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Mathematical programming formulations for the orthogonal 2d knapsack problem

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The 2d orthogonal packing problem consist in selecting a maximal value subset of items that can be packed in a unit square. Each item, $i = 1, \dots, n$, is defined by its value, $v_i \geq 0$, its width, $0 < w_i \leq 1$, and its height, $0 < h_i \leq 1$. A subset of items, $S \subset \{1, \dots, n\}$ admits a feasible orthogonal packing iff [10] there exists lower-left corner coordinates (x_i, y_i) for each $i \in S$ such that : **(Closeness)** each item remains within the unit square, i.e., $x_i + w_i \leq 1$ and $y_i + h_i \leq 1$ for each $i \in S$; **(No-overlap)** No two items overlap, i.e., $x_i + w_i \leq x_j$ or $x_j + w_j \leq x_i$ or $y_i + h_i \leq y_j$ or $y_j + h_j \leq y_i$. The term “orthogonal” comes from the cutting stock terminology : it means that the items could be cut from a unit square stock piece using orthogonal cuts, i.e., cuts that are made in direction orthogonal the the stock piece edges. It is assumed that the items cannot be rotated to exchange their width and height. The generalization to a non unit square is trivial. The problem is actually strongly NP-Hard, and in practice much harder than its 1-d counter part : the combinatoric not only concern item selection decisions but also their relative position to form a feasible packing. Moreover, there is an inherent symmetry between solutions that can be obtained by permuting the positions of items. The aim of this paper is to review the existing integer programming formulations for this problem, comparing the size and the resulting linear programming relaxation bound. This analysis of existing approaches should highlight the scope for further progress in math programming approaches to solving the 2-d orthogonal packing problem. Such approaches can also be combined/hybridized with exact algorithms based on combinatorial dual bounds [10, 11, 5] and intelligent enumeration schemes [7, 8, 12, 14], or constraint programming [9].

A straightforward formulation that implements directly the above definition of feasible packing is to use variables δ_{ixy} indicating whether item i has its lower left corner in position (x, y) [3]. The model assumes a discretisation of the X and Y dimensions to yield a finite number of possible coordinates (x, y) but this entails a pseudopolynomial size formulation :

$$\max\left\{\sum_{i,x,y} v_i \delta_{ixy} : \sum_i \sum_{\nu=x-w_i+1}^x \sum_{\tau=y-h_i+1}^y \delta_{i\nu\tau} \leq 1 \forall (x,y) \in X \times Y, \sum_{(x,y)} \delta_{ixy} \leq 1 \forall i, \delta_{ixy} \in \{0,1\} \forall i, x, y\right\}.$$

Besides its exponential size, this formulation also suffers a symmetry drawback : symmetric solution are represented by different δ value. Beyond this initial model, many alternative models have been developed, some of them do not enforce true placement and hence model a combinatorial relaxation for the problem. Some retains the discretization idea : [6] proposed a formulation in variable δ_{ix} , δ_{iy} and $\delta_{xy} = 1$ iff position (x, y) is unoccupied (it was latter shown to model only a relaxation). A polynomial formulation is used in [14] where coordinates are seen as continuous variables and binary variables model the relative position of items : $l_{ij} = 1$ iff item i is to the left of j and $b_{ij} = 1$ iff item i is below j . All of these approaches suffer the same symmetry drawback.

Column generation formulations have also been used : [16] generate solutions of the knapsack problem defined by the item height (they define vertical filling patterns) and model the covering

of the selected item width by vertical strips (the decision variables are λ_v is the width of vertical pattern v); [15] extended this idea to generating both horizontal and vertical filling patterns; [4] combined the latter with the use of relative position variables $x_{ij} = 1$ (resp. $y_{ij} = 1$) iff item i and j are on each others side along the X axis (resp. Y axis). Column generation approaches have also been combined with coordinates allocation approaches : [13] make use of variables are $\lambda_{vx} = 1$ iff vertical pattern v appears in coordinate x . Finally, another approach consists in reducing the 2-d knapsack to a 1-d problem by defining item weighs as their surface area. The relaxation can then be improved by adding cuts (dual feasible cuts [2, 5, 10], cover cuts [2, 14], or cuts on the objective [1]), using dominance rules, or reduced cost fixing [2]. In our presentation we shall provide the detailed formulations, analyze their relative advantages, and extend the ideas to new ways of modeling the 2-dimensional knapsack problem.

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