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A note on the Chvátal-rank of clique family inequalities

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Abstract

Clique family inequalities $a \sum_{v \in W} x_v + (a - 1) \sum_{v \in W'} x_v \leq a\delta$ form an intriguing class of valid inequalities for the stable set polytopes of all graphs. We prove firstly that their Chvátal-rank is at most a , which provides an alternative proof for the validity of clique family inequalities, involving only standard rounding arguments. Secondly, we strengthen the upper bound further and discuss consequences regarding the Chvátal-rank of subclasses of claw-free graphs.

For any polyhedron P , let P^I denote the convex hull of all integer points in P . Chvátal [4] (and implicitly Gomory [9]) introduced a method to obtain approximations of P^I outgoing from P as follows. If $\sum a_i x_i \leq b$ is valid for P and has integer coefficients only, then $\sum a_i x_i \leq \lfloor b \rfloor$ is a Chvátal-Gomory cut for P . Define P' to be the set of points satisfying all Chvátal-Gomory cuts for P , and let $P^0 = P$ and $P^{t+1} = (P^t)'$ for non-negative integers t . Obviously $P^I \subseteq P^t \subseteq P$ for every t . An inequality $\sum a_i x_i \leq b$ is said to have Chvátal-rank at most t if it is a valid inequality for the polytope P^t . Chvátal showed in [4] that for each polyhedron P there exists a finite $t \geq 0$ with $P^t = P^I$; the smallest such t is the *Chvátal-rank* of P .

The fractional matching polytope is a famous example of a polytope with Chvátal-rank one [4]. In this note, we consider the Chvátal-rank of the fractional stable set polytope $P = \text{QSTAB}(G)$. In particular, P^I is the stable set polytope $\text{STAB}(G)$.

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The *stable set polytope* $\text{STAB}(G)$ of a graph G is defined as the convex hull of the incidence vectors of all its stable sets (in a stable set all nodes are mutually nonadjacent). A canonical relaxation of $\text{STAB}(G)$ is the *fractional stable set polytope* $\text{QSTAB}(G)$ given by all “trivial” facets, the nonnegativity constraints $x_v \geq 0$ for all nodes v of G , and by the *clique constraints* $\sum_{v \in Q} x_v \leq 1$ for all cliques $Q \subseteq G$ (in a clique all nodes are mutually adjacent). Clearly, $\text{STAB}(G) \subseteq \text{QSTAB}(G)$ and $\text{STAB}(G) = \text{QSTAB}(G)^I$ holds for all graphs G . We say that a graph class \mathcal{G} has Chvátal-rank t if t is the minimum value such that $\text{QSTAB}(G)^t = \text{STAB}(G)$ for all $G \in \mathcal{G}$. We have $\text{STAB}(G) = \text{QSTAB}(G)$ if and only if G is perfect [5], that is perfect graphs form exactly the class of graphs with Chvátal-rank zero.

To describe the stable set polytopes of imperfect graphs, we consider two natural generalizations of clique constraints: 0/1-constraints associated with arbitrary induced subgraphs, and $a/(a-1)$ -valued constraints associated with families of cliques. *Rank constraints* are 0/1-inequalities

$$\sum_{v \in G'} x_v \leq \alpha(G')$$

associated with induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G' . *Clique family inequalities* (\mathcal{Q}, p)

$$a \sum_{v \in V_p} x_v + (a-1) \sum_{v \in V_{p-1}} x_v \leq a\delta \quad (1)$$

rely on the intersection of cliques within a family \mathcal{Q} , where V_p (resp. V_{p-1}) contains all nodes belonging to at least p (resp. exactly $p-1$) cliques in \mathcal{Q} , and $a = p - r$ with $r = |\mathcal{Q}| \bmod p$ and $\delta = \lfloor \frac{|\mathcal{Q}|}{p} \rfloor$ holds.

Both types of inequalities are valid for the stable set polytopes of all graphs: rank constraints by the choice of the right hand side, and clique family inequalities by [1, 11].

It is known from [6] that the Chvátal-rank of rank constraints of a graph with n nodes is $\Omega((n/\log n)^{\frac{1}{2}})$ and from [7] that the split rank of clique family inequalities is one, that is, clique family inequalities are simple split cuts (split cuts were studied in [2]).

The aim of this note is to establish $\min\{r, p-r\}$ as upper bound of the Chvátal-rank for general clique family inequalities. We close with remarks regarding Chvátal-ranks of *quasi-line graphs* (where the neighbors of any node split into two cliques), as their stable set polytopes are completely described by nonnegativity, clique, and clique family inequalities [7].

The Chvátal-rank of clique family inequalities. The following observation will be crucial for the proofs: summing up the clique inequalities corresponding to the cliques in \mathcal{Q} and possibly adding nonnegativity constraints $-x_v \leq 0$ for those nodes $v \in V_p$ which are contained in more than p cliques, we obtain that

$$p \sum_{v \in V_p} x_v + (p-1) \sum_{v \in V_{p-1}} x_v \leq p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \quad (2)$$

is valid for $\text{QSTAB}(G)$.

Theorem 1 *Let (\mathcal{Q}, p) be a clique family inequality and let $r = |\mathcal{Q}| \pmod{p}$. For every $1 \leq i \leq p - r$, the inequality $h(i)$*

$$i \sum_{v \in V_p} x_v + (i-1) \sum_{v \in V_{p-1}} x_v \leq i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

has Chvátal-rank at most i and, thus, (\mathcal{Q}, p) has Chvátal-rank at most $p - r$.

Proof: For every $1 \leq i \leq p - r$, let $H(i)$ be the assertion : "The inequality $h(i)$ has Chvátal-rank at most i ." The proof is performed by induction on i :

$H(1)$ is true: Inequality (2) implies that $\sum_{v \in V_p} x_v \leq \frac{|\mathcal{Q}|}{p}$ is valid for $\text{QSTAB}(G)$, hence $\sum_{v \in V_p} x_v \leq \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$ has Chvátal-rank 1, as required.

Induction step: assume that $H(i)$ is true and $i < p - r$. To prove that $H(i+1)$ holds, we show that $h(i+1)$ is a Chvátal-Gomory cut from $h(i)$ and Inequality (2). Therefore, we have to find a pair of solutions (λ, μ) to the following system of equations:

$$\begin{aligned} \lambda i + \mu p &= i + 1 \\ \lambda(i-1) + \mu(p-1) &= i \\ \left\lfloor \lambda i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \right) \right\rfloor &= (i+1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \end{aligned}$$

Indeed, $\lambda = \frac{p-i-1}{p-i}$, $\mu = \frac{1}{p-i}$ are solutions, as $\left\lfloor \lambda i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \right) \right\rfloor = \left\lfloor (\lambda i + \mu p) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \frac{r}{p-i} \right\rfloor = (i+1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$, since $0 \leq r/(p-i) < 1$. \square

Note that the proof of Theorem 1 yields an alternative proof for the validity of clique family inequalities for the stable set polytope of any graph, involving only standard rounding arguments.

Furthermore, we obtain that every *rank* clique family inequality has Chvátal-rank one. This is particularly nice, as neither general rank constraints nor general clique family inequalities have this property [6, 11], but the combination of both.

However, the upper bound established in Theorem 1 gets weaker if r gets smaller; we therefore improve the upper bound for $r < p/2$.

Theorem 2 *Every clique family inequality (\mathcal{Q}, p) with $r = |\mathcal{Q}| \pmod{p}$ has Chvátal-rank at most r if $0 \leq r < p - r$.*

Proof: For every $0 \leq i \leq r$, let $G(i)$ be the assertion : "The inequality $g(i): (p - i) \sum_{v \in V_p} x_v + (p - i - 1) \sum_{v \in V_{p-1}} x_v \leq (p - i) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i$ has Chvátal-rank at most i ." The proof is performed by induction on i :

$G(0)$ is true due to Inequality (2).

Induction step: assume that $G(i)$ is true and $i < r$. To prove that $G(i+1)$ holds, we show that $g(i+1)$ is a Chvátal-Gomory cut from $g(i)$ and $h(i)$. Therefore, we have to find a pair of solutions (λ, μ) to the following system of equations:

$$\begin{aligned} \lambda(p - i) + \mu i &= p - i - 1 \\ \lambda(p - i - 1) + \mu(i - 1) &= p - i - 2 \\ \left[\lambda \left[(p - i) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i \right] + \mu i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \right] &= (p - i - 1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1 \end{aligned}$$

Indeed, $\lambda = \frac{p-2i-1}{p-2i}$, $\mu = \frac{1}{p-2i}$ are solutions as $\left[\lambda \left[(p - i) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i \right] + \mu i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \right] = \left[(\lambda(p - i) + \mu i) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \lambda(r - i) \right] = \left[(p - i - 1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1 + \frac{p-i-r}{p-2i} \right] = (p - i - 1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1$ since $0 \leq \frac{p-i-r}{p-2i} < 1$. \square

Thus, Theorem 1 and Theorem 2 together imply:

Corollary 3 *Every clique family inequality (\mathcal{Q}, p) has Chvátal-rank at most $\min\{r, p-r\}$ where $r = |\mathcal{Q}| \pmod{p}$. In particular, a clique family inequality (\mathcal{Q}, p) has Chvátal-rank at most $\frac{p}{2}$.*

Consequences for quasi-line graphs. We now discuss consequences of the above results for quasi-line graphs, as all non-trivial, non-clique facets of their stable set polytopes are clique family inequalities according to [7].

Calling a graph G *rank-perfect* if $\text{STAB}(G)$ has rank constraints as only non-trivial facets, Theorem 1 implies that rank-perfect subclasses of quasi-line graphs have Chvátal-rank 1. This verifies Edmond's conjecture that the

Chvátal-rank of claw-free graphs is one for the class of semi-line graphs, as they are rank-perfect [3].

A *semi-line graph* is a line graph or a quasi-line graph without a representation as fuzzy circular interval graph. A *line graph* $L(G)$ is obtained by turning adjacent edges of a root graph G into adjacent nodes of $L(G)$. Fuzzy circular interval graphs are defined as follows. Let \mathcal{C} be a circle, \mathcal{I} a collection of intervals in \mathcal{C} without proper containments and common endpoints, and V a finite multiset of points in \mathcal{C} . The *fuzzy circular interval graph* $G(V, \mathcal{I})$ has node set V and two nodes are adjacent if both belong to one interval $I \in \mathcal{I}$, where edges between different endpoints of the same interval may be omitted.

As the only not rank-perfect quasi-line graphs are fuzzy circular interval, it suffices to restrict to this class in order to discuss the Chvátal-rank for quasi-line graphs. Giles and Trotter [8] exhibited a fuzzy circular interval graph with a clique family \mathcal{Q} of size 37 such that $(\mathcal{Q}, 8)$ induces a facet. Oriolo noticed in [11] that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at least* 2. This example disproves Edmonds' conjecture for fuzzy circular interval graphs. On the other hand, Theorem 1 shows that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at most* 3, since $r = 5$ and so $p - r = 3$.

Furthermore, Giles and Trotter [8] introduced a sequence of fuzzy circular interval graphs G^k for $k \geq 1$ and showed that each of them admits a clique family facet $(\mathcal{Q}, k + 2)$ with $|\mathcal{Q}| = 2k(k + 2) + 1$ and coefficients k and $k + 1$; Theorem 2 ensures that these facets have Chvátal-rank 1 since $r = 1$ holds in all cases.

Webs W_n^k are special fuzzy circular interval graphs with nodes $0, \dots, n - 1$ and edges ij iff $\min\{|i - j|, n - |i - j|\} < k$. Liebling et al. [10] exhibited a sequence of webs $W_{(2a+3)^2}^{2(a+2)}$ for $a \geq 1$, each with a $(a + 1)/a$ -valued clique family facet $(\mathcal{Q}, a + 2)$ with $|\mathcal{Q}| = (a + 2)(2a + 3)$. Since $(a + 2)(2a + 3) \equiv 1 \pmod{a + 2}$, Theorem 2 shows that also these facets have Chvátal-rank 1.

The authors conjectured in [12] and Stauffer proved in [13] that all non-rank facets of webs W_n^k are clique family inequalities $(\mathcal{Q}, k' + 1)$ associated with subwebs $W_{n'}^{k'} \subset W_n^k$ where the maximum cliques $\{i, \dots, i + k\}$ of W_n^k starting in nodes i of the subweb $W_{n'}^{k'}$ yield the clique family \mathcal{Q} of size n' where $(k' + 1) \nmid n'$ and $k' < k$. Thus, for any *fixed* k , the Chvátal-rank of all webs W_n^k is at most $\frac{k-1}{2}$. However, it is very likely that there exist sequences of webs inducing clique family facets (\mathcal{Q}, p) with arbitrarily high p and $2p = |\mathcal{Q}|$ having Chvátal-rank $\frac{p}{2}$. Thus, also the Chvátal-rank of webs and, therefore, of quasi-line graphs could be arbitrarily large, as for general claw-free graphs [6].

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References

- [1] A. Ben Rebea, *Etude des Stables dans les Graphes Quasi-Adjoints*. Ph.D. Thesis, Univ. of Grenoble (1981)
- [2] W. Cook, R. Kannan and A. Schrijver *Chvátal closures for mixed integer programming problems*. Math. Program. 47 (1990) 155–174
- [3] M. Chudnovsky and P. Seymour, *Claw-free graphs VI. The structure of quasi-line graphs*. manuscript (2004).
- [4] V. Chvátal, *Edmonds polytopes and a hierarchy of combinatorial problems*. Discrete Math. 4 (1973) 305–337.
- [5] V. Chvátal, *On Certain Polytopes Associated with Graphs*. J. Combin. Theory (B) 18 (1975) 138–154.
- [6] V. Chvátal, W. Cook and M. Hartmann, *On cutting-plane proofs in combinatorial optimization*. Linear Algebra and its Applications 114/115 (1989) 455–499
- [7] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura, *Circular One Matrices and the Stable Set Polytope of Quasi-Line Graphs*. Lecture Notes in Computer Science 3509 (2005) 291–305.
- [8] R. Giles and L.E. Trotter, jr., *On Stable Set Polyhedra for $K_{1,3}$ -free Graphs*. J. Comb. Theory B 31 (1981) 313–326.
- [9] R.E. Gomory, *Outline of an algorithm for integer solutions to linear programs*. Bul. of the American Math. Soc. 64 (1958) 27–278.
- [10] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer, *On Non-Rank Facets of the Stable Set Polytope of Claw-Free Graphs and Circulant Graphs*. Math. Methods Oper. Res. 59, No. 1, (2004) 25–35
- [11] G. Oriolo, *On the Stable Set Polytope for Quasi-Line Graphs*. In: Special issue on stability problems, Discrete Applied Math. 132 (2003) 185–201.
- [12] A. Pêcher and A. Wagler, *Almost all webs are not rank-perfect*, Math. Programming B 105 (2006) 311–328.
- [13] G. Stauffer, *On the Stable Set Polytope of Claw-free Graphs*, PhD thesis, EPF Lausanne, 2005.