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On classes of minimal circular-imperfect graphs

Arnaud Pêcher ¹

*Laboratoire Bordelais de Recherche Informatique (LaBRI)
351 cours de la Libération
33405 Talence, France*

Annegret K. Wagler ²

*Otto-von-Guericke-University Magdeburg, Faculty of Mathematics
Institute for Mathematical Optimization
Universitätsplatz 2
39106 Magdeburg, Germany*

Abstract

Circular-perfect graphs form a natural superclass of perfect graphs: on the one hand due to their definition by means of a more general coloring concept, on the other hand as an important class of χ -bound graphs with the smallest non-trivial χ -binding function $\chi(G) \leq \omega(G) + 1$.

The Strong Perfect Graph Conjecture, recently settled by Chudnovsky et al. [4], provides a characterization of perfect graphs by means of forbidden subgraphs. It is, therefore, natural to ask for an analogous conjecture for circular-perfect graphs, that is for a characterization of all minimal circular-imperfect graphs.

At present, not many minimal circular-imperfect graphs are known. This paper studies the circular-(im)perfection of some families of graphs: normalized circular cliques, partitionable graphs, planar graphs, and complete joins. We thereby exhibit classes of minimal circular-imperfect graphs, namely, certain partitionable webs, a subclass of planar graphs, and odd wheels and odd antiwheels. As those classes appear to be very different from a structural point of view, we infer that formulating

an appropriate conjecture for circular-perfect graphs, as analogue to the Strong Perfect Graph Theorem, seems to be difficult.

Keywords: circular coloring, circular-perfection, minimal circular-imperfect graph

1 Introduction

Coloring the vertices of a graph is an important concept with a large variety of applications. Let $G = (V, E)$ be a graph with finite vertex set V and simple edge set E . A k -coloring of G is a mapping $f : V \rightarrow \{1, \dots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices of G receive different colors. The minimum k for which G admits a k -coloring is called the *chromatic number* of G and denoted by $\chi(G)$. Calculating $\chi(G)$ is an NP-hard problem in general. In a set of k pairwise adjacent vertices, called *clique* K_k , all k vertices have to be colored differently. Thus the size of a largest clique in G , the *clique number* $\omega(G)$, is a trivial lower bound on $\chi(G)$. This bound can be arbitrarily bad [11] and is hard to evaluate as well.

Berge [1] proposed to call a graph G *perfect* if each induced subgraph $G' \subseteq G$ admits an $\omega(G')$ -coloring. Perfect graphs turned out to be an interesting and important class of graphs with a rich structure, see [15] for a recent survey. In particular, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [6].

Recently, the famous Strong Perfect Graph Conjecture of Berge [1] on characterizing perfect graphs by means of forbidden subgraphs has been settled by Chudnovsky, Robertson, Seymour, and Thomas [4]: Berge [1] observed that chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, and their complements $\overline{C_{2k+1}}$, the *odd antiholes*, are imperfect as clique and chromatic number differ. (The *complement* \overline{G} of a graph G has the same vertex set as G and two vertices are adjacent in \overline{G} if and only if they are non-adjacent in G .) Berge's famous conjecture was that odd holes and odd antiholes are the only minimal forbidden subgraphs in perfect graphs, i.e., the only *minimally imperfect graphs*. Considerable effort has been spent over the years to verify or falsify this conjecture revealing deep structural properties of minimally

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¹ Email: arnaud.pecher@labri.fr

² Email: wagler@imo.math.uni-magdeburg.de

imperfect graphs [15]. Finally, Chudnovsky, Robertson, Seymour, and Thomas [4] succeeded in turning the conjecture into the Strong Perfect Graph Theorem and exhibited many structural properties of perfect graphs, that were not known before.

As generalization of perfect graphs, X. Zhu [21,22] introduced recently the class of circular-perfect graphs based on the following more general coloring concept.

Define a (k, d) -coloring of a graph G , as a mapping $f : V \rightarrow \{0, \dots, k-1\}$ such that for each edge xy of G , $d \leq |f(x) - f(y)| \leq k-d$. The *circular chromatic number* is:

$$\chi_c(G) = \inf \left\{ \frac{k}{d} : G \text{ has a } (k, d) \text{-coloring} \right\}$$

From the definition, we immediately obtain $\chi_c(G) \leq \chi(G)$ because a usual k -coloring of G is a $(k, 1)$ -coloring. (Note that $\chi_c(G)$ is sometimes called the star chromatic number in the literature, see [3,17,20].)

In order to obtain a lower bound on $\chi_c(G)$, we generalize cliques as follows: Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k vertices $0, \dots, k-1$ and edges ij if and only if $d \leq |i-j| \leq k-d$. Such graphs $K_{k/d}$ are called *circular cliques* (note that they are also known as antiwebs in the literature, see [16,18]). A circular clique $K_{k/d}$ with $\gcd(k, d) = 1$ is said to be *prime*. Prime circular cliques include all cliques $K_k = K_{k/1}$ as well as all odd antiholes $\overline{C_{2k+1}} = K_{2k+1/2}$ and all odd holes $C_{2k+1} = K_{2k+1/k}$, see Figure 1. The *circular clique number* is

$$\omega_c(G) = \max \left\{ \frac{k}{d} : K_{k/d} \subseteq G, \gcd(k, d) = 1 \right\}$$

and we immediately obtain $\omega(G) \leq \omega_c(G)$.

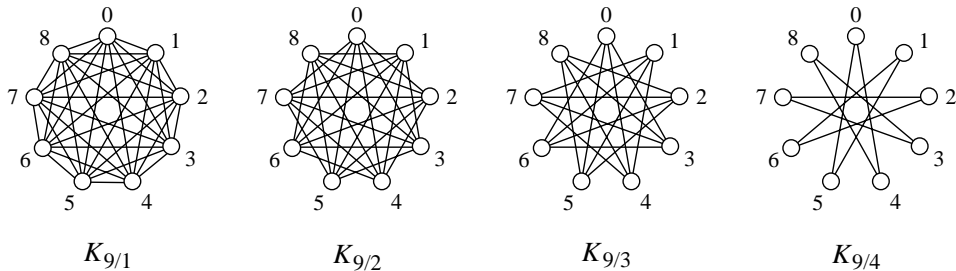


Fig. 1. The circular cliques on nine vertices.

Remark 1.1 Colorings can also be interpreted as homomorphisms from a graph to a clique.

Let h be a homomorphism from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ where $h : V_1 \rightarrow V_2$ such that $h(u)h(v) \in E_2$ if $uv \in E_1$; we write $G_1 \preceq G_2$. Any k -coloring of a graph G is equivalent to a homomorphism from G to K_k . Then the circular chromatic number can be written as $\chi_c(G) = \inf \left\{ \frac{k}{d} : G \preceq K_{k/d} \right\}$ and the circular clique number as $\omega_c(G) = \sup \left\{ \frac{k}{d} : K_{k/d} \preceq G, \gcd(k, d) = 1 \right\}$ [22].

Every circular clique $K_{k/d}$ clearly admits a (k, d) -coloring (simply take the vertex numbers as colors, as in Figure 1) but no (k', d') -coloring with $\frac{k'}{d'} < \frac{k}{d}$ by [3]. Thus we obtain, for any graph G , the following chain of inequalities:

$$(1) \quad \omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G).$$

A graph G is called *circular-perfect* if, for each induced subgraph $G' \subseteq G$, the circular clique number $\omega_c(G')$ and the circular chromatic number $\chi_c(G')$ coincide. Obviously, every perfect graph has this property by (1) as $\omega(G')$ equals $\chi(G')$. Moreover, it was proved in [22] that any circular clique is circular-perfect as well. Thus circular-perfect graphs constitute a proper superclass of perfect graphs. In contrary to perfect graphs, the class of circular-perfect graphs is not stable under complementation.

Another natural extension of perfect graphs was introduced by Gyárfás [7] as χ -bound graphs: A family \mathcal{G} of graphs is called χ -bound with χ -binding function b if $\chi(G') \leq b(\omega(G'))$ holds for all induced subgraphs G' of $G \in \mathcal{G}$. Thus this concept uses functions in $\omega(G)$ as *upper* bound on $\chi(G)$. Since it is known for any graph G that $\omega(G) = \lfloor \omega_c(G) \rfloor$ by [22] and $\chi(G) = \lceil \chi_c(G) \rceil$ by [17], we obtain that circular-perfect graphs G satisfy the following Vizing-like property

$$(2) \quad \omega(G) \leq \chi(G) \leq \omega(G) + 1.$$

Thus, circular-perfect graphs are a class of χ -bound graphs with the smallest non-trivial χ -binding function. In particular, this χ -binding function is best possible for a *proper* superclass of perfect graphs implying that circular-perfect graphs admit coloring properties almost as nice as perfect graphs.

The aim of this paper is to look for other parallels between the classes of perfect and circular-perfect graphs. As analogue to the Strong Perfect Graph Theorem, one might be tempted to ask for an appealing conjecture on minimal forbidden subgraphs in circular-perfect graphs. We call a graph G *minimal circular-imperfect* if G is not circular-perfect but every proper induced subgraph is. The hope is to identify all classes of minimal circular-imperfect graphs in order to characterize circular-perfect graphs by means of forbidden subgraphs.

The main contribution of this paper is to characterize all minimal circular-imperfect graphs in the classes of normalized circular cliques, partitionable graphs, and complete joins, as well as to exhibit a class of minimal circular-imperfect planar graphs. However, at first sight there is no straightforward common structure in these graphs, hence formulating an analogue to the Strong Perfect Graph Theorem for circular-perfect graphs seems to be difficult.

2 Results

2.1 Normalized circular cliques and partitionable graphs

Given a graph G , an edge e of G is called *indifferent* if e is not contained in any maximum clique of G . The *normalized subgraph norm* $\text{norm}(G)$ of G is obtained from G by deleting all indifferent edges.

A graph G is called (p, q) -*partitionable* if $|V(G)| = pq + 1$ and, for each vertex v of G , the subgraph $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p . A graph is *partitionable* if it is (p, q) -partitionable for some $p, q \geq 2$.

The complement of a circular clique (or antiweb) $K_{n/q}$ is a *web* C_n^q , and any circular clique $K_{n/q}$ (and its complement) with $n = \omega q + 1$ is a partitionable graph.

We characterize all circular cliques whose normalized subgraph is circular-imperfect, and show which of them are minimal with respect to this property.

Theorem 2.1 *Let $K_{p/q}$ be any prime circular clique. Then $\text{norm}(K_{p/q})$ is*

- (i) *circular-imperfect if and only if $p \not\equiv -1 \pmod{q}$ and $\lfloor p/q \rfloor \geq 3$;*
- (ii) *minimal circular-imperfect if and only if $p = 3q + 1$ and $q \geq 3$;*
- (iii) *isomorphic to $\overline{K}_{p/3}$ if $p = 3q + 1$ and $q \geq 3$.*

The above results imply:

Corollary 2.2 *The partitionable webs C_{3q+1}^3 are minimal circular-imperfect for all $q \geq 3$.*

Originally, Lovász [10] and Padberg [12] introduced partitionable graphs as a tool to study properties of minimal imperfect graphs, as every minimal imperfect graph is in particular partitionable. Since circular-perfect graphs include all perfect graphs and all minimal imperfect graphs, one might expect that some subclasses of partitionable graphs are circular-perfect. To support

this feeling further, every partitionable graph G satisfies the Vizing-like property $\chi(G) \leq \omega(G) + 1$, as every circular-perfect graph. This motivates to study circular-(im)perfection of partitionable graphs.

The above corollary shows, however, that the circular cliques whose normalized subgraphs are minimal circular-imperfect, are partitionable graphs with clique number 3. Therefore, we cannot expect anymore the circular-perfection of all partitionable graphs. Even worse, Theorem 2.3 below states that most partitionable graphs are in fact circular-imperfect:

Theorem 2.3 *All partitionable graphs apart from circular cliques are circular-imperfect.*

This implies further:

Corollary 2.4 *All normalized partitionable graphs apart from odd holes and odd antiholes are circular-imperfect.*

2.2 Planarity and Circular-perfection

Computer checks for small minimal circular-imperfect graphs showed that there exist planar ones (e.g. the 5-wheel); this suggests to check circular-(im)perfection of planar graphs.

Our first result introduces an interesting class of circular-perfect graphs: planar graphs where all vertices lie on the outer face, i.e., *outerplanar graphs*.

Theorem 2.5 *Outerplanar graphs are circular-perfect.*

As a by-product of Theorem 2.5, the circular chromatic number of an outerplanar graph is equal to 2 if all cycles have even size, or $2 + \frac{1}{d}$ where $2d + 1$ is the size of the smallest odd cycle. This gives a different proof of a recent result by Kemnitz and Wellmann [9].

Outgoing from the circular-perfection of outerplanar graphs, it is easy to introduce a simple class of minimal circular-imperfect planar graphs: for every positive integers k and l such that $(k, l) \neq (1, 1)$, let $T_{k,l}$ denote the planar graph with $2l+1$ inner faces $F_1, F_2, \dots, F_{2l+1}$ of size $2k+1$ arranged in a circular fashion around a central vertex, where all other vertices lie on the outer face, as depicted in Figure 2.2. We show circular-imperfection, minimality follows from Theorem 2.5 as the removal of any vertex yields an outerplanar graph.

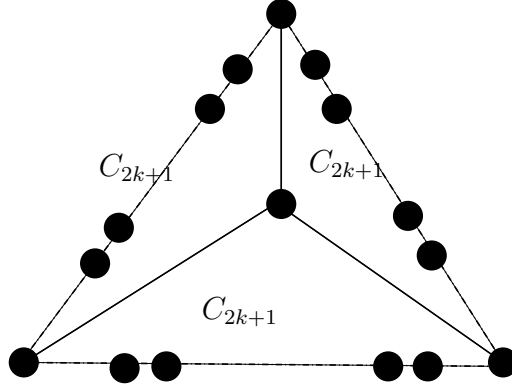


Fig. 2. Example of a graph $T_{k,1}$

2.3 Complete joins and circular-imperfection

At last, our third class of minimal circular-imperfect graphs involves *odd wheels* (complete joins of odd holes and *one* vertex) and *odd antiwheels* (complete joins of odd antiholes and *one* vertex); a complete join of two graphs G_1 and G_2 is the union of G_1 and G_2 , and all edges between G_1 and G_2 . We completely characterized complete joins w.r.t. circular-(im)perfection as follows:

Theorem 2.6 *The complete join $G * G'$ of two graphs G and G' is*

- (i) *circular-perfect if and only if both G and G' are perfect;*
- (ii) *minimal circular-imperfect if and only if G is an odd hole or odd antihole and G' is a single vertex (or vice versa), that is if and only if $G * G'$ is an odd wheel or an odd antiwheel.*

Notice that odd wheels are the same as graphs $T_{1,l}$, that is a class of planar minimal circular-imperfect graphs. Odd antiwheels are examples of minimal circular-imperfect graphs with arbitrarily large clique and chromatic number.

Corollary 2.7 *The complete join of more than two graphs is never minimal circular-imperfect.*

3 Normalized circular cliques and partitionable graphs

3.1 Proof of Theorem 2.1

We shall prove that the normalized subgraph norm $(K_{p/q})$ of a prime circular clique $K_{p/q}$ is

- circular-imperfect iff $p \not\equiv -1 \pmod{q}$ and $\lfloor p/q \rfloor \geq 3$ (assertion (i));

- minimal w.r.t. this property iff $p = 3q + 1$ for all $q \geq 3$ (assertion (ii));
- equal to $\overline{K}_{p/3}$ if $p = 3q + 1$ and $q \geq 3$ (assertion (iii)).

Given an integer p and a subset of integers S of $[0, p-1]$, the *circulant graph* $C(p, S)$ is the graph with vertex set $\{0, \dots, p-1\}$ and edge set $\{ij \mid i - j \in S\}$ with arithmetics performed modulo p .

We first state the following observation which relates the normalized subgraph of a partitionable circular-clique to its complement.

Lemma 3.1 *If $p = \omega q + 1$, then $\text{norm}(K_{p/q})$ is isomorphic to the complement $\overline{K}_{p/\omega} = C_p^\omega$ of $K_{p/\omega}$.*

Proof. Both $\text{norm}(K_{p/q})$ and $\overline{K}_{p/\omega}$ are circulant graphs on the vertex set $V = \{0, 1, \dots, p-1\}$. The former has generating set

$$S = \{q, q+1, 2q, 2q+1, \dots, (\omega-1)q, (\omega-1)q+1\}$$

and the latter has generating set

$$S' = \{1, 2, \dots, \omega-1, p-1, p-2, \dots, p-\omega+1\}.$$

It is easy to verify that $f : V \rightarrow V$ defined as $f(i) = iq \pmod{p}$ has the property $f(S') = S$. Hence f is an isomorphism from $\overline{K}_{p/\omega}$ to $\text{norm}(K_{p/q})$. \square

We shall now proceed to the proof of Theorem 2.1.

Proof. In the following, we denote by G the circular clique $K_{p/q}$ and let H denote the normalized subgraph $\text{norm}(K_{p/q})$ of G .

A *proper variant* of G is a subgraph H' of G obtained by removing a non-empty set of indifferent edges (i.e., any graph H' with $H \subseteq H' \subsetneq G$).

Let $p = \omega q + r$, where $0 \leq r \leq q-1$.

Claim 3.2 *The normalized subgraph H of G is the circulant graph $C[p, S]$, where $S = \{q, q+1, \dots, q+r, 2q, 2q+1, \dots, 2q+r, \dots, (\omega-1)q, (\omega-1)q+1, \dots, (\omega-1)q+r\}$.*

Consider an edge $0t$. We have $t = kq + r'$, with $1 \leq k \leq \omega-1$ and $0 \leq r' \leq q-1$.

If $0 \leq r' \leq r$, then the set $\{0, q+r', 2q+r', \dots, (\omega-1)q+r'\}$ induces a maximum clique containing the edge $0t$, and so the edge $0t$ is not indifferent.

Conversely, if $r+1 \leq r' \leq q-1$, then let K be a clique containing $0, t$. The other vertices of K belong to the intervals $[q, (k-1)q+r']$ and

$[(k+1)q+r', (\omega-1)q+r]$. Therefore, K has at most $\omega-1$ vertices, namely, at most $k-1$ vertices in the interval $[q, (k-1)q+r']$ and at most $\omega-k-2$ vertices from the interval $[(k+1)q+r', (\omega-1)q+r]$. Thus K is not a maximum clique and so $0t$ is an indifferent edge. \diamond

In particular, due to Lemma 3.1 if $p = 3q+1$ then $\text{norm}(K_{p/q})$ is isomorphic to $\overline{K}_{p/3}$, which proves assertion (iii).

Claim 3.3 *Suppose I is a maximal stable set of H and $i, i+t \in I$ for some $t \leq r+1$. Then $i+j \in I$ for all $0 \leq j \leq t$.*

If x is adjacent to $i+j$ in H for some $0 \leq j \leq t$, then x is adjacent to either i or $i+t$ in H . \diamond

Claim 3.4 *Suppose I is a stable set of H . There is a vertex i of H such that $i+j \notin I$ for any $1 \leq j \leq r$.*

Otherwise, Claim 3.3 would imply that all vertices of H belong to a maximal stable set I' containing I , an obvious contradiction. \diamond

Claim 3.5 *If I is a stable set of H , then $|I| \leq q$.*

As H is a circulant graph, by Claim 3.4, we may assume without loss of generality that $S \cap I = \emptyset$, where $S = \{\omega q, \omega q + 1, \dots, \omega q + r - 1\}$.

But $V(H) - S$ can be decomposed into the disjoint union of q cliques of H , namely, $K_i = \{i, i+q, i+2q, \dots, i+(\omega-1)q\}$, for $i = 0, 1, \dots, q-1$. As $|I \cap K_i| \leq 1$ for each $i \in \{0, 1, \dots, q-1\}$, so $|I| \leq q$. \diamond

Claim 3.6 *We have $\chi_c(H) = \chi_c(K_{p/q}) = p/q$.*

Since $\chi_c(K_{p/q}) = p/q$, we have $\chi_c(H) \leq p/q$. On the other hand, $\chi_c(H) \geq \chi_f(H) = |V(H)|/\alpha(H) \geq p/q$ due to Claim 3.5 (where χ_f denotes the fractional chromatic number, a lower bound of the circular chromatic number [20]). So equality holds everywhere. \diamond

Therefore the removal of indifferent edges of a circular clique does not alter its circular chromatic number, but clearly its circular clique number. This implies that normalization destroys circular-perfection:

Claim 3.7 *If $p \not\equiv -1 \pmod{q}$ and $\lfloor p/q \rfloor \geq 3$ then $K_{p/q}$ is not normalized and every of its proper variants is circular-imperfect.*

We denote by $\Delta(G)$ the maximum degree of a graph G . We have $\Delta(K_{p/q}) = p - (2q - 1)$ and $\Delta(H) = (r+1)(\omega-1)$, where $p = \omega q + r$ and r is the remainder modulo q , by Claim 3.2. Therefore, if $K_{p/q}$ is normalized (i.e., if $K_{p/q} = \text{norm}(K_{p/q})$) then $p - (2q - 1) = (r+1)(\omega-1)$, that is $(\omega-2)q = (r+1)(\omega-2)$. Since $\omega = \lfloor p/q \rfloor \geq 3$, this implies that $r = q-1$, and so $p = -1$

(mod q), a contradiction.

Hence $K_{p/q}$ is not normalized and the result follows from Claim 3.6: if H' is any proper variant of $K_{p/q}$ then

$$\omega_c(H') < p/q = \chi_c(H) = \chi_c(H').$$

◇

This completes the proof of the "if part" of Theorem 2.1 (i). We now treat the "only if part" of assertion (i).

Claim 3.8 *If $\lfloor p/q \rfloor < 3$ or $p = -1 \pmod{q}$ then $\text{norm}(K_{p/q})$ is circular-perfect.*

Notice that $\omega = \lfloor p/q \rfloor$ is the clique number of $K_{p/q}$. Therefore, if $\omega < 3$ then $\text{norm}(K_{p/q}) = K_{p/q}$. Thus $\text{norm}(K_{p/q})$ is circular-perfect.

If $p = -1 \pmod{q}$ then $\text{norm}(K_{p/q}) = K_{p/q}$ follows due to the description of $\text{norm}(K_{p/q})$ for general p and q in Claim 3.2. Thus $\text{norm}(K_{p/q})$ is circular-perfect. ◇

This completes the proof of Theorem 2.1 (i). We now treat the "only if part" of assertion (ii).

Claim 3.9 *If $p \neq 1, -1 \pmod{q}$ and $\omega = \lfloor p/q \rfloor \geq 3$ then $K_{p/q}$ has a circular clique $K_{(\omega q' + 1)/q'}$ as an induced subgraph with at least one indifferent edge of $K_{p/q}$, and $q' \geq 3$.*

Let G denote the circular clique $K_{p/q}$ and let $2 \leq r \leq q - 2$ such that $p = q\omega + r$. Notice that $q \neq 2r$ as p and q are relatively prime.

Case 1. If $r < \frac{q}{2}$ then let $q' = \lceil \frac{q}{r} \rceil$. We have $q' \geq 3$. For every $0 \leq i < \omega$, let $X_i = \{iq, iq + r, \dots, iq + (q' - 1)r\}$ and define $X = (\bigcup_{0 \leq i < \omega} X_i) \cup \{\omega q\}$. We first show that X induces a circular clique $K_{(\omega q' + 1)/q'} \subseteq G$.

For every $0 \leq x < p$, we denote by S_x the maximum stable set $\{x, x + 1, \dots, x + q - 1\}$ of G (arithmetics performed modulo p). Due to Trotter [8], it is enough to check that for every $x \in X$, S_x meets X in exactly q' vertices.

Let $x \in X$: by the definition of X , there exist $0 \leq i \leq \omega$ and $0 \leq \delta < q'$ such that $x = iq + \delta r$.

- If $i < \omega - 1$ then notice that $S_x \subseteq S_{iq} \cup S_{(i+1)q}$. Hence

$$\begin{aligned} S_x \cap X &= (S_{iq} \cap S_x \cap X) \cup (S_{(i+1)q} \cap S_x \cap X) \\ &= \{iq + \lambda r \mid \delta \leq \lambda < q'\} \cup \{(i+1)q + \lambda r \mid 0 \leq \lambda < \delta\} \end{aligned}$$

as for every $0 \leq \lambda < q'$, we have $(i+1)q + \lambda r \in S_x$ if and only if

$$0 \leq (i+1)q + \lambda r - x = q + (\lambda - \delta)r < q \text{ holds.}$$

Therefore S_x meets X in exactly q' vertices.

- If $i = \omega - 1$ and $\delta = 0$ then

$$S_x \cap X = S_{iq} \cap X = \{iq + \lambda r \mid 0 \leq \lambda < q'\}$$

holds and, again S_x meets X in exactly q' vertices.

- If $i = \omega - 1$ and $\delta > 0$ then $x = (\omega - 1)q + \delta r$. We have $S_x = \{(\omega - 1)q + \delta r, (\omega - 1)q + \delta r + 1, \dots, (\omega - 1)q + \delta r + q - 1\}$ (with arithmetics performed modulo p). Hence S_x is the disjoint union $S'_x \cup S''_x$ where $S'_x = \{(\omega - 1)q + \delta r, (\omega - 1)q + \delta r + 1, \dots, \omega q + r - 1\}$ and $S''_x = \{0, 1, \dots, (\delta - 1)r - 1\}$ ($S''_x = \emptyset$ if $\delta = 1$).

We have

$$\begin{aligned} X \cap S_x &= (X_{\omega-1} \cup X_0 \cup \{\omega q\}) \cap S_x \\ &= (X_{\omega-1} \cap S'_x) \cup (X_0 \cap S''_x) \cup \{\omega q\} \end{aligned}$$

and thus, $X \cap S$ is of size q' as

- $X_{\omega-1} \cap S'_x = \{(\omega - 1)q + \lambda r \mid \delta \leq \lambda < q'\}$ is of size $q' - \delta$;
- $X_0 \cap S''_x = \{\lambda r \mid 0 \leq \lambda < \delta - 1\}$ is of size $\delta - 1$.

Therefore S_x meets X in exactly q' vertices.

- If $i = \omega$ and $\delta = 0$ then $x = \omega q$. We have

$$\begin{aligned} S_x \cap X &= (\{\omega q, \omega q + 1, \dots, \omega q + r - 1\} \cap X) \\ &\quad \cup (\{0, 1, \dots, q - r - 1\} \cap X) \\ &= \{\omega q\} \cup \{\lambda r \mid 0 \leq \lambda r \leq q - r - 1 \text{ and } 0 \leq \lambda < q'\} \\ &= \{\omega q\} \cup \{\lambda r \mid 0 \leq \lambda \leq \lfloor q/r \rfloor - 1 = q' - 2 \text{ and } 0 \leq \lambda < q'\} \\ &= \{\omega q\} \cup \{\lambda r \mid 0 \leq \lambda \leq q' - 2\} \end{aligned}$$

which also implies that S_x meets X in exactly q' vertices.

Hence S_x always meets X in exactly q' vertices and so X induces a circular clique $G' = K_{(\omega q' + 1)/q'}$ of G according to [8]. As $\omega \geq 3$ and $0 < r < q/2$, we have $q + r < q + 2r < 2q$. Since $q' \geq 3$, the vertex $q + 2r$ belongs to G' . Hence the edge $\{0, q + 2r\}$ of G' is an indifferent edge of G by Claim 3.2.

Case 2. If $r > \frac{q}{2}$ then we show that $K_{(3\omega+1)/3}$ is an induced subgraph of G .

For $j = 0, 1, \dots, 3\omega$, let $x_j = \lfloor pj / (3\omega + 1) \rfloor$. Let $X = \{x_0, x_1, \dots, x_{3\omega}\}$.

We show that X induces a circular clique $K_{(3\omega+1)/3}$ of G : this is equivalent to show that for every $0 \leq i, j \leq 3\omega$, $\{x_i, x_j\}$ is an edge of G if and only if $3 \leq |i - j| \leq 3\omega - 2$.

To prove this, we shall use the following simple observation several times: if a and b are reals and δ is an integer such that $a - b \geq \delta$ then $\lfloor a \rfloor - \lfloor b \rfloor \geq \delta$.

- Let $0 \leq i, j \leq 3\omega$ such that $\{x_i, x_j\}$ is an edge of G and assume w.l.o.g. that $i < j$. We have $x_i < x_j$ and $q \leq x_j - x_i \leq p - q$.

If $j - i \leq 2$, then $pj/(3\omega + 1) - pi/(3\omega + 1) \leq 2(q\omega + r)/(3\omega + 1)$ follows. If $2(q\omega + r)/(3\omega + 1) > q - 1$ then as $\omega \geq 3$ and $q \geq r + 2$, a short computation gives $r < 1$ a contradiction. Thus $2(q\omega + r)/(3\omega + 1) \leq q - 1$ and so $x_j - x_i \leq q - 1$, a contradiction. Hence $j - i \geq 3$.

If $j - i \geq 3\omega - 1$, then $pj/(3\omega + 1) - pi/(3\omega + 1) \geq (3\omega - 1)(q\omega + r)/(3\omega + 1) \geq p - q + 1$ follows. Thus $x_j - x_i \geq p - q + 1$, a contradiction.

Therefore, we infer $3 \leq j - i \leq 3\omega - 2$.

- Conversely, let $0 \leq i, j \leq 3\omega$ such that $3 \leq j - i \leq 3\omega - 2$ and assume w.l.o.g. that $i < j$. We have $x_i < x_j$ and we need to check that $\{x_i, x_j\}$ is an edge of G .

On the one hand, $j - i \geq 3$ and $3r \geq q$ imply

$$pj/(3\omega + 1) - pi/(3\omega + 1) \geq 3(q\omega + r)/(3\omega + 1) \geq q$$

and, hence, $x_j - x_i \geq q$ follows.

On the other hand, $j - i \leq 3\omega - 2$ yields

$$pj/(3\omega + 1) - pi/(3\omega + 1) \leq (3\omega - 2)(q\omega + r)/(3\omega + 1) \leq p - q$$

and shows $x_j - x_i \leq p - q$.

Therefore $\{x_i, x_j\}$ is an edge of G , as required; and X induces a circular clique $G' = K_{(3\omega+1)/3}$ of G .

At last, we need to exhibit an indifferent edge of G in G' .

By Claim 3.2, the neighbours of 0 in $\text{norm}(G)$ are the vertices in $S = \{q, q+1, \dots, q+r, 2q, 2q+1, \dots, 2q+r, \dots, (\omega-1)q, (\omega-1)q+1, \dots, (\omega-1)q+r\}$.

We have $2q - 5p/(3\omega + 1) = (\omega q + 2q - 5r)/(3\omega + 1) > 0$ as $\omega \geq 3$ and $r \leq q - 2$. Hence $x_5 < 2q$.

If $x_5 \geq q + r + 1$ then $x_5 \notin S$ and $\{x_0, x_5\}$ is an edge of G' which is also an indifferent edge of G .

It remains to check the case $x_5 \leq q + r$: identifying an edge of G' which is also an indifferent edge of G is more difficult to handle. We are going to exhibit one in an induced circular clique G'' sharing all vertices but one with G' .

For $t = 1, 2, \dots, \omega - 2$, let $\delta_t = x_{3t+2} - (tq + r + 1)$. As $x_5 \leq q + r$, we have $\delta_1 < 0$.

We first check that $\delta_{\omega-2} \geq 0$: we have $\frac{p(3\omega-4)}{3\omega+1} - (\omega - 2)q - r - 1 = 2q - 1 - \frac{5p}{3\omega+1}$. If $5p/(3\omega + 1) > 2q - 1$ then $5q - 10 > \omega q - 3\omega + 2q - 1$ (as $r \leq q - 2$) which is equivalent to $0 > (q - 3)(\omega - 3)$. This is a contradiction as both q and ω are at least 3. Hence $\frac{p(3\omega-4)}{3\omega+1} - (\omega - 2)q - r - 1 \geq 0$ and therefore

$\delta_{\omega-2} \geq 0$.

Let t^* be the largest index such that $\delta_{t^*} < 0$: we have $1 \leq t^* < \omega - 2$. Let $x'_{3t^*+2} = t^*q + r + 1$ and let $X' = (X - \{x_{3t^*+2}\}) \cup \{x'_{3t^*+2}\}$. Let G'' be the induced subgraph of G by X' . To prove that G'' is an induced circular clique $K_{(3\omega+1)/3}$ of G , we have to check that the neighborhood of x'_{3t^*+2} in G'' is the same than the one of x_{3t^*+2} in G , namely $\{x_0, x_1, \dots, x_{3t^*-1}\} \cup \{x_{3t^*+5}, x_{3t^*+6}, \dots, x_{3\omega}\}$.

If $\frac{(3t^*+5)p}{3\omega+1} - (t^*q+r+1) < q$ then we have $\frac{(3(t^*+1)+2)p}{3\omega+1} - ((t^*+1)q+r+1) < 0$. Thus we infer $\delta_{t^*+1} < 0$, in contradiction with the maximality of t^* . Hence $x_{3t^*+2} \leq x'_{3t^*+2} \leq x_{3t^*+5} - q$, and so x'_{3t^*+2} is adjacent to $\{x_0, x_1, \dots, x_{3t^*-1}\} \cup \{x_{3t^*+5}, x_{3t^*+6}, \dots, x_{3\omega}\}$ and x'_{3t^*+2} is not adjacent to x_{3t^*+3} and x_{3t^*+4} .

We have $t^*q + r + 1 - \frac{p3t^*}{3\omega+1} = r + 1 + \frac{t^*(q-3r)}{3\omega+1} < q$ as $r \leq q - 2$ and $r > q/3$. Hence x'_{3t^*+2} is not adjacent to x_{3t^*} and x_{3t^*+1} .

Therefore G'' induces a circular clique $K_{(3\omega+1)/3}$ of G . As $t^*q + r < x'_{3t^*+2} = t^*q + r + 1 < (t^* + 1)q$ the edge $\{x_0, x'_{3t^*+2}\}$ of G'' is an indifferent edge of $K_{p/q}$. This finished the second case.

Thus in both cases $K_{p/q}$ contains an induced circular clique $K_{(\omega q'+1)/q'}$ with $q' \geq 3$ and an indifferent edge of $K_{p/q}$. \diamond

Claim 3.10 *If $H = \text{norm}(K_{p/q})$ is minimal circular-imperfect then H is a partitionable web $C_{\omega q+1}^\omega$, and $q \geq 3$.*

Since H is circular-imperfect we have $p \not\equiv -1 \pmod{q}$ and $\omega \geq 3$ due to Claim 3.8.

If H is not partitionable then $p \not\equiv 1 \pmod{q}$. By the previous claim, $K_{p/q}$ has an induced subgraph $K_{(\omega q'+1)/q'}$ with $q' \geq 3$ and vertex set W , containing an indifferent edge. As all non-indifferent edges of $K_{(\omega q'+1)/q'}$ are non-indifferent edges of $K_{p/q}$ (since these two graphs have same maximum clique size), the subgraph $H[W]$ of G , which is induced by W , is a proper variant of $K_{(\omega q'+1)/q'}$, and is, therefore, circular-imperfect by Claim 3.7. Hence $K_{p/q} = K_{(\omega q'+1)/q'}$, and $q = q' \geq 3$.

This implies that H is partitionable.

It follows that $q \geq 3$ (as $q = 2$ implies that H is an odd antihole and, therefore, circular-perfect, a contradiction). Due to Claim 3.1, this shows that H is a partitionable web $C_{\omega q+1}^\omega$ with $q \geq 3$. \diamond

Claim 3.11 *A claw-free graph does not contain any circular cliques different from cliques, odd holes, and odd antiholes.*

Assume $K_{p/q}$ is a circular clique different from a clique, an odd hole, and an odd antihole. Then $q \geq 3$ and $p \geq 2q + 2$. Thus $\{1, q + 1, q + 2, q + 3\}$

induces a claw. \diamond

Claim 3.12 *If $H = \text{norm}(K_{p/q})$ is a minimal circular-imperfect graph, then H has clique number 3.*

We first recall the following result of Trotter [8]: let $C_{n'}^{\omega'}$ ($2k' \leq n'$) and C_n^ω ($2k \leq n$) be two webs, then $C_{n'}^{\omega'}$ is an induced subgraph of C_n^ω if and only if holds

$$(3) \quad \frac{\omega' - 1}{\omega - 1}n \leq n' \leq \frac{\omega'}{\omega}n$$

By Claim 3.10, $H = \text{norm}(K_{p/q})$ is a partitionable web $C_{\omega q+1}^\omega$, with $q \geq 3$. If $\omega \leq 2$ then H is a stable set or an odd hole and is therefore circular-perfect, a contradiction. Hence $\omega \geq 3$.

Assume that $\omega \geq 4$.

Due to Trotter's inequality (3), the web C_{3q-1}^3 is an induced subweb of H if and only if holds

$$\frac{2}{\omega - 1}(q\omega + 1) \leq 3q - 1 \leq \frac{3}{\omega}(q\omega + 1)$$

Since the right inequality is always satisfied, this may be restated as $\frac{2}{\omega-1}(q\omega + 1) \leq 3q - 1$ which is equivalent to $1 + 4/(\omega - 3) \leq q$.

If $q \geq 5$ (resp. $\omega \geq 5$) then $q \geq 1 + 4/(\omega - 3)$ as $4/(\omega - 3) \leq 4$ (resp. $q \geq 3$ and $4/(\omega - 3) \leq 2$). Hence C_{3q-1}^3 is a proper induced subweb of H . If C_{2k+1}^k is any induced odd antihole of C_{3q-1}^3 then $k < 3$ due to Trotter's inequality (3). Hence the previous claim implies that $\omega_c(C_{3q-1}^3) = 3$. If C_{3q-1}^3 is 3-colorable, then it admits a partition in 3 stable sets of size at most $q - 1 = \lfloor (3q - 1)/3 \rfloor$, a contradiction. Hence $\chi(C_{3q-1}^3) \geq 4$ and so $\chi_c(C_{3q-1}^3) > 3 = \omega_c(C_{3q-1}^3)$. Thus C_{3q-1}^3 is a proper induced circular-imperfect graph of H , a contradiction.

Therefore, $\omega = 4$ and ($q = 3$ or $q = 4$), that is $H = C_{13}^4$ or $H = C_{17}^4$:

- C_{13}^4 is not minimal circular-imperfect as the subgraph induced by vertices $\{1, 2, 4, 5, 7, 9, 10, 12\}$ is circular-imperfect, since it has circular-clique number 3 and is not 3-colorable;
- C_{17}^4 is not minimal circular-imperfect as the subgraph induced by vertices $\{1, 2, 3, 5, 6, 8, 9, 11, 13, 14, 16\}$ is circular-imperfect, since it has circular-clique number 3 and is not 3-colorable.

In both cases, we get a contradiction and infer, therefore, $\omega = 3$. \diamond

This completes the proof of the "only if part" of assertion (ii). We now proceed to the proof of the "if part".

Claim 3.13 *Webs C_{3q+1}^3 with $q \geq 3$ are minimal circular-imperfect.*

Let $q \geq 3$. The web C_{3q+1}^3 is circular-imperfect by Claim 3.7.

If C_{3q+1}^3 is not minimal circular-imperfect, then there exists a proper induced subgraph W , which is minimal circular-imperfect. Let v be a vertex of C_{3q+1}^3 not in W .

If $\omega(W) = 3$ then $\omega(W) = 3 \leq \omega_c(W) \leq \chi_c(W) \leq \chi(C_{3q+1}^3 \setminus \{v\}) = 3$, a contradiction with the fact that W is minimal circular-imperfect.

If $\omega(W) = 2$ then let w be any vertex of W . If w is of degree at least 3 then w belongs to a triangle of W , as the neighborhood of any vertex of C_{3q+1}^3 can be covered with only 2 cliques (i.e. C_{3q+1}^3 is a quasi-line graph), a contradiction. Therefore, the degree of W is at most 2 and so W is a disjoint union of cycles and paths, and thus is circular-perfect, a contradiction.

Hence C_{3q+1}^3 is minimal circular-imperfect. \diamond

This finally proves Theorem 2.1. \square

3.2 Proof of Theorem 2.3

Proof. Let G be a partitionable graph. We shall prove that G is circular-imperfect unless G is a circular clique. If $\omega_c(G) = \omega(G)$, then we have $\chi_c(G) > \omega(G) = \omega_c(G)$ by $\chi(G) = \omega(G) + 1$, therefore G is circular-imperfect.

Assume that $\omega_c(G) = p/q > \omega$ and let $\{0, \dots, p-1\}$ be the vertices of an induced circular clique $K_{p/q}$ (where the vertices are labeled the usual way). For every $0 \leq i < \omega$, let Q_i be the maximum clique $\{jq \mid 0 \leq j \leq i\} \cup \{jq+1 \mid i < j < \omega\}$. Obviously $Q_0, \dots, Q_{\omega-1}$ are ω distinct maximum cliques of G containing the vertex 0.

If $p > \omega q + 1$ then the set $(Q_0 \setminus \{(\omega-1)q+1\}) \cup \{(\omega-1)q+2\}$ is another maximum clique containing 0, a contradiction as 0 belongs to exactly ω maximum cliques of G [2]. Hence $p = \omega q + 1$. This means that G contains the partitionable circular clique $K_{(\omega q+1)/q}$ as an induced subgraph. Hence G is the circular clique $K_{(\omega q+1)/q}$. \square

3.3 Proof of Corollary 2.4

Proof. Let G be a circular-perfect normalized partitionable graph. We conclude that G is an odd hole or odd antihole. By Theorem 2.3, G is a circular clique $K_{p/q}$. If $\omega(G) \geq 3$, since $p \equiv 1 \pmod{q}$ (as G is partitionable) and G is circular-perfect, it follows from Theorem 2.1 (i) that $p \equiv -1 \pmod{q}$, and so $q = 2$. Hence G is an odd antihole. If $\omega(G) = 2$ then G is an odd hole. \square

4 Some minimal circular-imperfect planar graphs

4.1 Proof of Theorem 2.5

Proof. In order to show the circular-perfection of outerplanar graphs, we first discuss the circular clique number of planar graphs.

Claim 4.1 *The circular clique number of a planar graph G is equal to*

- 1, if G is a stable set,
- 2, if G is bipartite,
- 4, if G has an induced K_4 ,
- else $2 + \frac{1}{d}$ where $2d + 1$ is the odd girth of G , i.e. $2d + 1$ is the size of a shortest chordless odd cycle in G .

This claim follows from the easy to prove fact that the only planar circular cliques are odd holes and cliques of size at most 4 (see [14] for instance). \diamond

It is well known that the identification of two disjoint perfect graphs G_1 and G_2 in a clique yields a perfect graph G again [5] (if $Q_1 \subseteq G_1 = (V_1, E_1)$ and $Q_2 \subseteq G_2 = (V_2, E_2)$ are two cliques of same size and ϕ is any bijection from Q_2 onto Q_1 , the identification of G_1 and G_2 in Q_1 w.r.t. ϕ is the graph $G = (V, E)$ where $V = (V_1 \cup V_2) \setminus Q_2$ and $E = E_1 \cup (E_2 \setminus \{ij | \{i, j\} \cap Q_2 \neq \emptyset\}) \cup \{\phi(i)j | ij \in E_2, |i \in Q_2, j \notin Q_2\}$).

We prove that the same holds for circular-perfect planar graphs.

Claim 4.2 *If G_1 and G_2 are two planar circular-perfect graphs, then identifying G_1 and G_2 in a clique K yields a circular-perfect graph G .*

If G_1 and G_2 are both bipartite then G is perfect and therefore circular-perfect. Hence we may assume that G_1 is not bipartite. In particular, $\omega(G) > 1$.

All we have to prove is that $\omega_c(G) = \chi_c(G)$.

If $\omega(G) = 4$ then $\omega_c(G) = \chi_c(G) = 4$ as $\omega(G) = 4 \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) = 4$. Hence we may assume that $\omega(G) \leq 3$.

If $\omega(G) = 3$ then G is 3-colorable as both G_1 and G_2 are 3-colorable. Hence $\omega_c(G) = \chi_c(G) = 3$ as $3 = \omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) = 3$.

It remains to handle the case $\omega(G) = 2$. Then the clique K is of size at most 2.

If G_2 is bipartite then it is homomorphic to an edge, and so G is homomorphic to G_1 . Hence $\chi_c(G) \leq \chi_c(G_1)$ and so $\omega_c(G_1) \leq \omega_c(G) \leq \chi_c(G) \leq \chi_c(G_1) = \omega_c(G_1)$.

If G_2 is not bipartite then let $2d_1 + 1$ be the odd girth of G_1 and let $2d_2 + 1$ be the odd girth G_2 . W.l.o.g. assume that $2d_1 + 1 \leq 2d_2 + 1$. There exists an homomorphism f_1 (resp. f_2) from G_1 (resp. G_2) into C_{2d_1+1} .

If K is of size 2 (resp. of size 1) then let q_1 and q_2 be the vertices of K (resp. let q be the vertex of K). Let σ be an automorphism of C_{2d_1+1} such that $f_1(q_1) = \sigma(f_2(q_1))$ and $f_1(q_2) = \sigma(f_2(q_2))$ (there is one as $\{f_1(q_1), f_1(q_2)\}$ and $\{f_2(q_1), f_2(q_2)\}$ are two edges of C_{2d_1+1}) (resp. such that $f_1(q) = \sigma(f_2(q))$). Then the application f which maps a vertex x of G onto $f_1(x)$ if $x \in G_1$, $\sigma(f_2(x))$ if $x \in G_2$ is a homomorphism from G into C_{2d_1+1} . Therefore, we have $\omega_c(G) = 2 + \frac{1}{d_1} \leq \chi_c(G) \leq 2 + \frac{1}{d_1}$. \diamond

A connected outerplanar graph different from a cycle is always obtained by identifying two strictly smaller outerplanar graphs in one vertex or one edge. Therefore, the previous claim and the fact that cycles are circular-perfect imply circular-perfection of outerplanar graphs. \square

It remains to show that the graphs $T_{k,l}$ are minimal circular-imperfect.

Lemma 4.3 *For every positive integers k and l such that $(k, l) \neq (1, 1)$, the graph $T_{k,l}$ is minimal circular-imperfect.*

Proof. If the graph $T_{k,l}$ has a $(2k + 1, k)$ -coloring then assume without loss of generality that the central vertex gets the color 0. Every neighbour of the central vertex is colored with k or $k + 1$, and two such neighbours belonging to a common inner face must have distinct colors (a $(2k + 1, k)$ -coloring can be seen as a homomorphism h to the odd hole C_{2k+1} , see Remark 1.1; the restriction of h to an odd hole of size $2k + 1$, e.g. any inner face of $T_{k,l}$, is bijective). Since the central vertex has an odd number of neighbours on the outer face, we get a contradiction. Hence graphs $T_{k,l}$ have $\omega_c(T_{k,l}) = 2 + 1/k$ (as $(k, l) \neq (1, 1)$) which is strictly less than $\chi_c(T_{k,l})$ and so are circular-imperfect.

Minimal circular-imperfection follows then from Theorem 2.5 as the removal of any vertex yields an outerplanar graph. \square

5 Complete joins and minimal circular-imperfection

5.1 Proof of Theorem 2.6

Proof. Our goal is to show that a complete join $G * G'$ is circular-perfect iff both G and G' are perfect and minimal circular-imperfect iff $G * G'$ is an odd wheel or odd antiwheel.

Claim 5.1 *An odd wheel $C_{2k+1} * v$ is minimal circular-imperfect if $k \geq 2$.*

This follows from the fact that the odd wheels $C_{2k+1} * v$ are precisely the graphs $T_{1,k}$. \diamond

Claim 5.2 *An odd antiwheel $\overline{C}_{2k+1} * v$ is minimal circular-imperfect if $k \geq 2$.*

Since \overline{C}_{2k+1} is an odd antihole for $k \geq 2$, we have $\omega(\overline{C}_{2k+1} * v) = k + 1$ and $\chi(\overline{C}_{2k+1} * v) = k + 2$. Moreover, $\omega_c(\overline{C}_{2k+1} * v) = \max\{k + 1, k + \frac{1}{2}\} = k + 1$ and $\chi_c(\overline{C}_{2k+1} * v) > \chi(\overline{C}_{2k+1} * v) - 1 = k + 1$. Thus $\omega_c(\overline{C}_{2k+1} * v) < \chi_c(\overline{C}_{2k+1} * v)$ implies that $\overline{C}_{2k+1} * v$ is circular-imperfect. Minimality follows since removing any vertex yields a perfect graph or \overline{C}_{2k+1} , hence all proper induced subgraphs of $\overline{C}_{2k+1} * v$ are circular-perfect. \diamond

This implies the following for the complete joins of an imperfect graph with a single vertex:

Claim 5.3 *If G is an imperfect graph, then $G * v$ is circular-imperfect and minimal if and only if G is an odd hole or odd antihole.*

Due to the Strong Perfect Graph Theorem, G contains an odd hole or odd antihole C as induced subgraph. Thus $G * v$ has $C * v$ as induced subgraph which is circular-imperfect by Claim 5.1 or Claim 5.2. $G * v$ is, therefore, circular-imperfect as well and minimal if and only if $C * v = G * v$ (i.e. $C = G$). \diamond

This proves assertion (ii), provided assertion (i) holds true.

Claim 5.4 *If both graphs G and G' are imperfect, then $G * G'$ is circular-imperfect but never minimal.*

Let v' be a vertex of G' . Then $G * v'$ is a proper induced subgraph of $G * G'$ and circular-imperfect by Claim 5.3. Thus $G * G'$ is circular-imperfect but never minimal. \diamond

Consider the complete join $G * G'$ of two graphs G and G' . If both graphs G and G' are perfect, then $G * G'$ is perfect as well. If one of G and G' is imperfect, then $G * G'$ is circular-imperfect by Claim 5.3. This proves assertion (i). \square

6 Concluding remarks and further work

We shortly summarize the results obtained in this paper:

- Theorem 2.1 studies the circular-imperfection of normalized circular cliques; we conclude that the webs C_{3q+1}^3 with $q \geq 3$ are the only minimal circular-imperfect graphs in this class (Theorem 2.1 and Corollary 2.2).
- Theorem 2.3 shows that no partitionable graphs different from circular cliques are circular-perfect.
- In Theorem 2.5, we prove that outerplanar graphs are circular-perfect and use them to build our second class of minimal circular-imperfect graphs, the planar graphs $T_{k,l}$ with $(k,l) \neq (1,1)$.
- At last, in Theorem 2.6, we study circular-imperfection of complete joins and prove that the minimal circular-imperfect complete joins are precisely odd wheels and odd antiwheels.

The last two families were independently found by B. Xu [19]; since these results are easy consequences of our considerations on planar graphs and complete joins, we have included our (short) proofs in this paper.

At first sight there is no straightforward common structure in the presented families of minimal circular-imperfect graphs, hence formulating an analogue to the Strong Perfect Graph Theorem for circular-perfect graphs seems to be difficult.

The Strong Perfect Graph Conjecture is equivalent to "every minimal imperfect graph or its complement has clique number 2". As every known minimal circular-imperfect graph or its complement has clique number 2 or 3, one might be tempted to ask whether it holds for every minimal circular-imperfect graph. However, Pan and Zhu [13] found recently a way to construct minimal circular-imperfect graphs with arbitrarily large clique and stability number.

This adds further support to the believe that characterizing circular-perfect graphs by means of forbidden subgraphs is, indeed, a difficult task.

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