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# THE CYLINDER OVER THE KORAS-RUSSELL CUBIC THREEFOLD HAS A TRIVIAL MAKAR-LIMANOV INVARIANT

ADRIEN DUBOULOZ

ABSTRACT. We show that the cylinder  $X \times \mathbb{A}^1$  over the Koras-Russell cubic threefold

$$X = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{A}^4$$

has a trivial Makar-Limanov invariant  $\text{ML}(X \times \mathbb{A}^1) = \mathbb{C}$ . This means equivalently that the only regular functions on  $X \times \mathbb{A}^1$  that are invariant under *all* algebraic actions of the additive group  $\mathbb{G}_a$  on  $X \times \mathbb{A}^1$  are constants.

## INTRODUCTION

The *Koras-Russell cubic threefold* is the subvariety  $X = \text{Spec}(A)$  of the affine space  $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$  defined by the equation  $x + x^2y + z^2 + t^3 = 0$ . It first appeared in the work of Koras and Russell [9, 10] on the linearization problem for algebraic actions of the multiplicative group  $\mathbb{G}_m$  on the affine space  $\mathbb{A}^3$ . The question at that time was to decide if  $X$  is algebraically isomorphic to  $\mathbb{A}^3$  or not, and a positive answer would have led to an example of a non linearizable  $\mathbb{G}_m$ -action on  $\mathbb{A}^3$ . One of the difficulties is that when equipped with the euclidean topology  $X$  is diffeomorphic to the euclidean space  $\mathbb{R}^6$  (see e.g. [1]). So it is impossible to distinguish  $X$  from  $\mathbb{A}^3$  by topological invariants. Actually, it turned out that all classical algebraic invariants fail to distinguish  $X$  from  $\mathbb{A}^3$ .

Nowadays, the fact that  $X$  is not algebraically isomorphic to  $\mathbb{A}^3$  can be derived from a result of Kaliman [5] which says that if the general fibers of regular function  $f : \mathbb{A}^3 \rightarrow \mathbb{A}^1$  are isomorphic to the affine plane  $\mathbb{A}^2$ , then *all* the closed fibers of  $f$  are isomorphic to  $\mathbb{A}^2$ . On the other hand, it is easily seen that the closed fibers of the projection  $\text{pr}_x : X \rightarrow \mathbb{A}^1$  are isomorphic to  $\mathbb{A}^2$  except for  $\text{pr}_x^{-1}(0)$  which is isomorphic to the cylinder  $C \times \mathbb{A}^1$  over the cuspidal cubic curve  $C \simeq \text{Spec}(\mathbb{C}[z, t] / (z^2 + t^3))$ .

But the problem was originally solved by Makar-Limanov [12] by a different method, based on the study of algebraic actions of the additive group  $\mathbb{G}_a$  on  $X$ . He established that  $X$  is not algebraically isomorphic to  $\mathbb{A}^3$  because it admits “fewer” algebraic  $\mathbb{G}_a$ -actions than  $\mathbb{A}^3$ . More precisely, Makar-Limanov introduced a new invariant of affine algebraic varieties  $V$  defined as the sub-algebra  $\text{ML}(V)$  of the coordinate ring of  $V$  consisting of regular functions on  $V$  which are invariants under *all* algebraic  $\mathbb{G}_a$ -actions on  $V$ . For affine spaces, this invariant consists of constants only. In contrast, Makar-Limanov established that  $\text{ML}(X)$  is isomorphic to the polynomial ring  $\mathbb{C}[x]$ . To compute  $\text{ML}(X)$ , Makar-Limanov used the correspondence between algebraic  $\mathbb{G}_a$ -actions on an affine variety  $V$  and *locally nilpotent  $\mathbb{C}$ -derivations* of its coordinate ring  $\mathbb{C}[V]$ , that is, derivations  $\partial : \mathbb{C}[V] \rightarrow \mathbb{C}[V]$  such that every element of  $\mathbb{C}[V]$  is annihilated by a suitable power of  $\partial$ . Under this correspondence,  $\mathbb{G}_a$ -invariant regular functions coincide with the elements of the kernel  $\text{Ker} \partial$  of the associated locally nilpotent derivation, and  $\text{ML}(V)$  can be equivalently defined as the intersection in  $\mathbb{C}[V]$  of the kernels of all locally nilpotent derivations of  $\mathbb{C}[V]$ .

It is easy to see that  $\text{ML}(X) \subset \mathbb{C}[x]$ . For instance, the locally nilpotent derivations  $x^2\partial_z - 2z\partial_y$  and  $x^2\partial_t - 3t^2\partial_y$  of  $\mathbb{C}[x, y, z, t]$  annihilate the defining equation  $x + x^2y + z^2 + t^3 = 0$  of  $X$  and induce non trivial locally nilpotent derivations  $\partial_1$  and  $\partial_2$  of the coordinate ring  $A$  of  $X$  such that  $\text{Ker}(\partial_1) \cap \text{Ker}(\partial_2) = \mathbb{C}[x]$ . The main achievement of Makar-Limanov was to show that  $\partial(x) = 0$  for every locally nilpotent derivation of  $A$ . The original proof has been simplified and generalized by many authors, but the key arguments remain quite elaborate and depend on techniques of equivariant deformations to reduce the problem to the study of homogeneous  $\mathbb{G}_a$ -actions on certain affine cones associated with  $X$  (see e.g., [6], [7] and [14]).

Now, given a new variable  $w$ , we can identify  $X \times \mathbb{A}^1$  with the subvariety of  $\mathbb{A}^5 = \text{Spec}(\mathbb{C}[x, y, z, t, w])$  defined by the equation  $x + x^2y + z^2 + t^3 = 0$ . Again, it is not difficult to see that  $\text{ML}(X \times \mathbb{A}^1) \subset \mathbb{C}[x]$ , and it is natural to ask if  $\text{ML}(X \times \mathbb{A}^1) \neq \mathbb{C}$  or not. It turned out that Makar-Limanov techniques are inefficient in this context, and very few progress has been made on this particular problem since the late nineties. In this note, we prove the following result.

**Theorem.**  $\text{ML}(X \times \mathbb{A}^1) = \mathbb{C}$ .

A consequence of this result is that the Makar-Limanov invariant carries no useful information to decide if  $X \times \mathbb{A}^1$  is an *exotic*  $\mathbb{A}^4$ , i.e. a variety diffeomorphic to  $\mathbb{R}^8$  but not algebraically isomorphic to  $\mathbb{A}^4$ .

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## 1. A DANIELEWSKI TRICK PROOF THAT $\text{ML}(X \times \mathbb{A}^1) = \mathbb{C}$

Before giving the proof, we find it enlightening to review Danielewski's classical counter-example to the Zariski Cancellation Problem. Indeed, it formally contains, in a simpler form, all the ingredients needed for the proof of the Theorem.

### 1.1. Danielewski's construction.

**1.1.** Danielewski [2] established that the smooth affine surfaces  $S_1 = \{xz = y^2 - 1\}$  and  $S_2 = \{x^2z = y^2 - 1\}$  in  $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$  provide a counter example to the generalized Cancellation Problem, that is,  $S_1 \times \mathbb{A}^1$  is isomorphic to  $S_2 \times \mathbb{A}^1$  but  $S_1$  is not isomorphic to  $S_2$ . To show that  $S_1 \times \mathbb{A}^1$  is isomorphic to  $S_2 \times \mathbb{A}^1$ , he exploited the fact that  $S_1$  and  $S_2$  can be equipped with set-theoretically free  $\mathbb{G}_a$ -actions induced by the  $\mathbb{G}_a$ -actions on  $\mathbb{A}^3$  associated with the locally nilpotent  $\mathbb{C}[x]$ -derivations  $x\partial_y + 2y\partial_z$  and  $x^2\partial_y + 2y\partial_z$  of  $\mathbb{C}[x, y, z]$  respectively. The fibers of the  $\mathbb{G}_a$ -invariant projections  $\pi_i = \text{pr}_x|_{S_i}: S_i \rightarrow \mathbb{A}^1 = \text{Spec}(\mathbb{C}[x])$ ,  $i = 1, 2$  coincide with the orbits of the  $\mathbb{G}_a$ -actions except  $\pi_i^{-1}(0)$  which consists of the disjoint union of two distinct orbits. In particular,  $\pi_i: S_i \rightarrow \mathbb{A}^1$  is not a  $\mathbb{G}_a$ -bundle. However, Danielewski observed that the  $\pi_i$ 's factor through Zariski locally trivial  $\mathbb{G}_a$ -bundles  $\rho_i: S_i \rightarrow \tilde{\mathbb{A}}^1$ ,  $i = 1, 2$ , over the affine line with a double origin, obtained from  $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[x])$  by replacing its origin by two closed points, one for each of the connected components of  $\pi_i^{-1}(0)$ .

**1.2.** In turn, this implies that there exists a cartesian diagram

$$\begin{array}{ccc} & S_1 \times_{\tilde{\mathbb{A}}^1} S_2 & \\ \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\ S_1 & & S_2 \\ \rho_1 \searrow & & \swarrow \rho_2 \\ & \tilde{\mathbb{A}}^1 & \end{array}$$

where  $S_1 \times_{\tilde{\mathbb{A}}^1} S_2$  is a  $\mathbb{G}_a$ -bundle over  $S_1$  and  $S_2$  via the first and the second projections respectively. Since  $S_1$  and  $S_2$  are both affine, it follows that  $S_1 \times_{\tilde{\mathbb{A}}^1} S_2$  is simultaneously isomorphic to the trivial  $\mathbb{G}_a$ -bundles  $S_1 \times \mathbb{A}^1$  and  $S_2 \times \mathbb{A}^1$  over  $S_1$  and  $S_2$  respectively (see e.g., XI.5.3 in [4]). This implies the existence of an isomorphism  $\Theta: S_1 \times \mathbb{A}^1 \xrightarrow{\sim} S_2 \times \mathbb{A}^1$  of  $\mathbb{A}^2$ -bundles over  $\tilde{\mathbb{A}}^1$ , whence of schemes over  $\mathbb{A}^1 \simeq \text{Spec}(\Gamma(\tilde{\mathbb{A}}^1, \mathcal{O}_{\tilde{\mathbb{A}}^1})) = \text{Spec}(\mathbb{C}[x])$ .

**1.3.** Although Danielewski argument was different, the fact that  $S_2$  and  $S_1$  are not isomorphic can be deduced from a result of Makar-Limanov [13] asserting that  $\text{ML}(S_2) = \mathbb{C}[x]$ , together with the observation that due to the symmetry between the variables  $x$  and  $z$  in the defining equation of  $S_1$ , one has  $\text{ML}(S_1) = \mathbb{C}$ . Since  $S_2 \times \mathbb{A}^1$  is isomorphic to  $S_1 \times \mathbb{A}^1$ , it follows in particular that  $\text{ML}(S_2 \times \mathbb{A}^1) \simeq \text{ML}(S_1 \times \mathbb{A}^1) = \mathbb{C}$ . This can be reinterpreted more explicitly as follows. Certainly, one has  $\text{ML}(S_2 \times \mathbb{A}^1) \subset \mathbb{C}[x]$ . On the other hand, the locally nilpotent derivation  $z\partial_y + 2y\partial_x$  of  $\mathbb{C}[x, y, z, w]$  induces a locally nilpotent derivation  $\delta_1$  of the coordinate ring  $\mathbb{C}[x, y, z, w]/(xz - y^2 + 1)$  of  $S_1 \times \mathbb{A}^1$  such that  $\delta_1(x) \neq 0$ . Since  $\Theta: S_1 \times \mathbb{A}^1 \xrightarrow{\sim} S_2 \times \mathbb{A}^1$  is an isomorphism of schemes over  $\text{Spec}(\mathbb{C}[x])$ , it follows that  $(\Theta^*)^{-1}\delta_1\Theta^*$  is a locally nilpotent derivation  $\delta$  of the coordinate ring of  $S_2 \times \mathbb{A}^1$  such that  $\delta(x) \neq 0$ , and so,  $\text{ML}(S_2 \times \mathbb{A}^1) = \mathbb{C}$ .

### 1.2. Proof of the Theorem.

**1.4.** For our purpose, it is more convenient to rewrite the defining equation of  $X = \text{Spec}(A)$  as  $x^2z = y^2 + x - t^3$ . This corresponds to making the coordinate change  $(x, y, z, t) \mapsto (-x, z, iy, t)$  on the ambient space  $\mathbb{A}^4$ . As observed in the introduction, one has certainly  $\text{ML}(X \times \mathbb{A}^1) \subset \mathbb{C}[x]$ . So  $\text{ML}(X \times \mathbb{A}^1) = \mathbb{C}$  provided that we can find a locally nilpotent derivation  $\partial$  of the coordinate ring  $A[w]$  of  $X \times \mathbb{A}^1$  such that  $\partial x \neq 0$ . We may even suppose that we are looking for such a derivation with the additional property that  $\partial t = 0$ . With this hypothesis, we can further reduce the problem to finding a locally nilpotent  $\mathbb{C}[t, t^{-1}]$ -derivation  $\delta$  of

$$A[w] \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \simeq \mathbb{C}[x, y, z, t^{\pm 1}][w] / (x^2z - y^2 - x + t^3)$$

such that  $\delta(x) \neq 0$ . Indeed, since  $A[w]$  is a finitely generated algebra, for a suitably chosen  $k \geq 0$ ,  $t^k\delta$  will extend to a locally nilpotent derivation  $\partial$  of  $A[w]$  such that  $\partial(x) \neq 0$ .

**1.5.** We let  $Y_* = \text{Spec}(\mathbb{C}[x, t^{\pm 1}]) \simeq \mathbb{A}^1 \times_{\mathbb{A}^*} \mathbb{A}^1$  and we consider the affine varieties  $X_1 = \text{Spec}(B_1)$  and  $X_2 = X \setminus \{t = 0\} = \text{Spec}(B_2)$  where

$$B_1 = \mathbb{C}[x, y, z, t^{\pm 1}] / (xz - y^2 + t^3) \quad \text{and} \quad B_2 = \mathbb{C}[x, y, z, t^{\pm 1}] / (x^2z - y^2 - x + t^3)$$

The locally nilpotent  $\mathbb{C}[x, t^{\pm 1}]$ -derivations  $x\partial_y + 2y\partial_z$  and  $x^2\partial_y + 2y\partial_z$  of  $\mathbb{C}[x, y, z, t^{\pm 1}]$  induce locally nilpotent derivations of  $B_1$  and  $B_2$  respectively, defining set-theoretically free  $\mathbb{G}_a$ -actions  $m_i : \mathbb{G}_a \times X_i \rightarrow X_i$ ,  $i = 1, 2$ . The  $\mathbb{G}_a$ -equivariant projections  $\pi_i = \text{pr}_{x,t} |_{X_i} : X_i \rightarrow Y_*$  restrict to trivial  $\mathbb{G}_a$ -bundles over  $Y_* \setminus \{x = 0\}$ . In contrast, the fibers of the  $\pi_i$ 's over every closed point of the punctured line  $\{x = 0\} \subset Y_*$  consist of the disjoint union of two  $\mathbb{G}_a$ -orbits, and their fiber over the point  $(x) \in Y_* = \text{Spec}(\mathbb{C}[x, t^{\pm 1}])$  is isomorphic to the affine line over the Galois extension  $\mathbb{C}(t)[y] / (y^2 - t^3) \simeq \mathbb{C}(t)[\mu] / (\mu^2 - t)$  of the residue field  $\kappa((x)) = \mathbb{C}(t)$ . Informally, this indicates that  $\pi_i : X_i \rightarrow Y_*$  should factor through a  $\mathbb{G}_a$ -bundle  $\rho_i : X_i \rightarrow \mathfrak{S}$ ,  $i = 1, 2$  over a geometric object  $\mathfrak{S}$  obtained from  $Y_*$  by replacing the point  $(x)$ , i.e., the punctured line  $\{x = 0\} \subset Y_*$ , not by two disjoint copies of itself as in Danielewski's construction, but rather by a nontrivial étale double covering of itself.

**1.6.** Clearly, an object  $\mathfrak{S}$  with the required property cannot exist in the category of schemes. However, one can construct such an  $\mathfrak{S}$  in the larger category of algebraic spaces as follows. We let  $Z_* = \text{Spec}(\mathbb{C}[x, \mu^{\pm 1}])$  and we let  $\mathfrak{S}$  be the quotient of  $Z_*$  by the étale equivalence relation  $(x, \mu) \sim (x, -\mu)$  if  $x \neq 0$ . More formally, this means that  $\mathfrak{S} = Z_*/R$  where  $(s, t) : R \rightarrow Z_* \times Z_*$  is the étale equivalence relation defined by

$$(s, t) : R = Z_* \sqcup Z_* \setminus \{x = 0\} \rightarrow Z_* \times Z_* \quad (x, \mu) \mapsto \begin{cases} ((x, \mu), (x, \mu)) & \text{if } (x, \mu) \in Z_* \subset R \\ ((x, \mu), (x, -\mu)) & \text{if } (x, \mu) \in Z_* \setminus \{x = 0\} \subset R \end{cases}$$

Now the  $R$ -invariant morphism  $Z_* \rightarrow Y_*$ ,  $(x, \mu) \mapsto (x, \mu^2)$  descends to a morphism  $\psi : \mathfrak{S} \rightarrow Y_*$  restricting to an isomorphism outside  $\{x = 0\}$  and with fiber over  $(x)$  isomorphic to  $\text{Spec}(\mathbb{C}(t)[\mu] / (\mu^2 - t))$  as desired.

*Remark 1.7.* An alternative construction of  $\mathfrak{S}$  is the following : First we let  $W$  be the scheme obtained by gluing two copies  $W_{\pm}$  of  $Z_* = \text{Spec}(\mathbb{C}[x, \mu^{\pm 1}])$  by the identity outside the punctured line  $\{x = 0\}$ . The group  $\mathbb{Z}_2$  acts freely on  $W$  by  $W_{\pm} \ni (x, \mu) \mapsto (x, -\mu) \in W_{\mp}$ , and  $\mathfrak{S}$  coincides with the quotient  $W/\mathbb{Z}_2$  taken in the category of algebraic spaces. Note that this  $\mathbb{Z}_2$ -action is properly discontinuous in the analytic topology on  $W$ , so that  $\mathfrak{S}$  equipped with the quotient analytic topology has the structure of a locally separated analytic space.

**1.8.** Let us assume for a moment that we have factorizations

$$\pi_i = \psi \circ \rho_i : X_i \xrightarrow{\rho_i} \mathfrak{S} \xrightarrow{\psi} Y_*, \quad i = 1, 2$$

where  $\rho_i : X_i \rightarrow \mathfrak{S}$ ,  $i = 1, 2$  is an étale locally trivial  $\mathbb{G}_a$ -bundle. Then  $X_1 \times_{\mathfrak{S}} X_2$  is an étale locally trivial  $\mathbb{G}_a$ -bundle over  $X_1$  and  $X_2$  via the first and the second projection respectively. Again, these bundles are both trivial as  $X_1$  and  $X_2$  are affine, and we obtain isomorphisms  $X_1 \times \mathbb{A}^1 \xrightarrow{\sim} X_1 \times_{\mathfrak{S}} X_2 \xleftarrow{\sim} X_2 \times \mathbb{A}^1$ . The induced isomorphism  $\Theta : X_1 \times \mathbb{A}^1 \xrightarrow{\sim} X_2 \times \mathbb{A}^1$  is an isomorphism of étale locally trivial  $\mathbb{A}^2$ -bundles over  $\mathfrak{S}$ , whence, in particular, of schemes over  $Y_*$ . Now the locally nilpotent  $\mathbb{C}[t^{\pm 1}]$ -derivation  $2y\partial_x + z\partial_y$  of  $\mathbb{C}[x, y, z, t^{\pm 1}, w]$  induces a locally nilpotent derivation  $d$  of the coordinate ring  $B_1[w]$  of  $X_1 \times \mathbb{A}^1$  such that  $d(x) \neq 0$ . Since  $\Theta : X_1 \times \mathbb{A}^1 \xrightarrow{\sim} X_2 \times \mathbb{A}^1$  is an isomorphism of schemes over  $Y_* = \text{Spec}(\mathbb{C}[x, t^{\pm 1}])$ , it follows that  $\delta = (\Theta^*)^{-1} d \Theta^*$  is a locally nilpotent  $\mathbb{C}[t^{\pm 1}]$ -derivation of the coordinate ring  $B_2[w]$  of  $X_2 \times \mathbb{A}^1$  such that  $\delta(x) \neq 0$ . Combined with the previous discussion, this shows that  $\text{ML}(X \times \mathbb{A}^1) = \mathbb{C}$ .

**1.9.** So it remains to check that the  $\mathbb{G}_a$ -invariant morphisms  $\pi_i : X_i \rightarrow Y_*$ ,  $i = 1, 2$ , admit the required factorization. It is a standard fact that a set-theoretically free  $\mathbb{G}_a$ -action on a scheme  $V$  admits a categorical quotient in the form of a  $\mathbb{G}_a$ -bundle  $\rho : V \rightarrow V/\mathbb{G}_a$  over an algebraic space  $V/\mathbb{G}_a$  (see e.g. 10.4 in [11]). Thus we only need to check that  $X_i/\mathbb{G}_a \simeq \mathfrak{S}$ ,  $i = 1, 2$ , and that the morphisms  $\bar{\pi}_i : X_i/\mathbb{G}_a \rightarrow Y_*$  induced by the  $\mathbb{G}_a$ -invariant morphisms  $\pi_i : X_i \rightarrow Y_*$  coincide with  $\psi : \mathfrak{S} \rightarrow Y_*$ . This can be seen as follows. Letting  $U = \mathbb{G}_a \times Z_* = \text{Spec}(\mathbb{C}[v][x, \mu^{\pm 1}])$ , one checks first that the  $\mathbb{G}_a$ -equivariant morphisms

$$\begin{aligned} \phi_1 : \mathbb{G}_a \times Z_* &\rightarrow X_1, (v, x, \mu) \mapsto (x, \mu^3 + xv, 2\mu^3v + xv^2, \mu^2) \\ \phi_2 : \mathbb{G}_a \times Z_* &\rightarrow X_2, (v, x, \mu) \mapsto \left( x, \mu^3 - \frac{1}{2\mu^3}x + x^2v, \frac{1}{4\mu^6} + (2\mu^3 - \mu^{-3}x)v + x^2v^2, \mu^2 \right) \end{aligned}$$

define étale trivializations of the  $\mathbb{G}_a$ -actions  $m_i : \mathbb{G}_a \times X_i \rightarrow X_i$  on  $X_i$ ,  $i = 1, 2$ . Then one checks easily that we have  $\mathbb{G}_a$ -equivariant isomorphisms

$$\begin{aligned} \xi_1 : \mathbb{G}_a \times R &\xrightarrow{\sim} U \times_{X_1} U \\ (x, \mu) &\mapsto \begin{cases} (\phi_1(v, x, \mu), \phi_1(v, x, \mu)) & \text{if } (v, x, \mu) \in \mathbb{G}_a \times Z_* \subset R \\ (\phi_1(v, x, \mu), \phi_1(v + 2\mu^3x^{-1}, x, -\mu)) & \text{if } (v, x, \mu) \in \mathbb{G}_a \times Z_* \setminus \{x = 0\} \subset R \end{cases} \end{aligned}$$

and

$$\xi_2 : \mathbb{G}_a \times R \xrightarrow{\sim} U \times_{X_2} U$$

$$(v, x, \mu) \mapsto \begin{cases} (\phi_2(v, x, \mu), \phi_2(v, x, \mu)) & \text{if } (v, x, \mu) \in \mathbb{G}_a \times Z_* \subset R \\ (\phi_2(v, x, \mu), \phi_2(v - \mu^{-3}x^{-1} + 2\mu^3x^{-2}, x, -\mu)) & \text{if } (v, x, \mu) \in \mathbb{G}_a \times Z_* \setminus \{x=0\} \subset R. \end{cases}$$

By construction, the projections  $(\text{pr}_1, \text{pr}_2) : U \times_{X_i} U \rightrightarrows U = \mathbb{G}_a \times Z_*$  are étale and descend to the ones  $(s, t) : R \simeq U \times_{X_i} U/\mathbb{G}_a \rightrightarrows Z_* = \mathbb{G}_a \times Z_*/\mathbb{G}_a$  in such a way that we have a cartesian diagram

$$\begin{array}{ccc} U \times_{X_i} U & \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \rightrightarrows \\ \xrightarrow{\text{pr}_2} \end{array} & U \\ \downarrow & & \downarrow \\ R = U \times_{X_i} U/\mathbb{G}_a & \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xrightarrow{t} \end{array} & Z_* = U/\mathbb{G}_a \end{array}$$

Since  $X_i$  coincides with the quotient of  $U \times_{X_i} U$  by the étale equivalence relation  $(\text{pr}_1, \text{pr}_2) : U \times_{X_i} U \rightrightarrows U$ , it follows from I.5.8 in [8] that the  $\mathbb{G}_a$ -bundle  $U \rightarrow Z_* = U/\mathbb{G}_a$  descends to a morphism of algebraic spaces  $\rho_i : X_i \rightarrow \mathfrak{S} = Z_*/R$ , and that we have a commutative diagram

$$\begin{array}{ccccc} U \times_{X_i} U & \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \rightrightarrows \\ \xrightarrow{\text{pr}_2} \end{array} & U & \longrightarrow & X_i \\ \downarrow & & \downarrow & & \downarrow \rho_i \\ R & \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xrightarrow{t} \end{array} & Z_* = U/\mathbb{G}_a & \longrightarrow & \mathfrak{S} = Z_*/R \end{array}$$

in which the right hand side square is cartesian. This implies that  $\rho_i : X_i \rightarrow \mathfrak{S}$  is an étale locally trivial  $\mathbb{G}_a$ -bundle, which shows that  $\mathfrak{S}$  is isomorphic to  $X_i/\mathbb{G}_a$ ,  $i = 1, 2$  as desired. Now the fact that  $\pi_i : X_i \rightarrow Y_*$  factors as  $\psi \circ \rho_i$ ,  $i = 1, 2$ , follows trivially from the construction.

*Remark 1.10.* The maps  $\rho_i : X_i \rightarrow \mathfrak{S}$ ,  $i = 1, 2$ , are holomorphic  $\mathbb{G}_a$ -bundles when the  $X_i$ 's and  $\mathfrak{S}$  are equipped with the analytic topology. Indeed, one can check that the  $\mathbb{G}_a$ -invariant maps  $\text{pr}_2|_{\tilde{X}_i} : \tilde{X}_i = X_i \times_{Y_*} Z_* \rightarrow Z_*$  obtained from the base change by the étale Galois covering  $Z_* \rightarrow Y_*$ ,  $(x, \mu) \mapsto (x, \mu^2)$  factor through  $\mathbb{Z}_2$ -equivariant holomorphic  $\mathbb{G}_a$ -bundles  $\tilde{\rho}_i : \tilde{X}_i \rightarrow W$  such that  $\rho_i = \tilde{\rho}_i/\mathbb{Z}_2 : X_i \simeq \tilde{X}_i/\mathbb{Z}_2 \rightarrow W/\mathbb{Z}_2 \simeq \mathfrak{S}$ ,  $i = 1, 2$ .

*Remark 1.11.* The above descriptions imply that the isomorphism classes of the  $\mathbb{G}_a$ -bundles  $\rho_1 : X_1 \rightarrow \mathfrak{S}$  and  $\rho_2 : X_2 \rightarrow \mathfrak{S}$  in  $H_{\text{ét}}^1(\mathfrak{S}, \mathbb{G}_a) \simeq H_{\text{ét}}^1(\mathfrak{S}, \mathcal{O}_{\mathfrak{S}})$  are represented by the non cohomologous Čech 1-cocycles

$$\{0, 2\mu^3x^{-1}\} \in \Gamma(R, \mathcal{O}_R) \quad \text{and} \quad \{0, -\mu^{-3}x^{-1} + 2\mu^3x^{-2}\} \in \Gamma(R, \mathcal{O}_R)$$

for the étale covering  $Z_* \rightarrow \mathfrak{S}$ . So the varieties  $X_1$  and  $X_2$  are not isomorphic as  $\mathbb{G}_a$ -bundles over  $\mathfrak{S}$ . Actually, one can check that  $\text{ML}(X_1) = \mathbb{C}[t^{\pm 1}]$  whereas  $\text{ML}(X_2) = \mathbb{C}[t^{\pm 1}][x]$ , so that  $X_1$  and  $X_2$  are not even isomorphic as abstract affine varieties. Thus they provide a counter-example to the Cancellation Problem for factorial affine threefolds (see [3] for other counter-examples).

## REFERENCES

1. A. Choudary and A. Dimca, *Complex hypersurfaces diffeomorphic to affine spaces*, Kodai Math. J. **17** (1994), 171–178.
2. W. Danielewski, *On a cancellation problem and automorphism groups of affine algebraic varieties*, Preprint Warsaw, 1989.
3. D. Finston and S. Maubach, *The automorphism group of certain factorial threefolds and a cancellation problem*, Israel J. Math. **163** (2008), no. 1, 369–382.
4. A. Grothendieck et al., *SGA1. Revêtements étales et Groupe Fondamental*, Lecture Notes in Math., vol. 224, Springer-Verlag, 1971.
5. S. Kaliman, *Polynomials with general  $\mathbb{C}^2$  fibers are variables*, Pacific J. Math. **203** (2002), 161–190.
6. S. Kaliman and L. Makar-Limanov, *On the Russel-Koras contractible threefolds*, J. Algebraic Geom. **6** (1997), 247–268.
7. S. Kaliman and L. Makar-Limanov, *AK-invariant of affine domains*, Affine Algebraic Geometry (in honor of M. Miyanishi) (T. Hibi, ed.), Osaka University Press, 2007, pp. 231–255.
8. D. Knutson, *Algebraic spaces*, Lecture Notes in Maths, vol. 203, Springer Verlag, 1971.
9. M. Koras and P. Russell, *On linearizing “good”  $\mathbb{C}^*$ -action on  $\mathbb{C}^3$* , Canadian Math. Society Conference Proceedings **10** (1989), 93–102.
10. M. Koras and P. Russell, *Contractible threefolds and  $\mathbb{C}^*$ -actions on  $\mathbb{C}^3$* , J. of Algebraic Geom. **6** (1997), 671–695.
11. G. Laumon and L. Moret-Bailly, *Champs algébriques*, Springer Verlag, 1999.
12. L. Makar-Limanov, *On the hypersurface  $x + x^2y + z^2 + t^3 = 0$  in  $\mathbb{C}^4$  or a  $\mathbb{C}^3$ -like threefold which is not  $\mathbb{C}^3$* , Israel J. Math. **96** (1996), 419–429.
13. L. Makar-Limanov, *On the group of automorphisms of a surface  $x^n y = p(z)$* , Israel J. Math. **121** (2001), 113–123.
14. M. Zaidenberg, *On exotic algebraic structures on affine spaces*, Algebra and Analysis. St. Petersburg Math J. **10** (1999), no. 5, 3–73.

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