On the Circular Chromatic Number of Circular Partitionable Graphs

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Abstract

This paper studies the circular chromatic number of a class of circular partitionable graphs. We prove that an infinite family of circular partitionable graphs G has $\chi_c(G) = \chi(G)$. A consequence of this result is that we obtain an infinite family of graphs G with the rare property that the deletion of each vertex decreases its circular chromatic number by exactly 1.

Key words: circular coloring, chromatic number, partitionable graph

1 Introduction

Suppose G is a graph and k and d, $k \geq 2d$, are positive integers. A (k,d)-coloring of G is a mapping $f : V(G) \to \mathbb{Z}_k$ such that for each edge xy of G, $d \leq |f(x) - f(y)| \leq k - d$. The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ has a } (k,d)\text{-coloring}\}.$$

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As a $(k,1)$-coloring of G is equivalent to a k-coloring of G, we have $\chi_c(G) \leq \chi(G)$. On the other hand, it is known [11,12] that $\chi_c(G) > \chi(G) - 1$. Therefore $\chi(G) = \lceil \chi_c(G) \rceil$. So the parameter $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

If G is vertex $\chi-$critical (i.e., the deletion of any vertex decreases its chromatic number), then deleting any vertex from G decreases its chromatic number by 1. However, for a vertex χ_c-critical graph G (i.e., the deletion of any vertex decreases its circular chromatic number), the decrease of the circular chromatic number by the deletion of one vertex could be any rational number $0 < r < 2$. Indeed, it seems very rare that a graph G has the property that the deletion of each vertex decrease its circular chromatic number by exactly 1. The question of characterizing such graphs was raised in [12]. Currently, complete graphs, the direct sum of vertex χ-critical graphs and a few isolated example graphs are known to have this property. In this paper, we study the circular chromatic number of partitionable graphs. We prove that for an infinite family of partitionable graphs G we have $\chi_c(G) = \chi(G)$. Since partitionable graphs G have the property for any vertex v, $\chi(G-v) = \omega(G-v) = \chi(G) - 1$, which implies that $\chi_c(G - v) = \chi(G) - 1$, it follows that for these partitionable graphs G, the deletion of any vertex decreases its circular chromatic number by exactly 1.

Another motivation for the study of the circular chromatic number of partitionable graphs concerns circular perfect graphs. Given positive integers k and d, $k \geq 2d$, the *circular complete graph* $K_{k/d}$ has vertex set $\mathbb{Z}_k$ in which ij is an edge if $d \leq |i-j| \leq k-d$. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y)$ is an edge of H whenever xy is an edge of G. Then a k-coloring of a graph G is equivalent to a homomorphism from G to K_k and a (k,d)-coloring of G is equivalent to a homomorphism from G to $K_{k/d}$. So in the study of circular chromatic number of graphs, the circular complete graphs $K_{k/d}$ play the role of complete graphs as in the study of chromatic number of graphs. The *circular clique number* of a graph G is defined as

$$\omega_c(G) = \sup\{k/d : K_{k/d} \text{ admits a homomorphism to } G\}.$$

It was shown in [13] that $\omega_c(G)$ is equal to the maximum of those k/d for which $K_{k/d}$ is an induced subgraph of G.

A graph G is called *circular perfect* [13] if for every induced subgraph H of G, $\chi_c(H) = \omega_c(H)$. Since $\omega(H) \leq \omega_c(H) \leq \chi_c(H) \leq \chi(H)$, every perfect graph is circular perfect. On the other hand, there are circular perfect graphs which are not perfect. In particular, odd cycles and the complement of odd cycles are circular perfect graphs. So the class of circular perfect graphs is strictly larger than the class of perfect graphs, but it is still a very restrictive class. A sufficient condition for a graph to be circular perfect is given in [13], and this sufficient condition is used to prove an analogue of Hajós Theorem for circular chromatic number. We call a graph G *minimal circular imperfect*, if G is not circular perfect but every

proper induced subgraph of G is circular perfect. As an analogue to the study of perfect graphs, it would be nice to have an appealing conjecture on the structure of minimal circular imperfect graphs. Since all the minimal imperfect graphs are circular perfect, we need to search a larger area for minimal circular imperfect graphs. The class of partitionable graphs is a natural candidate. By studying the circular chromatic number of a subclass of circular partitionable graphs, we prove that all the partitionable graphs in this subclass are circular imperfect. The question whether they are minimal circular imperfect remains an open question.

2 Circular partitionable graphs and the main result

Suppose $p, q \geq 2$ are integers. A graph G is a (p, q)-*partitionable graph* if $|V(G)| = pq+1$, and for each vertex v of G, $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p. A graph is *partitionable* if it is a (p, q)-partitionable graph for some $p, q \geq 2$. Partitionable graphs were introduced by Lovász [9] and Padberg [10] as a tool in the study of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, we have $\chi(H) = \omega(H)$. Here $\omega(H)$ is the *clique number* of G, which is the cardinality of a maximum clique of G. A graph G is *minimal imperfect* if G is not perfect, but every proper induced subgraph of G is perfect. The Strong Perfect Graph Theorem, which was conjectured by Berge [2] in 1961, proved by Chudnovsky, Robertson, Seymour and Thomas [5] in 2002, says that odd cycles of length at least 5 and their complements are the only minimal imperfect graphs. Before the proof of Berge's conjecture, it was shown by Lovász [9] and Padberg [10] that every minimal imperfect graph is a partitionable graph. Thus to prove Berge's conjecture, it suffices to show that none of the partitionable graph is a counterexample. Although the final proof of Berge's conjecture given by Chudnovsky, Robertson, Seymour and Thomas [5] takes a different route, the class of partitionable graphs has been studied thoroughly in the literature, and this turns out to be an interesting class of graphs. The understanding of the structure of this class of graphs may be helpful in the study of other graph theory problems. It is known (cf. [3]) that every (p, q)-partitionable graph has the following properties:

(1) p is the maximum cardinality of a stable set of G, and q is the maximum cardinality of a clique of G;

(2) G has exactly n stable sets of cardinality p and exactly n cliques of cardinality q, where n is the number of vertices of G;

(3) For each maximum clique C of G, there is a unique maximum stable set S such that $C \cap S = \emptyset$; and similarly, for each maximum stable set S there is a unique maximum clique C such that $C \cap S = \emptyset$;

(4) Each vertex belongs to exactly q maximum cliques, and belongs to p maximum stable sets.

In the study of partitionable graphs, some recursive constructions of sub-families of partitionable graphs are discussed in the literature [6,4]. The class of circular partitionable graphs was introduced by Chvátal, Graham, Perold and Whitesides [6].

For two sets of integers X, Y, let $X + Y$ denote the set $\{x + y : x \in X,\ y \in Y\}$. If $X = \{x\}$ is a singleton, we write $x + Y$ instead of $\{x\} + Y$.

Suppose $m_i \geq 2$ ($i = 1, 2, \cdots, 2r$) are integers. Define integers μ_i (for $i = 0, 1, \cdots, 2r$), sets M_i (for $i = 1, 2, \cdots, 2r$), and sets C, S as follows:

$$\begin{aligned}
\mu_i &= m_1 m_2 \cdots m_i \ (\mu_0 = 1),\\
M_i &= \{0, \mu_{i-1}, 2\mu_{i-1}, \cdots, (m_i - 1)\mu_{i-1}\},\\
C &= M_1 + M_3 + \cdots + M_{2r-1},\\
S &= M_2 + M_4 + \cdots + M_{2r}.
\end{aligned}$$

Let $n = m_1 m_2 \cdots m_{2r} + 1$. We denote by $C[m_1, m_2, \cdots, m_{2r}]$ the circulant graph with vertex set $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$, where xy is an edge if and only if $x \neq y$ and $(x - y)$ modulo n is equal to the difference of two elements of C.

Note that $\mu_i > \sum_{j=1}^{i-1} \max M_j$. This implies that $|C| = m_1 m_3 \cdots m_{2r-1}$ and $|S| = m_2 m_4 \cdots m_{2r}$. Let $\omega = |C|$ and $\alpha = |S|$. Then $n = \omega\alpha + 1$. Suppose X is a subset of $\mathbb{Z}_n$. A *circular shift* of X is a set of the form $i + X = \{i + x \pmod n : x \in X\}$.

Theorem 1 *[6] Suppose $m_i \geq 2$ are integers for $i = 1, 2, \cdots, 2r$. Then $G = C[m_1, m_2, \cdots, m_{2r}]$ is an (α, ω)-partitionable graph. Moreover, the n maximum cliques of G are the n circular shifts of C, and the n maximum stable sets of G are the n circular shifts of S.*

As an example, we consider the graph $C[2, 2, 2, 2]$. Then

$$\begin{aligned}
\mu_i &= 2^i, \ \ i = 0, 1, 2, 3, 4;\\
M_1 &= \{0, 1\},\ M_2 = \{0, 2\},\ M_3 = \{0, 4\},\ M_4 = \{0, 8\};\\
C &= \{0, 1, 4, 5\};\\
S &= \{0, 2, 8, 10\}.
\end{aligned}$$

The vertex set of $C[2, 2, 2, 2]$ is $\mathbb{Z}_{17}$, and ij is an edge if $|i - j| \in \{1, 3, 4, 5\}$. The graph is depicted in Figure 2.

The following is the main result of this paper.

Theorem 2 *Let $x_1, \ldots, x_p$ ($p \geq 2$) be integers such that $x_i \geq 2$ for every $1 \leq i \leq$*

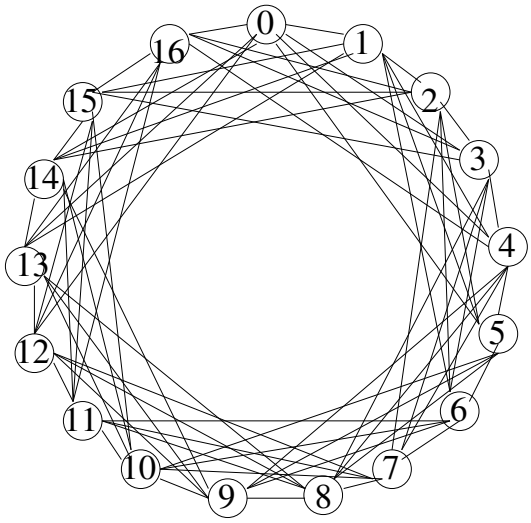

Fig. 1. The circular partitionable graph $C[2,2,2,2]$

p. Let $\delta = \max x_i$ *and let* $G = C[x_1, 2, \ldots, x_p, 2]$. *If* $p = 2$ *or* $x_1 x_2 \ldots x_p \geq 2^{p+1}\delta$ *then* $\chi_c(G) = \chi(G)$.

The proof of Theorem 2 is left to Section 4.

Observe that for $G = C[x_1, 2, \ldots, x_p, 2]$, $\alpha = 2^p$, $\omega = x_1 x_2 \cdots x_p$ and $|V(G)| = n = 2^p x_1 x_2 \ldots x_p + 1$. Therefore

$$\chi_f(G) = \frac{n}{\alpha} = \omega + \frac{1}{2^p} < \chi_c(G) = \chi(G) = \omega + 1.$$

There are a few papers devoted to the study of the circular chromatic number of circulant graphs [7,8]. It is known [12] that for any graph G, $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$. A graph G is called *star extremal* if $\chi_f(G) = \chi_c(G)$. In [7,8], classes of star extremal circulant graphs are investigated. Theorem 2 provides a class of circulant graphs of another kind of extremality, i.e., circulant graphs with $\chi_c(G) = \chi(G)$.

Corollary 3 *Suppose* $G = C[x_1, 2, \ldots, x_p, 2]$ *satisfies the condition of Theorem 2. Then* G *is circular imperfect.*

PROOF. It is known [13] that $\omega(G) \leq \omega_c(G) < \omega(G) + 1$. As $\chi_c(G) = \chi(G) = \omega(G) + 1$, it follows that $\chi_c(G) > \omega_c(G)$. $\square$

It is unknown if any of the graphs $G = C[x_1, 2, \ldots, x_p, 2]$ are minimal circular imperfect. A computer search shows that $C[2,2,2,2]$ is not minimal circular imperfect. The subgraph of $C[2,2,2,2]$ induced by the set $X = \{2, 3, 4, 5, 6, 9, 11, 12, 13, 14, 16\}$ has circular chromatic number 4 and circular clique number 3, and hence is circular imperfect.

Since the graphs $G = C[x_1, 2, x_2, 2, \cdots, x_p, 2]$ are partitionable, $\chi(G - x) = \omega(G - x) = \chi_c(G - x) = \chi(G) - 1$ for each vertex x. Therefore we have the following corollary.

Corollary 4 *Suppose $G = C[x_1, 2, \ldots, x_p, 2]$ satisfies the condition of Theorem 2. Then $\chi_c(G - x) = \chi_c(G) - 1$ for each vertex x of G.*

In other words, the circulant graphs satisfying the condition of Theorem 2 have the property that the deletion of each vertex decreases its circular chromatic number by exactly 1. Not many such graphs were known before, and the problem of characterizing and constructing such graphs was raised in [12].

3 Structural properties of G

In the remainder of this paper, $G = C[x_1, 2, x_2, \ldots, x_p, 2]$. Let $\omega = x_1 \ldots x_p$, $\alpha = 2^p$, $n = \alpha\omega + 1$.

In this section, we shall be interested in elements of $\mathbb{Z}_n$ only. If $a \equiv b \pmod{n}$, then a, b are treated as the same. However, by an abuse of notation, we also use the natural order of integers in the following sense: If $a < b$, we denote $[a, b]$ the set of integers $a \leq x \leq b$. Note that it is possible that $a, b \notin \{0, 1, \cdots, n-1\}$. However, $[a, b]$ always denote a subset of $\{0, 1, \cdots, n-1\}$, by means of taking modulo n. For example, $[-1, 1] = \{-1, 0, 1\} = \{n-1, 0, 1\}$. For a set A of integers, let $aA = \{ax : x \in A\}$, $a + A = \{a + x : x \in A\}$. Again the multiplications and additions are modulo n.

Let $c_0 = 1$, and for $i = 1, \ldots, p$, let

$$c_i = 2^{i-1} x_1 x_2 \ldots x_i, \text{ and } d_i = \sum_{j=1}^{i} c_j.$$

First we derive an explicit expression for the maximum cliques and stable sets of G. It follows from the definition that $\mu_0 = 1$ and for $1 \leq i \leq p$,

$$\mu_{2i-1} = 2^{i-1} x_1 x_2 \ldots x_i = c_i, \text{ and } \mu_{2i} = 2^i x_1 x_2 \ldots x_i = 2c_i.$$

Therefore for $1 \leq i \leq p$,

$$M_{2i-1} = 2c_{i-1}[0, x_i - 1], \text{ and } M_{2i} = \{0, c_i\}.$$

So

$$C = \sum_{j=1}^{p} M_{2j-1} = \sum_{j=1}^{p} 2c_{j-1}[0, x_j - 1],$$
$$S = \sum_{j=1}^{p} M_{2j} = \sum_{j=1}^{p} \{0, c_j\}.$$

By Theorem 1, the maximum cliques of G are the n circular shifts $C + i$ of C ($i \in \mathbb{Z}_n$), and the maximum stable sets of G are the n circular shifts $S + i$ of S ($i \in \mathbb{Z}_n$).

In the following, we consider the intersection $S \cap (S + i)$ of two maximum stable sets of G.

Lemma 5 *For $2 \leq i \leq p$, $c_i \notin [-2d_{i-1}, 2d_{i-1}]$.*

PROOF. Since $c_2 = 2x_1x_2 > 2x_1 = 2d_1$, and $n - c_2 \geq 4x_1x_2 + 1 - 2x_1x_2 > 2x_1x_2 > 2d_1$, we conclude that $c_2 \notin [-2d_1, 2d_1]$. Assume $i \geq 3$ and $c_{i-1} \notin [-2d_{i-2}, 2d_{i-2}]$. Since $c_i = 2x_ic_{i-1}$ and $x_i \geq 2$, we have

$$c_i \geq 2c_{i-1} + 2c_{i-1} > 2(2d_{i-2}) + 2c_{i-1} > 2d_{i-1}.$$

Furthermore,

$$n - c_i = 2^p x_1x_2 \cdots x_p + 1 - c_i > 2c_i - c_i = c_i > 2d_{i-1}.$$

Therefore $c_i \notin [-2d_{i-1}, 2d_{i-1}]$. $\square$

For $i = 1, 2, \cdots, p$, let

$$S_i = \sum_{j=1}^{i} M_{2j} = \sum_{j=1}^{i} \{0, c_j\}.$$

Lemma 6 *For every x in $\mathbb{Z}_n$ and for every $2 \leq i \leq p$,*

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x))$$
or
$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)).$$

PROOF. By definition, $S_i = S_{i-1} \cup (S_{i-1} + c_i)$. Hence $S_i + x = (S_{i-1} + x) \cup (S_{i-1} + c_i + x)$. Therefore

$$\begin{aligned} S_i \cap (S_i + x) = &(S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)) \\ &\cup (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)). \end{aligned}$$

If $S_{i-1} \cap (S_{i-1}+x) \neq \emptyset$, then $x \in [-d_{i-1}, d_{i-1}]$. By Lemma 5, $c_i \notin [-2d_{i-1}, 2d_{i-1}]$. Therefore $c_i + x, c_i - x \notin [-d_{i-1}, d_{i-1}]$. This implies that $S_{i-1} \cap (S_{i-1} + c_i + x) = \emptyset$ and $(S_{i-1} + c_i) \cap (S_{i-1} + x) = \emptyset$. Therefore

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)).$$

If $S_{i-1} \cap (S_{i-1} + x) = \emptyset$, then $(S_{i-1} + c_i) \cap (S_{i-1} + c_i + x) = \emptyset$, and hence

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)).$$

□

For $1 \leq j \leq i \leq p$, let

$$B_{i,j} = \sum_{t \in \{1,2,\cdots,i\} \setminus \{j\}} M_{2t} = \sum_{t \in \{1,2,\cdots,i\} \setminus \{j\}} \{0, c_t\}.$$

Then $S_i = B_{i,j} + \{0, c_j\} = B_{i,j} \cup (B_{i,j} + c_j)$. For convenience, let $B_j = B_{p,j}$.

Lemma 7 *For every $1 \leq i \leq p$ and for every $x \in \mathbb{Z}_n,\ x \neq 0$, we have $|S_i \cap (S_i + x)| \leq 2^{i-1}$. Moreover, if $|S_i \cap (S_i + x)| = 2^{i-1}$, then there is a unique index $j \leq i$ such that $x = \pm c_j$ and $S_i \cap (S_i + x) = B_{i,j}$ or $B_{i,j} + c_j$, depending on $x = -c_j$ or $x = c_j$. In particular, for any $x \in \mathbb{Z}_n$, $|S \cap (S + x)| \leq 2^{p-1}$, and if equality holds then there is a unique index j such that $x = \pm c_j$ and $S \cap (S + x) = B_j$ or $B_j + c_j$, depending on whether $x = -c_j$ or $x = c_j$.*

PROOF. We prove this lemma by induction on i. It is obvious that for every $x \in \mathbb{Z}_n,\ x \neq 0$, we have $|S_1 \cap (S_1 + x)| \leq 1$, and equality holds only if $x = \pm c_1$. Let $i \geq 2$ and suppose the lemma is true for $i' \leq i - 1$.

By Lemma 6,

$$\begin{aligned} S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)) \\ &\text{or} \\ S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)) \end{aligned}$$

First we consider the case that

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)).$$

By induction hypothesis, $|S_{i-1} \cap (S_{i-1} + x)| \leq 2^{i-1}$ and $|((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x))| \leq 2^{i-1}$. Therefore

$$|S_i \cap (S_i + x)| \leq 2^{i-1} + 2^{i-1} = 2^i.$$

Moreover, if $|S_i \cap (S_i + x)| = 2^i$, then $|S_{i-1} \cap (S_{i-1} + x)| = 2^{i-1}$ and hence there is a unique index $j \leq i-1$ such that $x = \pm c_j$ and $S_{i-1} \cap (S_{i-1} + x) = B_{i-1,j}$ or $B_{i-1,j} + c_j$. Then $S_i \cap (S_i + x) = B_{i-1,j} + \{0, c_i\} = B_{i,j}$ or $S_i \cap (S_i + x) = B_{i-1,j} + c_j + \{0, c_i\} = B_{i,j} + c_j$

Next we assume that

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)).$$

If $x \neq \pm c_i$ then the same argument as in the previous case works. Assume $x = c_i$. Then

$$|S_i \cap (S_i + x)| = |S_{i-1} + c_i| + |S_{i-1} \cap (S_{i-1} + 2c_i)|.$$

Note that if $i \leq p-1$, then $2c_i \in C$. If $i = p$, then $2c_p = -1$. In any case, $2c_i$ is equal to the difference of two integers of C. Therefore $\{0, 2c_i\}$ is an edge of G. Hence $2c_i \notin S_{i-1} - S_{i-1}$ (as S_{i-1} is a stable set of G). This implies that

$$S_{i-1} \cap (S_{i-1} + 2c_i) = \emptyset.$$

Hence

$$|S_i \cap (S_i + x)| = |S_{i-1}| = 2^{i-1},$$

and

$$S_i \cap (S_i + x) = B_{i,i} + c_i$$

as $B_{i,i} \cap (B_{i,i} + 2c_i) = \emptyset$ ($B_{i,i}$ is a stable set and $\{0, 2c_i\}$ is an edge).

If $x = -c_i$, then the same argument shows that

$$S_i \cap (S_i + x) = B_{i,i}.$$

□

4 Proof of Theorem 2

Assume that $\chi_c(G) = k/d$, where $\gcd(k, d) = 1$. Let f be a (k, d)-coloring of G, which is viewed as a homomorphism from G to $K_{k/d}$. For $i \in \mathbb{Z}_k$, let $X_i = f^{-1}(i)$ be the set of vertices of G of color i. Let $Y_i = X_i \cup X_{i+1} \cup \cdots \cup X_{i+d-1}$. Then Y_i is a stable set. It is known (see Lemma 1.3 of [12]) that for each i, $X_i \neq \emptyset$ and moreover, for each i, there is a vertex $x \in X_i$ and a vertex $y \in X_{i+d}$ such that xy is an edge of G. We need to prove that $d = 1$.

Lemma 8 *If $\chi_c(G) = k/d$ and $\gcd(k, d) = 1$, then $d \leq 2$.*

PROOF. Assume to the contrary that $d \geq 3$. We consider two cases.

- **Case 1:** $p = 2$

  Note that in this case $\alpha = 4$.

  Assume that there is an index i for which $|X_i| \geq 2$. Since $d \geq 3$, $X_i \cup X_{i+1} \cup X_{i+2} \subseteq Y_i$ and $X_{i-1} \cup X_i \cup X_{i+1} \subseteq Y_{i-1}$. However, $|Y_{i-1}|, |Y_i| \leq \alpha = 4$, and $|X_{i-1} \cup X_i \cup X_{i+1}|, |X_i \cup X_{i+1} \cup X_{i+2}| \geq 4$ (as $|X_i| \geq 2$, and each $|X_j| \geq 1$). Therefore $|Y_{i-1}| = |Y_i| = 4$, and hence Y_{i-1}, Y_i are maximum stable sets of G. However, $|Y_{i-1} \cap Y_i| = |X_i| + |X_{i+1}| \geq 3 > \alpha/2$, contrary to Lemma 7.

  Hence, for every $i \in \mathbb{Z}_k$, we have $|X_i| = 1$. In particular, $n = k = 4\omega + 1$. From $\frac{k}{d} \leq \chi = \omega + 1$ and $\omega > 2$, we get $d \geq 4$. Since Y_0 is a stable set of size d, we have $d = 4$. Thus Y_0 and Y_1 are two maximum stable sets of G sharing $d - 1$ vertices, contrary again to Lemma 7.

  This completes the proof of Case 1.

- **Case 2:** $\omega \geq 2^{p+1}\delta$

  If there exists $i \in \mathbb{Z}_k$ such that $|Y_i| + |Y_{i+1}| + |Y_{i+2}| = 3\alpha$ then $|Y_i| = |Y_{i+1}| = |Y_{i+2}| = \alpha$. By Claim 7, we have $|Y_i \cap Y_{i+1}| \leq \frac{\alpha}{2}$. Hence $|Y_{i+1} \setminus Y_i| \geq \frac{\alpha}{2}$. As $Y_{i+1} \setminus Y_i = X_{i+d}$, we get $|X_{i+d}| \geq \frac{\alpha}{2}$. As $d \geq 3$ and $X_{i+2} \neq \emptyset$, we conclude that $|X_{i+2} \cup X_{i+d}| > \frac{\alpha}{2}$. However $X_{i+2} \cup X_{i+d} \subseteq Y_{i+1} \cap Y_{i+2}$. So Y_{i+1} and Y_{i+2} are two distinct maximum stable sets with $|Y_{i+1} \cap Y_{i+2}| > \frac{\alpha}{2}$, contrary to Lemma 7.

  Thus for every $i \in \mathbb{Z}_k$, we have $|Y_i| + |Y_{i+1}| + |Y_{i+2}| < 3\alpha$. Since $\sum_{i \in \mathbb{Z}_k} |Y_i| = dn$, we have $\sum_{i \in \mathbb{Z}_k} (|Y_i| + |Y_{i+1}| + |Y_{i+2}|) = 3dn \leq (3\alpha - 1)k$.

  As $n = \alpha\omega + 1$ and $\frac{k}{d} < \omega + 1$, we have

$$3n = 3(\alpha\omega + 1) < (3\alpha - 1)(\omega + 1).$$

  It follows that $\omega < 3\alpha - 4$. Hence

$$2^{p+1}\delta \leq \omega < 3\alpha - 4,$$

  which is a contradiction, as $\alpha = 2^p$ and $\delta \geq 2$.

  □

By Lemma 8, we have $d = 1$ or 2. If $d = 1$, we are done. Thus we assume $d = 2$, and we shall derive a contradiction. As $\omega < \chi_c(G) \leq \chi(G) = \omega + 1$ and $d = 2$, we have $k = 2\omega + 1$.

Lemma 9 *There exists an $i \in \mathbb{Z}_k$ such that $Y_i, Y_{i+1}, \ldots, Y_{i+2\delta}$ are all of size α.*

PROOF.

- **Case 1:** $p = 2$

  If there exists $i \in \mathbb{Z}_k$ such that $|X_i| \geq 3$ then Y_i and Y_{i-1} are two maximum stable sets sharing $|X_i| > \alpha/2$ vertices, contrary to Lemma 7. Thus, for every $i \in \mathbb{Z}_k$, we have $|X_i| \leq 2$. Since $\sum_{i \in \mathbb{Z}_k} |X_i| = n = 4x_1x_2 + 1 = 2k - 1$, it follows that

there exists a unique $j \in \mathbb{Z}_k$, such that $|X_j| = 1$. Thus $Y_{j+1}, Y_{j+2}, \ldots, Y_{j+n-2}$ are $n-2$ stable sets of size $\alpha = 4$. As $n-2 = 4x_1x_2 - 1 > 2\delta + 1$, we are done.

- **Case 2:** $\omega \geq 2^{p+1}\delta$

  Assume to the contrary that for every $i \in \mathbb{Z}_k$, there exists one stable set in $Y_i, Y_{i+1}, \ldots, Y_{i+2\delta}$ of size strictly less than α, then

$$\sum_{i=0}^{k-1}(|Y_i| + |Y_{i+1}| + \cdots + |Y_{i+2\delta}|) \leq ((2\delta+1)\alpha - 1)k.$$

  On the other hand, as $\sum_{i=0}^{k-1} |Y_i| = 2n$, we have

$$\sum_{i=0}^{k}(|Y_i| + |Y_{i+1}| + \cdots + |Y_{i+2\delta}|) = (2\delta+1)2n.$$

  Therefore

$$(2\delta+1)2n \leq ((2\delta+1)\alpha - 1)k.$$

  Since $k = 2\omega + 1$ and $n = \omega\alpha + 1$, straightforward calculation shows that

$$4\delta + 2 \leq (2\delta+1)\alpha - 2\omega - 1.$$

  As $\omega \geq 2^{p+1}\delta$ and $\alpha = 2^p$, easy calculation derives a contradiction.

  □

In the remainder of this section, let i be an index such that $Y_i, Y_{i+1}, \ldots, Y_{i+2\delta}$ are all of size α.

Lemma 10 *For every $j = i+1, i+2, \ldots, i+2\delta$, we have $|X_j| = \frac{\alpha}{2}$.*

PROOF. By Lemma 7, $|X_j| = |Y_{j-1} \cap Y_j| \leq \frac{\alpha}{2}$. As $\alpha = |Y_j| = |X_j| + |X_{j+1}|$, it follows that $|X_j| = \alpha - |X_{j+1}| \geq \frac{\alpha}{2}$ if $i+1 \leq j \leq i+2\delta-1$. Hence $|X_j| = \frac{\alpha}{2}$. If $j = i+2\delta$, then since $\alpha = |Y_{j-1}| = |X_{j-1}| + |X_j|$, we also have $|X_j| = \frac{\alpha}{2}$. □

Lemma 11 *There exists an index $t \in \{1, 2, \cdots, p\}$ and $a \in \mathbb{Z}_n$, such that either for all $j = 1, 2, \ldots, 2\delta$,*

$$X_{i+j} = X_i + jc_t = B_t + a + jc_t, \text{ and } Y_{i+j-1} = S + a + (j-1)c_t$$

or for all $j = 1, 2, \ldots, 2\delta$,

$$X_{i+j} = X_i - jc_t = B_t + a - jc_t, \text{ and } Y_{i+j-1} = S + a - (j-1)c_t.$$

PROOF.

Let a be the element in $\mathbb{Z}_n$ such that $Y_i = S + a$.

By definition, for any $1 \le t \le p$, $S = B_t + \{0, c_t\} = B_t \cup (B_t + c_t)$. By Lemma 10, $|Y_i \cap Y_{i+1}| = \frac{\alpha}{2}$. By Lemma 7, $Y_{i+1} = Y_i \pm c_t$ for some $1 \le t \le p$. First we consider the case that $Y_{i+1} = Y_i + c_t$. Since $Y_i = S + a = (B_t + a) \cup (B_t + a + c_t)$, we have

$$X_{i+1} = (Y_i \cap Y_{i+1}) = (S \cap (S + c_t)) + a = B_t + c_t + a.$$

Moreover,

$$X_{i+2} = (Y_{i+1} \setminus Y_i) = ((S + c_t) \setminus S) + a = B_t + 2c_t + a.$$

Assume $3 \le j \le 2\delta$, $X_{i+j-1} = B_t + a + (j-1)c_t$ and $Y_{i+j-2} = S + a + (j-2)c_t$.

Assume $Y_{i+j-1} = S + j'$ for some $j' \in \mathbb{Z}_n$. Since

$$Y_{i+j-2} = S + a + (j-2)c_t$$

and

$$Y_{i+j-1} \cap Y_{i+j-2} = X_{i+j-1} = B_t + a + (j-1)c_t,$$

it follows that

$$S \cap (S + (j' - a - (j-2)c_t)) = B_t + c_t.$$

By Lemma 7, $j' - a - (j-2)c_t = c_t$. Thus we conclude that $Y_{i+j-1} = S + a + (j-1)c_t$ and $X_{i+j} = B_t + a + jc_t$, as $X_{i+j} = Y_{i+j-1} \setminus X_{i+j-1}$.

If $Y_{i+1} = Y_i - c_t$, then the same argument shows that for $j = 1, 2, \cdots, 2\delta$, $X_{i+j} = B_t + a - jc_t$ and $Y_{i+j-1} = S + a - (j-1)c_t$. $\square$

Now we derive the final contradiction. Let t be the index given in Lemma 11. Without loss of generality, we assume that for $j = 1, 2, \cdots, 2\delta$, $X_{i+j} = B_t + a + jc_t$. If $t < p$, then $2x_{t+1}c_t = c_{t+1}$. Then $X_{i+2x_{t+1}} = X_i + 2x_{t+1}c_t = B_t + a + c_{t+1}$. By definition, $c_{t+1} \in B_t$. Thus $a + c_{t+1} \in B_t + a = X_i$. On the other hand, $0 \in B_t$, and hence $a + c_{t+1} \in X_{i+2x_{t+1}}$. This is a contradiction, as $2x_{t+1} \le 2\delta < \omega < k$, which implies that $X_i \cap X_{i+2x_{t+1}} = \emptyset$. If $t = p$, then

$$2x_1c_t = (2^p x_1 x_2 \cdots x_p)x_1 = (n-1)x_1 = -x_1.$$

Then $X_{i+2x_1} = X_i + 2x_1c_t = B_t + a - x_1$. By definition, $x_1 \in B_t$. Thus $a \in B_t + a - x_1 = X_{i+2x_1}$. On the other hand, $0 \in B_t$, and hence $a \in B_t + a = X_i$. This is a contradiction, as $2x_1 < k$ and hence $X_i \cap X_{i+2x_1} = \emptyset$. This completes the proof of Theorem 2.

5 Open question

Theorem 2 gives the circular chromatic number of some circular partitionable graphs such that their stability number is a power of two (these graphs are said to be of type 1 or 2 in [1]).

However, we believe that our result is likely to hold for most of the circular partitionable graphs: e.g., is it true that every graph $C[m_1, m_2, \ldots, m_{2r}]$ with $r \geq 2$ has its circular chromatic number equal to its chromatic number?

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