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Convex shapes and convergence speed of discrete tangent estimators

Jacques-Olivier Lachaud and François de Vieilleville

LaBRI, Univ. Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France

Abstract. Discrete geometric estimators aim at estimating geometric characteristics of a shape with only its digitization as input data. Such an estimator is multigrid convergent when its estimates tend toward the geometric characteristics of the shape as the digitization step h tends toward 0. This paper studies the multigrid convergence of tangent estimators based on maximal digital straight segment recognition. We show that such estimators are multigrid convergent for some family of convex shapes and that their speed of convergence is on average $\mathcal{O}(h^{\frac{2}{3}})$. Experiments confirm this result and suggest that the bound is tight.

1 Introduction

The problem of estimating geometric quantities of digitized shapes like area, perimeter, tangents or curvatures is an important and active research field. Discrete geometric estimators have indeed many applications in vision, shape analysis and pattern recognition. It is however difficult to compare objectively their respective accuracy, since for a given shape there exists infinitely many shapes with the same digitization. In this paper, we are mainly interested by the *multigrid convergence* property of some estimators, which is one of the few existing objective criteria. This property ensures that a better resolution brings a better approximation. For concrete applications, the *speed of convergence* is a very important criterion too, since it has a huge impact on their accuracy at standard resolutions.

Formally, taking the definitions of [9], let Dig_h be some digitization process of step h. Let $\mathbb{F}$ be a family of shapes in $\mathbb{R}^2$ and let G be a geometric quantity defined for all $X \in \mathbb{F}$. A *discrete estimator of* G is a map that associates to a digitization $\mathrm{Dig}_h(X)$ an estimation of $G(X)$. A discrete estimator $\mathcal{G}$ is *multigrid convergent* toward G for $\mathbb{F}$ and Dig iff, for any $X \in \mathbb{F}$, there exists some $h_X > 0$ for which

$$\forall 0 < h < h_X, |\mathcal{G}(\mathrm{Dig}_h(X)) - G(X)| \leq \tau(h) \ ,$$

where $\tau : \mathbb{R}^+ \to \mathbb{R}^{+*}$ has limit value 0 at $h = 0$. This function defines the *speed of convergence* of $\mathcal{G}$ toward G.

For instance, for the family of plane convex bodies with C^3-boundary and positive curvatures, denoted later on by $\mathbb{F}_c^3$, the best known speed of convergence for an area and or a moment estimator is $\mathcal{O}(h^{\frac{15}{11}})$ [8,10], for a perimeter estimator it is $\mathcal{O}(h)$ [16]. There are fewer results concerning local geometric quantities like

tangent direction or curvature. The first works on this topic were presented in [4], where some evidence of the convergence of tangent estimators based on digital straight segment (DSS) recognition were given. We report the recent work of [13] which establishes the multigrid convergence of tangent direction estimators based on maximal DSS recognition, with an average speed of convergence of $\mathcal{O}(h^{\frac{1}{3}})$. The recent result of [5] has confirmed that there is yet no curvature estimator proven to be multigrid convergent.

In this paper, we prove a new upper bound for the average speed of convergence of discrete tangent estimators which are based on maximal DSS recognition around the point of interest [6,12]. This new bound of $\mathcal{O}(h^{\frac{2}{3}})$ was suggested by the experimental study of [12] and enhances the previous bound of $\mathcal{O}(h^{\frac{1}{3}})$ [13]. The proof of this enhanced result, obtained for shapes in $\mathbb{F}_c^3$, follows these steps:

Section 2 Digitizations of convex shapes are digital convex polygons (CDP). We achieve thus a better localization of the shape boundary with respect to the digitized boundary (Proposition 1).

Section 3 We first claim that the DSS characteristics have an average asymptotic behaviour depending on their length (Claim 2). The asymptotic edge length of CDP is recalled (Theorem 1), which induces a superlinear localization of the shape boundary in $\mathcal{O}(h^{\frac{4}{3}})$ (Proposition 2). With these relations, the slope of digital edges of CDP is shown to be multigrid convergent to the tangent direction of a nearby boundary point with an average speed of $\mathcal{O}(h^{\frac{2}{3}})$ (Proposition 3).

Section 4 The behaviour of maximal DSS is identical to the behavior of digital edges, since any maximal DSS contains at least one digital edge (Proposition 4). The average speed of convergence of tangent estimators based on maximal DSS is thus achieved (Theorem 3).

We will then conclude and open some perspectives in Section 5.

2 Preliminary definitions and first properties

2.1 Digitization and convex digital polygon

Let S be some subset of $\mathbb{R}^2$. Its *Gauss digitization* of grid step h is defined as $\mathrm{D}_h^G(S) = S \cap h\mathbb{Z} \times h\mathbb{Z}$. Thus, the considered digitized objects are subsets of the rescaled digital plane $h\mathbb{Z} \times h\mathbb{Z}$.

A *convex digital polygon (CDP)* Γ is a subset of the digital plane $h\mathbb{Z} \times h\mathbb{Z}$ with a single 4-connected component equal to the Gauss digitization of its convex hull, i.e. $\Gamma = \mathrm{D}_h^G(\mathrm{conv}(\Gamma))$. Its *vertices* $(V_i)_{i=1..e}$ form the minimal subset for which $\Gamma = \mathrm{D}_h^G(\mathrm{conv}(V_1, \ldots, V_e))$ (vertices are ordered clockwise). The points of Γ which are 8-adjacent to some point not in Γ form the *border* of Γ. It is a 4-connected digital path that visits every V_i in order. When moving clockwise, any subpath $\Gamma_{V_i,V_{i+1}}$ is called a *digital edge* of Γ, while the Euclidean straight segment V_iV_{i+1} is an *edge* of Γ.

For small enough grid steps, Gauss digitizations of any shape in $\mathbb{F}_c^3$ are convex digital polygons. This result will be admitted throughout the paper.

2.2 Standard line, digital straight segment, maximal segments

Definition 1. *[15] The set of points (x, y) of the digital plane $\mathbb{Z}^2$ verifying $\mu \leq ax - by < \mu + |a| + |b|$, with a, b and μ integer numbers, is called the* standard line *with slope a/b and shift μ. Standard lines are the 4-connected discrete lines.*

The quantity $ax - by$ is called the *remainder* of the line. Points whose remainder is μ (resp. $\mu + |a| + |b| - 1$) are called upper (resp. lower) leaning points. Any finite connected portion of a standard line is called a *digital straight segment (DSS)*. Its *characteristics* are the slope a/b and the shift μ of the standard line containing it with smallest $|a| + |b|$.

Most of the results demonstrated here are directly transferable for 8-connected curves since there is a natural bijective transformation between standard and naive digital lines. In the paper, all the reasoning is made in the first octant, but it extends naturally to the whole digital plane.

2.3 Recursive decomposition of DSS

We here recall a few properties about *patterns* composing DSS and their close relations with continued fractions. They constitute a powerful tool to describe discrete lines with rational slopes [2,7]. Without loss of generality all definitions and propositions stated below hold for standard lines and DSS with slopes in the first octant (e.g. $\frac{a}{b}$ with $0 \leq a \leq b$). In the first octant, only two Freeman moves are possible, 0 is a step to the right and 1 is a step up, 4-connected digital paths can be expressed as words of $\{0, 1\}^*$.

Definition 2. *Given a standard line (a, b, μ), we call* pattern *of characteristics (a, b) the word that is the succession of Freeman moves between any two consecutive upper leaning points. The Freeman moves defined between any two consecutive lower leaning points is the previous word read from back to front and is called the* reversed pattern.

Since a DSS has at least either two upper or two lower leaning points, a DSS (a, b, μ) contains at least one *pattern* or one *reversed pattern* of characteristics (a, b). There exists recursive transformations for computing the *pattern* of a standard line from the *simple continued fraction* of its slope (see [2], [18] Chap. 4 and [9] Chap. 9), here Berstel approach better suits our purpose.

A rational slope z in $]0, 1]$ can be written uniquely as the continued fraction

$$z = 0 + \cfrac{1}{u_1 + \cfrac{1}{\ldots + \cfrac{1}{u_n}}}$$

and is conveniently denoted $[0, u_1, \ldots, u_n]$. The u_i are called the *partial coefficients* and the continued fraction formed with the k first *partial coefficient* is said to be a *k-th convergent* of z and is a rational numbers denoted by z_k. The

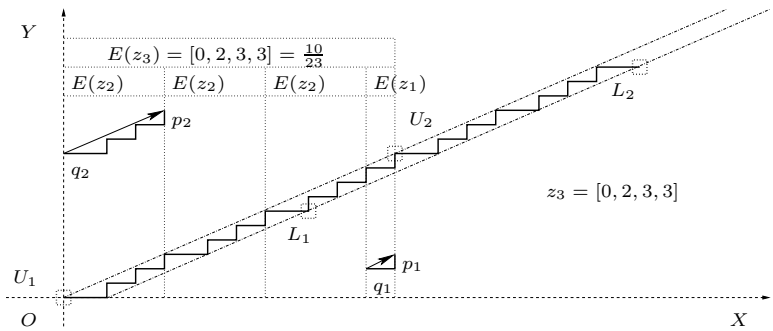

Fig. 1. A digital straight segment of characteristics $(10, 23, 0)$ with an odd depth slope, taken between the origin and its second lower leaning point

depth of a *k-th convergent* equals k. We conveniently denote p_k the numerator (resp. q_k the denominator) of a *k-th convergent*.

We recall a few more relations regarding the way convergents are related and which will be used later on in this paper:

$$\forall k \geq 1 \quad p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1} \tag{1}$$

$$p_0 = 0 \quad p_{-1} = 1 \qquad \forall k \geq 1 \quad p_k = u_k p_{k-1} + p_{k-2} \tag{2}$$

$$q_0 = 1 \quad q_{-1} = 0 \qquad \forall k \geq 1 \quad q_k = u_k q_{k-1} + q_{k-2} \tag{3}$$

Given a rational slope between 0 and 1 its continued fraction is finite and for each i, u_i is a strictly positive integer. In order to have a unique writing we consider that the last *partial coefficient* is greater or equal to two; except for slope $1 = [0, 1]$.

Let us now explain how to compute the *pattern* associated with a rational slope z in the first octant. Let us define E a mapping from the set of positive rational number smaller than one onto the Freeman-move's words. More precisely: $E(z_0) = 0$, $E(z_1) = 0^{u_1}1$ and the other values are expressed recursively:

$$E(z_{2i+1}) = E(z_{2i})^{u_{2i+1}} E(z_{2i-1}) \tag{4}$$

$$E(z_{2i}) = E(z_{2i-2}) E(z_{2i-1})^{u_{2i}} \tag{5}$$

It has been shown that this mapping constructs the pattern (a, b) for any rational slope $z = \frac{a}{b}$. Fig. 1 illustrates the construction of an odd pattern using the mapping E. The Minkowski $\mathcal{L}^1$ length of $E(z_k)$ equals $p_k + q_k$ and can be computed recursively using Eq. (2) and (3). Moreover we recall that any digital edge is a pattern or a succession of the same pattern, its *digital parameters* are its slope, denoted by $\frac{p}{q}$, and the number δ of repetitions of the pattern $E(\frac{p}{q})$.

2.4 Localization accuracy of digitized convex shapes

One can expect that the boundary of a convex shape is approximately at distance h from the border of its digitization of grid step h. In fact, for a convex shape

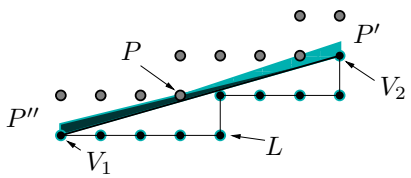

Fig. 2. Two local constraints for the real underlying convex shape

S, its *boundary* ∂S is much closer to the convex hull of $\mathrm{D}_h^G(S)$ than h. A better localization of the shape is thus possible, as stated below and illustrated in Fig. 2:

Proposition 1. *Let S be a convex shape such that $\mathrm{D}_h^G(S)$ is a CDP Γ for some h. Consider an edge V_iV_{i+1} of Γ with slope in the first octant. Then any point of the boundary ∂S above the straight segment V_iV_{i+1} has a vertical distance to it no greater than $\frac{h}{q_{n-1}}$, where the slope of this edge is the irreducible fraction $\frac{p_n}{q_n}$.*

The proof of this proposition can be found in [11].

3 Asymptotic behaviour of edges of digitized shapes

We study here asymptotic properties of edges of digitized convex shapes. Their average length is first exhibited and an experimental study of the digital parameters of edges is presented. The direction of digital edges is shown to converge toward the tangent direction.

3.1 Asymptotic number and digital parameters of edges

Let S be some shape of $\mathbb{F}_c^3$. We have the following theorem:

Theorem 1. *(Adapted from [1], Theorem 2) For a small enough h, the Gauss digitization $\mathrm{D}_h^G(S)$ of S is a CDP and its number of edges $n_e(\mathrm{D}_h^G(S))$ satisfies:*

$$c_1(S)\frac{1}{h^{\frac{2}{3}}} \leq n_e(\mathrm{D}_h^G(S)) \leq c_2(S)\frac{1}{h^{\frac{2}{3}}}$$

where the constants $c_1(S)$ and $c_2(S)$ depend on extremal bounds of the curvatures along S. Hence for a disc c_1 and c_2 are absolute constants.

As an immediate corollary, the average Minkowski $\mathcal{L}^1$ length of edges grows as $\Theta(h^{\frac{2}{3}})$. The question is: what is the average behavior of the digital parameters p, q and δ of edges, knowing that the average digital length $l = \delta(p+q)$ satisfies $\Theta(h^{\frac{2}{3}}/h) = \Theta(1/h^{\frac{1}{3}})$? Since the slope of edges should tend toward the slope of points on the boundary of S and since almost all these points have irrational slope, p and q should tend toward infinity almost everywhere with a bounded δ. This observation is confirmed by experiments, as illustrated on Fig. 3, which plots the means and standard deviations of $\frac{q}{l}$ and δ for edges on finer and finer digitizations of a disk. It is clear that q (but also p) satisfies the same asymptotic law as l while δ remains bounded on average. We make hence the following claim:

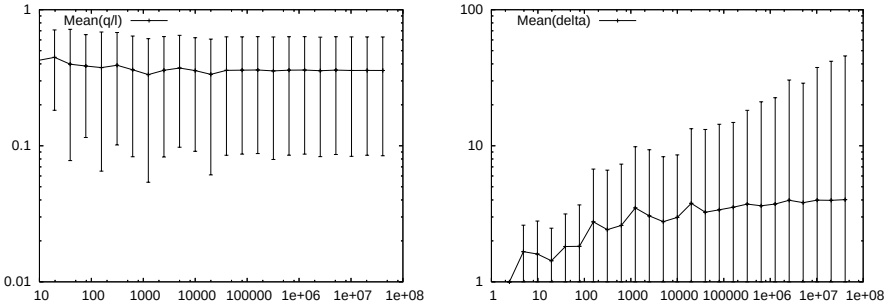

Fig. 3. Plots in log-space of the means of digital parameters $\frac{q}{l}$ (left) and δ (right) for the edges of Gauss digitizations of a disk, as functions of the inverse $\frac{1}{h}$ of the grid step. Standard deviations are symbolized with errorbars

Claim 2. *On average, the digital parameters p, q of the edges of $\mathrm{D}_h^G(S)$ with $S \in \mathbb{F}_c^3$ grow as $\Theta(\frac{1}{h^{\frac{1}{3}}})$, whereas δ is bounded, when h tends toward 0.*

This claim induces a first result about the convergence speed of the localization constraints of Proposition 1 (q_{n-1} has the same asymptotic law as q_n).

Proposition 2. *Let $S \in \mathbb{F}_c^3$ and $\mathrm{D}_h^G(S)$ its Gauss digitization. For a small enough h, in each octant, the vertical distance between any edge of $\mathrm{D}_h^G(S)$ and ∂S is bounded. On average, this bound is $\mathcal{O}(h^{\frac{4}{3}})$ (and never worse than $\mathcal{O}(h)$).*

3.2 Convergence of tangent estimation based on edge direction

Let M be a point on the boundary of a shape $S \in \mathbb{F}_c^3$ and let $\theta_M(S)$ be the tangent direction of ∂S at M. We propose to estimate $\theta_M(S)$ as the arctangent of the slope of the digital edge of the CDP $\mathrm{D}_h^G(S)$ lying below M (for the first octant; definitions for other octants are straightforward). We denote this discrete estimator by $\mathcal{T}_M^{\mathrm{edge}}$. Assuming Claim 2, we have the multigrid convergence of this tangent estimator:

Proposition 3. *Let M be a point in the plane and let $\mathbb{F}_c^3(M)$ be the subset of shapes S of $\mathbb{F}_c^3$ with $M \in \partial S$. The discrete estimator $\mathcal{T}_M^{\mathrm{edge}}$ at point M is multigrid convergent toward the tangent direction θ_M for $\mathbb{F}_c^3(M)$ and Gauss digitization. Furthermore, its speed of convergence is on average $\mathcal{O}(h^{\frac{2}{3}})$.*

Proof. We have to prove that for a shape $S \in \mathbb{F}_c^3(M)$, there exists some grid step h_S for which $\forall 0 < h < h_S, |\mathcal{T}_M^{\mathrm{edge}}(\mathrm{D}_h^G S) - \theta_M(S)| \leq \tau(h)$. Without loss of generality, we assume that the tangent direction $\theta_M(S)$ is in the first octant and we locally parameterize ∂S as $(x, f(x))$, setting M as the origin of the coordinate axes ($(x_M, y_M) = (0, 0)$). Let h_0 be the digitization step below which $\mathrm{D}_h^G(S)$ is a CDP.

Since the slope of ∂S around M is in the first octant, there exists some $h_S < h_0$ such that for any $0 < h < h_S$, the vertical straight segment going down from M intersect some edge V_iV_{i+1} of $\mathrm{D}_h^G(S)$. Let δ, p and q be the digital parameters of this edge, let z be its slope $\frac{p}{q}$ and let n be the depth of z (i.e. $z = \frac{p_n}{q_n}$). The horizontal distance between M and either V_i or V_{i+1} is necessarily greater than half the horizontal length of V_iV_{i+1}. We assume V_{i+1} maximizes this distance, without loss of generality. We get:

$$\frac{h\delta q_n}{2} \leq x_{v_{i+1}} \leq h\delta q_n \tag{6}$$

From Proposition 1 we have for any $x \in [x_{v_i}, x_{v_{i+1}}]$:

$$zx - \frac{h}{q_{n-1}} \leq f(x) \leq zx + \frac{h}{q_{n-1}} \ . \tag{7}$$

Inserting Taylor expansion of $f(x)$ about $x = 0$ in Eq. (7) induces:

$$|z - f'(0)| \leq \frac{h}{xq_{n-1}} + \mathcal{O}(x) \ . \tag{8}$$

Setting $x = x_{v_{i+1}}$ in Eq. (8) and using both sides of Eq. (6) gives the inequality

$$|z - f'(0)| \leq \frac{2h}{h\delta q_n q_{n-1}} + \mathcal{O}(h\delta q_n) \ . \tag{9}$$

We notice that $z = \tan(\mathcal{T}_M^{\mathrm{edge}}(\mathrm{D}_h^G(S)))$ and that $f'(0) = \tan(\theta_M(S))$. Moreover, for any angle $s, t \in [0, \frac{\pi}{4}]$, we have $|s - t| \leq |\tan(s) - \tan(t)|$. With these two remarks, Eq. (9) implies:

$$|\mathcal{T}_M^{\mathrm{edge}}(\mathrm{D}_h^G(S)) - \theta_M(S)| \leq \frac{2}{\delta q_n q_{n-1}} + \mathcal{O}(h\delta q_n) \ . \tag{10}$$

Since δq_n is no smaller than half the edge length, it follows from Theorem 1 that the dominant term $\frac{2}{\delta q_n q_{n-1}}$ is at least some $\mathcal{O}(h^{\frac{1}{3}})$ on average. In [13] it is also shown that there is no edge of bounded length as h tends toward 0. Then the right part of Eq. (10) tends toward 0. The multigrid convergence is thus shown. At last, assuming Claim 2, Eq. (10) then induces

$$|\mathcal{T}_M^{\mathrm{edge}}(\mathrm{D}_h^G(S)) - \theta_M(S)| \leq \mathcal{O}(h^{\frac{2}{3}}) \ , \tag{11}$$

which indicates that the average speed of convergence of $\mathcal{T}_M^{\mathrm{edge}}$ is $\mathcal{O}(h^{\frac{2}{3}})$. $\square$

4 Tangent estimators based on maximal DSS recognition

This section discusses the convergence speed of discrete tangent estimators based on maximal digital straight segment (maximal segment) recognition. Along a digital path, maximal segments are the inextensible digital straight segments, otherwise said adding the next point to the front or to the back constructs a set of digital points that no standard line contains. The set of all maximal segments of a digital path can be extracted efficiently in time linear with its number of points [6]. Maximal segments have deep links with edges of convex hulls [5,14].

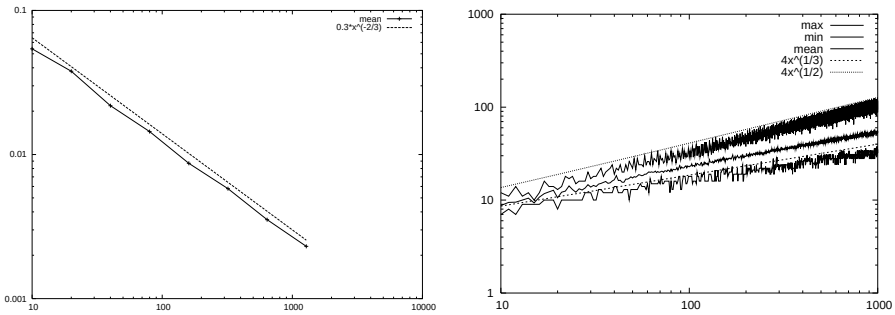

Fig. 4. For both plots, the digitized shape is a disk of radius 1 and the abscissa is the inverse of the digitization step. Left: plot in log-space of the mean absolute error between the λ-MST tangent direction and the theoretical one (mean taken over all digital points). The convergence speed on this shape is likely to be in $\Theta(h^{\frac{2}{3}})$. Right: plot in log-space of the digital length of maximal segments, which is on average $\Theta(1/h^{\frac{1}{3}})$

Estimating the tangent direction at some point is then achieved by considering specific DSS [17] or maximal segments [6] containing this point. A recent experimental evaluation [12] has shown that tangent estimators based on maximal segments are accurate and preserve convexity properties of the real shape.

Let M be a point on the boundary of a shape $S \in \mathbb{F}_c^3$ and let $\theta_M(S)$ be the tangent direction of ∂S at M. We propose to estimate $\theta_M(S)$ as the arctangent of the slope of any maximal segment of the CDP $\mathrm{D}_h^G(S)$ lying below M (for the first octant; definitions for other octants are straightforward). We denote this discrete estimator by $\mathcal{T}_M^{\mathrm{ms}}$. We shall prove in Theorem 3 that this estimator is multigrid convergent with average convergence speed of $\mathcal{O}(h^{\frac{2}{3}})$. As a corollary, the Feschet-Tougne tangent estimator [6], which uses the most centered maximal segment around M, and the λ-MST estimator [12], which makes a convex combination of the directions of all maximal segments around M, are also multigrid convergent with same average speed.

Before proving this theorem, note first that the experimental evaluation of the λ-MST estimator, whose absolute error is displayed in Fig. 4, left, suggests that its average convergence speed is indeed likely to be in $\mathcal{O}(h^{\frac{2}{3}})$. Secondly, the average Minkowski length of maximal segments behave as the average length of the edges, that is in $\mathcal{O}(h^{\frac{2}{3}})$ as shown on Fig. 4, right (the Minkowski length is h times the digital length). Thirdly, every maximal segment contains a digital edge as stated in the following proposition whose proof can be found in [11]:

Proposition 4. *On the border of a CDP Γ, any maximal segment, whose slope is some $\frac{p_n}{q_n}$, contains at least one digital edge that has either the same slope or its $n-1$-convergent $\frac{p_{n-1}}{q_{n-1}}$.*

We may now state our main result, which relies also on Claim 2.

Theorem 3. *Let M be a point in the plane and let $\mathbb{F}_c^3(M)$ be the subset of shapes S of $\mathbb{F}_c^3$ with $M \in \partial S$. The discrete estimator $\mathcal{T}_M^{\mathrm{ms}}$ at point M is multigrid convergent toward the tangent direction θ_M for $\mathbb{F}_c^3(M)$ and Gauss digitization. Furthermore, its speed of convergence is on average $\mathcal{O}(h^{\frac{2}{3}})$.*

Proof. We take the same notations as in the proof of Proposition 3. We consider a maximal segment MS below M with slope z_n. According to Proposition 4, it contains a digital edge VV' of slope z_n or z_{n-1}, which may not be below M. Worst case is for a digital edge with slope z_{n-1} and this is the one considered below. We have:

$$\begin{aligned}|\mathcal{T}_M^{\mathrm{ms}}(S) - \theta_M(S)| &\leq |z_n - f'(0)| \\ &\leq |z_{n-1} - f'(0)| + |z_n - z_{n-1}| \\ &\leq |z_{n-1} - f'(x)| + \mathcal{O}(x) + \tfrac{1}{q_n q_{n-1}} \ ,\end{aligned}$$

using Taylor relation and Eq. (1) to get the last inequality.

If we choose some abscissa x such that $(x, f(x))$ is above the edge VV', then the proof of Proposition 3 indicates that $|z_{n-1} - f'(x)| \leq \mathcal{O}(h^{\frac{2}{3}})$ on average. Since $(x, f(x))$ is above MS too, x cannot be greater than the length of MS, which is also some $\mathcal{O}(h^{\frac{2}{3}})$ on average. At last, Claim 2 similarly provides $\frac{1}{q_n q_{n-1}} = \Theta(h^{\frac{2}{3}})$. Summing all these bounds concludes the proof. $\square$

As an immediate corollary, the λ-MST estimator [12] and the Feschet-Tougne tangent estimator [6] have the same asymptotic behaviour.

5 Conclusion

We have studied properties of digital convex polygons and exhibited several new results (Proposition 1 and Proposition 4). We have also examined the asymptotic properties of digital edges on digitized convex shapes (Claim 2), which has led to a position estimator of average convergence speed $\mathcal{O}(h^{\frac{4}{3}})$ (Proposition 2) and to a tangent direction estimator of average convergence speed $\mathcal{O}(h^{\frac{2}{3}})$ (Proposition 3). At last we have shown the new bound of $\mathcal{O}(h^{\frac{2}{3}})$ for the average convergence speed of tangent estimators based on maximal segments (Theorem 3), which matches experimental evaluation.

These results indicate that curvature estimators relying on surrounding DSS have most probably an error of $\mathcal{O}(h^0)$: position $\equiv f(x)$ in $\mathcal{O}(h^{\frac{4}{3}})$, tangent direction $\equiv f'(x)$ in $\mathcal{O}(h^{\frac{2}{3}})$, curvature $\equiv f''(x)$ probably in $\mathcal{O}(1)$ (uncertainty on tangent further divided by $\mathcal{O}(h^{\frac{2}{3}})$). The problem of exhibiting a multigrid convergent curvature estimator is thus still open.

Another straightforward extension of this work would be to investigate the properties of discrete surfaces and estimators based on digital plane recognition. However since the problem of finding an enclosing polyhedron with a minimal number of 2-facets has been proven to be NP-hard (see [3]), the problem would get much harder than the two dimensional case.

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