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► **To cite this version:**

Louis Esperet, Pascal Ochem. Oriented coloring of 2-outerplanar graphs. Information Processing Letters, 2007, 101, pp.215–219. hal-00307156

**HAL Id: hal-00307156**

**<https://hal.science/hal-00307156>**

Submitted on 28 Jul 2008

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# Oriented coloring of 2-outerplanar graphs

Louis Esperet\*, and Pascal Ochem†  
LaBRI UMR CNRS 5800, Université Bordeaux I,  
33405 Talence Cedex  
FRANCE.

19th January 2006

## Abstract

A graph  $G$  is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar. The oriented chromatic number of an oriented graph  $H$  is defined as the minimum order of an oriented graph  $H'$  such that  $H$  has a homomorphism to  $H'$ . In this paper, we prove that 2-outerplanar graphs are 4-degenerate. We also show that oriented 2-outerplanar graphs have a homomorphism to the Paley tournament  $QR_{67}$ , which implies that their (strong) oriented chromatic number is at most 67.

**Keywords:** combinatorial problems, oriented coloring, 2-outerplanar graphs.

## 1 Introduction

Oriented graphs are directed graphs without opposite arcs. In other words an oriented graph is an orientation of an undirected graph, obtained by assigning to every edge one of the two possible orientations. If  $G$  is a graph,  $V(G)$  denotes its vertex set,  $E(G)$  denotes its set of edges. A homomorphism from an oriented graph  $G$  to an oriented graph  $H$  is a mapping  $\varphi$  from  $V(G)$  to  $V(H)$  which preserves the arcs, that is  $(x, y) \in E(G) \implies (\varphi(x), \varphi(y)) \in E(H)$ . We say that  $H$  is a *target graph* of  $G$  if there exists a homomorphism from  $G$  to  $H$ . The oriented chromatic number  $\chi_o(G)$  of an oriented graph  $G$  is defined as the minimum order of a target graph of  $G$ . The oriented chromatic number  $\chi_o(G)$  of an undirected graph  $G$  is then defined as the maximum oriented chromatic number of its orientations. Nešetřil and Raspaud introduced in [4] the *strong oriented chromatic number* of an oriented graph  $G$  (denoted by  $\chi_s(G)$ ), which definition differs from that of  $\chi_o(G)$  by requiring that the target graph is an oriented Cayley graph. Upper bounds on the (strong) oriented chromatic number have been found for various subclasses of planar graphs. In particular:

1. if  $G$  is a planar graph, then  $\chi_o(G) \leq 80$  [6].
2. if  $G$  is an outerplanar graph, then  $\chi_s(G) \leq 7$  [7].

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\* [esperet@labri.fr](mailto:esperet@labri.fr)

† [ochem@labri.fr](mailto:ochem@labri.fr)

A graph  $G$  is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar. The second author proved that 2-outerplanar graphs have an acyclic partition into three independent sets and an outerplanar graph [5]. By Theorem 1 in [1], the oriented chromatic number of a 2-outerplanar graph is thus at most  $2^{4-1} \times (1 + 1 + 1 + 7) = 80$ . The same result follows from the bound of Raspaud and Sopena [6] holding for planar graphs.

In Section 2, we prove among other results that any 2-outerplanar graph  $G$  is 4-degenerate, *i.e.* every subgraph  $H$  of  $G$  has minimum degree at most 4. In Section 3, we use these results to show that 2-outerplanar graphs have a homomorphism to  $QR_{67}$ , which improves the previous bounds of 80.

## 2 Structural properties of 2-outerplanar graphs

**Definition 1** A 2-outerplanar graph embedded in the plane is said to be a block if its external face is an induced cycle.

**Theorem 1** If  $G$  is a 2-outerplanar graph, then it contains a  $\leq 4$ -vertex.

**Proof.** Let  $G$  be a 2-outerplanar graph embedded in the plane. We consider the subgraph  $H$  induced by the external face of  $G$ .  $H$  is an outerplanar graph, so it contains an internal face  $F$  incident to at most one other internal face of  $H$  (see Proof of Lemma 2 in [3]). Let  $B$  be the subgraph of  $G$  induced by the vertices of  $F$  and the vertices inside  $F$ . By construction, the graph  $B$  obtained is a block. Moreover,  $B$  contains only two vertices  $x$  and  $x'$  such that the degree of  $x$  and  $x'$  in  $G$  may be higher than their degree in  $B$ . By construction,  $x$  and  $x'$  are two adjacent vertices belonging to the external face of  $B$  (see Figure 1).

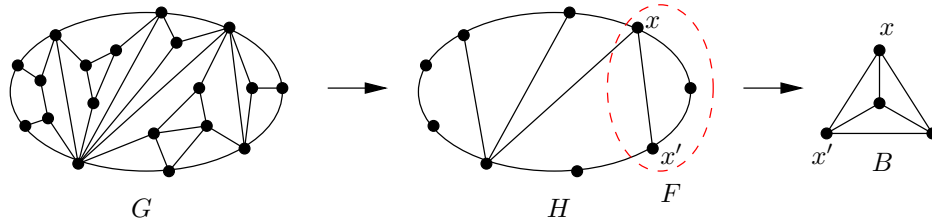


Figure 1: The decomposition of a 2-outerplanar graph into blocks.

Let  $B_c$  be the graph induced by the external face of  $B$ , and  $B_o$  be the graph obtained from  $B$  by removing the vertices of  $B_c$ . By definition of 2-outerplanar graphs,  $B_o$  is outerplanar. So it contains two non-adjacent 2-vertices  $u$  and  $v$  (see Figure 2).

As mentioned above, vertices of  $B_o$  have the same degree in  $B$  and in  $G$ , so  $d_B(u) = d_G(u)$  and  $d_B(v) = d_G(v)$ . Let us find a  $\leq 4$ -vertex in  $B$ . If  $B_o$  contains a  $\leq 4$ -vertex, it is done. Else, it means that  $B_o$  contains only  $\geq 5$ -vertices; in particular  $u$  (resp.  $v$ ) is adjacent to three vertices  $u_1, u_2, u_3$  (resp.  $v_1, v_2, v_3$ ), where  $u_1 u_2 u_3$  (resp.  $v_1 v_2 v_3$ ) is an induced  $P_3$  of  $B_c$  (see Figure 3).

We now use the fact that  $B$  contains only two vertices  $x$  and  $x'$  having a degree in  $G$  possibly higher than their degree in  $B$ . As  $xx'$  is an edge of  $B_c$ , this means that  $u_2$  or  $v_2$  have

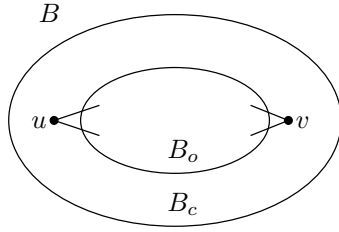


Figure 2: The decomposition of  $B$  into  $B_c$  and  $B_o$ .

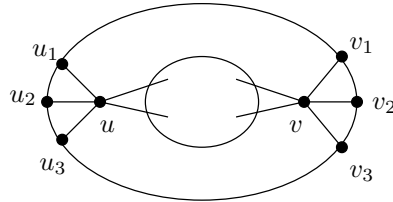


Figure 3:  $u$  and  $v$  have three neighbors in  $B_c$ .

the same degree in  $B$  and in  $G$ , i.e.  $d_G(u_2) = d_B(u_2) = 3$  or  $d_G(v_2) = d_B(v_2) = 3$ . Hence  $B$  always contains a vertex with degree at most 4 in  $G$ .  $\square$

We now prove that outerplanar graphs have properties stronger than 2-degeneration, in order to find more precise configurations in 2-outerplanar graphs.

**Lemma 1** *Let  $G$  be an outerplanar graph.  $G$  contains either a 1-vertex, two adjacent 2-vertices, a 2-vertex adjacent to a 3-vertex as depicted in Figure 4.a, or two 2-vertices adjacent to a 4-vertex as depicted in Figure 4.b.*

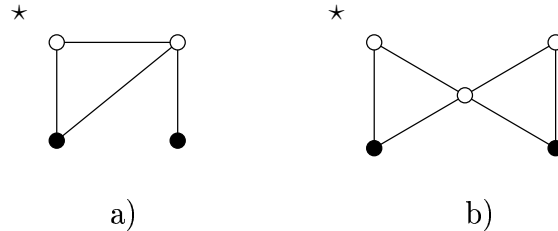


Figure 4: Unavoidable configurations in an outerplanar graph without two adjacent 2-vertices. The star symbol indicates the external face.

**Proof.** We prove this lemma by induction. Let  $G$  be an outerplanar graph, and let  $v$  be a 2-vertex of  $G$  ( $v$  exists, see [3] for details). The graph  $H = G \setminus v$  is outerplanar, and smaller than  $G$ . By induction,  $H$  contains either two adjacent 2-vertices, or the configurations of Figure 4. If  $v$  is not adjacent to such a configuration of  $H$ , then it is a configuration of  $G$ , and the induction is finished. Else  $v$  is adjacent to a configuration, and we have to make the distinction between various cases. Notice that the neighbors of  $v$  must be adjacent in  $H$  in

order to obtain an outerplanar graph.

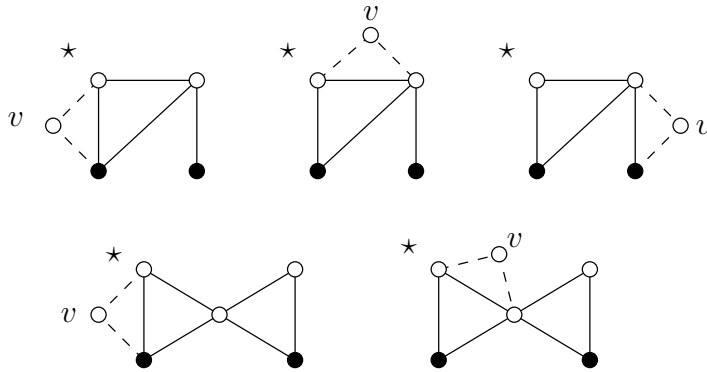


Figure 5: Induction step in the proof of Lemma 1.

- If  $H$  contains two adjacent 2-vertices, we obtain the configuration of Figure 4.a.
- If  $H$  contains a configuration of Figure 4, we obtain either the configuration of Figure 4.a, or the configuration of Figure 4.b (see Figure 5).

In any case,  $G$  contains one of the three configurations described earlier. □

We now use Lemma 1 to prove a key structural theorem on 2-outerplanar graphs admitting a block embedding in the plane. The following result can be extended to the whole class of 2-outerplanar graphs by using the same kind of proof as in Theorem 1.

**Theorem 2** *Let  $G$  be a 2-outerplanar graph admitting a block embedding in the plane.  $G$  contains either a  $\leq 3$ -vertex, two adjacent 4-vertices, or the configuration depicted in Figure 6.*

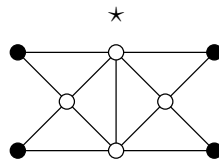


Figure 6: Unavoidable configuration in a 2-outerplanar block containing neither a  $\leq 3$ -vertex nor two adjacent 4-vertices.

**Proof.** We consider a block embedding of  $G$  in the plane. Then the subgraph induced by the external face is a cycle. Let  $G_c$  be this cycle and let  $G_o$  be the graph obtained from  $G$  by removing the vertices of  $G_c$ . By definition of  $G$  and  $G_c$ , the graph  $G_o$  is outerplanar. We then know by Lemma 1 that  $G_o$  contains either two adjacent 2-vertices, a 2-vertex having a neighbor of degree 3 as depicted in Figure 4.a, or two 2-vertices having a common neighbor of degree 4 as depicted in Figure 4.b.

- If  $G_o$  contains a 1-vertex or two adjacent 2-vertices, we easily find a  $\leq 3$ -vertex or two adjacent 4-vertices in  $G$ .
- If  $G_o$  contains a 2-vertex  $v$  adjacent to a 3-vertex  $u$ , we can prove that either  $d_G(v) = 4$  or there is a vertex of degree 3 in  $G$  (which is a neighbor of  $v$  belonging to the external face). This is done by applying the same method as in the previous proof. Thus  $G$  must contain the configuration depicted in Figure 7. Notice that  $u$  and  $w$  are neighbors, else one of them would have degree at most 3. For reasons of planarity, if  $u$  is adjacent to another vertex of  $G_c$ ,  $w$  cannot be adjacent to another vertex of  $G_o$ . Conversely, if  $w$  is adjacent to another vertex of  $G_o$ ,  $u$  cannot be adjacent to a vertex of  $G_c$ . This proves that either  $u$  or  $w$  has degree 4 in  $G$ , say  $u$ . If there is no 3-vertex in  $G$ , we found two adjacent 4-vertices:  $u$  and  $v$ .

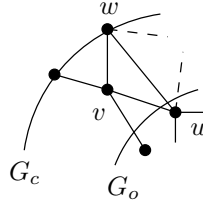


Figure 7:  $G_o$  contains a 2-vertex  $v$  adjacent to a 3-vertex  $u$ .

- If  $G_o$  contains two 2-vertices  $v$  and  $v'$  both adjacent to a 4-vertex  $u$  as depicted in Figure 4.b, we first prove that either  $v$  and  $v'$  have degree 4 in  $G$ , or  $G$  contains a 3-vertex (in which case the proof is finished). Let  $v_1$  and  $v_2$  (resp.  $v'_1$  and  $v'_2$ ) be the neighbors of  $v$  (resp.  $v'$ ) belonging to the external face. As depicted in Figure 8, we have to make a distinction between two cases :  $\{v_1, v_2\}$  and  $\{v'_1, v'_2\}$  are disjoint (case 1), or they have a vertex in common, say  $v_2 = v'_1$  (case 2).

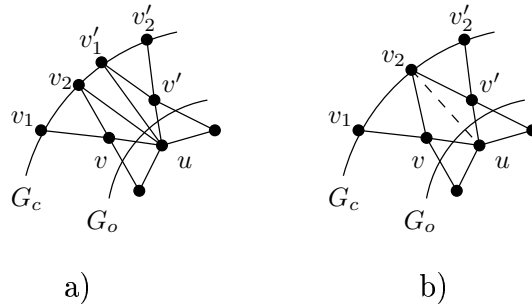


Figure 8:  $G_o$  contains two 2-vertices  $v$  and  $v'$  adjacent to a common 4-vertex  $u$ .

**case 1** (see Figure 8.a) If  $v_2$  and  $v'_1$  have degree at least 4 in  $G$ , they both have to be adjacent to  $u$ , in which case  $d_G(v_2) = d_G(v'_1) = 4$ , and we found two adjacent 4-vertices in  $G$ .

**case 2** (see Figure 8.b) If  $u$  is adjacent to  $v_2 = v'_1$ , we obtain exactly the configuration depicted in Figure 6. Otherwise, we simply have two adjacent 4-vertices ( $v$  and  $v_2$ ).

□

### 3 Strong oriented coloring of 2-outerplanar graphs

**Theorem 3** *If  $G$  is a 2-outerplanar graph, then  $\chi_s(G) \leq 67$ .*

For a prime power  $q \equiv 3 \pmod{4}$ , the vertices of the Paley tournament  $QR_q$  are the elements of  $\mathbb{F}_q$  and  $(i, j)$  is an arc in  $QR_q$  if and only if  $j - i$  is a non-zero quadratic residue of  $\mathbb{F}_q$ . An *orientation vector* of size  $k$  is a sequence  $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$  in  $\{0, 1\}^k$ . Let  $G$  be an oriented graph and  $X = (x_1, x_2, \dots, x_k)$  be a sequence of pairwise distinct vertices of  $G$ . A vertex  $y$  of  $G$  is said to be an  $\alpha$ -*successor* of  $X$  if for every  $i$ ,  $1 \leq i \leq k$ , we have  $\alpha_i = 1 \Rightarrow (x_i, y) \in E(G)$  and  $\alpha_i = 0 \Rightarrow (y, x_i) \in E(G)$ . The graph  $G$  satisfies property  $S_{k,n}$  if for every sequence  $X = (s_1, s_2, \dots, s_k)$  of  $k$  pairwise distinct vertices of  $G$ , and for every orientation vector  $\alpha$  of size  $k$ , there exist at least  $n$  vertices in  $V(G)$  which are  $\alpha$ -successors of  $X$ .

A computer check proves the following lemma:

**Lemma 2** *The tournament  $QR_{67}$  satisfies properties  $S_{3,6}$  and  $S_{4,1}$ .*

We use the method of reducible configurations to show that every 2-outerplanar graph is  $QR_{67}$ -colorable. We define the partial order  $\prec$  for the set of all graphs. Let  $n_3(G)$  be the number of  $\geq 3$ -vertices in  $G$ . For any two graphs  $G_1$  and  $G_2$ , we have  $G_1 \prec G_2$  if and only if at least one of the following conditions hold:

- $G_1$  is a proper subgraph of  $G_2$ .
- $n_3(G_1) < n_3(G_2)$ .

Note that this partial order is well-defined, since if  $G_1$  is a proper subgraph of  $G_2$ , then  $n_3(G_1) \leq n_3(G_2)$ . So  $\prec$  is a partial linear extension of the subgraph poset.

Let  $G$  be a 2-outerplanar graph having no homomorphism to  $QR_{67}$ , which is minimal with this property according to  $\prec$ .

**Lemma 3**  *$G$  is 2-connected and does not contain a cut consisting in two adjacent vertices.*

**Proof.** If  $G$  is not 2-connected, then we can obtain a  $QR_{67}$ -coloring of  $G$  from the coloring of its 2-connected components, since  $QR_{67}$  is a circular tournament. Moreover  $G$  cannot contain a cut set consisting of two adjacent vertices, since  $QR_{67}$  is an arc-transitive tournament. □

Notice that Lemma 3 implies that every 2-outerplanar embedding of  $G$  is a block.

**Lemma 4**

1. *The graph  $G$  does not contain any  $\leq 3$ -vertex.*
2. *The graph  $G$  does not contain two adjacent 4-vertices.*
3. *The graph  $G$  does not contain the configuration depicted in Figure 6.*

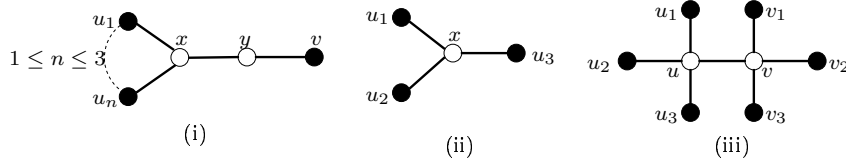


Figure 9: Forbidden configurations for Lemma 4.

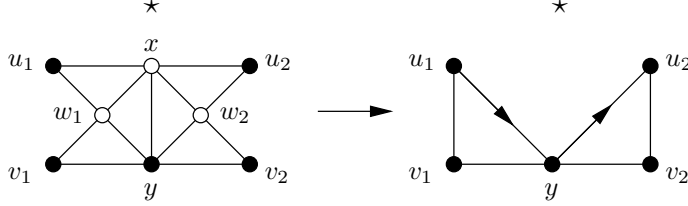


Figure 10: Construction of  $G'$  in the proof of Lemma 4.

**Proof.**

1. Consider configuration (i) in Figure 9. Let  $f$  be any  $QR_{67}$ -coloring of  $G \setminus \{y\}$ . By property  $S_{3,6}$ , we can choose  $f$  such that  $f(x) \neq f(v)$  and extend this coloring to  $G$ . Consider now configuration (ii) in Figure 9. Notice that  $u_1, u_2$ , and  $u_3$  are  $\geq 3$ -vertices, since configuration (i) with  $n = 2$  is forbidden. Since  $QR_{67}$  is self-reverse, we assume w.l.o.g. that  $d^-(x) \leq d^+(x)$  by considering either  $G$  or  $G^R$ . We have  $d^-(x) \neq 0$ , since otherwise we could extend any  $QR_{67}$ -coloring of  $G \setminus \{x\}$  to  $G$ . Suppose now  $d^-(x) = 1$ , which is the only remaining case. Let us set  $N^-(x) = \{u_1\}$ ,  $N^+(x) = \{u_2, u_3\}$ . We now consider the graph  $G'$  obtained from  $G \setminus \{x\}$  by adding directed 2-paths joining respectively  $u_1$  and  $u_2$ , and  $u_1$  and  $u_3$ . Notice that if  $G$  is a block, then  $G'$  is a block. Moreover  $G' \prec G$  since  $n_3(G') = n_3(G) - 1$ . Any  $QR_{67}$ -coloring  $f$  of  $G'$  induces a coloring of  $G \setminus \{x\}$  such that  $f(u_1) \neq f(u_2)$  and  $f(u_1) \neq f(u_3)$ , which can be extended to  $G$ .
2. Consider configuration (iii) in Figure 9. Let  $f$  be any  $QR_{67}$ -coloring of  $G \setminus \{uv\}$  (that is we delete the edge  $uv$ ). By property  $S_{3,6}$ , we can choose  $f$  such that  $f(u) \notin \{f(v_1), f(v_2), f(v_3)\}$ . Now by property  $S_{4,1}$ , we can choose  $f$  such that  $f(v) \notin \{f(u), f(u_1), f(u_2), f(u_3)\}$  and extend this coloring to  $G$ .
3. Consider the configuration depicted in Figure 6. Let  $G'$  be the graph obtained from  $G \setminus \{w_1, w_2, x\}$  by adding the arcs  $\overrightarrow{u_1y}$  and  $\overrightarrow{yu_2}$ , and the arc  $\overrightarrow{u_1v_1}$  (resp.  $\overrightarrow{u_2v_2}$ ) if  $u_1$  and  $v_1$  (resp.  $u_2$  and  $v_2$ ) are not adjacent in  $G$ . This construction is depicted in Figure 10. Notice that if  $G$  is a block, then  $G'$  is a block. Moreover  $G' \prec G$ , since  $n_3(G') = n_3(G) - 3$ . Thus  $G'$  admits a  $QR_{67}$ -coloring which induces a  $QR_{67}$ -coloring  $f$  of  $G \setminus \{w_1, w_2, x\}$  such that  $f(u_1), f(v_1), f(y)$  (resp.  $f(u_2), f(v_2), f(y)$ ; resp.  $f(u_1), f(u_2), f(y)$ ) are pairwise distinct. By Property  $S_{3,6}$ , we can assign  $x$  a color  $f(x) \notin \{f(u_1), f(u_2), f(y)\}$ . By Property  $S_{4,1}$ , we can assign  $w_1$  a color  $f(w_1) \notin \{f(u_1), f(v_1), f(y), f(x)\}$  and assign  $w_2$  a color  $f(w_2) \notin \{f(u_2), f(v_2), f(y), f(x)\}$ . We thus obtain a  $QR_{67}$ -coloring of  $G$ , which is a contradiction.



□

By Lemma 3  $G$  is a block. Using Theorem 2,  $G$  must contain one of the configurations that are forbidden by Lemma 4. This contradiction completes the proof of Theorem 3.

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