldots = x_n = 1$, then the edge $x_2 = x_3 = \ldots = x_n = 1$ and after that the plane $x_3 = x_4 = \ldots = x_n = 1$ and so on, we obtain a variety $\widetilde{p}_n : \widetilde{X}_n \to \mathbb{A}^n$. There are natural local coordinates $(s_1, \ldots, s_n))$ on $\widetilde{X}_n$ such that the coordinates on $\mathbb{A}^n$ defined by $y_i = 1 - x_i$ satisfy:

$$y_1 = s_1, \quad y_2 = s_1 s_2, \quad \ldots, \quad y_i = s_1 s_2 \cdots s_i, \quad \ldots, \quad y_n = s_1 s_2 \cdots s_n.$$

In the y_i-coordinates the stratum $x_j = x_{j+1} = \ldots = x_n = 1$ is $y_j = y_{j+1} = \ldots = y_n = 0$ and its preimage in $\widetilde{X}_n$ is given by $s_j = 0$.

For any permutation s of $[\![1, n]\!]$ we could apply the same construction, that is blowing the point $x_{s(1)} = x_{s(2)} = \ldots = x_{s(n)} = 1$ then the edge $x_{s(2)} = x_{s(3)} = \ldots = x_{s(n)} = 1$ and so on, and have a variety $\widetilde{p}^s_n : \widetilde{X}^s_n \to \mathbb{A}^n$. The preimage of D_n

in $\widetilde{X}_n^s$ will be denoted by $\widetilde{D}_n^s$, $\widetilde{B}_n^s$ will denote the preimage of B_n and $\widetilde{A}_n^s$ is $\widetilde{D}_n^s \setminus \widetilde{B}_n^s$. To prove that $\widehat{A}_n$ does not intersect the boundary of $\widehat{C}_n$ in $X_n(\mathbb{R})$ it is enough to show that for any permutation s $\widetilde{A}_n^s$ does not intersect, in $\widetilde{X}_n^s(\mathbb{R})$, the boundary of the preimage of C_n. It is then enought to show that the proper transforms of the divisors A_I do not intersect the boundary of $\tilde{X}_n^s(\mathbb{R})$ because it will then be the same for the irreducible components of their intersections as for the proper transforms of those components by the remaining blow-up used to reach X_n. By symmetry, it is enough to show it when s is the identity map and then in the case of $\widetilde{X}_n$. Let $\widetilde{C}_n$ be the preimage of C_n in $\widetilde{X}_n$.

Let A_I be a codimension 1 stratum of A_n, I being the set $\{i_0, \dots, i_p\}$ and suppose that $i_0 < \ldots < i_p$. We want to show that the closure $\widetilde{A}_I$ of the preimage of $A_I \setminus B_n$ in $\widetilde{X}_n$ does not intersect the boundary of $\widetilde{C}_n$. The k-th symmetric function will be denoted by σ_k with the following convention

$$\sigma_0 = 1, \qquad \sigma_k(X_1, \dots, X_l) = 0 \text{ if } l > k$$

The stratum A_I is defined in the x_i-coordinates by $1 - x_{i_0} \cdots x_{i_p} = 0$ and in the y_i coordinates by

$$0 = \sum_{k=1}^{p+1} (-1)^{k-1} \sigma_k(y_{i_0}, y_{i_1} \dots, y_{i_p}). \tag{20}$$

Before giving an explicit expression of $\widetilde{A}_I$ with the s_i coordinates, we define the set J_0 as $\{1, \dots, i_0\}$ and the sets $J_1, \dots, J_p$ by

$$J_k = \{i_0 + 1, i_0 + 2, \dots, i_k\}$$

for all k in $[\![1, p]\!]$

For any subset J of $[\![1, n]\!]$, $\Pi^J \mathbf{s}$ will denote the product $\prod_{j \in J} s_j$ and we have the following relations

$$y_{i_0} = \Pi^{J_0} \mathbf{s} \qquad \text{and} \qquad \forall k \in [\![1, p]\!], \quad y_{i_k} = \Pi^{J_0} \mathbf{s} \Pi^{J_k} \mathbf{s}$$

The RHS of the equation (20) can be written, using the change of variables $y_i = s_1 \cdots s_i$ as

$$\sum_{k=1}^{p+1} (-1)^{k-1} \sigma_k(\Pi^{J_0} \mathbf{s}, \Pi^{J_0} \mathbf{s} \Pi^{J_1} \mathbf{s}, \dots, \Pi^{J_0} \mathbf{s} \Pi^{J_p} \mathbf{s}). \tag{21}$$

For any indeterminate λ one have, for any k,

$$\sigma_k(\lambda, \lambda X_1, \lambda X_2, \dots, \lambda X_p) = \lambda^k (\sigma_{k-1}(X_1 \dots, X_p) + \sigma_k(X_1, \dots, X_p)).$$

Then the expression (21) is equal to

$$\begin{aligned} \Pi^{J_0} \mathbf{s} \Big[& 1 + \sigma_1(\Pi^{J_1} \mathbf{s}, \dots, \Pi^{J_p} \mathbf{s}) \\ & + \sum_{k=1}^{p-1} \left((-1)^k (\Pi^{J_0} \mathbf{s})^k \left(\sigma_k(\Pi^{J_1} \mathbf{s}, \dots, \Pi^{J_p} \mathbf{s}) + \sigma_{k+1}(\Pi^{J_1} \mathbf{s}, \dots, \Pi^{J_p} \mathbf{s}) \right) \right) \\ & + (-1)^p \sigma_p(\Pi^{J_1} \mathbf{s}, \dots, \Pi^{J_p} \mathbf{s}) \Big] \end{aligned}$$

The expression of $\widetilde{A}_I$ in the s_i-coordinates is then

$$0 = 1 + \sigma_1(\Pi^{J_1}\mathbf{s}, \dots, \Pi^{J_p}\mathbf{s}) + \sum_{k=1}^{p-1} \left((-1)^k (\Pi^{J_0}\mathbf{s})^k \left(\sigma_k(\Pi^{J_1}\mathbf{s}, \dots, \Pi^{J_p}\mathbf{s}) + \sigma_{k+1}(\Pi^{J_1}\mathbf{s}, \dots, \Pi^{J_p}\mathbf{s})\right)\right) + (-1)^p \sigma_p(\Pi^{J_1}\mathbf{s}, \dots, \Pi^{J_p}\mathbf{s}) \tag{22}$$

The closure of $\widetilde{C}_n$ is given, in the s_i coordinates, by $s_1 \in [0,1]$ and for any $i \in [\![1,n]\!]$ $s_1 \cdots s_i \in [0,1]$. It is enough to look the intersection of $\widetilde{A}_I$ with codimension 1 strata of the boundary of $\widetilde{C}_n$.

Suppose that $s_{i_0} = 0$ for some $i_0 \in J_0$ then the RHS of (22) become

$$1 + \sigma_1(\Pi^{J_1}\mathbf{s}, \dots, \Pi^{J_p}\mathbf{s})$$

which is strictly positive if for any i, $s_i \geqslant 0$. So the divisor $\widetilde{A}_I$ does not intersect any component of the form $s_{i_0} = 0$ for i_0 in J_0.

Then, we can suppose that $s_i \neq 0$ for all $i \in J_0$ in order to study the intersection of $\widetilde{A}_I$ with the boundary of $\widetilde{C}_n$ and the RHS of (22) can be written

$$\frac{1}{\Pi^{J_0}\mathbf{s}} \left(1 - \prod_{j=1}^{p} \left(1 - \Pi^{J_0}\mathbf{s}\Pi^{J_j}\mathbf{s}\right)\right) + \prod_{j=1}^{p} \left(1 - \Pi^{J_0}\mathbf{s}\Pi^{J_j}\mathbf{s}\right).$$

Suppose that a point $x = (s_1, \dots, s_n)$ with $s_i > 0$ for all i in J_0, lies in the closure of $\widetilde{C}$. That is, for all i in $[\![1,n]\!]$ the product $s_1 s_2 \cdots s_i$ is between 0 and 1 which means all the product $\Pi^{J_0}\mathbf{s}\Pi^{J_j}\mathbf{s}$ are between 0 and 1 for j in $[\![1,p]\!]$ and then one find the following inequalities

$$0 \leqslant \frac{1}{\Pi^{J_0}\mathbf{s}} \left(1 - \prod_{j=1}^{p} \left(1 - \Pi^{J_0}\mathbf{s}\Pi^{J_j}\mathbf{s}\right)\right) \leqslant \frac{1}{\Pi^{J_0}\mathbf{s}},$$

$$0 \leqslant \prod_{j=1}^{p} \left(1 - \Pi^{J_0}\mathbf{s}\Pi^{J_j}\mathbf{s}\right) \leqslant 1.$$

Both term can not be equal to 0 together, thus $\widetilde{A}_I$ does not intersect the boundary of $\widetilde{C}_n$ the s_i being strictly positive for i in J_0 and the proposition is proved.

□

4.3. **An alternative definition for motivic MZV.** Both propositions 4.16 and 4.19 lead to the following theorem and to an alternative definition for motivic multiple zeta values.

Theorem 4.20. *Let $\mathbf{k} = (k_1, \dots, k_p)$ be a tuple of integers with $k_1 \geqslant 2$ and $k_1 + \dots + k_p = n$ and let s be a permutation of $[\![1,n]\!]$. Let $\widehat{A}^s_{\mathbf{k}}$ be the divisor of singularities of the differential form $\omega^s_{\mathbf{k}}$. Then there exist a mixed Tate motive*

$$\mathrm{H}^n(X_n \setminus \widehat{A}^s_{\mathbf{k}}; \widehat{B}_n^{\widehat{A}^s_{\mathbf{k}}}).$$

The differential form $\omega^s_{\mathbf{k}}$ and the preimage $\widehat{C}_n$ of the real n-dimensional cube in X_n give two non zero elements

$$[\omega^s_{\mathbf{k}}] \in \mathrm{Gr}^W_{2n} \mathrm{H}^n(X_n \setminus \widehat{A}^s_{\mathbf{k}}; \widehat{B}_n^{\widehat{A}^s_{\mathbf{k}}}) \qquad \text{and} \qquad [\widehat{C}_n] \in \left(\mathrm{Gr}^W_0 \mathrm{H}^n(X_n \setminus \widehat{A}^s_{\mathbf{k}}; \widehat{B}_n^{\widehat{A}^s_{\mathbf{k}}})\right)^\vee$$

The periods of the n-framed mixed Tate motive

$$\zeta^{fr.,\mathcal{M}}(\mathbf{k}, s) = \left[\mathrm{H}^n(X_n \setminus \widehat{A}^s_{\mathbf{k}}; \widehat{B}_n^{\widehat{A}^s_{\mathbf{k}}}); [\omega^s_{\mathbf{k}}], [\widehat{C}_n]\right]$$

is equal to $\zeta(k_1,\dots,k_n)$.

Moreover, let $(\mathcal{F},i_1,\dots,i_p)$ *and* $(\mathcal{F}',i'_1,\dots,i'_q)$ *are two distinguished flags of length* i_p *and* i'_q *with* $|i_1|\geqslant 2$, $|i'_1|\geqslant 2$ *and* I_{i_p}, I'_{i_q} *being a partition of* $[\![1,n]\!]$ *and* $\widehat{A}^{\mathcal{F}|\mathcal{F}'}_{i_1,\dots,l_p|i'_1,\dots,i'_q}$ *be the divisor of singularities of* $\omega^{\mathcal{F}}_{i_1,\dots,i_p}\wedge\omega^{\mathcal{F}'}_{i'_1,\dots,i'_q}$. *There exists an* n*-framed mixed Tate motive*

$$\zeta^{fr.,\mathcal{M}}(\mathcal{F},i_1,\dots,i_p|\mathcal{F}',i'_1,\dots,i'_q)=\mathrm{H}^n(X_n\setminus\widehat{A}^{\mathcal{F}|\mathcal{F}'}_{i_1,\dots,l_p|i'_1,\dots,i'_q};\widehat{B}_n^{\widehat{A}^{\mathcal{F}|\mathcal{F}'}_{i_1,\dots,l_p|i'_1,\dots,i'_q}}),$$

the frames being given by $[\omega^{\mathcal{F}}_{i_1,\dots,i_p}\wedge\omega^{\mathcal{F}'}_{i'_1,\dots,i'_q}]$ *and* $[\widehat{C}_n]$.

Proof. We want to apply theorem 3.6 in [Gon02] to our particular case. As $\widehat{D}_n$ is a normal crossing divisor and as proposition 4.19 ensures that $\widehat{A}_n$ does not intersect $[\widehat{C}_n]$,using Proposition 4.16, the only thing that remained to show is that we have a Tate stratification of X_n which is ensured by Lemma 4.22.

The computation of the period follows from the fact that integrating over $\widehat{C}_n$ is the same as integrating over the real cube. □

The following lemma is the key to prove Lemma 4.22.

Lemma 4.21. *Let* $I_1,\dots,I_r$ *be* r *subsets of* $[\![1,n]\!]$ *and* X *the intersection* $A_{I_1}\cap\cdots\cap A_{I_r}\subset\mathbb{A}^n$. *Then,* X *and its irreducible componnents are Tate varieties.*

Proof. We can assume that the equations defining the A_{I_i} are independent. If $|I_1\cup\cdots\cup I_r|=a<n$ then X is isomorphic to $(A'_{I_1}\cap\cdots\cap A'_{I_r})\times\mathbb{A}^{n-a}\subset\mathbb{A}^a\times\mathbb{A}^{n-a}$ where the A'_{I_i} are defined by the same equations, $1-\prod_{j\in I_i}x_j=0$, that define A_{I_i} but view in $\mathbb{A}^a$ instead of $\mathbb{A}^n$.

Thus, using the Künneth formula, it is enough to prove the lemma when $I_1\cup\cdots\cup I_r=[\![1,n]\!]$. We will now construct two finite morphisms

$$\mathbb{G}_m^{n-r}\times\prod_{k=1}^{r-1}\{x^{d_{n-k}a_k}=1\}\xrightarrow{f}X\xrightarrow{g}\mathbb{G}_m^{n-r}.$$

Let $\bar{k}$ be an algebraically closed field. The system of equations

$$E_i\quad:\quad 1-\prod_{j\in I_i}x_j=0$$

can be reduce in the following way.

The variable x_n is by assumption in some I_i, and we can assume without loss of generality that $i=1$, so E_1 can be written

$$x_n=\frac{1}{\prod_{j\in I_1}x_j}=\prod_{j<n}x_j^{\beta_j^{(1)}}\qquad 0\leqslant|\beta_j^{(1)}|\leqslant 1\text{ and }\beta_j^{(1)}\in\mathbb{Z}.$$

Substituting x_n by this product into the other equations, we obtain E'_2, ..., E'_r, $r-1$ independent equations of the form $\prod_{j<n}x_i^{c_j}$. At least one variable appears in those equations and we can suppose that it is x_{n-1} in E'_2. We have then

$$x_{n-1}^{d_{n-1}}=\prod_{j<n-1}x_j^{\alpha_j^{(2)}}\qquad\alpha_j^{(2)}\in\mathbb{Z}\text{ and }d_{n-1}\in\mathbb{N}^*.$$

There exists $\zeta_{d_{n-1}}$ such that $\zeta_{d_{n-1}}^{d_{n-1}}=1$ and

$$x_{n-1}=\zeta_{d_{n-1}}\prod_{j<n-1}x_j^{\beta_j^{(2)}}\qquad\beta_j^{(2)}\in\mathbb{Q}.$$

Substituting x_{n-1} into the remaining equations we obtain new equations of the form $\prod_{j<n-1} x_i^{c'_j}$, now with the c'_j in $\mathbb{Q}$. We can then apply the same operation to another variable, let's say

$$x_{n-2}^{\tilde{d}_{n-2}} = \prod_{j<n-2} x_j^{\tilde{\alpha}_j^{(3)}} \qquad \tilde{\alpha}_j^{(3)} \in \mathbb{Q} \text{ and } \tilde{d}_{n-2} \in \mathbb{Q}^*.$$

If $\tilde{d}_{n-2}$ is not in $\mathbb{N}^*$, raising the equation to some power and taking the inverse, we can rewrite the previous equation as

$$x_{n-2}^{d_{n-2}} = \prod_{j<n-2} x_i^{\alpha_j^{(3)}} \qquad \alpha_j^{(3)} \in \mathbb{Q} \text{ and } d_{n-2} \in \mathbb{N}^*.$$

We obtain an expression for x_{n-2} (as with x_{n-1}) which is a root of unity, $\zeta_{d_{n-2}}$, times the product of powers (possibly negative) of roots of the x_j for $j < n-2$. We can substitute this expression into the remaining equations. Continuing the process, we find r variables, $x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-r+1}$, and $r-1$ roots of unity, $\zeta_{d_{n-1}}, \ldots, \zeta_{d_{n-r+1}}$, that allow us to write the system of equations $E_1, \ldots, E_r$ as a triangular system :

$$\begin{aligned}
x_n &= \prod_{j<n} x_j^{\beta_j^{(1)}} \\
x_{n-1} &= \zeta_{d_{n-1}} \prod_{j<n-1} x_j^{\beta_j^{(2)}} \\
&\cdots \\
x_{n-i} &= \zeta_{d_{n-i}} \prod_{j<n-i} x_j^{\beta_j^{(i)}} \\
&\cdots \\
x_{n-r+1} &= \zeta_{d_{n-r+1}} \prod_{j<n-r+1} x_j^{\beta_j^{(r)}}
\end{aligned}$$

such that for all $i \in \{0, \ldots, r-1\}$ $\zeta_{d_{n-i}}^{d_{n-i}} = 1$. Finally, solving this triangular system, we find integers p_{ij} and p'_{ik} and positive integers q_{ij} and q'_{ik} for all i in $\{0, \ldots, r-1\}$, all k in $\{1, \ldots, r-1\}$ and all j in $\{1, \ldots, n-r\}$ such that :

$$\forall i \in \{0, \ldots, r-1\} \qquad x_{n-i} = \zeta_{d_{n-i}} \prod_{k>i} \zeta_{d_{n-k}}^{p'_{ik}/q'_{ik}} \prod_{j=1}^{n-r} x_j^{p_{ij}/q_{ij}}.$$

Now, setting

$$\begin{aligned}
\forall j \in \{1, \ldots, n-r\} \qquad & a_j = \mathrm{lcm}_i(q_{ij}) \\
\forall k \in \{1, \ldots, r-1\} \qquad & b_k = \mathrm{lcm}_i(q'_{ij})
\end{aligned}$$

we define

$$\mathbb{G}_m^{n-r} \times \textstyle\prod_{k=1}^{r-1} \{x^{d_{n-k} a_k} = 1\} \xrightarrow{\qquad f \qquad} X$$

$$(\lambda_1, \ldots, \lambda_{n-r}) \times (\zeta_{n-1}, \ldots, \zeta_{n-r+1}) \longmapsto (f_1, \ldots, f_n)$$

with

$$\forall i \in \{1, \ldots, n-r\} \qquad f_i(\lambda_1, \ldots, \lambda_{n-r}, \zeta_{n-1}, \ldots, \zeta_{n-r+1}) = \lambda_i^{a_i}$$

and

$$\forall i \in \{0,\ldots,r-1\} \qquad f_{n-i}(\lambda_1,\ldots,\lambda_{n-r},\zeta_{n-1},\ldots,\zeta_{n-r+1}) = \zeta_{n-i}^{b_i}\prod_{k>i}\zeta_{n-k}^{p'_{ik}b_k/q'_{ik}}\prod_{j=1}^{n-r}\lambda_j^{p_{ij}a_j/q_{ij}}.$$

We also define the morphism $g : X \longrightarrow \mathbb{G}_m^{n-r}$ to be the projection onto the first $n-r$ factor (X is embedded in $\mathbb{A}^n$). The morphism g is finite. Moreover the composition $g \circ f$ raise each λ_i to some power and is therefore finite. As we can solve the system over any algebraically closed field, the morphism f is surjective and therefore finite, because g and $g \circ f$ are.

Moreover, there is a one to one map between the set of the irreducible components of X the set of those of $\prod_{k=1}^{r-1}\{x^{d_{n-k}a_k}=1\}$ and the irreducible components of X are disjoint

In the Hodge-De Rham realisation, we then have the following composition

$$\mathrm{H}^*(\mathbb{G}_m^{n-r}\times\prod_{k=1}^{r-1}\{x^{d_{n-k}a_k}=1\},\mathbb{Q}) \xleftarrow{f^*} \mathrm{H}^*(X,\mathbb{Q}) \xleftarrow{g^*} \mathrm{H}^*(\mathbb{G}_m^{n-r},\mathbb{Q})\ .$$

with the composite arrow labelled $(g\circ f)^*$.

As f and g are finite, both f^* and g^* are injective. The morphism of mixed Hodge structures, f^*, is strict [Del71], therfore $\mathrm{H}^*(X,\mathbb{Q})$ is a direct sum of Hodge-Tate structures $\mathbb{Q}(m)$ (for different m) because the cohomology of $\mathbb{G}_m^{n-r}\times\prod_{k=1}^{r-1}\{x^{d_{n-k}a_k}=1\}$ is. The irreducible components of X being disjoint, each is a Tate variety

Finally, using the fact that the Hodge-De Rham realisation is fully faithful, we can conclude that the motive of X is a direct sum of Tate motives, in other words X is a Tate variety. □

Lemma 4.22. *The divisor $\widehat{D}_n = \widehat{B}_n^0 \cup \widehat{D}_n^1$ provides X_n with a Tate stratification.*

Proof. We first need to show that all the strata of $\widehat{D}_n^1$ and X_n are Tate but using the proposition 4.4, it is enough to show that all the strata of D_n^1 are Tate ($\mathbb{A}^n$ being Tate). A stratum $A_{I_1}\cap\cdots\cap A_{I_k}$ of D_n^1 is a Tate variety by Lemma 4.21. So X_n and all the strata of $\widehat{D}_n^1$ are Tate.

Note that the previous discussion tells us that for any $k \geqslant 2$, X_k and all the strata of $\widehat{D}_k^1$ are Tate varieties.

Let $\widehat{S}$ be the intersection of certain codimension 1 strata of $\widehat{B}_n^0$; it is the proper transform of the corresponding intersection, say $S = \cap_{j\in J}\{x_j = 0\}$ for some $J \subset [\![1,n]\!]$, in B_n^0. That is, $\widehat{S}$ is isomorphic to

$$\mathrm{Bl}_{S\cap D\,:\,D\in\mathcal{D}_n^1}\, S. \tag{23}$$

The intersection S is isomorphic to $\mathbb{A}^d$ for $d = n - |J|$ and hence is Tate and if I is a subset of $[\![1,n]\!]$ then $S\cap A_I$ is either empty ($I\cap J \neq \emptyset$) or, if $I \cap J = \emptyset$, isomorphic to the subvariety of $\mathbb{A}^d$ given by $\{1-\prod_{i\in I}x_i = 0\}$ (up to renumbering). Thus, the proper transform $\widehat{S}$ is isomorphic to X_d which is Tate by the discussion above.

Now, if $\widehat{S}_i$ is some irreducible codimension 1 stratum of $\widehat{D}_n^1$ that have a non-empty intersection with $\widehat{S}$ then, as $\widehat{S}_i$ is the exceptional divisor of some of the blow-ups in the construction of X_n, this intersection $\widehat{S}\cap\widehat{S}_i$ is the exceptional divisor in the blow-up sequence (23) that leads to $\widehat{S}$. As a consequence, the intersection $\widehat{S}\cap\widehat{S}_i$ is isomorphic to some irreducible stratum of $\widehat{D}_d^1$ in X_d and we can conclude that any possible intersection of strata in $\widehat{D}_n^1$ with $\widehat{S}$ is isomorphic to an intersection of strata in $\widehat{D}_d^1$ inside $X_d \simeq \widehat{S}$ and so is Tate by the above discussion. □

4.4. **Motivic Stuffle.** Let $\mathbf{k} = (k_1, \dots, k_p)$ and $\mathbf{l} = (l_1, \dots, l_q)$ be respectively a p-tuple and a q-tuple of integers with $k_1, l_1 \geqslant 2$, $\sum k_i = n$ and $\sum l_j = m$. In this section, as in section 1.1 and 1.3, if σ is a term of the formal sum $\mathbf{k} * \mathbf{l}$ with all coefficients being equal to 1, we will write $\sigma \in \mathrm{st}(\mathbf{k}, \mathbf{l})$. The map δ defined at proposition 2.3 extends to:

$$\overline{\mathcal{M}_{0,n+m+3}} \quad \xrightarrow{\delta} \quad \overline{\mathcal{M}_{0,n+3}} \times \overline{\mathcal{M}_{0,m+3}}.$$

Let $A_{\mathbf{k}}$ (resp. $A_{\mathbf{l}}$) be the divisor of singularities of the meromorphic differential form $\omega_{\mathbf{k}}$ on $\overline{\mathcal{M}_{0,n+3}}$ (resp. $\omega_{\mathbf{l}}$ on $\overline{\mathcal{M}_{0,m+3}}$) given in simplicial coordinates by $\omega_{\overline{k}}$ (resp. $\omega_{\overline{l}}$) (cf. 3) and in the cubical coordinates by $f_{k_1,\dots,k_p}$ (resp. $f_{l_1,\dots,l_q}$). For all σ in $\mathrm{st}(\mathbf{k}, \mathbf{l})$, let A_σ be the divisor of singularities of the form ω_σ. As in section 3.2, let Φ_n, Φ_m and Φ_{n+m} denote respectively the standard cells in $\overline{\mathcal{M}_{0,n+3}}(\mathbb{R})$, $\overline{\mathcal{M}_{0,m+3}}(\mathbb{R})$ and $\overline{\mathcal{M}_{0,n+m+3}}(\mathbb{R})$ and B_n, B_m and B_{n+m} be the Zariski closure of the boundary of respectively Φ_n, Φ_m and Φ_{n+m}.

Proposition 4.23. *We have an equality of framed motives:*

$$\left[\mathrm{H}^n\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B_n^{A_{\mathbf{k}}}\right); [\omega_{\mathbf{k}}]; [\Phi_n]\right] \cdot \left[\mathrm{H}^m\left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}; B_m^{A_{\mathbf{l}}}\right); [\omega_{\mathbf{l}}]; [\Phi_m]\right] = \sum_{\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})} \left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_\sigma; B_{n+m}^{A_\sigma}\right); [\omega_\sigma]; [\Phi_{n+m}]\right].$$

Proof. Let A_0 be the Zariski closure of $\partial\overline{\mathcal{M}_{0,n+m+3}} \setminus B_{n+m}$, $B_{n,m}$ be the Zariski closure of the boundary of $\Phi_n \times \Phi_m$ and A' be the boundary of $\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}\right) \times \left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}\right)$. As the map δ maps B_{n+m} onto $B_{n,m}$, we have an induced map

$$\delta : \left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_{n+m}^{A_0}\right) \longrightarrow \left(\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}\right) \times \left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}\right); B_{n,m}^{A'}\right).$$

Using the Künneth formula, we have maps of mixed Tate motives

$$\begin{aligned}(24)\quad \mathrm{H}^n\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}; B_n^{A_{\mathbf{k}}}\right) \otimes \mathrm{H}^m\left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}; B_m^{A_{\mathbf{l}}}\right) &\longrightarrow \\ \mathrm{H}^{n+m}\left(\left(\overline{\mathcal{M}_{0,n+3}} \setminus A_{\mathbf{k}}\right) \times \left(\overline{\mathcal{M}_{0,m+3}} \setminus A_{\mathbf{l}}\right); B_{n,m}^{A'}\right) &\longrightarrow \\ \mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_{n+m}^{A_0}\right)&\end{aligned}$$

which are both compatible with the respective frames $[\omega_{\mathbf{k}}] \otimes [\omega_{\mathbf{l}}]; [\Phi_n] \otimes [\Phi_m]$, $[\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}]; [\Phi_n \times \Phi_m]$ and $[\delta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}})]; [\Phi_{n+m}]$.

We now need to show that

$$\left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B_{n+m}^{A_0}\right); [\delta^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}})], [\Phi_{n+m}]\right] = \sum_{\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})} \left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_\sigma; B_{n+m}^{A_\sigma}\right); [\omega_\sigma]; [\Phi_{n+m}]\right].$$

As A_σ is included in A_0, using lemma 3.6 it enough to prove the previous equality with A_0 instead of A_σ in the RHS. The two following lemma tell us that it is enough to work with X_{n+m} (cf. section 4.2) instead of $\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0$.

Lemma 4.24. *Let $r \geqslant 2$ be an integer and let $\tilde{\delta_r} : \overline{\mathcal{M}_{0,r+3}} \to (\mathbb{P}^1)^r$ be the map given on the open set by*

$$(0, z_1, \dots, z_r, 1, \infty) \longmapsto (0, z_1, z_2, \infty) \times (0, z_2, z_3, \infty) \times \cdots \times (0, z_{r-1}, z_r, \infty) \times (0, z_r, 1, \infty).$$

Let A_r be the union of the codimension 1 irreducible components of $\partial\overline{\mathcal{M}_{0,r+3}}$ that are send by $\tilde{\delta_r}$ into $(\mathbb{P}^1)^r \setminus \mathbb{A}^r$.

Then, $A_r \subset A_0$ and there exist a sequence of flags $\mathcal{F}_1, \dots, \mathcal{F}_N$, of elements of $\mathcal{D}^1_r$ (Lemma 4.8) satisfying condition of Corollary 4.3 such that

$$X_r = \mathrm{Bl}_{\mathcal{F}_N,\dots,\mathcal{F}_1} \mathbb{A}^r \xrightarrow{\alpha_r} \overline{\mathcal{M}_{0,r+3}} \setminus A_r = \mathrm{Bl}_{\mathcal{F}_r,\dots,\mathcal{F}_1} \mathbb{A}^r \xrightarrow{\tilde{\delta_r}} \mathbb{A}^r \tag{25}$$

Proof. The map $\tilde{\delta_r}$ is given in the cubical coordinates on $\overline{\mathcal{M}_{0,r+3}}$ by $x_i = u_i$ where the x_i denotes the standard affine coordinates on $(\mathbb{P}^1)^r$ and send components of B_r into hyperplanes $x_i = 0$ or $x_i = 1$.

The induced map $\overline{\mathcal{M}_{0,r+3}} \setminus A_r \to \mathbb{A}^r$ is the blow-up along the strata

$$\{x_i = x_{i+1} = \ldots = x_j = 1\} \tag{26}$$

which are all elements of $\mathcal{D}^1_r$.

The beginning $\mathcal{F}_1, \dots, \mathcal{F}_r$ of the sequence of flags is given by

$$\mathcal{F}_1 = \{\{x_1 = x_2 = \cdots = x_n = 1\}, \{x_1 = x_2 = \cdots = x_{n-1} = 1\}, \dots, \{x_1 = 1\}\}$$
$$\mathcal{F}_2 = \{\{x_2 = x_3 = \cdots = x_n = 1\}, \{x_2 = x_3 = \cdots = x_{n-1} = 1\}, \dots, \{x_2 = 1\}\}$$
$$\dots$$
$$\mathcal{F}_i = \{\{x_i = x_{i+1} = \cdots = x_n = 1\}, \{x_i = x_{i+1} = \cdots = x_{n-1} = 1\}, \dots, \{x_i = 1\}\}$$
$$\dots$$
$$\mathcal{F}_r = \{\{x_r = 1\}\}$$

That part of the sequence satisfies condition (2) of Corollary 4.3. Then the easiest way to complete the sequence is to take flags with just one element beginning with the rank 1 strata of $\mathcal{D}^1_r$ (the only stratum of rank 0 is $\{x_1 = x_2 = \ldots = x_n = 1\}$), then the rank 2 strata and so on.

Now that the sequence of flags exists, Corollary 4.3 ensures that the morphisms in (25) holds.

Indeed, the usual map $\overline{\mathcal{M}_{0,r+3}} \to (\mathbb{P}^1)^r$ which maps $(0, z_1, \dots, z_r, 1, \infty)$ to $(z_1, \dots, z_r)$ sends Φ_r to the standard simplex $\Delta_r = \{0 < t_1 < \ldots < t_r < 1\}$ and maps B_r to the algebraic boundary of Δ_r. A first sequence of blow-up along the subvarieties $\{0 = t_1 = \ldots t_i\}$ corresponds to the change of variable from the simplicial to the cubical coordinates (6). In order to recover B_r, the blow-up along the proper transform of the subvarieties $\{t_i = t_{i+1} = \ldots = t_j\}$ and $\{t_i = t_{i+1} = \ldots = t_r = 1\}$ has still to be performed. The expression of these subvarieties in the cubical coordinates is $\{x_i = x_{i+1} = \ldots = x_j = 1\}$. The fact that it seems that we are blowing up less strata in order to recover $\overline{\mathcal{M}_{0,r+3}}$ from $(\mathbb{P}^1)^r$ using $\tilde{\delta_r}$ (25) comes from the fact that we are only looking at $\overline{\mathcal{M}_{0,r+3}} \setminus A_r$. □

From the previous lemma we deduce

Corollary 4.25. (1) *Let $\mathbf{a} = (a_1, \dots, a_b)$ be a b-tuple of integer with $a_1 \geqslant 2$ with $a_1 + \cdots + a_b = n + m$. Using the previous convention we have the following equality of framed mixed Tate motives*

$$\zeta^{fr.\mathcal{M}}(\mathbf{a}, \mathrm{id}) = \left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0; B^{A_0}_{n+m}\right); [\omega_{\mathbf{a}}], [\Phi_{n+m}]\right].$$

(2) *Let $\mathbf{k}$ and $\mathbf{l}$ be as in proposition 4.23, then there exists two distinguished flag $(\mathcal{F}, i_1, \dots, i_p)$ and $(\mathcal{F}', j_1, \dots, j_q)$ with $i_1, j_1 \geqslant 2$ and I_{i_p}, I_{j_q} being a partition of $[\![1, n]\!]$ such that the following equality of framed mixed Tate motives holds*

$$\zeta^{fr.\mathcal{M}}(\mathcal{F}, i_1, \dots i_p | \mathcal{F}', j_1, \dots j_q) = \left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0, B^{A_0}_{n+m}\right); [\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}], [\Phi_{n+m}]\right]$$

As a consequence, for all $\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})$, the framed mixed Tate motives

$$\left[\mathrm{H}^{n+m}\left(\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0, B_{n+m}^{A_0}\right); [\omega_\sigma], [\Phi_{n+m}]\right]$$

is equal to it counterpart in X_{n+m}, $\zeta^{fr.\mathcal{M}}(\sigma)$.

Proof. In 1. and 2., the map on the underlying vector space is given by α_{n+m}^* (cf. (25)). As $\widehat{C}_{n+m}$ is map to Φ_{n+m}, knowing the behaviour of α_{n+m}^* with respect to the form $\omega_{\mathbf{a}}$ and $\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}$ is enough to deduce that α_{n+m}^* respect the frames.

(1) As the the map α_{n+m}^* as no effect on the cubical coordinates, u_i on $\overline{\mathcal{M}_{0,n+m+3}} \setminus A_0$, we have $\alpha_{n+m}^*(\omega_{\mathbf{a}}) = \omega_{\mathbf{a}}^{\mathrm{id}}$, and so the equality of framed mixed Tate motives.

(2) Writing down in cubical coordinates the expression

$$\omega_{\mathbf{k}} = f_{\mathbf{k}}(u_1, \dots, u_n)d^n u \quad \text{and} \quad \omega_{\mathbf{l}} = f_{\mathbf{k}}(u_{n+1}, \dots, u_{n+m})d^m u$$

leads to the definition of two distinguished flag

$$(\mathcal{F}, i_1, \dots, i_p) \quad \text{and} \quad (\mathcal{F}', j_1, \dots, j_q),$$

as in remark 4.14 with $s = \mathrm{id}$. The fact that α_{n+m}^* respects the frames come from the equality

$$\omega_{i_1,\dots,i_p}^{\mathcal{F}'} \wedge \omega_{j_1,\dots,j_q}^{\mathcal{F}'} = \alpha_{n+m}^*(\omega_{\mathbf{k}} \wedge \omega_{\mathbf{l}}).$$

□

The only thing that remain to be checked to complete the proof of proposition 4.23 is using the notation of the previous lemma

$$\zeta^{fr.\mathcal{M}}(\mathcal{F}, i_1, \dots i_p | \mathcal{F}', j_1, \dots j_q) = \sum_{\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})} \zeta^{fr.\mathcal{M}}(\sigma, \mathrm{id}).$$

Using the computation of section 1.3, in particular the proposition 1.5, we have that for each $\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})$ there exists a permutation s_σ such that

$$[\omega_{i_1,\dots,i_p}^{\mathcal{F}'} \wedge \omega_{j_1,\dots,j_q}^{\mathcal{F}'}] = \sum_{\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})} [\omega_{\sigma,s_\sigma}].$$

As the divisor $A_{\mathcal{F}',j_1,\cdots,j_q}^{\mathcal{F},i_1,\dots,i_p}$ of $\omega_{i_1,\dots,i_p}^{\mathcal{F}'} \wedge \omega_{j_1,\dots,j_q}^{\mathcal{F}'}$ and the divisors A_{σ,s_σ} are in $\widehat{A}_{n+m}$, lemma 3.6 and an analogue of lemma 3.3 show that

$$\zeta^{fr.\mathcal{M}}(\mathcal{F}, i_1, \dots i_p | \mathcal{F}', j_1, \dots j_q) = \sum_{\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})} \zeta^{fr.\mathcal{M}}(\sigma, s_\sigma). \tag{27}$$

Permuting the variables give a well defined morphism $X_{n+m} \to X_{n+m}$ that preserve $\widehat{C}_{n+m}$ and its algebraic boundary $\widehat{B}_{n+m}$. It leads, on each term of the RHS of 27, to an equality

$$\zeta^{fr.\mathcal{M}}(\sigma, s_\sigma) = \zeta^{fr.\mathcal{M}}(\sigma, \mathrm{id}),$$

and hence to

$$\zeta^{fr.\mathcal{M}}(\mathcal{F}, i_1, \dots i_p | \mathcal{F}', j_1, \dots j_q) = \sum_{\sigma \in \mathrm{st}(\mathbf{k},\mathbf{l})} \zeta^{fr.\mathcal{M}}(\sigma, \mathrm{id}).$$

and the proposition 4.23

□

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