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# Ratchet phenomena in a spatially periodic harmonic potential

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## Abstract

Transport of a particle in a spatially periodic harmonic potential under the influence of a slowly time-dependent unbiased periodic external force is studied. The equations of motion are the same as in the problem of a slowly forced nonlinear pendulum. Using methods of the adiabatic perturbation theory we show that for a periodic external force of general kind the system demonstrates directed (ratchet) transport and obtain a formula for the average velocity of this transport.

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In recent years, studies of transport phenomena in nonlinear systems have been attracting a growing interest. In particular, a large and constantly growing number of papers are devoted to dynamics in systems which allow for directed [on average] motion under unbiased external forces and are referred to as ratchet systems. Their intensive study was motivated by problems of motion of Brownian particles in spatially periodic potentials, unidirectional transport of molecular motors in biological systems, and recognition of “ratchet effects” in quantum physics (see review [1] and references therein). Generally speaking, ratchet phenomena occur due to lack of symmetry in the spatially periodic potential and/or the external forcing. It is interesting, however, to understand microscopic mechanisms leading to these phenomena. A possible approach is to neglect dissipation and noise terms and arrive at a Hamiltonian system with deterministic forcing. Thus, one can make use of results obtained and methods developed in the theory of Hamiltonian chaos. Many papers studying such Hamiltonian ratchets appeared in the last years (see, e.g. [2, 3, 4, 5, 6, 7]). In particular, in [6] the ratchet current is estimated in the case when there are stability islands in the chaotic domain in the phase space of the system. The borders of such islands are “sticky” [8] and this stickiness together with desymmetrization of the islands are responsible for occurrence of the ratchet transport.

In the present paper, we consider the problem of motion of a particle in a spatially periodic harmonic potential  $U(q) = U(q + 2\pi)$  under the influence of unbiased external forcing. The equations are the same as in the paradigmatic model of a nonlinear pendulum under the action of external torque with zero time average. We study the case when the external forcing varies slowly and periodically in time at a rate of order  $\varepsilon$ ,  $0 < \varepsilon \ll 1$ , and use results and methods of the adiabatic perturbation theory. If  $\varepsilon$  is small enough, there are no stability islands in the domain of chaotic dynamics (see [9, 10]). Thus, the mechanism of ratchet transport in this system differs from one suggested in [6].

Our main objective is to find a formula for the average velocity  $V_q$  of transport in  $q$ -direction. Chaotization of dynamics in the system is a result of multiple separatrix crossings due to slow variation of the external forcing. At each crossing the adiabatic invariant (“action” of the system) undergoes a quasi-random jump (see [11, 12, 13]). We show that these jumps result in effective mixing and uniform distribution of the action along a trajectory in the chaotic domain. On the other hand, velocity of the  $q$ -motion depends on the immediate value of the action. Thus, to find the average velocity of transport on time intervals of order

or larger than the mixing time, we find formulas for displacement in  $q$  at a given value of the action and then integrate them over the interval of values of the action corresponding to the chaotic domain. We demonstrate that for an external force of general kind, there is directed transport in the system and obtain an analytic formula for the average velocity  $V_q$  of this transport.

We start with a basic equation describing a nonlinear pendulum under the influence of an external force  $\tilde{f}(t)$ :

$$\ddot{q} + \omega_0^2 \sin q = \tilde{f}(t). \quad (1)$$

We assume that  $\tilde{f}(t)$  is small in amplitude and a slowly periodic function of time:  $\tilde{f}(t) \equiv \varepsilon f(\tau) = \varepsilon f(\tau + 2\pi)$ , where  $\tau = \varepsilon t$ . Moreover, we assume that  $\tilde{f}(t)$  is a function with zero time average. This system is Hamiltonian, with time-dependent Hamiltonian function

$$H = \frac{p^2}{2} - \omega_0^2 \cos q - \varepsilon f(\varepsilon t)q. \quad (2)$$

We make a canonical transformation of variables  $(p, q) \mapsto (\bar{p}, \bar{q})$  using generating function  $W = (\bar{p} - F(\varepsilon t))q$ , where  $F(\tau) = -\int f(\tau)d\tau$ . Thus,  $F(\tau)$  is a periodic function defined up to an additive constant, which we are free to choose. To make the following presentation more clear, we choose this constant in such a way that the minimal value of  $F$  is  $F_{min} > 4\omega_0/\pi$ . Note that  $\bar{q} \equiv q$ . After this transformation, Hamiltonian of the system acquires the form (bars over  $q$  are omitted):

$$H = \frac{(\bar{p} - F(\tau))^2}{2} - \omega_0^2 \cos q. \quad (3)$$

Phase portrait of the system at a frozen value of  $\tau$  (we call it the unperturbed system) is shown in Fig. 1. The separatrix divides the phase space into the domains of direct rotations (above the upper branch of the separatrix), oscillations (between the separatrix branches), and reverse rotations (below the lower branch of the separatrix). Introduce the ‘‘action’’  $I$  associated with a phase trajectory of the unperturbed system on this portrait. In the domains of rotation,  $I$  equals an area between the trajectory, the lines  $q = -\pi$ ,  $q = \pi$ , and the axis  $\bar{p} = 0$ , divided by  $2\pi$ ; in the domain of oscillations, this is an area surrounded by the trajectory divided by  $2\pi$ . As  $\tau$  slowly varies, the separatrix on this portrait slowly periodically moves up and down according to behavior of function  $F(\tau)$ . Thus, a phase trajectory can cross the separatrix and switch its regime of motion from direct rotations to reverse rotations and vice versa. It is known that  $I$  is an adiabatic invariant of this system:

far from the separatrix its value is preserved along a phase trajectory with the accuracy of order  $\varepsilon$  on long time intervals (see, e.g., [14]).

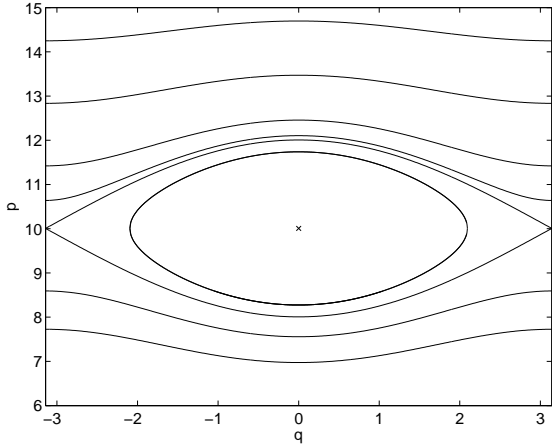


FIG. 1: Phase trajectories of the system (3) with sample function  $F(\tau) = A(1 + 2 \exp[-\alpha(\sin \tau)^2])$ ,  $A = 10$ ,  $\alpha = 8$ ,  $\tau = \pi/2$ ,  $\omega_0 = 1$ .

Location of the separatrix on the  $(q, \bar{p})$ -plane depends on the value of  $F(\tau)$ . As  $\tau$  slowly varies, the separatrix slowly moves up and down, and phase trajectories of the system cross the separatrix. Recall known results on variation of the adiabatic invariant when a phase trajectory crosses the separatrix. The area surrounded by the separatrix is constant, and hence, capture into the domain of oscillations is impossible in the first approximation (in the exact system, only a small measure of initial conditions correspond to phase trajectories that spend significant time in this domain). To be definite, consider the situation when the separatrix contour on the phase portrait slowly moves down. Thus, phase trajectories cross the separatrix and change the regime of motion from reverse rotation to direct rotation. Let the action before the separatrix crossing at a distance of order 1 from the separatrix be  $I = I_-$  and let the action after the crossing (also at a distance of order 1 from the separatrix) be  $I = I_+$ . In the first approximation, we have  $I_+ = I_- + 8\omega_0/\pi$ , i.e. the action increases by the value of the area enclosed by the separatrix divided by  $2\pi$  (see, e.g., [15, 16]). We shall call this change in the action a “geometric jump”. If the separatrix contour slowly moves up, and a trajectory goes from the regime of direct rotation to the regime of reverse rotation, the corresponding value of adiabatic invariant is decreased by the same value  $8\omega_0/\pi$ . Thus, in this approximation, the picture of motion looks as follows. While a phase trajectory is in the domain of reverse rotation, the value of  $I$  along this trajectory stays constant:

$I = I_-$ . After transition to the domain of direct rotation, this value changes by the value of the geometric jump. The transition itself in this approximation occurs instantaneously. After the next separatrix crossing, the adiabatic invariant changes again by the value of the geometric jump, with the opposite sign, and returns to its initial value  $I_-$ . We call this approximation adiabatic.

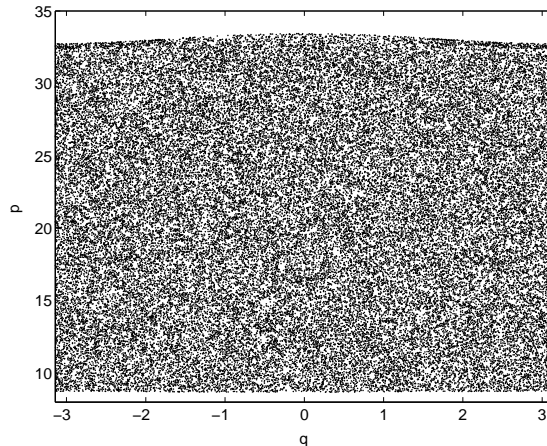
In the next approximation, the value of action at the separatrix crossing undergoes a small additional jump. Consider for definiteness the case when the separatrix contour on the phase portrait moves down, and  $I_-$  and  $I_+$  are measured when it is in its uppermost and lowermost positions, accordingly. For the jump in the adiabatic invariant we find the following formula (see [12, 13]):

$$\begin{aligned} 2\pi(I_+ - I_-) &= 16\omega_0 + 2a(1 - \xi)\varepsilon\Theta\ln(\varepsilon\Theta) \\ &+ a\varepsilon\Theta\ln\frac{2\pi(1 - \xi)}{\Gamma^2(\xi)} - 2b\varepsilon\Theta(1 - \xi), \end{aligned}$$

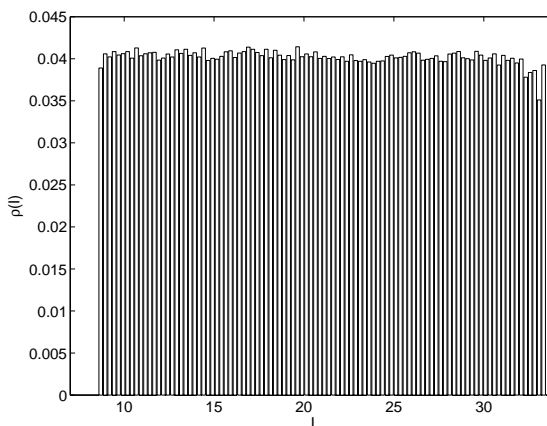
where  $a = \omega_0^{-1}$ ,  $b = \omega_0^{-1}\ln(32\omega_0^2)$ ,  $\Theta = 2\pi F'(\tau_*)$ . Here  $F'$  is the  $\tau$ -derivative of  $F$ ,  $\tau_*$  is the value of  $\tau$  at the separatrix crossing found in the adiabatic approximation. Value  $\xi$  is a so-called pseudo-phase of the separatrix crossing; it strongly depends on the initial conditions and can be considered as a random variable uniformly distributed on interval  $(0, 1)$  (see, e.g., [13]). Thus, value of the jump in the adiabatic invariant at the separatrix crossings has a quasi-random component of order  $\varepsilon\ln\varepsilon$ .

Accumulation of small quasi-random jumps due to multiple separatrix crossings produces diffusion of adiabatic invariant. Indeed, a heuristic reasoning shows that variations in the value of action can be represented as a random walk (see, e.g., [13]). On time interval  $4\pi/\varepsilon$  (after two separatrix crossings) the action changes by a value of order  $\varepsilon\ln\varepsilon$ . Hence, after  $N \sim \varepsilon^{-2}(\ln\varepsilon)^{-2}$  separatrix crossings the adiabatic invariant varies by a value of order one. As a result, in time of order  $t_{diff} \sim \varepsilon^{-3}(\ln\varepsilon)^{-2}$  the value of adiabatic invariant is distributed in all the range of values corresponding to the domain swept by the separatrix on the phase plane, and its distribution is close to the uniform one. We have checked this fact numerically for a sample function  $F(\tau)$  at various parameter values. Poincaré sections and distribution histograms of  $I$  in all the cases look similar; see an example in Fig. 2.

Our aim is to find a formula for average velocity  $V_q$  in the  $q$ -direction along a phase trajectory on time intervals of order  $t_{diff}$  or larger. We first only take into consideration the geometric jumps, and afterwards, to obtain the final result, we take into account the mixing



(a)



(b)

FIG. 2: a) Poincaré section at  $\tau = 0 \bmod 2\pi$  of a long phase trajectory. All the points are mapped onto the interval  $q \in (-\pi, \pi)$ .  $F(\tau) = A(1 + 2\exp[-\alpha(\sin \tau)^2])$  with  $A = 10, \alpha = 16, \varepsilon = 0.05, \omega_0 = 1$ . (b) Histogram of  $I$  on the segment  $(I_{min}, I_{max})$  along the same phase trajectory.

due to small quasi-random jumps. To simplify the consideration, assume that function  $F(\tau)$  has one local minimum  $F_{min}$  and one local maximum  $F_{max}$  on the interval  $(0, 2\pi)$ . The main results are valid without this assumption.

Introduce  $\tilde{I}$ , defined in the domains of rotation as follows: it equals the area bordered by the trajectory, the line  $\bar{p} = F(\tau)$ , and the lines  $q = -\pi, q = \pi$ , divided by  $2\pi$ . Thus,  $\tilde{I} = |F(\tau) - I|$ . Frequency of motion in the domains of rotation is  $\omega(\tilde{I})$ , where  $\omega(\tilde{I})$  at  $\tilde{I} > 4\omega_0/\pi$  is the frequency of rotation of a standard nonlinear pendulum with Hamiltonian  $H_0 = p^2/2 - \omega_0^2 \cos q$ , expressed in terms of its action variable  $\tilde{I}$ . We do not need an explicit expression for function  $\omega(\tilde{I})$ . From Hamiltonian (3) we find  $\dot{q} = \bar{p} - F(\tau)$ . Consider a phase

trajectory of the system frozen at  $\tau = \bar{\tau}$  in a domain of rotation. Let the value of action on this trajectory be  $I = I_0$ . Then the value of  $\dot{q}$  averaged over a period  $T$  of rotation equals  $\int_0^T \dot{q} dt/T = 2\pi/T = \omega(|F(\bar{\tau}) - I_0|)$ .

Now consider a long phase trajectory in the case of slowly varying  $\tau$ . Let on the interval  $(\tau_1, \tau_2)$  a phase trajectory of the system (3) be below the separatrix contour. In the adiabatic approximation, the value  $I_0$  of the adiabatic invariant along this trajectory is preserved on this interval. Hence, at  $\tau \in (\tau_1, \tau_2)$  we have

$$2\pi F(\tau) - 2\pi I_0 \geq 8\omega_0, \quad (4)$$

and the equality here takes place at  $\tau = \tau_1$  and  $\tau = \tau_2$ . In the process of motion on this time interval,  $q$  changes by a value

$$\Delta q_-(I_0) = - \int_{\tau_1}^{\tau_2} \omega(F(\tau) - I_0) d\tau. \quad (5)$$

On the interval  $(\tau_2, \tau_1 + 2\pi)$  the phase trajectory is above the separatrix contour, and the value of the adiabatic invariant equals  $\hat{I}_0 = I_0 + 8\omega_0/\pi$  due to the geometric jump. On this interval we have

$$2\pi F(\tau) - 2\pi I_0 \leq 8\omega_0. \quad (6)$$

In the process of motion on this time interval,  $q$  changes by a value

$$\Delta q_+(I_0) = \int_{\tau_2}^{\tau_1+2\pi} \omega(|F(\tau) - \hat{I}_0|) d\tau. \quad (7)$$

Total displacement in  $q$  on the interval  $(\tau_1, \tau_1 + 2\pi)$  equals  $\Delta q(I_0) = \Delta q_-(I_0) + \Delta q_+(I_0)$ , and the average velocity in  $q$ -direction on this interval is  $\Delta q(I_0)/(2\pi)$ .

Consider now the motion on a long enough time period  $\Delta t \sim t_{diff}$ . Due to the diffusion in the adiabatic invariant described above, in this time period values of  $I_0$ , defined as a value of  $I$  when the phase point is *below* the separatrix contour, cover the interval  $(I_{min}, I_{max} - 8\omega_0/\pi)$ . Here  $I_{min} = F_{min} - 4\omega_0/\pi$  and  $I_{max} = F_{max} + 4\omega_0/\pi$ . Assuming that the distribution of  $I$  on this interval is uniform, to find the average velocity, we integrate  $\Delta q(I_0)/(2\pi)$  over this interval. Integrating (5) over  $I_0$  and changing the order of integration we find

$$\begin{aligned} \int_{I_{min}}^{I_{max}-8\omega_0/\pi} \Delta q_- dI_0 &= - \int_0^{2\pi} d\tau \int_{I_{min}}^{F(\tau)-4\omega_0/\pi} \omega(F(\tau) - I_0) dI_0 \\ &= - \int_0^{2\pi} d\tau \int_{4\omega_0/\pi}^{F(\tau)-I_{min}} \omega(\eta) d\eta. \end{aligned}$$



Now we take into account a well known equality  $\omega(\tilde{I}) = \partial H_0(\tilde{I})/\partial \tilde{I}$  (recall that  $H_0(\tilde{I})$  is the Hamiltonian of a nonlinear pendulum as a function of its action variable) and obtain

$$-\int_0^{2\pi} d\tau \int_{4\omega_0/\pi}^{F(\tau)-I_{min}} \omega(\eta) d\eta = -\int_0^{2\pi} (H_0(F(\tau)-I_{min}) - H_0^s) d\tau, \quad (8)$$

where  $H_0^s$  is the value of  $H_0$  on the separatrix. Similarly, integrating (7) we obtain

$$\int_{I_{min}}^{I_{max}-8\omega_0/\pi} \Delta q_+ dI_0 = \int_0^{2\pi} (H_0(I_{max}-F(\tau)) - H_0^s) d\tau. \quad (9)$$

Adding (8) to (9) and dividing by  $2\pi(F_{max} - F_{min})$  we find the expression for the average velocity  $V_q$  of transport in  $q$ -direction on long time intervals:

$$V_q = \frac{1}{2\pi(F_{max} - F_{min})} \times \int_0^{2\pi} (H_0(I_{max}-F(\tau)) - H_0(F(\tau)-I_{min})) d\tau. \quad (10)$$

In (10),  $H_0(I)$  can be found as the inverse function to  $\tilde{I}(h)$ , which defines action as a function of energy in domains of rotation of a nonlinear pendulum. For the latter function, the following formula holds (see e.g. [17]):

$$\tilde{I}(h) = \frac{4}{\pi} \omega_0 \kappa \mathcal{E}(1/\kappa), \quad \kappa \geq 1, \quad (11)$$

where  $\kappa^2 = (1 + h/\omega_0^2)/2$ ,  $\mathcal{E}(\cdot)$  is the complete elliptic integral of the second kind. If function  $F(\tau)$  has several local extremes on the interval  $(0, 2\pi)$ ,  $F_{min}$  and  $F_{max}$  in (10) are the smallest and largest values of  $F$  respectively.

It can be seen from (10) that for function  $F(\tau)$  of general type  $V_q$  is not zero, and hence there is the directed transport in the system. We checked this formula numerically for a sample function  $F(\tau) = A(1 + 2 \exp[-\alpha(\sin \tau)^2])$ ,  $\alpha > 0$  at various values of parameters  $\varepsilon$  and  $\alpha$ . The results are represented in Table I. To find numerical values of  $V_q$  presented in the table, we integrated the system with Hamiltonian (3) on a long time interval  $\Delta t = 2\pi \cdot 10^6/\varepsilon$  with a constant time step of  $\pi/100$  (5th order symplectic scheme). The table demonstrates satisfactory agreement between the formula and the numerics.

To summarize, we have described the phenomenon of the directed transport in a spatially periodic harmonic potential influenced by a small slow-periodic in time unbiased external force. We have shown that for the external force of a general kind the system exhibits directed transport on long time intervals. Direction of the transport is independent of initial

	$\alpha = 4$	$\alpha = 8$	$\alpha = 16$
$\varepsilon = 0.1$	4.721	6.756	8.363
$\varepsilon = 0.05$	4.446	6.681	8.076
$\varepsilon = 0.01$	4.298	6.211	7.442
$\varepsilon = 0.005$	4.598	6.702	8.202
$V_q^{theor}$	4.393	6.679	8.110

TABLE I: Numerically found values of  $V_q$  corresponding to various values of parameters  $\varepsilon$ ,  $\alpha$  (four upper rows,  $A = 10$ ,  $\omega_0 = 1$ ) and theoretical values  $V_q^{theor}$  obtained according to (10) (the bottom row).

conditions and is determined by properties of the external force. We have obtained an approximate formula for the average velocity of the transport and checked it numerically.

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