Almost all webs are not rank-perfect

Arnaud Pecher, Annegret K. Wagler

Arnaud Pêcher[1] · Annegret K. Wagler[2]

Almost all webs are not rank-perfect

the date of receipt and acceptance should be inserted later

Abstract. Graphs with circular symmetry, called webs, are relevant w.r.t. describing the stable set polytopes of two larger graph classes, quasi-line graphs [8,12] and claw-free graphs [7,8]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [9]. Ben Rebea conjectured a description for quasi-line graphs, see [12]; Chudnovsky and Seymour [2] verified this conjecture recently for quasi-line graphs not belonging to the subclass of fuzzy circular interval graphs and showed that rank facets are required in this case only. Fuzzy circular interval graphs contain all webs and even the problem of finding all facets of their stable set polytopes is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [5,16] while there are examples with clique number ≥ 4 having non-rank facets [10–12,14].

In this paper we prove, building on a construction for non-rank facets from [15], that the stable set polytopes of almost all webs with clique number ≥ 5 admit non-rank facets. This adds support to the belief that these graphs are indeed the core of Ben Rebea's conjecture. Finally, we present a conjecture how to construct all facets of the stable set polytopes of webs.

Key words. web – rank-perfect graph – stable set polytope – (non-)rank facet

1. Introduction

Graphs with circular symmetry of their maximum cliques and stable sets are called *webs*: a web W_n^k is a graph with vertices $1, \ldots, n$ where ij is an edge if i and j differ by at most $k \pmod n$ and $i \neq j$. The webs W_9^k on nine vertices are depicted in Figure 1. Notice that webs are also called circulant graphs C_n^k in [4] and that similar graphs $W(n, k)$ were introduced in [16].
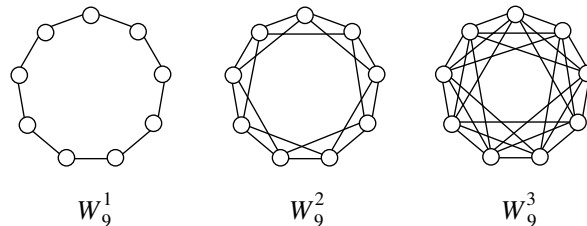
Fig. 1. The webs W_9^k on nine vertices

Webs and line graphs belong to the classes of quasi-line graphs and claw-free graphs and are relevant w.r.t. describing the stable set polytopes of those larger graph classes [7, 8,12]. (The *line graph* of a graph H is obtained by taking the edges of H as nodes and connecting two nodes iff the corresponding edges of H are incident. A graph is *quasi-line*

Arnaud Pêcher: Laboratoire Bordelais de Recherche Informatique (LaBRI)
351 cours de la Libération, 33405 Talence, France, e-mail: `pecher@labri.fr`

Annegret K. Wagler: Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)
Takustr. 7, 14195 Berlin, Germany, e-mail: `wagler@zib.de`

Correspondence to: wagler@zib.de

(resp. *claw-free*) if the neighborhood of any node can be partitioned into two cliques (resp. does not contain any stable set of size 3).)

The *stable set polytope* $\mathrm{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of the graph G. In order to describe $\mathrm{STAB}(G)$ by means of facet-defining inequalities, the "trivial" facets $x_i \geq 0$ for all vertices i of G and the clique constraints

$$\sum_{i \in Q} x_i \leq 1$$

for all cliques $Q \subseteq G$ are necessary. These two types of facets are sufficient to describe $\mathrm{STAB}(G)$ for perfect graphs G only [3]. That are precisely the graphs without odd holes W_{2k+1}^1 and odd antiholes W_{2k+1}^{k-1} as induced subgraphs.

A natural way to generalize clique constraints is to investigate *rank constraints*, that are 0/1-constraints of the form

$$\sum_{i \in G'} x_i \leq \alpha(G')$$

associated with arbitrary induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G' (note $\alpha(G') = 1$ holds iff G' is a clique). A graph is *rank-perfect* if all non-trivial facets of its stable set polytope are rank constraints. The class of rank-perfect graphs contains all perfect graphs [3], odd holes and odd antiholes [13], line graphs [6], and the complements of webs [17].

A characterization of the *rank* facets in stable set polytopes of claw-free graphs was given by Galluccio and Sassano [7]. They showed that all rank facets can be constructed by means of standard operations from rank constraints associated with cliques, certain webs, and special line graphs. Finding *all* facets of their stable set polytopes is a long-standing problem (Grötschel, Lovász and Schrijver [9]), as claw-free graphs are not rank-perfect: Giles and Trotter [8], Oriolo [12], and Liebling et al. [11] found non-rank facets which occur even in the stable set polytopes of quasi-line graphs.

A famous conjecture due to Ben Rebea (see [12]) claims that the stable set polytopes of quasi-line graphs admit only one type of non-trivial facets, so-called clique family inequalities. Let $G = (V, E)$ be a graph, $\mathcal{F}$ be a family of (at least three inclusion-wise) maximal cliques of G, $p \leq |\mathcal{F}|$ be an integer, and define two sets as follows:

$$\begin{aligned} I(\mathcal{F}, p) &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\} \\ O(\mathcal{F}, p) &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p - 1\} \end{aligned}$$

The *clique family inequality* $(\mathcal{F}, p)$ is

$$(p - r) \sum_{i \in I(\mathcal{F}, p)} x_i + (p - r - 1) \sum_{i \in O(\mathcal{F}, p)} x_i \leq (p - r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \tag{1}$$

with $r = |\mathcal{F}| \bmod p$ and $r > 0$.

Oriolo [12] verified Ben Rebea's conjecture for line graphs and webs W_n^2 (note: the latter graphs are rank-perfect due to [5]). Chudnovsky and Seymour introduced recently the class of so-called fuzzy circular interval graphs and verified the conjecture for all quasi-line graphs which are *not* fuzzy circular interval graphs.

Let $\mathcal{C}$ be a circle and $\mathcal{I} = \{I_1, \ldots, I_m\}$ be a collection of intervals $I_k = [l_k, r_k]$ in $\mathcal{C}$ s.t. no interval in $\mathcal{I}$ is properly contained in another one and no two intervals in $\mathcal{I}$ share an endpoint. Moreover, let $V = \{v_1, \ldots, v_n\}$ be a finite multiset of points in $\mathcal{C}$ (i.e. $v_i \in \mathcal{C}$ may occur in V with a multiplicity > 1). The *fuzzy circular interval graph* $G(V, \mathcal{I}) = (V, E_1 \cup E_2)$ has node set V and edge set $E_1 \cup E_2$ where

$$\begin{aligned} E_1 &= \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i, v_j \in I_k \text{ and } \{v_i, v_j\} \neq \{l_k, r_k\}\} \\ E_2 &\subseteq \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i = l_k, v_j = r_k\} \end{aligned}$$

(i.e., different endpoints of one interval are not necessarily joined by an edge).

Chudnovsky and Seymour recently proved that nonnegativity constraints, clique constraints, and rank constraints coming from clique family inequalities $(\mathcal{F}, 2)$ with $|\mathcal{F}|$ odd are the only necessary inequalities to describe stable set polytopes of quasi-line graphs which are not fuzzy circular interval graphs.

Webs are obviously quasi-line graphs as well as fuzzy circular interval graphs; the problem of describing their stable set polytopes is still open. So far, it is only known that webs W_n^1 are as holes perfect or rank-perfect [3,13]; the webs W_n^2 are rank-perfect by Dahl [5]. On the other hand, Kind [10] found (by means of the PORTA software[1]) examples of webs with clique number > 4 which are *not* rank-perfect, e.g., $W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9$. Oriolo [12], Liebling et al. [11], and Pêcher and Wagler [14,15] presented further examples of such webs.

In this paper we prove, with the help of a construction for non-rank facets from [15], that there are only finitely many rank-perfect webs W_n^k for all $k \geq 4$. Together with a result from [14] showing the same for the case $k = 3$ we obtain that, for any $k \geq 3$, almost all webs W_n^k are not rank-perfect. This adds support to the belief that webs as subclass of fuzzy circular interval graphs are the core of Ben Rebea's conjecture.

The paper is organized as follows: the next section describes as our main results the construction of infinite sequences of not rank-perfect webs and discusses consequences. The three following sections are devoted to the proofs of the three main theorems. We close with a conjecture which clique family inequalities give rise to facets in the stable set polytopes of webs.

2. Main results

For proving that almost all webs are not rank-perfect, we make use of a construction for non-rank facets from [15], introduced in the sequel. For that, we need the notion of *proper weak non-rank facets*. A facet $\mathbf{a}^T \mathbf{x} \leq c\alpha(G')$ of $\mathrm{STAB}(G)$ is a *weak rank facet* w.r.t. $G' \subseteq G$, if $a_i = c$ for every vertex i of G' and if G' is rank facet-producing (i.e., $\sum_{i \in V(G')} x_i \leq \alpha(G')$ defines a facet of $\mathrm{STAB}(G')$); any rank facet is a particular weak rank facet (with $a_i = c = 1$ for every $i \in V(G')$ and $a_i = 0$ otherwise). A weak rank facet is *proper* if G' is not a clique and *non-rank* if it cannot be scaled to have 0/1-coefficients only (i.e., it is not a rank-constraint).

Theorem 1. [15] *If* $\mathrm{STAB}(W_n^k)$ *possesses a proper weak non-rank facet then also* $\mathrm{STAB}(W_{n+k+1}^k)$ *has a proper weak non-rank facet.*

Therefore, if $\mathrm{STAB}(W_n^k)$ has a proper weak non-rank facet then all webs $W_{n+l(k+1)}^k$ $(l \geq 0)$ are not rank-perfect, too. Hence Theorem 1 implies the following corollary:

Corollary 1. [15] *If there are $k + 1$ webs $W_{n_0}^k, \ldots, W_{n_k}^k$ such that*

- $\mathrm{STAB}(W_{n_i}^k)$ *has a proper weak non-rank facet*
- $n_i = i \pmod{k+1}$ *for* $0 \leq i \leq k$

then all webs W_n^k with $n \geq \max\{n_0, \ldots, n_k\} - k$ are not rank-perfect.

For $k = 3$, such a set $W_{33}^3, W_{42}^3, W_{51}^3, W_{60}^3$ is presented in [14], as consequence of the following theorem.

Theorem 2. [14] *If $l = 2 \pmod 3$ and $l \geq 11$ then $\mathrm{STAB}(W_{3l}^3)$ has a proper weak non-rank facet.*

Thus, Theorem 2 and Corollary 1 imply that all webs W_n^3 with $n > 56$ are not rank-perfect. The aim of this paper is to provide such a set $W_{n_0}^k, \ldots, W_{n_k}^k$ of not rank-perfect webs for each value $k \geq 4$. For that, we consider special clique family inequalities giving rise to proper weak non-rank facets. A clique family inequality $(\mathcal{Q}, p)$ is *associated* with a proper subweb $W_{n'}^{k'}$ of a web W_n^k if $\mathcal{Q} = \{Q_i : i \in W_{n'}^{k'}\}$ is chosen as clique family and $p = k' + 1$, where $Q_i = \{i, \ldots, i+k\}$ denotes the maximum clique of W_n^k starting in vertex i. Note that the clique number of a web W_n^k is $k + 1$ and the stability number is $\lfloor \frac{n}{k+1} \rfloor$.

Lemma 1. [14] *Let $W_{n'}^{k'} \subset W_n^k$ be a proper induced subweb. The clique family inequality $(\mathcal{Q}, k' + 1)$ associated with $W_{n'}^{k'}$*

$$(k' + 1 - r) \sum_{i \in I(\mathcal{Q}, k'+1)} x_i + (k' - r) \sum_{i \in O(\mathcal{Q}, k'+1)} x_i \leq (k' + 1 - r)\, \alpha(W_{n'}^{k'}) \qquad (2)$$

where $r = n' \bmod (k' + 1)$, $0 < r < k' + 1$, is a valid inequality for STAB(W_n^k), such that $W_{n'}^{k'} \subseteq I(\mathcal{F}, p)$ holds.

For illustration, look at the smallest not rank-perfect web W_{25}^5. Its non-rank facets are clique family inequalities associated with induced subwebs $W_{10}^2 \subseteq W_{25}^5$ (note that the vertex sets 1, 2, 6, 7, 11, 12, 16, 17, 21, 22 and 1, 3, 6, 8, 11, 13, 16, 18, 21, 23 both induce a $W_{10}^2 \subseteq W_{25}^5$, see the black vertices in Figure 2).
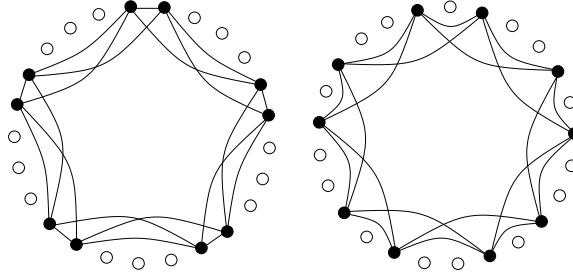

Fig. 2. : The induced subwebs $W_{10}^2 \subseteq W_{25}^5$

Choosing $\mathcal{Q} = \{Q_i : i \in W_{10}^2\}$ yields $p = \omega(W_{10}^2) = 3$ in both cases. All remaining vertices are covered 2 times, hence $O(\mathcal{Q}, p) = W_{25}^5 - W_{10}^2$ follows. The corresponding clique family inequality $(\mathcal{Q}, 3)$ is

$$2 \sum_{i \in W_{10}^2} x_i + 1 \sum_{i \notin W_{10}^2} x_i \leq 2\alpha(W_{10}^2)$$

due to $r = |\mathcal{Q}| \bmod p = 1$ and yields a non-rank facet of STAB(W_{25}^5).

The main results of this paper prove that several clique family inequalities $(\mathcal{Q}, k' + 1)$ associated with different regular subwebs $W_{n'}^{k'}$ induce proper weak non-rank facets (note that $(\mathcal{Q}, k' + 1)$ is a proper weak non-rank constraint if $r < k'$). A subweb $W_{n'}^{k'} \subset W_n^k$ is called $(b_1, w_1, \ldots, b_t, w_t)$-*regular*, if the vertices of $W_{n'}^{k'}$ occur in W_n^k in equal blocks where b_i consecutive vertices from $W_{n'}^{k'}$ alternate with w_i consecutive vertices outside $W_{n'}^{k'}$, for $1 \leq i \leq t$. The two subwebs $W_{10}^2 \subseteq W_{25}^5$ presented in Figure 2 show a (2,3)-regular and a (1,1,1,2)-regular subweb, resp. In Section 3, we show the following:

Theorem 3. *For any $k \geq 5$, consider a $(k', k - k')$-regular subweb $W_{lk'}^{k'} \subset W_{lk}^k$ with $2 \leq k' \leq k - 3$ and odd $l \geq 3$. The clique family inequality*

$$2 \sum_{i \in W_{lk'}^{k'}} x_i + 1 \sum_{i \notin W_{lk'}^{k'}} x_i \leq 2\alpha(W_{lk'}^{k'}) \qquad (3)$$

associated with $W_{lk'}^{k'}$ is a proper weak non-rank facet of $\mathrm{STAB}(W_{lk}^k)$ *if* $l = 2 \,(\mathrm{mod}\ k'+1)$ *and* $\alpha(W_{lk'}^{k'}) < \alpha(W_{lk}^k)$.

As a consequence, we obtain many different infinite sequences of not rank-perfect webs, among them the required base sets for all even values of $k \geq 6$ (but not for the odd values $k \geq 5$ since all webs in the latter sequences have an odd number of vertices). For any even $k \geq 6$, choosing $k' = \frac{k}{2}$ if $k = 0 \,(\mathrm{mod}\ 4)$ and $k' = \frac{k}{2} - 1$ if $k = 2$ $(\mathrm{mod}\ 4)$ and $l = (k' + 3) + (k' + 1)2j$ for $j \geq 1$ in both cases as odd values of l with $l = 2 \,(\mathrm{mod}\ k' + 1)$ satisfies the precondition of Theorem 3. Thus, we obtain the following infinite sequences of not rank-perfect webs:

Theorem 4. *Let $k \geq 6$ be even. Then for every integer $j \geq 1$ holds*

– $\mathrm{STAB}\left(W_{\left(\frac{k+6}{2} + (k+2)j\right)k}^k \right)$ *has a proper weak non-rank facet if $k = 0 \,(\mathrm{mod}\ 4)$;*

– $\mathrm{STAB}\left(W_{\left(\frac{k+4}{2} + kj\right)k}^k \right)$ *has a proper weak non-rank facet if $k = 2 \,(\mathrm{mod}\ 4)$.*

That means for, e.g., $k = 6$ that there is an infinite sequence W_{66}^6, W_{102}^6, W_{138}^6, W_{174}^6, W_{210}^6, W_{246}^6, W_{282}^6, ... of not rank-perfect webs. Corollary 1 implies, therefore, that all webs W_n^6 with $n \geq 276$ are not rank-perfect. More generally, for every $1 \leq j \leq k + 1$, we have

$$\left(\frac{k+6}{2} + (k+2)j \right) k = -\frac{k+6}{2} - j \quad (\mathrm{mod}\ k+1)$$

and

$$\left(\frac{k+4}{2} + kj \right) k = -\frac{k+4}{2} + j \quad (\mathrm{mod}\ k+1)$$

thus, the sequences contain the required base sets. Furthermore, if $k \geq 6$ then

$$\left(\frac{k+6}{2} + (k+2)(k+1) \right) k < 2(k+1)^3$$

follows and Theorem 4 and Corollary 1 imply together:

Corollary 2. *For any even $k \geq 6$, all webs W_n^k with $n \geq 2(k+1)^3$ are not rank-perfect.*

It remains to construct the required base sets for $k = 4$ and all odd values of $k \geq 5$. The case $k = 4$ is treated in Section 4 by constructing sequences of clique family inequalities associated with regular subwebs $W_l^2 \subset W_{2l}^4$:

Theorem 5. *The clique family inequality$(\mathcal{Q}, 3)$*

$$2 \sum_{i \in W_l^2} x_i + 1 \sum_{i \notin W_l^2} x_i \leq 2\,\alpha(W_l^2) \tag{4}$$

associated with a (1,1)-regular subweb $W_l^2 \subset W_{2l}^4$ is a proper weak non-rank facet of $\mathrm{STAB}(W_{2l}^4)$ *if $l = 1 \,(\mathrm{mod}\ 3)$ and $l \geq 13$.*

Due to Theorem 5, the stable set polytopes of the webs W_{26}^4, W_{32}^4, W_{38}^4, W_{44}^4, and W_{50}^4 have a proper weak non-rank facet. Hence Corollary 1 implies that all webs W_n^4 with $n > 45$ are not rank-perfect.

For each odd $k \geq 5$, we extend the result for $k = 3$ from [14] by considering the clique family inequality associated with the $(k - 1, 1)$-regular subweb $W_{l(k-1)}^{k-1} \subset W_{lk}^k$:

Theorem 6. *The clique family inequality* $(\mathcal{Q}, k)$

$$2 \sum_{i \in W^{k-1}_{l(k-1)}} x_i + 1 \sum_{i \notin W^{k-1}_{l(k-1)}} x_i \leq 2\,\alpha(W^{k-1}_{l(k-1)}) \tag{5}$$

associated with a $(k-1, 1)$-*regular subweb* $W^{k-1}_{l(k-1)}$ *is a proper weak non-rank facet of* $STAB(W^k_{lk})$ *for any odd* $k \geq 5$ *if* $l = 3k + 2$.

The sequence of the $k+1$ webs $W^k_{l'k+2}$ with $3 \leq k' \leq 3 + k$ webs is the required base set for any odd $k \geq 5$, as

$$k(l'k + 2) \bmod (k+1) = (l' - 2) \bmod (k+1)$$

Thus, Theorem 6 and Corollary 1 imply together:

Corollary 3. W^k_n *with* $n \geq ((k+3)k + 1)k$ *is not rank-perfect for any odd* $k \geq 5$.

In summary, all the above results show:

Corollary 4. *A web* W^k_n *is not rank-perfect if*

- *$k = 3$ and $n \geq 57$,*
- *$k = 4$ and $n \geq 46$,*
- *$k \geq 5$ is odd and $n \geq ((3+k)k + 1)k$,*
- *$k \geq 6$ is even and $n \geq 2(k+1)^3$.*

Thus, for any $k \geq 3$ there are only finitely many rank-perfect webs W^k_n implying:

Corollary 5. *Almost all webs with given clique size at least 4 are not rank-perfect.*

The following three sections contain the proofs of the main results Theorem 3, Theorem 5, and Theorem 6.

3. Proof of Theorem 3

For any $k \geq 5$, let $W^{k'}_{lk'}$ be a $(k', k - k')$-regular subweb of W^k_{lk} with $2 \leq k' \leq k - 3$ and odd $l \geq 3$. By assumption, we have $l = 2 \pmod{k' + 1}$ and $\alpha(W^{k'}_{lk'}) < \alpha(W^k_{lk})$.

In order to prove Theorem 3, we have to establish that the inequality (3)

$$2 \sum_{i \in W^{k'}_{lk'}} x_i + 1 \sum_{i \notin W^{k'}_{lk'}} x_i \leq 2\alpha(W^{k'}_{lk'})$$

is valid and facet-inducing for $STAB(W^k_{lk})$.

Validity follows from Lemma 1: since $l = 2 \pmod{k' + 1}$, we have $lk' = -2 \pmod{k' + 1}$ and therefore the remainder r of the division of lk' by $k' + 1$ is equal to $k' - 1$. Therefore the valid inequality (2) associated with the subweb $W^{k'}_{lk'}$ is

$$2 \sum_{i \in I(\mathcal{Q}, k'+1)} x_i + \sum_{i \in O(\mathcal{Q}, k'+1)} x_i \leq 2\,\alpha(W^{k'}_{lk'}) \tag{6}$$

where $W^{k'}_{lk'} \subseteq I(\mathcal{F}, p)$ holds. Therefore, inequality (3) is a valid inequality.

To prove that inequality (3) is facet-inducing, we may define the set of vertices V' of the $(k', k - k')$-regular subweb $W^{k'}_{lk'}$ w.l.o.g. as

$$V' = \bigcup_{0 \leq j < l} \{k \cdot j + 1, k \cdot j + 2, \ldots, k \cdot j + k'\}$$

(where $l \geq 5$ and $l = 2 \pmod{k'+1}$)

For convenience, we call the vertices in V' *black vertices* and all remaining vertices *white vertices*. A *black set* is a set of black vertices and likewise a *white set* is a set of white vertices.

The following lemma from [15] is essential for the proof. It provides a characterization when a valid inequality $\mathbf{a}^T\mathbf{x} \leq b$ is a facet of the stable set polytope of a general graph G. For that we need the following notions. A *root* of $\mathbf{a}^T\mathbf{x} \leq b$ is any stable set of G satisfying the inequality at equality. A pair i, j of vertices is $\mathbf{a}$-*critical* in G if there are two roots S_1 and S_2 of $\mathbf{a}^T\mathbf{x} \leq b$ such that $\{i\} = S_1 \setminus S_2$ and $\{j\} = S_2 \setminus S_1$. A subset V' of $V(G)$ is $\mathbf{a}$-*connected* if the graph with vertex set V' and edge set $\{ij|\ i, j \in V',\ ij\ \mathbf{a}\text{-critical in } G\}$ is connected.

All matrices in this paper have rational entries (in fact integer entries). If M is any square matrix, then $|M|$ stands for the determinant of M.

Lemma 2. [15] *Let $\mathbf{a}^T\mathbf{x} \leq b$ be a valid inequality for $\mathrm{STAB}(G)$ with $b \neq 0$. Consider a partition $V_1, \ldots, V_p$ of $V(G)$ such that V_i is $\mathbf{a}$-connected for every $1 \leq i \leq p$. The inequality $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining if and only if there are p roots $S_1, \ldots, S_p$ with*

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0$$

If the involved inequality $\mathbf{a}^T\mathbf{x} \leq b$ is the full rank-constraint $\mathbf{1}^T\mathbf{x} \leq \alpha$, we use the terms α-critical and α-connected instead of $\mathbf{a}$-critical and $\mathbf{a}$-connected respectively.

Notice that Chvátal [3] called α-critical edges simply "critical" and that Lemma 2 generalizes the well-known result of Chvátal [3] that a graph G is rank facet-producing if the set of its critical edges induces a conncted subgraph of G.

We now proceed to the proof that inequality (3) is facet-inducing,

Claim. The black set V' is $\mathbf{a}$-connected w.r.t. the valid inequality (3).

Proof.

If $lk' = 0 \pmod{k'+1}$ then $-l = 0 \pmod{k'+1}$ and therefore $l = 0 \pmod{k'+1}$, in contradiction with $l = 2 \pmod{k'+1}$, as $k' \geq 2$. Hence $k'+1$ is not a divisor of lk'. Hence we have $lk' = \alpha(G[V'])(k'+1)+r$ with $1 \leq r \leq k'$. Let $S_1 = \{1, 2+(k'+1), 2+2(k'+1), \ldots, 2+(\alpha(G[V'])-1)(k'+1)\}$ and $S_2 = \{2, 2+(k'+1), 2+2(k'+1), \ldots, 2+(\alpha(G[V'])-1)(k'+1)\}$. Since $2+(\alpha(G[V'])-1)(k'+1) = 2+(lk'-r)-(k'+1) \leq lk'-k'$, S_1 and S_2 are both maximum stable sets of $G[V']$. Hence, the edge $\{1, 2\}$ of $G[V']$ is α-critical. By circular symmetry of $G[V']$, this implies that $G[V']$ is α-connected. Since $\alpha(G[V']) = \alpha'$, this implies that V' is $\mathbf{a}$-connected. $\square$

Claim. We have $lk' > (\alpha'-2)(k'+1)+3k'$.

Proof. Since $l = 2 \pmod{k'+1}$, we have $lk' = k'-1 \pmod{k'+1}$. Hence $lk' - \alpha'(k'+1) = k'-1$. It follows that $lk' - \alpha'(k'+1) > 3k'-2(k'+1)$. Thus $lk' > (\alpha'-2)(k'+1)+3k'$. $\square$

Claim. We obtain $\alpha(W_{lk'}^{k'} \setminus [1, 3k']) \geq \alpha(W_{lk'}^{k'})-1$.

Proof. By the previous Claim, the set $S' := \{3k'+1, 3k'+(k'+1)+1, \ldots, 3k'+(\alpha'-2)(k'+1)+1\}$ is a stable set of size $\alpha'-1$ of $W_{lk'}^{k'} \setminus [1, 3k']$ and the result follows. $\square$

Claim. For every $0 \leq i < l$, the white set $V_i := ik + \{k'+1, \ldots, k\}$ is $\mathbf{a}$-connected.

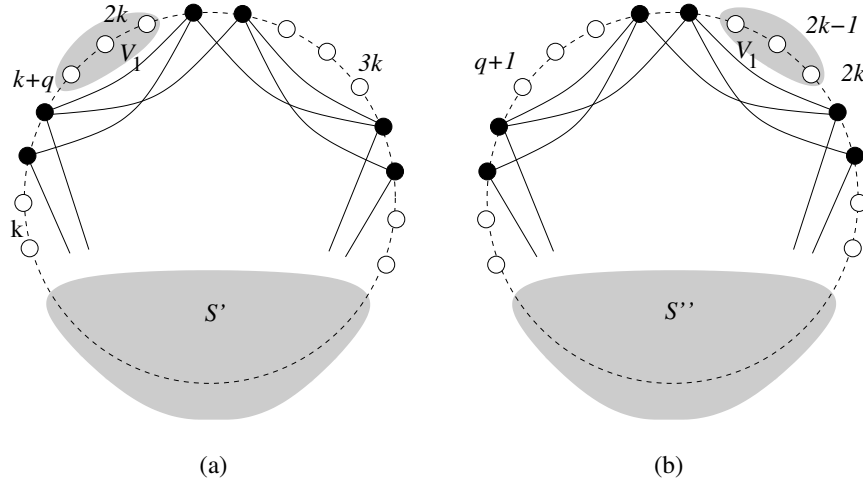(a)   (b)

Fig. 3. : The roots for the proof of the fourth Claim with $k = 5$ and $k' = 2$: (a) the roots $S_j = S' \cup \{3k, j\}$ (b) the stable sets $S_1 = S'' \cup \{k' + 1, 2k - 1\}$ and $S_2 = S'' \cup \{k' + 1, 2k\}$

Proof. We are going to prove that V_1 is **a**-connected. By the previous Claim, there is a black stable set S' of size $\alpha' - 1$ in $G \setminus [k + 1, 4k]$. For every $k + k' + 1 \leq j \leq 2k - 1$, the set $S_j := S' \cup \{3k, j\}$ is obviously a root of (3), hence the edges $\{k' + k + 1, k' + k + 2\}, \ldots, \{2k - 2, 2k - 1\}$, are **a**-critical (see Fig. 3(a)).

It remains to show that the edge $\{2k - 1, 2k\}$ is **a**-critical. By the previous Claim again, there exists a black stable set S'' of size $\alpha' - 1$ in $G \setminus [k' - k + 1, 3k]$. The set $S_1 := S'' \cup \{k' + 1\} \cup \{2k - 1\}$ is a root as $k' + 1 + k < 2k - 1$ (since $k' \leq k - 3$). The set $S_2 := S'' \cup \{k' + 1\} \cup \{2k\}$ is also a root (see Fig. 3(b)). Hence $\{2k - 1, 2k\}$ is **a**-critical and, therefore, V_1 is **a**-connected.

Likewise, the sets $V_0, V_2 \ldots, V_{l-1}$ are **a**-connected. □

Claim. For every $0 \leq i < l$ there exists a stable set S_i such that S_i meets V' in exactly $\alpha' - 1$ vertices, V_i in exactly one vertex, and V_{i+1} in also exactly one vertex.

Proof. For every $0 \leq i < l$, there exists a black stable set S_i' of size $\alpha' - 1$ in $G \setminus [ik + 1, (i + 3)k]$. Let S_i be the stable set $S_i' \cup \{ik + k' + 1\} \cup \{(i + 1)k + k' + 2\}$. Then $S_0, \ldots, S_{l-1}$ give the result. □

Let S' be a maximum stable set of $G[V']$.

Hence we have

$$
\begin{vmatrix}
|S' \cap V'| & |S' \cap V_0| & \cdots & |S' \cap V_{l-1}| \\
|S_0 \cap V'| & |S_0 \cap V_0| & \cdots & |S_0 \cap V_{l-1}| \\
\vdots & \vdots & & \vdots \\
|S_{l-1} \cap V'| & |S_{l-1} \cap V_0| & \cdots & |S_{l-1} \cap V_{l-1}|
\end{vmatrix}
=
\begin{vmatrix}
\alpha' & 0 \cdots 0 \\
\alpha' - 1 & \\
\vdots & C \\
\alpha' - 1 &
\end{vmatrix}
$$

where C is the $(2, l)$-circulant matrix with top row $(1, 1, 0, \ldots, 0)$ of size l. The matrix C is invertible as l is odd. Hence the above determinant is non-zero.

Therefore, the proof of Theorem 3 is done due to Lemma 2.

4. Proof of Theorem 5

In order to obtain an infinite sequence of not rank-perfect webs W_n^4 we consider, for any even $n = 2l$, the $(1,1)$-regular subweb $W_l^2 \subset W_{2l}^4$. In the remaining part of this section, let $V = \{1, \ldots, 2l\}$ denote the vertex set of W_{2l}^4 and

$$V_o = \{1, 3, \ldots, 2l - 1\}$$
$$V_e = \{2, 4, \ldots, 2l\}$$

denote the subsets of vertices with odd resp. even index in V. Then both sets V_o and V_e induce a subweb W_l^2 of W_{2l}^4, see Figure 4(a).
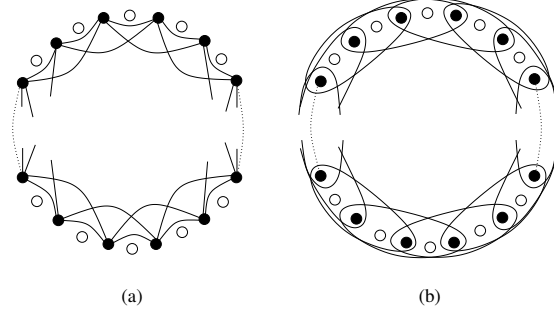

(a)   (b)

Fig. 4. The partition of W_{2l}^4

In the sequel, consider the clique family inequality $(\mathcal{Q}, 3)$ of STAB(W_{2l}^4) associated with the subweb W_l^2 induced by V_o. This means, we choose $\mathcal{Q} = \{Q_i : i \in V_o\}$ (the cliques of $\mathcal{Q}$ are represented in Figure 4(b) where the vertices from V_o are drawn in black) and obtain $I(\mathcal{Q}, 3) = V_o$ and $O(\mathcal{Q}, 3) = V_e$. The clique family inequality $(\mathcal{Q}, 3)$ associated with $W_l^2 = W_{2l}^4[V_o]$ is, therefore,

$$(3-r) \sum_{i \in V_o} x_i + (2-r) \sum_{i \in V_e} x_i \leq (3-r)\, \alpha(W_l^2)$$

and it is a *non*-rank constraint if $r = 1$. Hence, $(\mathcal{Q}, 3)$ corresponds to the studied inequality (4) if $l = 1 \pmod 3$. We prove that it is a facet for all $l \in \{13, 16, 19, \ldots\}$.

For that, we have to present $2l$ roots of (4) whose incidence vectors are linearly independent. (Recall that a root of (4) is a stable set of W_{2l}^4 satisfying (4) at equality.) It follows from [16] that a web W_n^k produces the full rank facet $\sum_{i \in W_n^k} x_i \leq \alpha(W_n^k)$ if and only if $(k+1) \nmid n$. Thus W_l^2 is facet-producing as $l = 1 \pmod 3$ and the maximum stable sets of W_l^2 yield, therefore, already l independent roots of (4). We need a set $\mathcal{S}$ of further l roots of (4) admitting vertices from V_o as well as from V_e, called *mixed roots*, and are independent, too.

We construct, for all $l \geq 13$ with $l = 1 \pmod 3$, a set $\mathcal{S}$ of l mixed roots S of size $\alpha_o + 1$ with $|S \cap V_o| = \alpha_o - 1$ and $|S \cap V_e| = 2$ where $\alpha_o = \alpha(W_l^2) = \lfloor \frac{l}{3} \rfloor$ (notice that $2(\alpha_o - 1) + 2 = 2\alpha_o$ according to the coefficients of (4)).

For that, we use the following representation of stable sets $S \subseteq W_{2l}^4$ of size $\alpha_o + 1$: choose a start vertex $i \in S$ and the distance vector $D = (d_1, \ldots, d_{\alpha_o+1})$ containing the distances between two consecutive vertices of S, i.e.,

$$S = S(i, D) = \{i, i + d_1, (i + d_1) + d_2, \ldots, (i + \sum_{j < \alpha_o} d_j) + d_{\alpha_o}\}$$

where

$$\sum_{j \leq \alpha_o + 1} d_j = 2l \ (\text{i.e. } i = i + \sum_{j \leq \alpha_o + 1} d_j \bmod 2l)$$

and $d_j > k = 4$ for $1 \leq j \leq \alpha_o + 1$ (ensuring that S is a stable set in W_{2l}^4).

Claim. Let $D = (d_1, \ldots, d_{\alpha_o+1})$ be a distance vector such that D has 4 entries d_j equal to 5 and $\alpha_o - 3$ entries d_j equal to 6. Then $S(i, D)$ is a stable set of W_{2l}^4 for every vertex i for all $l = 1 \bmod 3$.

Proof. We have to show that $\sum_{d_j \in D} d_j = 2l$ holds. Recalling $\alpha_o = \lfloor \frac{l}{3} \rfloor$ and $l = 1$ (mod 3), we obtain

$$5 \cdot 4 + 6(\alpha_o - 3) = 2 + 6\alpha_o = 2 + 6 \left\lfloor \frac{l}{3} \right\rfloor = 2 + 6\frac{l-1}{3} = 2 + 2(l-1) = 2l$$

as required. □

We define two different distance vectors with $\alpha_o + 1$ entries each by

$$D_1 = (5, 5, 5, 6, \ldots, 6, 5)$$
$$D_2 = (5, 6, 5, 5, 6, \ldots, 6, 5)$$

and show that they produce the studied mixed roots of (4).

Claim. For every $i \in V_e$, $S(i, D_1)$ (resp. $S(i, D_2)$) contains precisely the 2 vertices $i, i+10$ (resp. $i, i+16$) from V_e and $\alpha_o - 1$ vertices from V_o.

Proof. By the choice of D_1 and D_2 and the previous claim, both stable sets have size $\alpha_o + 1$ and start with a vertex in V_e (due to $i \in V_e$). The parity of the distances d_j in D_1 and D_2 implies that the third vertex $i + 10$ of $S(i, D_1)$ and the fourth vertex $i + 16$ of $S(i, D_2)$ is in V_e again, whereas all remaining $\alpha_o - 1$ vertices belong to V_o (see Figure 5, vertices in V_e (resp. V_o) are drawn in white (resp. black)). □
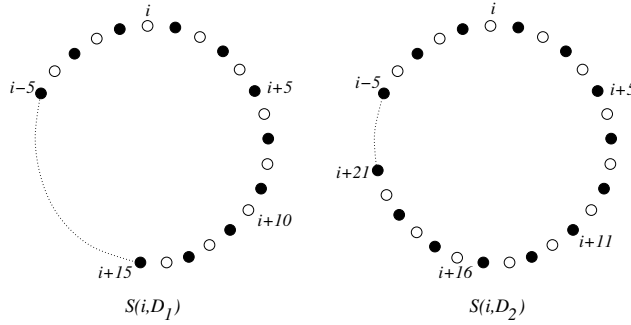

Fig. 5. The mixed roots $S(i, D_1)$ and $S(i, D_2)$

Hence, each set $S(i, D_1)$ and $S(i, D_2)$ with $i \in V_e$ is a mixed root of (4). We now have to choose a set $\mathcal{S}$ of l distinct mixed roots of (4) with linearly independent incidence vectors.

Assume that $\mathcal{S}$ is such a set and denote by $A_{\mathcal{S}}$ the square matrix containing the incidence vectors of l linearly independent maximum stable sets of $W_l^2 = W_{2l}^4[V_o]$ as first l rows and the incidence vectors of the l mixed roots in $\mathcal{S}$ as last l rows. Order the columns of $A_{\mathcal{S}}$ s.t. the first (resp. last) l columns correspond to the vertices in V_o (resp. V_e), both in increasing order. Then $A_{\mathcal{S}}$ has the block structure

$$A_{\mathcal{S}} = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right)$$

where the $l \times l$-matrix A_{11} is invertible since W_l^2 is facet-producing by [16] (in the considered case with $l = 1 \pmod 3$).

In the sequel, we provide a set $\mathcal{S}$ of l distinct mixed roots s.t. A_{22} (i.e. the intersection of the mixed roots with V_e) is an invertible $l \times l$-matrix (then $A_{\mathcal{S}}$ is invertible due to its block structure).

Claim. For every $l \geq 13$, there is a set $\mathcal{S}$ of l mixed roots of (4) containing precisely 2 vertices of V_e s.t. the $l \times l$-submatrix A_{22} of $A_{\mathcal{S}}$ is invertible.

	2 ... 12 ... 2l−26	2l−24 ... 2l−16 ... 2l−10 2l−8 ... 2l
1	1 1	
⋮	⋱ ⋱	⋱
⋮	⋱	⋱
⋮		
l−13	1	1
l−12		1 ⋱
⋮		⋱
⋮		⋱ ⋱
l−5		1 1
l−4		1 1
⋮		⋱ ⋱
l		1 1

Fig. 6. The $l \times l$-matrix A_{22}

Proof. For any $i \in V_e$, both $S(i, D_1)$ and $S(i, D_2)$ are roots of (4) by the previous claim. Chose $S(i, D_1)$ with $i \in \{2, 4, \ldots, 2l - 10\}$ as the first $l - 5$ roots in $\mathcal{S}$ and $S(i, D_2)$ with $i \in \{2l - 24, 2l - 22, \ldots, 2l - 16\}$ as the last 5 roots in $\mathcal{S}$. We have $S(i, D_1) \cap V_e = \{i, i + 10\}$ and $S(i, D_2) \cap V_e = \{i, i + 16\}$ by the previous claim.

Take the incidence vectors $\chi^{S(i, D_1)}$ for $i \in \{2, 4, \ldots, 2l - 10\}$ as the first $l - 5$ rows and $\chi^{S(i, D_2)}$ for $i \in \{2l - 24, \ldots, 2l - 16\}$ as the last 5 rows of $(A_{21} | A_{22})$. By construction, A_{22} is the $l \times l$-matrix shown in Figure 6 (0-entries are dropped and the columns represent the vertices in V_e).

A_{22} has only 1-entries on the main diagonal (coming from the first vertices in V_e of $S(i, D_1)$ for $i \in \{2, 4, \ldots, 2l - 10\}$ and from the second vertices in V_e of $S(i, D_2)$ for $i \in \{2l - 24, \ldots, 2l - 16\}$). The only non-zero entries of A_{22} below the main diagonal come from the first vertices in V_e of $S(i, D_2)$ for $i \in \{2l - 24, \ldots, 2l - 16\}$. Hence, A_{22} has the form

$$A_{22} = \left(\begin{array}{c|c} A'_{22} & \\ \hline 0 & A''_{22} \end{array} \right)$$

where both matrices A'_{22} and A''_{22} are invertible due to the following reasons. A'_{22} is an $(l - 13) \times (l - 13)$-matrix having 1-entries on the main diagonal and 0-entries below the main diagonal by construction; hence A'_{22} is clearly invertible. A''_{22} is the (2,13)-circulant matrix and, therefore, clearly invertible as well. (Note that $l = 13$ implies $A_{22} = A''_{22}$.)

This completes the proof that A_{22} is invertible for every $l \geq 13$ with $l = 1 \pmod 3$ if we choose the set $\mathcal{S}$ of l roots of (4) as constructed above. □

Remark. Note that there are no mixed roots of (4) in the case $l = 7$ (since $\alpha(W_7^2) = 2 = \alpha(W_{14}^4)$ implies that we cannot built stable sets of size $> \alpha(W_7^2)$ in W_{14}^4). In the case $l = 10$, there are only 5 mixed roots of size $> \alpha(W_{10}^2)$, namely $S(i, D_1)$ with $i \in \{2, 4, 6, 8, 10\}$ because of $S(i, D_1) = S(i + 10, D_1)$; the sets $S(i, D_2)$ can be constructed only if $l \geq 13$. Hence, (4) is neither a facet of $\mathrm{STAB}(W_{14}^4)$ nor of $\mathrm{STAB}(W_{20}^2)$. Moreover, in the case $l = 13$, we would obtain the same set $\mathcal{S}$ by choosing the roots $S(i, D_1)$ with $i \in \{2l - 8, \ldots, 2l\}$ instead of $S(i, D_2)$ with $i \in \{2l - 24, \ldots, 2l - 16\}$.

Hence, we have shown that, for every $l \geq 13$ with $l = 1 \pmod 3$, there are $2l$ roots of (4) whose incidence vectors are linearly independent, completeing the proof of Theorem 5.

5. Proof of Theorem 6

The aim of this section is to prove that the clique family inequality$(\mathcal{Q}, k)$ associated with any $(k - 1, 1)$-regular subweb $W_{l(k-1)}^{k-1}$ induces the facet (5) of $\mathrm{STAB}(W_{lk}^k)$ for every odd $k \geq 5$ whenever $l \geq 3k + 2$.

For that, let $l = l'k + 2$ with $l' \geq 3$ and $n = lk = l'k^2 + 2k$. Denote the set of vertices of W_n^k by $V = \{1, \ldots, n\}$ and the subset of all vertices $i \in V$ with $i \nmid k$ by V', i.e., let $V' = V \setminus \{k, 2k, \ldots, lk\}$ be the vertex set of $W_{l(k-1)}^{k-1}$. Then $\mathcal{Q} = \{Q_i : i \in V'\}$ obviously implies $I(\mathcal{Q}, k) = V'$. As

$$\alpha(W_{l(k-1)}^{k-1}) = \left\lfloor \frac{(k-1)l}{k} \right\rfloor = \left\lfloor l - l' - \frac{2}{k} \right\rfloor = l - l' - 1 = l'(k-1) + 1$$

holds, the clique family inequality $(\mathcal{Q}, k)$ reads

$$2 \sum_{i \in V'} x_i + \sum_{i \in V \setminus V'} x_i \leq 2l'(k-1) + 2$$

and is supposed to define a facet of $\mathrm{STAB}(W_{lk}^k)$ for any odd $k \geq 5$ if $l = l'k + 2$ and $l' \geq 3$.

In order to verify that we have to present lk roots, i.e., stable sets satisfying $(\mathcal{Q}, k)$ at equality, whose incidence vectors are linearly independent.

The maximum stable sets of $W_{l(k-1)}^{k-1}$ are independent by Trotter [16] as $k \nmid l(k-1)$ by $l = 2 \pmod{k}$ and k odd.

This provides us already $l(k-1)$ independent roots of $(\mathcal{Q}, k)$ containing vertices from V' only. We are going to build l further mixed roots containing vertices from V' as well as from $V \setminus V'$ such that their incidence vectors are linearly independent, too.

For any vertex $i \in V \setminus V'$, denote by $D_i = \{i, i+1, \ldots, i+k-1\}$ resp. $B_i = \{i, i+1, \ldots, i+k^2-1\}$ the subset of V consisting of k resp. k^2 consecutive vertices starting in vertex i (with arithmetics performed modulo n). Furthermore, define $S(B_i) \subset B_i$ by

$$S(B_i) = \{i + 1 + j(k+1) : 0 \leq j \leq k - 2\}$$

as the black vertices in Figure 7 (the squares stand for nodes in $V \setminus V'$). By construction, $S(B_i)$ is obviously a stable set of W_{lk}^k consisting of nodes from V' only.

Fig. 7. The stable set $S(B_i) \subset B_i$ in the case $k = 5$

To build the mixed roots, we are going to use two types of partitions of $V = \{1, \ldots, n\}$ into 2 subsets D_j of size k and l' subsets B_i of size k^2 (recall that $n = 2k + l'k^2$ holds). We pick the first vertex from each subset D_j and $S(B_i)$ from the involved subsets B_i and show that the so constructed vertex sets form roots of $(\mathcal{Q}, k)$.

For every $1 \leq i \leq l$, the vertex ki belongs to $V \setminus V'$ and

$$D_{ki} \cup B_{k(i+1)} \cup D_{k(i+1+k)} \cup \bigcup_{j=1,\ldots,l'-1} B_{k(i+2+jk)}$$

forms a partition of $V = \{1, \ldots, n\}$, as $k(i+2+(l'-1)k) + k^2 - 1 = ki - 1 \pmod{n}$ holds. Let

$$S_{ki} = \{ki\} \cup S(B_{k(i+1)}) \cup \{k(i+1+k)\} \cup \bigcup_{j=1,\ldots,l'-1} S(B_{k(i+2+jk)})$$

be the set consisting of the first vertices from D_{ki} and $D_{k(i+1+k)}$ and the stable sets $S(B_j)$ for the involved subsets B_j.

Furthermore, for every $1 \le i \le l$,

$$D_{ki} \cup \bigcup_{j=0,\dots,l'-3} B_{k(i+1+jk)} \cup D_{k(i+1+(l'-2)k)} \cup B_{k(i+2+(l'-2)k)} \cup B_{k(i+2+(l'-1)k)}$$

forms a second type of partition of $V = \{1, \dots, n\}$, as $k(i+2+(l'-2)k)+k^2-1 = ki-1$ (mod n). Let

$$S'_{ki} = \{ki\} \cup \bigcup_{j=0,\dots,l'-3} S(B_{k(i+1+jk)}) \cup \{k(i+1+(l'-2)k)\} \cup \bigcup_{j=l'-2,l'-1} S(B_{k(i+2+jk)})$$

be the corresponding set consisting of the first vertices from D_{ki} and $D_{k(i+1+(l'-2)k)}$ and the stable sets $S(B_j)$ for the involved subsets B_j. (Note that $S_{ki} = S'_{ki}$ iff $l' = 3$). We call ki the start vertex of S_{ki} resp. of S'_{ki}.

Claim. The sets S_{ki} and S'_{ki} are mixed roots of $(\mathcal{Q}, k)$ for $1 \le i \le l$.

Proof. By construction, we have $ki, k(i+1+k) \in S_{ki}$ and $ki, k(i+1+(l'-2)k) \in S'_{ki}$ (these vertices belong obviously to $V \setminus V'$). The remaining vertices of S_{ki} and S'_{ki} come from the sets $S(B_j)$ for each of the l' subsets B_j. Since $S(B_j)$ contains only $k-1$ vertices from V' by construction, we obtain $|S_{ki} \cap V'| = |S'_{ki} \cap V'| = l'(k-1)$ and $|S_{ki} \setminus V'| = |S'_{ki} \setminus V'| = 2$. Thus S_{ki} as well as S'_{ki} satisfy $(\mathcal{Q}, k)$ at equality for every $1 \le i \le l$.

It is left to show that S_{ki} and S'_{ki} are stable. For that, recall first that $S(B_j)$ is a stable set for any j. Second, the last k vertices of B_j do not belong to $S(B_j)$ by construction (see Figure 7), thus B_j can be followed by any subset without introducing adjacencies in S_{ki} or S'_{ki}. Finally, consider a subset D_{kj} followed by $B_{k(j+1)}$. By construction, the last $k-1$ vertices of D_{kj} as well as the first vertex of $B_{k(j+1)}$ do not belong to S_{ki} or S'_{ki}, thus no adjacencies are introduced again.

This implies that S_{ki} and S'_{ki} are stable sets satisfying $(\mathcal{Q}, k)$ at equality. □

We are now prepared to select a set of lk independent roots of $(\mathcal{Q}, k)$:

Claim. There are lk roots of $(\mathcal{Q}, k)$ whose incidence vectors are linearly independent: the $l(k-1)$ maximum stable sets of $W^{k-1}_{l(k-1)}$ and the l stable sets S_{ki} for $1 \le i \le l-(k+1)$ resp. S'_{ki} for $l-k \le i \le l$.

Proof. The maximum stable sets of $W^{k-1}_{l(k-1)}$ are linearly independent as mentioned above. Moreover, they contain only vertices from V' whereas the stable sets S_{ki} and S'_{ki} contain vertices from V' as well as vertices from $V \setminus V'$. Thus, we are done if we can show that the incidence vectors of S_{ki} for $1 \le i \le l-(k+1)$ and S'_{ki} for $l-k \le i \le l$ are linearly independent.

We construct an $(l \times lk)$-matrix M having the incidence vectors of $S_k, \dots, S_{k(l-(k+1))}$ as first $l-(k+1)$ rows and the incidence vectors of $S'_{k(l-k)}, \dots, S'_{kl}$ as last $k+1$ rows. We show that the $(l \times l)$-submatrix M' of M containing all columns corresponding to the vertices in $V \setminus V'$ is invertible. For that, choose an ordering of the columns of M s.t. the first l columns correspond to the vertices $k, \dots, lk$ in $V \setminus V'$ and the remaining $l(k-1)$ columns correspond to the vertices in V' (see Figure 5).

Each row of M' has a 1-entry on the main diagonal (since ki is the start vertex of S_{ki} as well as of S'_{ki} by construction), thus we have to discuss the second 1-entries of the rows coming from the vertices $k(i+1+k) \in S_{ki}$ resp. $k(i+1+(l'-2)k) \in S'_{ki}$ (see Figure 5).

Let $\bar{l} = l - (3k+1)$ (we have $\bar{l} \ge 1$ since $l = l'k+2$ and $l' \ge 3$). We show that $k\bar{l}$ is the first column with a 1-entry below the main diagonal, namely in the row corresponding to $S'_{k(l-k)}$.

	k	...	$k(l-1)$	kl	...	$k(l+k)$	$k(l+kl+1)$	...	kl	V'
S_k	1									⋮
⋮		⋱			⋱					
$S_{k(l-1)}$			1			1				
S_{kl}				1			1			
⋮					⋱			⋱		⋮
$S_{k(l-(k+1))}$						⋱			1	
$S'_{k(l-k)}$				1			⋱			
⋮					⋱			⋱		⋮
S'_{kl}						1			1	

Fig. 8. The $(l \times lk)$-matrix M

The first $l - (k + 1)$ rows of M' do not have any 1-entry below the main diagonal, since the second vertex of S_{kj} in $V \setminus V'$ is $k(j + 1 + k)$ and $k(j + 1 + k) \leq kl$ holds due to $j \leq l - (k + 1)$. (In fact, the row corresponding to $S_{k(l-(k+1))}$ has its 1-entries in the columns $k(l - (k + 1))$ and kl.)

Consider now the row $l - k$ corresponding to $S'_{k(l-k)}$. We have $S'_{k(l-k)} \setminus V' = \{k(l - k), k(l - k + 1 + (l' - 2)k)\}$ where

$$k(l - k + 1 + (l' - 2)k) = k(-3k + 1 + l'k) \bmod n = kl$$

as $l = l'k + 2$. Hence the row given by $S'_{k(l-k)}$ has indeed a 1-entry at column kl.

The matrix M' is invertible, if its $(3k + 2) \times (3k + 2)$-submatrix M'' consisting of the colums $kl, \ldots, kl$ of the rows corresponding to $S_{kl}, \ldots, S_{k(l-(k+1))}, S'_{k(l-k)}, \ldots, S'_{kl}$ is invertible (since the previous part of M' has 0-entries below the main diagonal only).

We complete the proof of this claim by showing that M'' is a $(3k + 2, 2)$-circulant matrix: The 1-entries below the main diagonal start in column kl, as seen above, and end in the last row in column $k(l - (2k + 1))$ since

$$k(l + 1 + (l' - 2)k) = kl + k(l'k + 2 - (2k + 1)) \bmod n = k(l - (2k + 1))$$

holds as $l = l'k + 2$, whereas the 1-entries above the main diagonal start in column $k(l - 2k)$ due to

$$k(l + 1 + k) = k(l - (3k + 1) + 1 + k) = k(l - 2k)$$

and end with kl in the row corresponding to $S_{k(l-(k+1))}$ as shown above. This implies that M'' is a $(3k + 2, 2)$-circulant matrix and, therefore, invertible as k is odd by our hypothesis. This complets the proof that the choosen stable sets are lk independent roots of $(\mathcal{Q}, k)$. □

Hence $(\mathcal{Q}, k)$ is, for any odd $k \geq 5$, a proper weak non-rank facet (5) of $\mathrm{STAB}(W_{kl}^k)$ if $l \geq 3k + 2$ completing the proof of Theorem 6.

6. Concluding remarks and open problems

In this paper, we presented infinite sequences of not rank-perfect webs W_n^k for $k = 4$ (Theorem 5), all even $k \geq 6$ (Theorem 4), and all odd $k \geq 5$ (Theorem 6). Before, the case $k = 3$ was settled in [14]. Applying the construction from [15] yields that there are only finitely many rank-perfect webs W_n^k for all values of $k \geq 3$ (Corollary 4), implying that almost all webs with fixed clique number at least 4 are not rank-perfect (Corollary 5).

For our construction, we used clique family inequalities associated with certain sub-webs yielding 1/2-valued facets; the construction from [15] does not change the involved coefficients and, therefore, the stable set polytopes of almost all webs admit 1/2-valued facets.

According to Ben Rebea's Conjecture [12], the stable set polytopes of quasi-line graphs (and therefore of webs) have clique family inequalities as only non-trivial facets. However, clique family inequalities constitute a large class of valid inequalities; even verifying Ben Rebea's Conjecture would provide no information about which inequalities are *essential* among them. The following conjecture addresses this problem for the subclass of webs.

Conjecture 1. Every facet of $STAB(W_n^k)$ belongs to one of the following classes:

(0) nonnegativity constraints,
 (i) clique constraints,
(ii) full rank constraint,
(iii) clique family inequalities $(\mathcal{Q}, k'+1)$ associated with proper subwebs $W_{n'}^{k'}$
     where $(k'+1) \nmid n'$ and $\alpha(W_{n'}^{k'}) < \alpha(W_n^k)$.

All non-rank facets known so far for webs are of type (iii). Note that a web W_n^k usually has subwebs $W_{n'}^{k'}$ for all values $1 \le k' < k$. Hence, we expect that the stable set polytope of W_n^k admits $(k-2)/(k-1)$-valued facets. The conjecture implies in particular, that the stable set polytopes of all webs W_n^3 have 1/2-valued facets only, where for all webs W_n^k with $k > 3$ larger coefficients are required. In fact, Liebling et al. [11] proved recently that, for any odd $k \ge 5$, the stable set polytope of $W_{k^2}^k$ has an $(k-2)/(k-1)$-valued facet. Hence, further effort is needed for having a complete description of stable set polytope of webs (and for the larger class of fuzzy circular interval graphs).

References

1. M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *Progress on Perfect Graphs.* Mathematical Programming B 97 (2003) 405–422
2. M. Chudnovsky and P. Seymour, *Claw-free graphs VI. The structure of quasi-line graphs.* manuscript (2004).
3. V. Chvátal, *On Certain Polytopes Associated with Graphs.* J. Combin. Theory B 18 (1975) 138–154
4. V. Chvátal, *On the Strong Perfect Graph Conjecture.* J. Combin. Theory B 20 (1976) 139–141
5. G. Dahl, *Stable Set Polytopes for a Class of Circulant Graphs.* SIAM J. Optim. 9 (1999) 493–503
6. J.R. Edmonds and W.R. Pulleyblank, *Facets of 1-Matching Polyhedra.* In: C. Berge and D.R. Chuadhuri (eds.) Hypergraph Seminar. Springer-Verlag, Heidelberg, (1974) 214–242
7. A. Galluccio and A. Sassano, *The Rank Facets of the Stable Set Polytope for Claw-Free Graphs.* J. Comb. Theory B 69 (1997) 1–38
8. G. Giles and L.E. Trotter, jr., *On Stable Set Polyhedra for $K_{1,3}$-free Graphs.* J. Comb. Theory B 31 (1981) 313–326
9. M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization.* Springer-Verlag, (1988)
10. J. Kind, *Mobilitätsmodelle für zellulare Mobilfunknetze: Produktformen und Blockierung.* PhD thesis, RWTH Aachen (2000)
11. T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer, *On Non-Rank Facets of the Stable Set Polytope of Claw-Free Graphs and Circulant Graphs.* Submitted to Math. Methods of Operations Research
12. G. Oriolo, *Clique Family Inequalities for the Stable Set Polytope for Quasi-Line Graphs.* In: Special Issue on Stability Problems, Discrete Applied Math. 132 (2003) 185–201
13. M.W. Padberg, *Perfect Zero-One Matrices.* Math. Programming 6 (1974) 180–196
14. A. Pêcher and A. Wagler, *On Non-Rank Facets of Stable Set Polytopes of Webs with Clique Number Four*, ZIB-Report ZR 03-01 (2003)
15. A. Pêcher and A. Wagler, *A construction for non-rank facets of stable set polytopes of webs*, LaBRI-Report RR-1304-03 (2003)
16. L.E. Trotter, jr., *A Class of Facet Producing Graphs for Vertex Packing Polyhedra.* Discrete Math. 12 (1975) 373–388
17. A. Wagler, *Antiwebs are rank-perfect.* to appear in: Quarterly Journal of the Belgian, French and Italian OR Societies