On Non-Rank Facets in Stable Set Polytopes of Webs with Clique Number Four

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Abstract

Graphs with circular symmetry, called webs, are relevant for describing the stable set polytopes of two larger graph classes, quasi-line graphs [6,10] and claw-free graphs [5,6]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [7]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [3,13] while there are examples with clique number > 4 having non-rank facets [8,10,9]. The aim of the present paper is to treat the remaining case with clique number $= 4$: we provide an infinite sequence of such webs whose stable set polytopes admit non-rank facets.

Key words: web, rank-perfect graph, stable set polytope, (non-)rank facet

1 Introduction

A natural generalization of odd holes and odd antiholes are graphs with circular symmetry of their maximum cliques and stable sets, called webs: a web W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge iff i and j differ by at most k (modulo

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n) and $i \neq j$. These graphs belong to the classes of quasi-line graphs and claw-free graphs and are, besides line graphs, relevant for describing the stable set polytopes of those larger graph classes [5,6,10]. (The line graph of a graph H is obtained by taking the edges of H as nodes and connecting two nodes iff the corresponding edges of H are incident. A graph is quasi-line (resp. claw-free) if the neighborhood of any node can be partitioned into two cliques (resp. does not contain any stable set of size 3).) All facets of the stable set polytope of line graphs are known from matching theory [4]. In contrary, providing all facets of the stable set polytopes of claw-free graphs is a long-standing problem [7] but we are even still far from having a complete description for the stable set polytopes of webs (and, therefore, of quasi-line and claw-free graphs, too).

In particular, as shown by Giles & Trotter [6], the stable set polytopes of claw-free graphs contain facets with a much more complex structure than those defining the matching polytope. Oriolo [10] discussed which of them occur in quasi-linegraphs. In particular, these non-rank facets rely on certain combinations of joined webs.

Several further authors studied the stable set polytopes of webs. Obviously, webs with clique number 2 are either even or odd holes (their stable set polytopes are known due to [1,11]). Dahl [3] studied webs with clique number 3 and showed that their stable set polytopes admit rank facets only. On the other hand, Kind [8] found (by means of the PORTA software [3]) examples of webs with clique number > 4 whose stable set polytopes have *non*-rank facets. Oriolo [10] and Liebling et al. [9] presented further examples of such webs. It is natural to ask whether the stable set polytopes of webs with clique number $= 4$ admit rank facets only.

The aim of the present paper is to answer that question by providing an infinite sequence of webs with clique number $= 4$ whose stable set polytopes have *non*-rank facets.

2 Results on Stable Set Polytopes

The *stable set polytope* $\text{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of the graph $G = (V, E)$ (a set $V' \subseteq V$ is a stable set if the nodes in V' are mutually non-adjacent). A linear inequality $a^T x \leq b$ is said to be *valid* for $\text{STAB}(G)$ if it holds for all $x \in \text{STAB}(G)$. We call a stable set S of G a *root* of $a^T x \leq b$ if its incidence vector χ^S satisfies $a^T \chi^S = b$. A valid inequality for $\text{STAB}(G)$ is a *facet* if and only if it has $|V|$ roots with affinely independent incidence vectors. (Note that the incidence vectors of the roots of $a^T x \leq b$ have to be *linearly* independent if $b > 0$.)

[3] By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

The aim is to find a system $Ax \leq b$ of valid inequalities s.t. STAB(G) = $\{x \in \mathbb{R}_+^{|G|} : Ax \leq b\}$ holds. Such a system is unknown for the most graphs and it is, therefore, of interest to study certain linear relaxations of STAB(G) and to investigate for which graphs G these relaxations coincide with STAB(G).

One relaxation of STAB(G) is the *fractional stable set polytope* QSTAB(G) given by all "trivial" facets, the *nonnegativity constraints*

$$x_i \;\geq\; 0 \tag{0}$$

for all nodes i of G and by the *clique constraints*

$$\sum_{i \in Q} x_i \;\leq\; 1 \tag{1}$$

for all cliques $Q \subseteq G$ (a set $V' \subseteq V$ is a clique if the nodes in V' are mutually adjacent). Obviously, a clique and a stable set have at most one node in common. Therefore, QSTAB(G) contains all incidence vectors of stable sets of G and STAB$(G) \subseteq$ QSTAB(G) holds for all graphs G. The two polytopes coincide precisely for perfect graphs [1,11].

A graph G is called *perfect* if, for each (node-induced) subgraph $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$. That is, for all $G' \subseteq G$, as many stable sets cover all nodes of G' as a maximum clique of G' has nodes (maximum cliques resp. maximum stable sets contain a maximal number of nodes).

In particular, for all imperfect graphs G follows STAB$(G) \subset$ QSTAB(G) and, therefore, further constraints are needed to describe their stable set polytopes. A natural way to generalize clique constraints is to investigate *rank constraints*

$$\sum_{i \in G'} x_i \;\leq\; \alpha(G') \tag{2}$$

associated with *arbitrary* (node-)induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the stability number of G', i.e., the cardinality of a maximum stable set in G' (note that $\alpha(G') = 1$ holds iff G' is a clique). For convenience, we often write (2) in the form $x(G') \leq \alpha(G')$.

Let RSTAB(G) denote the *rank polytope* of G given by all nonnegativity constraints (0) and all rank constraints (2). A graph G is called *rank-perfect* [14] if STAB(G) coincides with RSTAB(G).

By construction, every perfect graph is rank-perfect. Some further graphs are rank-perfect by definition: *near-perfect* [12] (resp. *t-perfect* [1], *h-perfect* [7]) graphs, where rank constraints associated with cliques and the graph itself (resp. edges and odd cycles, cliques and odd cycles) are allowed. Moreover, the result of Edmonds and Pulleyblank [4] implies that line graphs are rank-perfect as well (see [15] for a list with more examples).

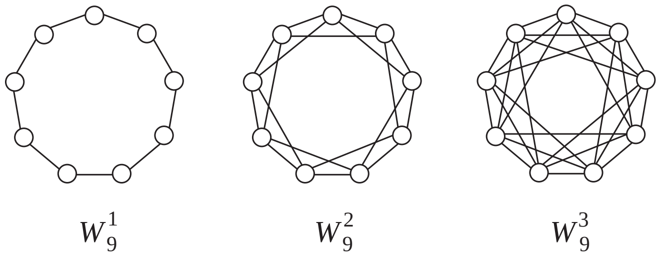

Figure 1

Recall that a web W_n^k is a graph with nodes $1, \dots, n$ where ij is an edge if i and j differ by at most k (i.e., if $|i-j| \leq k \bmod n$) and $i \neq j$. We assume $k \geq 1$ and $n \geq 2(k+1)$ in the sequel in order to exclude the degenerated cases when W_n^k is a stable set or a clique. W_n^1 is a hole and W_{2k+1}^{k-1} an odd antihole for $k \geq 2$. All webs W_9^k on nine nodes are depicted in Figure 1. It is easy to see that $\omega(W_n^k) = k+1$ and $\alpha(W_n^k) = \lfloor \frac{n}{k+1} \rfloor$ holds. Note that webs are also called circulant graphs C_n^k [2]. Furthermore, similar graphs $W(n,k)$ were introduced in [13].

So far, the following is known about stable set polytopes of webs. The webs W_n^1 are holes, hence they are perfect if n is even and near-perfect if n is odd (recall that we suppose $n \geq 2(k+1)$). Dahl [3] showed that all webs W_n^2 with clique number 3 are rank-perfect. But there are several webs with clique number > 4 known to be *not* rank-perfect [8,10,9], e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9; these results are summarized in Table 1.

Table 1: Known results on rank-perfectness of webs

	$\omega = 2$	$\omega = 3$	$\omega = 4$	$\omega \geq 5$
All webs rank-perfect?	Yes	Yes	?	No
Infinitely many not rank-perfect webs?	No	No	?	?

A conjecture due to Ben Rebea (see [10]) claims that the stable set polytopes of quasi-line graphs admit only one type of facets besides nonnegativity constraints (0) and clique constraints (1), so-called clique family inequalities: Let $G = (V, E)$ be a graph, $\mathcal{F}$ be a family of (at least three inclusion-wise) maximal cliques of G, $p \leq |\mathcal{F}|$ be an integer, and define two sets as follows:

$$I(\mathcal{F}, p) = \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\}$$
$$O(\mathcal{F}, p) = \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p-1\}$$

The *clique family inequality* $(\mathcal{F}, p)$

$$(p-r) \sum_{i \in I(\mathcal{F},p)} x_i + (p-r-1) \sum_{i \in O(\mathcal{F},p)} x_i \leq (p-r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \tag{3}$$

with $r = |\mathcal{F}| \bmod p$ and $r > 0$ is valid for the stable set polytope of *every* graph by

Oriolo [10]. Since webs are quasi-line graphs in particular, the stable set polytopes of webs should admit, according to Ben Rebea's conjecture, facets coming from cliques and clique family inequalities only.

In order to answer the question whether the webs with clique number $= 4$ are rank-perfect or not, we introduce clique family inequalities associated with certain subwebs and prove the following: the clique family inequality associated with $W^2_{2l} \subset W^3_{3l}$ induces a non-rank facet of $\mathrm{STAB}(W^3_{3l})$ if $l \geq 11$ and $2 = l \bmod 3$ (Theorem 6).

3 Non-Rank Facets of STAB(W^3_n)

Consider a web W^k_n. We say that a clique family inequality $(\mathcal{F}, p)$ of STAB(W^k_n) is *associated* with a proper subweb $W^{k'}_{n'} \subset W^k_n$ if $\mathcal{F} = \{Q_i : i \in W^{k'}_{n'}\}$ is chosen as clique family, $p = k' + 1$, and $Q_i = \{i, \ldots, i + k\}$ denotes the maximum clique of W^k_n starting in node i. In order to explore the special structure of such inequalities, we need the following fact from Trotter [13].

Observation 1 [13] *$W^{k'}_{n'}$ is an induced subweb of W^k_n if and only if there is a subset $V' = \{i_1, \ldots, i_{n'}\} \subseteq V(W^k_n)$ s.t. $|V' \cap Q_{i_j}| = k' + 1$ for every $1 \leq j \leq n'$.*

We now prove the following.

Lemma 2 *Let $W^{k'}_{n'} \subset W^k_n$ be any proper induced subweb. The clique family inequality $(\mathcal{F}, p)$ of STAB(W^k_n) associated with $W^{k'}_{n'}$ is*

$$(k' + 1 - r) \sum_{i \in I(\mathcal{F},p)} x_i + (k' - r) \sum_{i \in O(\mathcal{F},p)} x_i \leq (k' + 1 - r)\, \alpha(W^{k'}_{n'}) \qquad (4)$$

with $p = k' + 1$, $r = n' \bmod (k' + 1)$, $r > 0$; we have $W^{k'}_{n'} \subseteq I(\mathcal{F}, p)$ and the union of $I(\mathcal{F}, p)$ and $O(\mathcal{F}, p)$ covers all nodes of W^k_n.

PROOF. Let $W^{k'}_{n'}$ be a proper subweb of W^k_n and choose $\mathcal{F} = \{Q_i : i \in W^{k'}_{n'}\}$, $p = k' + 1$. Obviously $|\mathcal{F}| = |W^{k'}_{n'}| = n'$ follows. Let $V' = \{i_1, \ldots, i_{n'}\}$ be the node set of $W^{k'}_{n'}$ in W^k_n. Observation 1 implies that $Q_{i_j} = \{i_j, \ldots, i_j + k\}$ contains the nodes $i_j, \ldots, i_{j+k'}$ from V'. Obviously, the node $i_{j+k'}$ belongs exactly to the $(k' + 1)$ cliques $Q_{i_j}, \ldots, Q_{i_{j+k'}}$ from $\mathcal{F}$. Since all indices are taken modulo n, every node in $W^{k'}_{n'}$ is covered precisely $(k' + 1)$ times by $\mathcal{F}$ and $p = k' + 1$ yields, therefore, $W^{k'}_{n'} \subseteq I(\mathcal{F}, p)$. Furthermore, $|\mathcal{F}| = n'$ and $p = \omega(W^{k'}_{n'})$ implies $\lfloor \frac{|\mathcal{F}|}{p} \rfloor = \alpha(W^{k'}_{n'})$. Hence the clique family inequality given by $(\mathcal{F}, p)$ is (4) which finishes the proof. $\square$

Let us turn to the clique family inequality associated with $W_{2l}^2 \subset W_{3l}^3$, i.e. n is divisible by 3 (for some $l \geq 3$ by $n \geq 2(k+1)$). Observation 1 easily yields that every third node of W_{3l}^3 does not belong to the subweb W_{2l}^2 and that $W_{2l}^2 = I(\mathcal{F}, 3)$ holds if we choose $\mathcal{F} = \{Q_i : i \in W_{2l}^2\}$, see Figure 2.

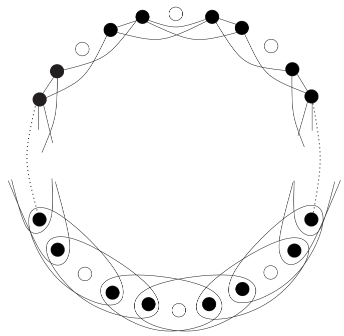

Fig. 2. The subweb $W_{2l}^2 \subset W_{3l}^3$

Furthermore, the nodes in $W_{3l}^3 - W_{2l}^2 = O(\mathcal{F}, 3)$ induce the hole W_{1l}^1. Thus, the clique family inequality $(\mathcal{F}, 3)$

$$(3-r)\, x(W_{2l}^2) + (2-r)\, x(W_{1l}^1) \leq (3-r)\, \alpha(W_{2l}^2)$$

associated with $W_{2l}^2 \subset W_{3l}^3$ is a *non*-rank constraint if $r = 1$ holds. The aim of this section is to prove that $(\mathcal{F}, 3)$ is a non-rank facet of STAB(W_{3l}^3) whenever $l \geq 11$ and $2 = l \bmod 3$ (note: $2 = l \bmod 3$ implies $r = 1 = 2l \bmod 3$).

For that, we have to present $3l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent. (Recall that a root of $(\mathcal{F}, 3)$ is a stable set of W_{3l}^3 satisfying $(\mathcal{F}, 3)$ at equality.)

It follows from [13] that a web W_n^k produces the full rank facet $x(W_n^k) \leq \alpha(W_n^k)$ iff $(k+1) \not| \, n$. Thus W_{2l}^2 is facet-producing if $2 = l \bmod 3$ and the maximum stable sets of W_{2l}^2 yield already $2l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent.

Let $V = V(W_{3l}^3)$ and $V' = V(W_{2l}^2)$. We need a set $\mathcal{S}$ of further l roots of $(\mathcal{F}, 3)$ which have a non-empty intersection with $V - V'$, called *mixed roots*, and are independent, in order to prove that $(\mathcal{F}, 3)$ is a facet of STAB(W_{3l}^3).

We show that there exists a set $\mathcal{S}$ of l mixed roots of $(\mathcal{F}, 3)$ whenever $l \geq 11$. Due to $2 = l \bmod 3$, we set $l = 2 + 3l'$ and obtain $|V| = 3l = 6 + 9l'$. Thus, V can be partitioned into 2 blocks D_1, D_2 with 3 nodes each and l' blocks $B_1, \ldots, B_{l'}$ with 9 nodes each s.t. every block ends with a node in $V - V'$ (this is possible since every third node of V belongs to $V - V'$ say $i \in V'$ if $3 \not| \, i$ and $i \in V - V'$ if $3|i$). Figure 3 shows a block D_i and a block B_j (where circles represent nodes in V' and squares represent nodes in $V - V'$). For the studied mixed roots of $(\mathcal{F}, 3)$ we choose the black filled nodes in Figure 3:

D_i B_j

○ ○ ■ | ○ ○ □ ● ○ □ ○ ● □

Fig. 3. A block D_i and a block B_j

Lemma 3 *The set S containing the 3rd node of the blocks D_1, D_2 as well as the 4th and 8th node of any block B_j is a root of $(\mathcal{F}, 3)$ with $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$ for every ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ of the blocks s.t. D_1, D_2 are not neighbored.*

PROOF. Consider a set S constructed that way. Since every block ends with a node in $V - V'$ by definition and every third node of V is in $V - V'$, we have that the last node of D_i and the 3rd, 6th, and 9th node of B_j belong to $V - V'$ while all other nodes are in V'. Thus, the two last nodes in D_1 and D_2 are the two studied nodes in $S \cap (V - V')$ and the 4th and 8th node in B_j for $1 \leq j \leq l'$ are the studied $2l'$ nodes in $S \cap V'$ (see Figure 3).

S is a stable set provided the two blocks D_1 and D_2 are not neighbored: Obviously, there is no edge between the 4th and 8th node of any block B_j. Thus, we only have to discuss what happens between two consecutive blocks. Since the first 3 nodes of every block B_j do not belong to S, there is no problem with having any block before B_j, i.e., $B_k B_j$ or $D_i B_j$. For the remaining case $B_j D_i$, notice that the last node of B_j and the first two nodes of D_i do not belong to S and there cannot be an edge between two nodes of S in that case, too.

This shows that S is a stable set satisfying $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$. Due to $\alpha(W_{2l}^2) = \lfloor \frac{2(2+3l')}{3} \rfloor = 2l' + 1$, the set S is finally a root of $(\mathcal{F}, 3)$. □

Lemma 3 implies that there exist mixed roots S of $(\mathcal{F}, 3)$ with $|S| = 2 + 2l'$ if $l' \geq 2$. The next step is to show that there are l such roots if $l' \geq 3$ (resp. $l \geq 11$).

In the sequel, we denote by $S_{i,m}$ the stable set constructed as in Lemma 3 when $D_1 = \{i-2, i-1, i\}$ and $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$. If there are more than $\lfloor \frac{l'}{2} \rfloor$ blocks between D_1 and D_2, there are less than $\lfloor \frac{l'}{2} \rfloor$ blocks between D_2 and D_1. Hence it suffices to consider $m \leq \lfloor \frac{l'}{2} \rfloor$.

By construction, $S_{i,m}$ contains a second node from $V - V'$, namely, the third node $i+9m+3$ of block D_2. If $2|l'$ and $m = \frac{l'}{2}$, then $(i+9m+3)+9m+3 = i+9l'+6 = i(\bmod n)$ and, therefore, $S_{i,m} = S_{i+9m+3,m}$ follows.

We are supposed to construct *distinct* mixed roots $S_{i,m}$ of $(\mathcal{F}, 3)$ with $2 + 2l'$ nodes, hence we choose orderings $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ and obtain easily:

Lemma 4 *If $l' \geq 3$, then the stable sets $S_{i,m}$ for each $i \in V - V'$ obtained from any ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ yield $|V - V'| = l$ roots of $(\mathcal{F}, 3)$ with $2 + 2l'$ nodes each.*

Consequently, we can always choose a set of $3l$ roots of $(\mathcal{F}, 3)$ if $l' \geq 3$ resp. $l \geq 11$.

If $\mathcal{S}$ is a set of l distinct mixed roots, denote by $A_{\mathcal{S}}$ the square matrix containing the incidence vectors of the $2l$ maximum stable sets of W_{2l}^2 and the l mixed roots in $\mathcal{S}$. $A_{\mathcal{S}}$ can be arranged s.t. the first $2l$ and the last l columns correspond to the nodes in W_{2l}^2 and W_{1l}^1, respectively, and the first $2l$ rows contain the incidence vectors of the maximum stable sets of W_{2l}^2 where the last rows contain the incidence vectors of the l mixed roots in $\mathcal{S}$. (Note that the nodes corresponding to the last l columns of $A_{\mathcal{S}}$ are $3, 6, \ldots, 3l$.) Then $A_{\mathcal{S}}$ has the block structure

$$A_{\mathcal{S}} = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right)$$

where the $2l \times 2l$-matrix A_{11} is invertible (recall: W_{2l}^2 is facet-producing by [13] in the considered case with $1 = 2l \bmod 3$ resp. $2 = l \bmod 3$).

It is left to find a set $\mathcal{S}$ of l distinct mixed roots s.t. A_{22} is an invertible $l \times l$-matrix (then $A_{\mathcal{S}}$ is invertible due to its block structure).

Lemma 5 *For every $l \geq 11$, there is a set $\mathcal{S}$ of l mixed roots of $(\mathcal{F}, 3)$ containing 2 nodes from $V - V'$ s.t. the $l \times l$-submatrix A_{22} of $A_{\mathcal{S}}$ is invertible.*

PROOF. Every root $S_{i,m}$ of $(\mathcal{F}, 3)$ corresponds to a row in $(A_{21}|A_{22})$ of $A_{\mathcal{S}}$ having precisely two 1-entries in the columns belonging to A_{22} (by $|S_{i,m} \cap (V - V')| = 2$ for all $i \in V - V'$). Lemma 4 ensures that no such roots coincide if $1 \leq m < \frac{l'}{2}$ for all $i \in V - V'$.

The idea of finding cases when A_{22} is invertible goes as follows: Let $S_{3j,1}$ for $1 \leq j \leq l - 4$ be the first $l - 4$ roots in $\mathcal{S}$ with $S_{3j,1} \cap (V - V') = \{3j, 3(j + 4)\}$. Choose as the remaining 4 roots in $\mathcal{S}$ the stable sets $S_{3j,2}$ for $l - 10 \leq j \leq l - 7$ with $S_{3j,2} \cap (V - V') = \{3j, 3(j + 7)\}$. Then take their incidence vectors $\chi^{S_{3j,1}}$ for $1 \leq j \leq l - 4$ as the first $l - 4$ rows and $\chi^{S_{3j,2}}$ for $l - 10 \leq j \leq l - 7$ as the last 4 rows of $(A_{21}|A_{22})$. By construction, A_{22} is the $l \times l$-matrix in Figure 4 (1-entries are shown only, the column i corresponds to the node $3i$).

A_{22} has only 1-entries on the main diagonal (coming from the first nodes in $V - V'$ of $S_{3j,1}$ for $1 \leq j \leq l - 4$ and from the second nodes in $V - V'$ of $S_{3j,2}$ for $l - 10 \leq j \leq l - 7$). The only non-zero entries of A_{22} below the main diagonal come from the first nodes in $V - V'$ of $S_{3j,2}$ for $l - 10 \leq j \leq l - 7$. Hence, A_{22}

$$
\begin{array}{c|ccccc|ccccccccccc}
 & 1 & \dots & 5 & \dots & l\text{–}11 & l\text{–}10 & l\text{–}9 & l\text{–}8 & l\text{–}7 & l\text{–}6 & l\text{–}5 & l\text{–}4 & l\text{–}3 & l\text{–}2 & l\text{–}1 & l \\
\hline
1 & 1 & & 1 & & & & & & & & & & & & & \\
\vdots & & \ddots & & & \ddots & & & & & & & & & & & \\
\vdots & & & & \ddots & & & \ddots & & & & & & & & & \\
l\text{–}11 & & & & & 1 & & & & 1 & & & & & & & \\
\hline
l\text{–}10 & & & & & & 1 & & & & 1 & & & & & & \\
l\text{–}9 & & & & & & & 1 & & & & 1 & & & & & \\
l\text{–}8 & & & & & & & & 1 & & & & 1 & & & & \\
l\text{–}7 & & & & & & & & & 1 & & & & 1 & & & \\
l\text{–}6 & & & & & & & & & & 1 & & & & 1 & & \\
l\text{–}5 & & & & & & & & & & & 1 & & & & 1 & \\
l\text{–}4 & & & & & & & & & & & & 1 & & & & 1 \\
l\text{–}3 & & & & & & 1 & & & & & & & 1 & & & \\
l\text{–}2 & & & & & & & 1 & & & & & & & 1 & & \\
l\text{–}1 & & & & & & & & 1 & & & & & & & 1 & \\
l & & & & & & & & & 1 & & & & & & & 1 \\
\hline
\end{array}
$$

Fig. 4. The $l \times l$-matrix A_{22}

has the form

$$A_{22} = \left(\begin{array}{c|c} A'_{22} & \\ \hline 0 & A''_{22} \end{array} \right)$$

where both A'_{22} and A''_{22} are invertible due to the following reasons:

A'_{22} is an $(l-11) \times (l-11)$-matrix having 1-entries on the main diagonal and 0-entries below the main diagonal by construction. Hence A'_{22} is clearly invertible.

A''_{22} is an 11×11-matrix which has obviously the circular 1's property. In other words, A''_{22} is equivalent to the matrix $A(\overline{C}_{11})$ containing the incidence vectors of the maximum stable sets of the odd antihole $\overline{C}_{11}$ as rows. Since $A(\overline{C}_{11})$ is invertible due to Padberg [11], the matrix A''_{22} is invertible, too. (Note that $l = 11$ implies $A_{22} = A''_{22}$.)

This completes the proof that A_{22} is invertible for every $l \geq 11$ if we choose the set $\mathcal{S}$ of l roots of $(\mathcal{F}, 3)$ as constructed above. $\square$

Finally, we have shown that, for every $l \geq 11$ with $2 = l \bmod 3$, there are $3l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent:

Theorem 6 *For any* $W^2_{2l} \subset W^3_{3l}$ *with* $2 = l \bmod 3$ *and* $l \geq 11$*, the clique family inequality*

$$2x(W^2_{2l}) + 1x(W^1_{1l}) \leq 2\alpha(W^2_{2l})$$

associated with W^2_{2l} *is a non-rank facet of STAB(*W^3_{3l}*).*

This gives us an infinite sequence of not rank-perfect webs W^3_{3l} with clique number 4, namely W^3_{33}, W^3_{42}, W^3_{51}, W^3_{60}, ... and answers the question whether the webs W^3_n with clique number 4 are rank-perfect negatively. Thus, we can update Table 1 as follows:

Table 2: Updated results on rank-perfectness of webs

	$\omega = 2$	$\omega = 3$	$\omega = 4$	$\omega \geq 5$
All webs rank-perfect?	Yes	Yes	No	No
Infinitely many not rank-perfect webs?	No	No	Yes	?

4 Concluding Remarks

It is open whether there exist, for each $\omega \geq 5$, infinitely many not rank-perfect webs, see Table 2. We beliefe that this is the case.

Assuming Ben Rebea's Conjecture as true, we conjecture further that all non-rank facets of STAB(W^k_n) are clique family inequalities $(\mathcal{F}, p)$

$$(k' + 1 - r) \sum_{i \in I(\mathcal{F},p)} x_i + (k' - r) \sum_{i \in O(\mathcal{F},p)} x_i \leq (k' + 1 - r)\, \alpha(W^{k'}_{n'})$$

associated with certain subwebs $W^{k'}_{n'} \subset W^k_n$. All non-rank facets would have, therefore, coefficients at most $k-1$ and $k-2$ (since $k' < k$ follows by $W^{k'}_{n'} \subset W^k_n$ and $(k'+1-r) \leq k'$ by $r > 0$). This would imply that the stable set polytopes of webs W^3_n could have non-rank facets with coefficients 2 and 1 only.

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