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# On the strong approximation and functional limit laws for the increments of the non-overlapping $k$ -spacings processes

Salim Bouzebda\* and Nabil Nessigha

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## Abstract

The first aim of the present paper, is to establish strong approximations of the uniform non-overlapping  $k$ -spacings process extending the results of [Aly \*et al.\* \(1984\)](#). Our methods rely on the invariance principle in [Mason and van Zwet \(1987\)](#). The second goal, is to generalize the [Dindar \(1997\)](#) results for the increments of the spacings quantile process to the uniform non-overlapping  $k$ -spacings quantile process. We apply the last result to characterize the limit laws of functionals of the increments  $k$ -spacings quantile process.

*Keywords:* Stochastic processes; Strong approximations; Gaussian processes; Functional laws of the iterated logarithm, Quantile processes.

*MSC 2010:* 62E20; 60F15.

## 1 Introduction

Let  $U_1, U_2, \dots$ , be independent and identically distributed (*i.i.d.*) uniform  $[0, 1]$  random variables (*r.v.s*) defined on the same probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Denote by  $0 =: U_{0,n} \leq U_{1,n} \leq \dots \leq U_{n-1,n} \leq U_{n,n} := 1$ , the order statistics of  $U_1, U_2, \dots, U_{n-1}$ , and  $0, 1$ . The corresponding non-overlapping  $k$ -spacings are then defined by

$$\begin{aligned} D_{i,n}^k &:= U_{im,n} - U_{(i-1)m,n}, \quad \text{for } 1 \leq i \leq N-1, \\ D_{N,n}^k &:= 1 - U_{(N-1)m,n}, \end{aligned} \tag{1.1}$$

where  $N = \lfloor n/k \rfloor$ , with  $\lfloor u \rfloor \leq u < \lfloor u \rfloor + 1$  denoting the integer part of  $u$ . When  $k = 1$ , i.e.,  $N = n$ , the  $k$ -spacings reduce to the usual 1-spacings (or simple spacings) defined by

$$D_{i,n}^1 = U_{i,n} - U_{i-1,n}, \quad \text{for } i = 1, \dots, n.$$

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Simple spacings have received a great deal of attention in the literature. We refer to Deheuvels (1986), Pyke (1965, 1972), Shorack (1972), Rao and Sethuraman (1975), Beirlant (1984) and Beirlant *et al.* (1991) for details. Throughout the sequel,  $k \geq 1$  will denote a fixed integer. In applications it is more convenient to use the normalized non-overlapping  $k$ -spacings  $\{kD_{i,n}^k : 1 \leq i \leq N\}$ . For a fixed  $k \geq 1$ , as  $n \rightarrow \infty$ , the distribution function of  $kD_{i,n}^k$  (which is independent of the index  $i$  with  $1 \leq i \leq N - 1$ ) converges to the distribution function  $F_k(\cdot)$ , of a standard gamma random variable with expectation  $k$ , given by

$$F_k(t) := \frac{1}{(k-1)!} \int_0^t x^{k-1} e^{-x} dx = \int_0^t F_k(t) dt, \quad \text{for } t \geq 0, \quad (1.2)$$

where

$$F_k(t) = \frac{t^{k-1} e^{-t}}{(k-1)!} \quad \text{and} \quad F_k(t) = 0, \quad \text{for } t < 0. \quad (1.3)$$

For each choice of  $k \geq 1$ , the empirical  $k$ -spacings process is defined by

$$\alpha_n(x) := N^{1/2} \left( \widehat{F}_n(x) - F_k(x) \right), \quad \text{for } x > 0, \quad (1.4)$$

where  $\widehat{F}_n(\cdot)$  is the empirical distribution function of  $\{kD_{i,n}^k : 1 \leq i \leq N\}$ , defined for  $n \geq m$ , by

$$\widehat{F}_n(x) := \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{kD_{i,n}^k \leq x\}}, \quad \text{for } x \in \mathbb{R}, \quad (1.5)$$

with  $\mathbb{1}_{\{A\}}$  denoting the indicator function of the event  $A$ . We will need the following additional notation and definitions. Let  $M_{1:n}^k \leq M_{2:n}^k \leq \dots \leq M_{N:n}^k$ , be the order statistics of  $\{D_{i,n}^k : 1 \leq i \leq N\}$ . The quantile  $k$ -spacings function is given by

$$\widehat{Q}_n(t) := \begin{cases} kM_{i,n}^k, & \text{if } \frac{i-1}{N} < t \leq \frac{i}{N}, \quad i = 1, 2, \dots, N, \\ 0, & \text{if } t = 0. \end{cases}$$

Let us introduce

$$Q_k(t) = \inf \{x \geq 0 : F_k(x) \geq t\}, \quad (1.6)$$

and

$$f_k(t) = \frac{d}{dt} F_k(t).$$

The quantile  $k$ -spacings process  $\{\gamma_n(t) : 0 \leq t \leq 1\}$  is then defined by

$$\gamma_n(t) := N^{1/2} f_k(Q_k(t)) \left( Q_k(t) - \widehat{Q}_n(t) \right), \quad \text{for } 0 \leq t \leq 1. \quad (1.7)$$

In Deheuvels (1985), more than 60 references are given on this subject, with statistical applications such as testing uniformity or goodness-of-fit tests. Weak convergence results for the process  $\{\alpha_n(x) : 0 \leq x < \infty, n \geq 1\}$  were obtained by Pyke (1965), Shorack (1972), Rao and Sethuraman (1975), Aly (1983) and Beirlant (1984). Further, Durbin (1975) obtained tables for the limiting distribution of the Kolmogorov-Smirnov (K-S) statistic based on  $\{\alpha_n(x) : 0 \leq x < \infty, n \geq 1\}$ . Here, we mention that Pyke (1965) was the first to suggest the use of the K-S and Cramér-von Mises functionals of  $\{\alpha_n(x) : 0 \leq$

$x < \infty, n \geq 1$ }. [del Pino \(1979\)](#) consider the  $k$ -spacings as in (1.1) and characterized the limiting distribution of the statistics

$$W_n(g, k) = N^{-1/2} \sum_{i=1}^N (g(NkD_{i,N}^k) - a),$$

where  $g(\cdot)$  is a smooth function,  $k$  is fixed and  $a = \mathbb{E}[g(Y)]$ ,  $Y$  is a *r.v.* with a density function  $f_k(y)$ . These statistics  $W_n(g, k)$  can be used for testing goodness-of-fit to a uniform distribution. For application of the spacing in statistical tests and others we may refer to [Ekström \(2013, 2008\)](#), [Tung and Jammalamadaka \(2012\)](#), [Tung and Rao Jammalamadaka \(2012\)](#), [Deheuvels and Derzko \(2006\)](#) and [Baryshnikov \*et al.\* \(2009\)](#). It is worth noticing that simple spacings have received a great deal of attention in the literature, we may refer to [Deheuvels \(2011\)](#) and [Alvarez-Andrade and Bouzebda \(2017\)](#) for details. In [Deheuvels \(2011\)](#), the author obtain an explicit description of the limiting Gaussian process generated by the sample spacings from a non-uniform distribution. [Aly \*et al.\* \(1984\)](#) obtained strong approximations results for the empirical process and the quantile process based on non-overlapping  $k$ -spacings and also the weak convergence of these processes in  $\|/q\|$ -metrics, refer to the last reference for definition.

The aim in this paper is to obtain a refinement of the strong approximation results for  $\{\alpha_n(x) : 0 \leq x < \infty, n \geq 1\}$  and  $\{\gamma_n(t) : 0 \leq t \leq 1, n \geq 1\}$  obtained by [Aly \*et al.\* \(1984\)](#). Their main tool is the well known (KMT) invariance principle introduced in [Komlós \*et al.\* \(1975\)](#). In our approach we shall make use the refinement of the KMT inequality for the Brownian bridge approximation of uniform empirical and quantile processes presented respectively in [Mason and van Zwet \(1987\)](#). This approach is based on the approximation of the  $k$ -spacings process in the interval  $[0, a]$ , with  $a \leq 1$ . In order to prove the invariance principle, we use the same method developed in [Aly \*et al.\* \(1984\)](#), which is based on the following representation of simple spacings given by [Pyke \(1965\)](#). In the sequel of this section, we use a notation similar to that used in [Aly \*et al.\* \(1984\)](#) including some changes absolutely necessary for our setting. Let  $E_1, E_2, \dots$  denote an *i.i.d.* sequence of exponential *r.v.*'s with mean 1 and set

$$S_n := \sum_{i=1}^n E_i.$$

Then, for each  $n > 1$ , we have the distributional identity

$$\{U_{i,n} - U_{i-1,n} : 1 \leq i \leq n\} \stackrel{d}{=} \left\{ \frac{E_i}{S_n} : 1 \leq i \leq n \right\}. \quad (1.8)$$

where  $\stackrel{d}{=}$  denote the distributional equality. Consequently we obtain the following representation of the non-overlapping  $k$ -spacings

$$\begin{aligned} & \{D_{i,n}^k, 1 \leq i \leq N-1, D_{N,n}^k\} \\ & \stackrel{d}{=} \left\{ \left( \sum_{\ell=i}^{i+k-1} E_\ell \right) / S_n, 1 = 1, k+1, \dots, \left( \left\lfloor \frac{n}{k} \right\rfloor - 1 \right) k + 1, \left( \sum_{\ell=k \lfloor \frac{n}{k} \rfloor + 1}^n E_\ell \right) / S_n \right\} \end{aligned} \quad (1.9)$$

In particular, if  $n = Nk$  is an integer multiple of  $k$ , then

$$\{D_{i,n}^k, 1 \leq i \leq N\} \stackrel{d}{=} \{Y_i/T_N, 1 \leq i \leq N\}, \quad (1.10)$$

where

$$Y_i := \sum_{\ell=(i-1)k+1}^{ik} E_\ell, \text{ for } i = 1, 2, \dots, N, \quad (1.11)$$

is a sequence of *i.i.d. r.v.'s* with distribution function  $F_k(\cdot)$  and

$$T_N = \sum_{i=1}^N Y_i.$$

Now, we denote by  $G_N(\cdot)$  the empirical distribution function and by  $K_N(\cdot)$  the empirical quantile function of the sequence  $Y_1, \dots, Y_N$ , respectively, defined by

$$G_N(x) := \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{Y_i \leq x\}}, \text{ for } x \in \mathbb{R}^+, \quad (1.12)$$

and

$$K_N(t) := \inf\{x : G_N(x) \geq t\}, \text{ for } 0 \leq t \leq 1. \quad (1.13)$$

Let  $\{\beta_N(x) : 0 \leq x < \infty, N \geq 1\}$  and  $\{\kappa_N(t) : 1 \leq t \leq 1, N \geq 1\}$  be the corresponding empirical and quantile processes, respectively, defined by

$$\beta_N(x) := \sqrt{N} (G_N(x) - F_k(x)), \text{ for } x \in \mathbb{R}^+, \quad (1.14)$$

and

$$\kappa_N(t) := \sqrt{N} f_k(Q_k(t)) (Q_k(t) - K_N(t)), \text{ for } 0 \leq t \leq 1. \quad (1.15)$$

By (1.10), we have the following representation

$$\{\alpha_{Nk}(x), 0 \leq x < \infty\} \stackrel{d}{=} \left\{ \alpha_N^1(x) = \beta_N \left( x \frac{T_N}{Nk} \right) + \mathcal{R}_N(x), 0 \leq x < \infty \right\}, \quad (1.16)$$

where

$$\mathcal{R}_N(x) = N^{1/2} \left( F_k \left( x \frac{T_N}{Nk} \right) - F_k(x) \right).$$

In the same way, by (1.10), and the definition of the empirical quantile function  $K_N(\cdot)$ , we have the following representation for the process  $\{\gamma_{Nk}(t) : 0 \leq t < 1, N \geq 1\}$

$$\{\gamma_{Nk}(t) : 0 \leq t < 1\} \stackrel{d}{=} \left\{ \gamma_N^1(t) = \frac{Nk}{T_N} \left( \kappa_N(t) + N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) \phi_k(t) \right) : 0 \leq t < 1 \right\}, \quad (1.17)$$

where

$$\phi_k(t) = f_k(Q_k(t))Q_k(t).$$

The methodology in the proof of the results of [Aly et al. \(1984\)](#) will be essential to obtain our main theorems. In the present work, in the derivation of our asymptotic results we give a rigorous proof with all mathematical details and give some constants explicitly.

The rest of the paper is organized as follows. We establish local strong approximations of the uniform non-overlapping  $k$ -spacings process in the forthcoming section. In Section 3, we establish the functional limit laws for the increments of the quantile process of non-overlapping  $k$ -spacings processes. We provide an application concerning the limit laws of functionals of the  $k$ -spacings process. To avoid interrupting the flow of the presentation, all mathematical developments are relegated to Section 4.

## 2 Local Strong Approximation

### 2.1 Preliminaries

Let us begin by introducing some Gaussian processes playing a central role in strong approximations theory. Let  $W = \{W(s) : s \geq 0\}$  and  $B = \{B(u) : u \in [0, 1]\}$  be the standard Wiener process and Brownian bridge, that is, the centered Gaussian processes with continuous sample paths, and covariance functions

$$\mathbb{E}(W(s)W(t)) = s \wedge t, \quad \text{for } s, t \geq 0$$

and

$$\mathbb{E}(B(u)B(v)) = u \wedge v - uv, \quad \text{for } u, v \in [0, 1].$$

The interested reader may refer to Csörgő and Révész (1981) for details on the Gaussian processes mentioned above. In the sequel, the underlying probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  is assumed to be rich enough, in the sense that an independent sequence of Gaussian processes, which is independent of the originally given i.i.d. sequence of random vectors, can be constructed on this probability space. This is a technical requirement which allows for the construction of the Gaussian processes in our following Theorems. Since one can expand the underlying probability space, this assumption is not restrictive, refer to de Acosta (1982), (Berkes and Philipp, 1979, Lemma A1) and Komlós *et al.* (1975) for further details. Throughout the paper we set  $\log_+(u) = \log(u \vee e)$ , for  $u \in \mathbb{R}$ . Let us recall the following theorem which is a weak version of the result in Mason and van Zwet (1987).

**Theorem 2.1.** (*Mason and van Zwet (1987)*). *There exists a sequence of empirical processes  $\beta_N$  based on  $Y_1, \dots, Y_N$  and a sequence of Brownian bridges  $\{B_N^{(1)}(t) : 0 \leq t \leq 1\}$  such that, for all  $\varepsilon > 0$  and  $0 \leq a \leq 1$ , we have*

$$\mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} |\beta_N(x) - B_N^{(1)}(F_k(x))| \geq \mathcal{A}N^{-1/2}(\log aN) \right) \leq \mathcal{B}N^{-\varepsilon}, \quad (2.1)$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are positive constants depending on  $\varepsilon$  and  $a$ .

A similar result is needed for the quantile process  $\{\kappa_n(t) : 0 \leq t \leq 1, n \geq 1\}$ . For this, we consider deviations between the quantile process  $\{\kappa_N(t) : 0 \leq t \leq 1, N \geq 1\}$  and the approximating Brownian bridges  $\{B_N^{(1)}(t), 0 \leq t \leq 1\}$  on  $[0, a]$ , instead of  $[0, 1]$ . We formulate this idea in the following theorem.

**Theorem 2.2.** Let  $\{B_N^{(1)}(t), 0 \leq t \leq 1\}$  be as in of Theorem 2.1. Then for all  $\varepsilon > 0$  and  $n \geq m$ , we have

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |\kappa_N(t) - B_N^{(1)}(t)| \geq A_1 N^{-1/4} (\log aN)^{3/4} \right) \leq B_1 N^{-\varepsilon}, \quad (2.2)$$

for all  $0 \leq a \leq 1$ , where  $A_1$  and  $B_1$  are positive constants.

The proof of Theorem 2.2 is postponed until Section 4.

## 2.2 Main results

Our next theorem describes the strong approximations of the process  $\{\gamma_{Nk}(t) : 0 \leq t < 1\}$ .

**Theorem 2.3.** There exists a sequence  $\{W_{Nk}(t) : 0 \leq t \leq 1, \geq 1\}$  of Gaussian processes, such that the following properties hold. We have

$$\mathbb{E}W_{Nk}(t) = 0, \text{ and } \mathbb{E}W_{Nk}(t)W_{Nk}(s) = \min(t, s) - ts - \frac{1}{n}\phi_k(t)\phi_k(s),$$

and

$$\phi_k(t) = f_k(Q_k(t))Q_k(t).$$

Moreover, for each  $\varepsilon > 0$ , there exists constants  $A_2 > 0$  and  $B_2 > 0$ , such that, for all  $n \geq k$  and  $a \in [0, 1]$  we have

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |\gamma_{Nk}(t) - W_{Nk}(t)| > A_2 N^{-1/4} (\log aN)^{3/4} \right) \leq B_2 N^{-\varepsilon}.$$

The proof of Theorem 2.3 is postponed until Section 4.

Our next theorem describes the strong approximations of the process  $\{\alpha_n(x) : 0 \leq x < \infty\}$ .

**Theorem 2.4.** There exists a sequence of Gaussian processes  $\{V_n(x) : 0 \leq x \leq \infty, n \geq 1\}$ , such that the following properties hold. We have

$$\mathbb{E}V_n(x) = 0, \quad (2.3)$$

and

$$\mathbb{E}V_n(x)V_n(y) = \min(F_k(x), F_k(y)) - F_k(x)F_k(y) - \frac{1}{k}xyF_k(x)F_k(y). \quad (2.4)$$

Moreover, for all  $\varepsilon > 0$  and  $a \in [0, 1]$  we have

$$\mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} |\alpha_n(x) - V_n(x)| \geq A_3 N^{-1/4} (\log aN) \right) \leq B_3 N^{-\varepsilon},$$

where  $A_3 > 0$  and  $B_3 > 0$  are positive constants.

The proof of Theorem 2.4 is postponed until Section 4.

Immediate consequences of Theorems 2.3 and 2.4 are upper bounds for the convergence of distributions of smooth functionals of  $\{\gamma_{Nk}(t) : 0 \leq t \leq 1, N \geq 1\}$  and  $\{\alpha_n(x) : 0 \leq x < \infty, n \geq 1\}$ . Notice that the following corollary is the analogous of the Corollary of Komlós *et al.* (1975) page 113. Let  $\mathcal{D}(\mathbb{A})$  be the space of right-continuous real-valued functions defined on  $\mathbb{A}$  which have left-hand limits, equipped with the Skorohod topology; refer to Billingsley (1968) for further details.

**Corollary 2.5.** *Let  $\Phi(\cdot)$  be a functional defined on the space  $\mathcal{D}(\mathbb{R})$ , satisfying a Lipschitz condition*

$$|\Phi(v) - \Phi(w)| \leq L \sup_{t \in \mathbb{R}} |v(t) - w(t)|.$$

*Assume further that the distribution of the r.v.  $\Phi(W_{Nk}(F(\cdot)))$  has a bounded density. Then, as  $n \rightarrow \infty$ ,*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}\{\Phi(\gamma_{Nk}(\cdot)) \leq x\} - \mathbb{P}\{\Phi(W_{Nk}(\cdot)) \leq x\}| = O(N^{-1/4}(\log aN)^{3/4}). \quad (2.5)$$

*Assume further that the distribution of the r.v.  $\Phi(V_n(F(\cdot)))$  has a bounded density. Then, as  $n \rightarrow \infty$ ,*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}\{\Phi(\alpha_n(\cdot)) \leq x\} - \mathbb{P}\{\Phi(V_n(\cdot)) \leq x\}| = O(N^{-1/4}(\log aN)^{3/4}). \quad (2.6)$$

The proof of Corollary 2.5 is postponed until Section 4.

*Remark 2.6.* By the Borel-Cantelli Lemma and Theorem 2.3 we have

$$\sup_{0 \leq t \leq a} |\gamma_{Nk}(t) - W_{Nk}(t)| \stackrel{a.s.}{=} O(N^{-1/4}(\log aN)^{3/4}). \quad (2.7)$$

Applying the Borel-Cantelli Lemma and Theorem 2.4 we have

$$\sup_{0 \leq x \leq Q_k(a)} |\alpha_n(x) - V_n(x)| \stackrel{a.s.}{=} O(N^{-1/4}(\log aN)^{3/4}). \quad (2.8)$$

For  $a = 1$ , our results reduce to the results of Aly *et al.* (1984).

**Theorem 2.7** (Aly *et al.* (1984)). *Given the process  $\{\alpha_n(x) : 0 \leq x < \infty\}$  constructed from a sequence  $U_1, U_2, \dots$  of i.i.d random variables of uniform law on  $[0, 1]$  and defined on a space of probability eventually enlarged version of  $(\Omega, \mathcal{A}, \mathbb{P})$ , there exists a sequence of Brownian bridges  $B_1, B_2, \dots$ , defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  such that, for all  $0 \leq s < \infty$ , if,*

$$V_n(x) = B_N(F_k(x)) - \frac{1}{k} x f_k(x) \int_0^\infty B_N(F_k(u)) du, \quad (2.9)$$

*then with probability 1,*

$$\sup_{0 \leq x < \infty} |\alpha_n(x) - \Gamma_N(x)| = O(n^{-1/4}(\log n)^{3/4}), \quad \text{as } n \rightarrow \infty. \quad (2.10)$$



Notice that the approximating Gaussian processes, for  $k = 1$ , is given by

$$\Gamma_n(x) = B_n(x) + (1-x) \log(1-x) \int_0^1 \frac{B_n(u)}{1-u} du.$$

A similar approximation is obtained for the process of quantiles  $\{\gamma_N(t) : 0 \leq t \leq 1, N \geq 1\}$ , in the special case where  $N = n/k$ . This approximation is the following one.

**Theorem 2.8** (Aly *et al.* (1984)). *given the process  $\{\gamma_N(t) : 0 \leq t \leq 1, N \geq 1\}$  constructed from a sequence  $U_1, U_2, \dots$  of i.i.d random variables of uniform law on  $[0, 1]$  and defined on a space of probability eventually enlarged version of  $(\Omega, A, \mathbb{P})$ , there exists a sequence of Brownian bridges  $B_1, B_2, \dots$ , defined on  $(\Omega, A, \mathbb{P})$  such that, for all  $0 \leq t \leq 1$ , if,*

$$W_{Nk}(t) = B_N(t) - \frac{1}{k} Q_k(t) f_k(Q_k(t)) \int_0^1 B_N(u) dQ_k(u), \quad (2.11)$$

then with probability 1,

$$\sup_{0 \leq t \leq 1} |\gamma_N(t) - W_{Nk}(t)| = O(N^{-1/4} (\log N)^{3/4}), \quad \text{as } n \rightarrow \infty. \quad (2.12)$$

### 3 Functional limit laws for the increments of the quantile process of non-overlapping $k$ -spacings processes

Hereafter, our study of the process of quantile of  $k$ -spacings  $\{\gamma_N(t) : 0 \leq t \leq 1\}$ , will be restricted to the particular case where  $N = n/k$ . The increments of the process  $\{\gamma_N(t) : 0 \leq t \leq 1\}$  are defined in the following way: for all  $0 < h < 1$  and  $n \geq k$

$$\eta_N(h, t; s) = \gamma_N(t + sh) - \gamma_N(t), \quad \text{for } 0 \leq t \leq 1 \text{ and } s \in \mathbb{R}. \quad (3.1)$$

In this section, we are interested in the study of the fluctuations of the process of quantile. We are going to give a functional law of the iterated logarithm for the increments  $\{\eta_N(h, t; s) : 0 \leq t \leq 1, -\infty < s < \infty\}$ . Dindar (1997) established the functional law of the iterated logarithm for the increments of the processes associated to the uniform spacing. Our aim, is to provide analogous results for the  $k$ -spacing quantile process. In particular, we are going to prove that the results obtained by Dindar (1997) for the process of quantile of the uniform spacings, are still valid in the general case of the uniform non-overlapping  $k$ -spacings by preserving the same hypotheses.

In the sequel, we fix the following notation: Let  $B[0, 1]$  denote the set of bounded functions defined on  $[0, 1]$ .  $B[0, 1]$  is endowed with the topology induced by the norm sup

$$\|f\| = \sup_{0 \leq s \leq 1} |f(s)|.$$

Let  $AC[0, 1]$  be the set of all absolutely continuous function on  $[0, 1]$ , and  $\dot{f}(s) = df(s)/ds$  the Lebesgue derivative of  $f(\cdot) \in AC[0, 1]$ . For any  $c \geq 0$ , we denote by

$$\mathcal{S}_c = \left\{ f(\cdot) \in AC[0, 1] : f(0) = 0 \text{ et } \int_0^1 \dot{f}^2(s) ds \leq c \right\}$$

the Strassen set (see e.g., [Strassen \(1964\)](#)). For any subset  $A \subset AC[0, 1]$  and  $\varepsilon > 0$ , let

$$A^\varepsilon = \{f(\cdot) \in B[0, 1] : \exists g(\cdot) \in A, \|f - g\| < \varepsilon\}.$$

The Hausdorff distance between the sets  $A$  and  $B$  is defined to be

$$\Delta(A, B) = \inf\{\varepsilon > 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}.$$

Consider a sequence of constants  $\{h_N : N \geq 1\}$  satisfying the following assumptions

**(H.1)**  $h_N \downarrow 0$  and  $Nh_N \uparrow \infty$  where  $N \uparrow \infty$  and  $0 < h_N < 1$ ,

**(H.2)**  $\log(1/h_N)/\log_2 N \rightarrow c \in [0, \infty]$ , where  $N \rightarrow \infty$ ,

**(H.3)**  $(1/h_N) = o(N^{1/2}(\log_2 N)/(\log N)^{3/2})$ , where  $N \rightarrow \infty$ .

Let  $\stackrel{\mathbb{P}}{=}$  denotes equality in probability. Let

$$\beta_N = (2h_N\{\log(1/h_N) + \log_2 N\})^{-1/2}, \text{ for } N \geq 3.$$

The following theorem constitute our main results of this section.

**Theorem 3.1.** *Assume that  $\{h_N : n \geq 1\}$  fulfills the limiting conditions, **(H.1)**, **(H.2)**, **(H.3)**. Then we have*

$$\lim_{N \rightarrow \infty} \Delta\left(\{\beta_N \eta_N(h_N, t; s) : 0 \leq t \leq 1 - h_N\}, \mathcal{S}_{\frac{c}{c+1}}\right) \stackrel{\mathbb{P}}{=} 0. \quad (3.2)$$

The proof of Theorem 3.1 is postponed until Section 4.

### 3.1 Limit laws of functional of the $k$ -spacings process

Let  $\Phi : B_0[0, 1] \rightarrow \mathbb{R}$  be a functional defined on a subset  $B_0[0, 1]$  of  $B[0, 1]$  such that

**(i)**  $\beta_N \eta_N(h_N, t; \cdot) \in B_0[0, 1] \quad \forall \quad 0 \leq t \leq 1 - h_N$ ,

**(ii)**  $\mathcal{S}_c \subseteq B_0[0, 1] \quad \forall \quad c > 0$ ,

**(iii)**  $\Phi$  is continuous for the norm sup in  $B_0[0, 1]$ .

For  $f(\cdot) \in B_0[0, 1]$ , examples of such functional are given by the following

- $\Phi_1(f) = |f(1)|$ ,
- $\Phi_2(f) = \sup_{0 \leq s \leq 1} |f(s)|$ ,
- $\Phi_3(f) = \pm \int_0^1 f(u) dv(u)$ , where we suppose that  $v(\cdot)$  has bounded variation with  $v(1) = 0$ .

*Remark 3.2.* We notice that  $\Phi_3(f)$  is not defined for all  $f(\cdot) \in B[0, 1]$ , which justify the introduction of  $B_0[0, 1]$ .

Let us consider the process defined by  $\{\Phi(\beta_N \eta_N(h_N, t; \cdot)) : 0 \leq t \leq 1 - h_N\}$ . We have the following results.

**Corollary 3.3.** *Assume that  $\{h_N : N \geq 1\}$  fulfills the limiting conditions, (H.1), (H.2), (H.3). Then we have*

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1 - h_N} \Phi(\beta_N \eta_N(h_N, t; \cdot)) \stackrel{\mathbb{P}}{=} \sup_{f \in \mathcal{S}_{\frac{c}{c+1}}} \Phi(f). \quad (3.3)$$

The proof of Corollary 3.3 is postponed until Section 4.

By applying Corollary 3.3 to the functionals  $\Phi$  given above in example, we obtain the following corollary.

**Corollary 3.4.** *Assume that  $\{h_N : N \geq 1\}$  fulfills the limiting conditions, (H.1), (H.2), (H.3). Then we have*

$$\lim_{n \rightarrow \infty} \left[ \sup_{0 \leq t \leq 1 - h_N} \beta_N |\gamma_N(t + h_N) - \gamma_N(t)| \right] \stackrel{\mathbb{P}}{=} \left( \frac{c}{c+1} \right)^{1/2}, \quad (3.4)$$

$$\lim_{n \rightarrow \infty} \left[ \sup_{0 \leq t \leq 1 - h_N} \sup_{0 \leq s \leq 1} \beta_N |\gamma_N(t + h_N s) - \gamma_N(t)| \right] \stackrel{\mathbb{P}}{=} \left( \frac{c}{c+1} \right)^{1/2}, \quad (3.5)$$

$$\lim_{n \rightarrow \infty} \left[ \sup_{0 \leq t \leq 1 - h_N} \pm \int_0^1 \beta_N \eta_N(h_N, t; s) dv(s) \right] \stackrel{\mathbb{P}}{=} \left( \frac{c}{c+1} \int_0^1 v^2(s) ds \right)^{1/2}. \quad (3.6)$$

The proof of Corollary 3.4 is postponed until Section 4.

*Remark 3.5.* The results given above for the increments of the process of quantile of the disjoint  $k$ -spacings, and the results achieved in Dindar (1997) for the empirical process of the uniform spacings, remain valid for the increments of the empirical process  $\{\alpha_n(x) : 0 \leq x < \infty, n \geq 1\}$ .

## 4 Proof

This section is devoted to the proofs of our results. The previously defined notation continues to be used in the following.

### Proof of Theorem 2.2.

Consider the sequence  $\xi_i = F_k(Y_i), i = 1, 2, \dots$ , of i.i.d.  $U[0, 1]$  r.v.s and construct the corresponding uniform quantile process defined by

$$U_N(t) = N^{1/2}(t - F_k(K_N(t))), \quad (4.1)$$

where  $Y_i$  and  $K_N(\cdot)$  are defined by (1.11) and (1.13) successively. A simple application of Theorem 1.1 of Csörgő *et al.* (1986) with  $a = d/n$  and  $x = \varepsilon\lambda^{-1} \log aN$ , we can find a sequence of Brownian bridges  $\{B_N^{(2)}(t) : 0 \leq t \leq 1\}$ , such that for all  $\varepsilon > 0$  we have

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |U_N(t) - B_N^{(2)}(t)| \geq A_4 N^{-1/2} (\log aN) \right) \leq B_4 N^{-\varepsilon}, \quad (4.2)$$

where  $A_4, B_4$  are positive constants depending on  $\varepsilon$  and  $a$ . Furthermore, we have for all  $0 \leq a \leq 1$ ,

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |B_N^{(2)}(t)| > x \right) \leq 2e^{-2x^2}, \quad x \geq 0. \quad (4.3)$$

The last inequality together with (4.2) implies that

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |U_N(t)| \geq \left( \frac{1}{2} \varepsilon (\log aN) \right)^{1/2} + A_4 N^{-1/2} (\log aN) \right) \leq (2 + B_4) N^{-\varepsilon}. \quad (4.4)$$

We will prove in the next lemma that  $\{U_N(t) : 0 \leq t \leq 1, N \geq 1\}$ , as defined in (4.1), can be approximated by  $\{B_N^{(1)}(t) : 0 \leq t \leq 1, N \geq 1\}$  as well.

**Lemma 4.1.** *There exists a sequence of Gaussian processes  $\{B_N^{(1)}(t) : 0 \leq t \leq 1, N \geq 1\}$ , such that, for all  $\varepsilon > 0$ , we have*

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |U_N(t) - B_N^{(1)}(t)| \geq A_5 N^{-1/2} (\log aN)^{3/4} \right) \leq B_5 N^{-\varepsilon}, \quad (4.5)$$

where  $A_5$  and  $B_5$  are positive constants.

**Proof of Lemma 4.1.** Let  $\xi_{1,N}, \dots, \xi_{N,N}$  denote the order statistics of  $\xi_1, \dots, \xi_N$ . By Theorem 2.1 and the fact that

$$\beta_N(Q_k(\xi_{i,N})) = U_N \left( \frac{i}{N} \right),$$

we have, for each  $0 < a \leq 1$ ,

$$\mathbb{P} \left( \max_{0 \leq i \leq aN} \left| U_N \left( \frac{i}{N} \right) - B_N^{(1)}(\xi_{i,N}) \right| > \mathcal{A} N^{-1/2} (\log aN) \right) \leq \mathcal{B} N^{-\varepsilon}. \quad (4.6)$$

On the other hand, from (4.4) we have

$$\mathbb{P} \left( \max_{0 \leq i \leq aN} \left| \frac{i}{N} - \xi_{i,N} \right| \geq N^{-1/2} \left( \frac{\varepsilon}{2} (\log aN) \right)^{1/2} + A_4 N^{-1} (\log aN) \right) \leq (2 + B_4) N^{-\varepsilon}. \quad (4.7)$$

An application of Lemma 1.2.1 in connection with Lemma 1.4.1 of Csörgő and Révész (1981) allow us to write

$$\begin{aligned} \mathbb{P} \left( \sup_{0 \leq i \leq N - N^{1/2}(\log aN)} \sup_{0 \leq s \leq N^{-1/2}(\log aN)} \left| B_N^{(1)} \left( \frac{i}{N} + s \right) - B_N^{(1)} \left( \frac{i}{N} \right) \right| \right. \\ \left. > A_6 N^{-1/4} (\log aN)^{3/4} \right) \leq B_6 N^{-\varepsilon}, \end{aligned}$$

This, when combined with (4.7), implies that

$$\mathbb{P} \left( \max_{0 \leq i \leq aN} |B_N^{(1)} \left( \frac{i}{N} \right) - B_N^{(1)}(\xi_{i,N})| > A_7 N^{-1/4} (\log aN)^{3/4} \right) \leq B_7 N^{-\varepsilon}. \quad (4.8)$$

Lemma 4.1 follows from the fact that

$$\left| U_N(t) - U_N \left( \frac{i}{N} \right) \right| \leq N^{-1/2} \quad \text{for} \quad \frac{i-1}{N} < t < \frac{i}{N}. \quad (4.9)$$

■

We return now to the proof of Theorem 2.2. Following Aly *et al.* (1984), we have

$$\sup_{0 < t < \infty} F_k(t)(1 - F_k(t)) \frac{|(f_k)'(t)|}{(f_k)^2(t)} \leq \gamma, \quad (4.10)$$

together with

$$\lim_{t \rightarrow \infty} F_k(t)(1 - F_k(t)) \frac{|(f_k)'(t)|}{(f_k)^2(t)} = 1, \quad (4.11)$$

$$\lim_{t \rightarrow 0} F_k(t)(1 - F_k(t)) \frac{|(f_k)'(t)|}{(f_k)^2(t)} = 1, \quad (4.12)$$

for some

$$\gamma = \gamma(k) < \infty.$$

By the mean value theorem, we readily obtain

$$\kappa_N(t) - U_N(t) = U_N(t) \left( \frac{f_k(Q_k(t))}{f_k(Q_k(\theta_{t,N}))} - 1 \right), \quad (4.13)$$

for some  $\theta_{t,N}$  such that

$$|\theta_{t,N} - t| < N^{-1/2} |U_N(t)|.$$

In Theorem 1.5.1 in Csörgő (1983), it is proved that

$$\begin{aligned} & \mathbb{P} \left( \sup_{c \leq t \leq 1-c} \left| \frac{f_k(Q_k(t))}{f_k(Q_k(\theta_{t,N}))} - 1 \right| > \delta \right) \\ & \leq 4([\gamma] + 1) \{ \exp(-N \text{ch}((1 + \delta)^{1/2([\gamma]+1)})) \\ & \quad + \exp(-N \text{ch}((1 + \delta)^{-1/2([\gamma]+1)})) \}, \end{aligned} \quad (4.14)$$

for all  $\delta > 0$ ,  $0 < c < 1$  and  $N \geq 1$ , where

$$h(x) = x + \log(1/x) - 1, \quad \text{for } x > 0.$$

Moreover, there exist a  $\delta_0 > 0$  such that

$$h((1 + \delta)^{\mp 1/2([\gamma]+1)}) \geq \frac{1}{8}([\gamma] + 1)^2 \delta^2, \quad \text{for } 0 < \delta < \delta_0. \quad (4.15)$$

Let

$$\delta_N := (8\varepsilon)^{1/2} ([\gamma] + 1)^{-1} N^{-1/4} (\log aN)^{1/2},$$

and

$$C^{(1)} := C_N^{(1)} := N^{-1/2}.$$

By the above inequality and (4.14) we obtain that, for  $N$  sufficiently large, that

$$\mathbb{P} \left( \sup_{C_N^{(1)} \leq t \leq 1 - C_N^{(1)}} \left| \frac{f_k(Q_k(t))}{f_k(Q_k(\theta_{t,N}))} - 1 \right| > \delta_N \right) \leq 8([\gamma] + 1)N^{-\varepsilon}. \quad (4.16)$$

Combining (4.13), (4.4) and (4.16), we obtain that, for  $N$  sufficiently large,

$$\mathbb{P} \left( \sup_{C_N^{(1)} \leq t \leq a - C_N^{(1)}} |\kappa_N(t) - U_N(t)| > A_8 N^{-1/4} (\log aN)^{3/4} \right) \leq B_8 N^{-\varepsilon}. \quad (4.17)$$

To complete the proof of Theorem 2.2, we replace  $\log N$  in the proof of the Theorem B of Aly *et al.* (1984) by  $(\log aN)$ . We omit the details, which essentially repeats, more or the less verbatim, the same arguments.  $\blacksquare$

The following technical Lemmas 4.2 and 4.3 will be instrumental in the proof of Theorem 2.3.

**Lemma 4.2.** *We have, for each  $\varepsilon > 0$ , and all  $n \geq m$  sufficiently large*

$$\mathbb{P} \left( \left| N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) - \frac{1}{k} \int_0^\infty tdB_N^{(1)}(F_k(t)) \right| > A_9 N^{-1/2} (\log aN)^2 \right) \leq B_9 N^{-\varepsilon}. \quad (4.18)$$

where  $A_9 = A_9(\varepsilon) = 4(1/2 + \varepsilon)\mathcal{A}$  and  $B_9 = 8\sqrt{2} + \mathcal{B}$  denote positive constants.

**Proof of Lemma 4.2.**

It is readily checked that,

$$\frac{T_N}{Nk} = \frac{1}{Nk} \sum_{i=1}^N Y_i = \frac{1}{k} \int_0^\infty tdG_N(t) \quad \text{and} \quad \int_0^\infty tdF_k(t) = k. \quad (4.19)$$

From which we obtain readily that

$$N^{1/2} \left( \frac{T_N}{N} - k \right) = \int_0^\infty td\beta_N(t) = - \int_0^\infty \beta_N(t) dt. \quad (4.20)$$

Let  $\lambda_N$  be a sequence of positive numbers. Making use of the triangle inequality, we infer that

$$\begin{aligned} \left| \int_0^\infty \beta_N(t) dt - \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| &\leq \int_0^{\lambda_N} |\beta_N(t) - B_N^{(1)}(F_k(t))| dt \\ &\quad + \int_{\lambda_N}^\infty |B_N^{(1)}(F_k(t))| dt \\ &\quad + \int_{\lambda_N}^\infty |\beta_N(t)| dt. \end{aligned}$$

Let us recall the following well known properties

$$\mathbb{E}(\beta_N(t)) = \mathbb{E}\left(B_N^{(1)}(F_k(t))\right) = 0, \quad (4.21)$$

$$\text{Var}(\beta_N(t)) = \mathbb{E}\left[(\beta_N(t))^2\right] = F_k(t)(1 - F_k(t)), \quad (4.22)$$

and

$$\text{Var}\left(B_N^{(1)}(F_k(t))\right) = \mathbb{E}\left[\left(B_N^{(1)}(F_k(t))\right)^2\right] = F_k(t)(1 - F_k(t)). \quad (4.23)$$

Making use of the Fubini theorem's, in combination with Cauchy-Schwartz inequality implies that

$$\begin{aligned} \mathbb{E} \int_{\lambda_N}^{\infty} |\beta_N(t)| dt &= \int_{\lambda_N}^{\infty} \mathbb{E} |\beta_N(t)| dt \\ &\leq \int_{\lambda_N}^{\infty} (F_k(t)(1 - F_k(t)))^{1/2} dt, \end{aligned} \quad (4.24)$$

and similarly

$$\begin{aligned} \mathbb{E} \int_{\lambda_N}^{\infty} |B_N^{(1)}(F_k(t))| dt &= \int_{\lambda_N}^{\infty} \mathbb{E} |B_N^{(1)}(F_k(t))| dt \\ &\leq \int_{\lambda_N}^{\infty} (F_k(t)(1 - F_k(t)))^{1/2} dt. \end{aligned} \quad (4.25)$$

By [Aly et al. \(1984\)](#), there exists  $t_0 > 0$  such that

$$1 - F_k(t) \leq 2 \exp\left(-\frac{t}{2}\right), \quad \text{if } t \geq t_0. \quad (4.26)$$

Hence, provided that  $\lambda_N \geq t_0$ , by (4.26) and the fact that

$$F_k(t) \leq 1 \quad \text{for all } t > 0, \quad (4.27)$$

the left hand sides of (4.24) and (4.25) are bounded above by  $4\sqrt{2} \exp(-\lambda_N/4)$ . Indeed, we have

$$\begin{aligned} \mathbb{E}\left(|B_N^{(1)}(F_k(t))|\right) &\leq (F_k(t)(1 - F_k(t)))^{1/2} \\ &\leq \sqrt{2} \exp(-t/4), \end{aligned}$$

and by using (4.25), we infer that

$$\begin{aligned} \mathbb{E}\left(\int_{\lambda_N}^{\infty} |B_N^{(1)}(F_k(t))| dt\right) &\leq \sqrt{2} \int_{\lambda_N}^{\infty} \exp(-t/4) dt \\ &= 4\sqrt{2} \exp(-\lambda_N/4). \end{aligned}$$

By using similar arguments shows likewise that

$$\mathbb{E}\left(\int_{\lambda_N}^{\infty} |\beta_N(t)| dt\right) \leq 4\sqrt{2} \exp(-\lambda_N/4).$$

By choosing  $\lambda_N = 4(\frac{1}{2} + \varepsilon)(\log aN)$ , and applying Markov inequality, we infer that

$$\mathbb{P} \left( \int_{4(\frac{1}{2} + \varepsilon)(\log aN)}^{\infty} |\beta_N(t)| dt > a^{-(1/2 + \varepsilon)} N^{-1/2} \right) \leq 4\sqrt{2} N^{-\varepsilon}, \quad (4.28)$$

and

$$\mathbb{P} \left( \int_{4(\frac{1}{2} + \varepsilon)(\log aN)}^{\infty} |B_N^{(1)}(F_k(t))| dt > a^{-(1/2 + \varepsilon)} N^{-1/2} \right) \leq 4\sqrt{2} N^{-\varepsilon}. \quad (4.29)$$

An application of Theorem 2.1 shows that,

$$\mathbb{P} \left( \int_0^{\lambda_N} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right| dt > \lambda_N \mathcal{A} N^{-1/2} (\log aN) \right) \leq \mathcal{B} N^{-\varepsilon}. \quad (4.30)$$

More precisely, we have the inequality

$$\begin{aligned} \int_0^{\lambda_N} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right| dt &\leq \sup_{0 \leq x \leq Q_k(a)} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right| \int_0^{\lambda_N} dt \\ &= \lambda_N \sup_{0 \leq x \leq Q_k(a)} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right|. \end{aligned}$$

Making use of Theorem 2.1, we obtain

$$\begin{aligned} &\mathbb{P} \left( \int_0^{\lambda_N} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right| dt > \lambda_N \mathcal{A} N^{-1/2} (\log aN) \right) \\ &\leq \mathbb{P} \left( \lambda_N \sup_{0 \leq x \leq Q_k(a)} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right| > \lambda_N \mathcal{A} N^{-1/2} (\log aN) \right) \\ &= \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| \beta_N(t) - B_N^{(1)}(F_k(t)) \right| > \mathcal{A} N^{-1/2} (\log aN) \right) \\ &\leq \mathcal{B} N^{-\varepsilon}. \end{aligned}$$

Let

$$\Lambda_1 = 2a^{-(1/2 + \varepsilon)} N^{-1/2},$$

and

$$\Lambda_2 = \lambda_N \mathcal{A} N^{-1/2} (\log aN) = 4(1/2 + \varepsilon) \mathcal{A} N^{-1/2} (\log aN)^2.$$

From this we infer that we have

$$\Lambda_1 + \Lambda_2 = 4(1/2 + \varepsilon) \mathcal{A} N^{-1/2} (\log aN)^2 (1 + o(1)).$$

Lemma 4.2 now follows by combining the above three inequalities (4.24), (4.25) and



(4.30). We have

$$\begin{aligned}
& \mathbb{P} \left( \left| \int_0^\infty \left( \beta_N(t) - B_N^{(1)}(F_k(t)) \right) dt \right| > \Lambda_1 + \Lambda_2 \right) \\
& \leq \mathbb{P} \left( \left| \int_0^\lambda \left( \beta_N(t) - B_N^{(1)}(F_k(t)) \right) dt \right| > 4(1/2 + \varepsilon) \mathcal{A} N^{-1/2} (\log_+ aN)^2 \right) \\
& \quad + \mathbb{P} \left( \left| \int_\lambda^\infty \left( B_N^{(1)}(F_k(t)) \right) dt \right| > a^{-(1/2+\varepsilon)} N^{-1/2} \right) \\
& \quad + \mathbb{P} \left( \left| \int_\lambda^\infty \left( \beta_N(t) \right) dt \right| > a^{-(1/2+\varepsilon)} N^{-1/2} \right) \\
& \leq 4\sqrt{2} N^{-\varepsilon} + 4\sqrt{2} N^{-\varepsilon} + \mathcal{B}.
\end{aligned}$$

By choosing  $A_9 = A_9(\varepsilon) = 4(1/2 + \varepsilon)\mathcal{A}$  and  $B_9 = 8\sqrt{2} + \mathcal{B}$ , the proof of Lemma 4.2 is completed.  $\blacksquare$

**Lemma 4.3.** *For each  $\varepsilon > 0$  and  $n \geq k$ , we have, uniformly over  $0 \leq a \leq 1$*

$$\begin{aligned}
& \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| B_N^{(1)} \left( F_k \left( x \frac{T_N}{Nk} \right) \right) - B_N^{(1)}(F_k(x)) \right| \right. \\
& \quad \left. > A_{10} N^{-1/4} (\log aN)^{3/4} \right) \leq B_{10} N^{-\varepsilon},
\end{aligned}$$

where  $A_{10}$  and  $B_{10}$  are positive constants.

**Proof of lemma 4.3.** The random variable  $\int_0^\infty B_N^{(1)}(F_k(t)) dt$  has a normal distribution, with expectation 0 and finite variance, given by

$$\sigma_1^2 = \mathbb{E} \left\{ \left( \int_0^\infty B_N^{(1)}(F_k(t)) dt \right)^2 \right\} < \infty. \quad (4.31)$$

Hence, we have the inequality

$$\mathbb{P} \left( \frac{1}{\sigma_1} \left| \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| > (2\varepsilon \log aN)^{1/2} \right) \leq 2N^{-\varepsilon}. \quad (4.32)$$

This inequality and Lemma 4.2 imply that

$$\mathbb{P} \left( \left| \frac{T_N}{Nk} - 1 \right| > A_{11} N^{-1/2} (\log aN)^{1/2} \right) \leq B_{11} N^{-\varepsilon}, \quad (4.33)$$

where

$$A_{11} = A_{11}(\varepsilon) = (2k^{-2} \sigma_1^2 \varepsilon)^{1/2} \quad \text{and} \quad B_{11} = 2 + B_9.$$

More precisely, we have

$$A_9 N^{-1/2} (\log aN)^{1/2} + (2k^{-2} \sigma_1^2 \varepsilon \log aN)^{1/2} = (2k^{-2} \sigma_1^2 \varepsilon)^{1/2} (\log aN)^{1/2} (1 + o(1)).$$

So the probability (4.33) is the same as

$$\mathbb{P} \left( \left| N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) \right| > A_9 N^{-1/2} (\log aN)^{1/2} + (2k^{-2} \sigma_1^2 \varepsilon \log aN)^{1/2} \right).$$

By combination of Lemma 4.2 and inequality (4.32), it follows that

$$\begin{aligned}
& \mathbb{P} \left( \left| N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) \right| > A_9 N^{-1/2} (\log aN)^{1/2} + (2k^{-2} \sigma_1^2 \varepsilon \log aN)^{1/2} \right) \\
& \leq \mathbb{P} \left( \left| N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) - \frac{1}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| > A_9 N^{-1/2} (\log aN)^{1/2} \right) \\
& \quad + \mathbb{P} \left( \left| \frac{1}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| > (2k^{-2} \sigma_1^2 \varepsilon \log aN)^{1/2} \right) \\
& = \mathbb{P} \left( \left| N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) - \frac{1}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| > A_9 N^{-1/2} (\log aN)^{1/2} \right) \\
& \quad + \mathbb{P} \left( \left| \frac{1}{\sigma_1} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| > (2\varepsilon \log aN)^{1/2} \right) \\
& \leq (B_9 + 2) N^{-\varepsilon}.
\end{aligned}$$

By Taylor expansions, we readily obtain

$$\left| F_k \left( x \frac{T_N}{Nk} \right) - F_k(x) \right| = x f_k(x_N) \left| \frac{T_N}{Nk} - 1 \right|, \quad (4.34)$$

where

$$|x_N - x| \leq x \left| \frac{T_N}{Nk} - 1 \right|.$$

Let  $0 < \delta < 1$  and define  $A_N(\delta)$  by

$$A_N(\delta) = \left\{ \omega : \left| \frac{T_N}{Nk} - 1 \right| \leq \delta \right\}. \quad (4.35)$$

Now, by choosing  $N$  sufficiently large so that  $A_{11} N^{-1/2} (\log aN)^{1/2} \leq \delta$ , and using (4.33) we get that

$$\mathbb{P}(A_N^c(\delta)) \leq B_{11} N^{-\varepsilon}.$$

In addition, we have for each  $x_N \in A_N(\delta)$ ,

$$x f_k(x_N) \leq \frac{(1 + \delta)^{m-1}}{\Gamma(k)} x^k e^{-(1-\delta)x}, \quad (4.36)$$

which is bounded on  $[0, \infty)$ . Now, we let

$$A_{12} = A_{11} \cdot \sup_{0 \leq x \leq Q_k(a)} \frac{(1 + \delta)^{k-1}}{\Gamma(k)} x^k e^{-(1-\delta)x}. \quad (4.37)$$

Recall the following elementary fact

$$\mathbb{P}(A) \leq \mathbb{P}(B^c) + \mathbb{P}(A \cap B),$$

then, for large enough  $N$ , we infer that we have

$$\begin{aligned}
& \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| F_k \left( x \frac{T_N}{Nk} \right) - F_k(x) \right| > A_{12} N^{-1/2} (\log aN)^{1/2} \right) \\
& \leq \mathbb{P} (A_N^c(\delta)) \\
& \quad + \mathbb{P} \left( A_N(\delta) \text{ and } \left\{ \sup_{0 \leq x \leq Q_k(a)} \left| F_k \left( x \frac{T_N}{Nk} \right) - F_k(x) \right| > A_{12} N^{-1/2} (\log aN)^{1/2} \right\} \right) \\
& \leq \mathbb{P} (A_N^c(\delta)) \\
& \quad + \mathbb{P} \left( A_N(\delta) \text{ and } \left\{ \sup_{0 \leq x \leq Q_k(a)} x F_k(x_N) \left| \frac{T_N}{Nk} - 1 \right| > A_{12} N^{-1/2} (\log aN)^{1/2} \right\} \right) \\
& \leq B_{11} N^{-\varepsilon} + \mathbb{P} \left( A_N(\delta) \text{ and } \left\{ \left| \frac{T_N}{Nk} - 1 \right| > A_{11} N^{-1/2} (\log aN)^{1/2} \right\} \right) \\
& \leq B_{11} N^{-\varepsilon}. \tag{4.38}
\end{aligned}$$

Now, (4.38) when combined with Lemma 1.1.1 of Csörgő and Révész (1981) implies that

$$\begin{aligned}
& \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| B_N^{(1)} \left( F_k \left( x \frac{T_N}{Nk} \right) \right) - B_N^{(1)}(F_k(x)) \right| > A_{10} N^{-1/4} (\log aN)^{3/4} \right) \\
& = \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| B_N^{(1)} \left( F_k(x) + F_k \left( x \frac{T_N}{Nk} \right) - F_k(x) \right) - B_N^{(1)}(F_k(x)) \right| \right. \\
& \quad \left. > A_{10} N^{-1/4} (\log aN)^{3/4} \right) \\
& \leq \mathbb{P} \left( \sup_{0 \leq t \leq 1 - A_{12} N^{-1/2} (\log aN)^{1/2}} \sup_{0 \leq s \leq A_{12} N^{-1/2} (\log aN)^{1/2}} \left| B_N^{(1)}(t+s) - B_N^{(1)}(t) \right| \right. \\
& \quad \left. > \frac{A_{10}}{\sqrt{A_{12}}} (\log aN)^{1/2} (A_{12} N^{-1/2} (\log aN)^{1/2})^{1/2} \right) + B_{11} N^{-\varepsilon} \\
& \leq B_{10} N^{-\varepsilon}. \tag{4.39}
\end{aligned}$$

This completes the proof of Lemma 4.3. ■

### Proof of Theorem 2.3.

By the representation (1.10) we get

$$\{\gamma_{Nk}(t), 0 \leq t < 1\} \stackrel{d}{=} \{\gamma_N^1(t), 0 \leq t < 1\}. \tag{4.40}$$

Recall that our aim is to prove the inequality

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |\gamma_N^1(t) - W_N^*(t)| > A_2 N^{-1/4} (\log aN)^{3/4} \right) \leq B_2 N^{-\varepsilon}, \tag{4.41}$$

where

$$W_N^*(t) := B_N^{(1)}(t) - \frac{\phi_k(t)}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt. \tag{4.42}$$

First, we observe that

$$\begin{aligned}
& \gamma_N^1(t) - \left( B_N^{(1)}(t) - \frac{\phi_k(t)}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right) \\
&= \kappa_N(t) - B_N^{(1)}(t) + \left( \left( \frac{T_N}{Nk} \right)^{-1} - 1 \right) \kappa_N(t) \\
&+ \phi_k(t) N^{1/2} \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \left( \left( \frac{T_N}{Nk} \right)^{-1} - 1 \right) \\
&- \frac{\phi_k(t)}{k} \left( N^{1/2} \left( k - \frac{T_N}{N} \right) - \int_0^\infty B_N^{(1)}(F_k(t)) dt \right). \tag{4.43}
\end{aligned}$$

Now, by Theorem 2.2 we infer that

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} |\kappa_N(t) - B_N^{(1)}(t)| \geq A_1 N^{-1/4} (\log aN)^{3/4} \right) \leq B_1 N^{-\varepsilon}. \tag{4.44}$$

Noting that

$$\sup_{0 \leq t \leq a} \phi_k(t) = \sup_{0 \leq x \leq Q(a)} x f_k(x) < \infty. \tag{4.45}$$

Let

$$A_{13} = A_9 \sup_{0 \leq x \leq Q(a)} x f_k(x).$$

By combining Lemma 4.2 with (4.45), we get

$$\begin{aligned}
\mathbb{P} \left( \sup_{0 \leq t \leq a} \left| N^{1/2} \left( 1 - \frac{T_N}{Nk} \right) \phi_k(t) - \frac{\phi_k(t)}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right| > A_{13} N^{-1/2} (\log aN)^2 \right) \\
\leq B_9 N^{-\varepsilon} \tag{4.46}
\end{aligned}$$

By using the following elementary fact

$$\left( \frac{1}{u} - 1 \right) = -(u-1) + \frac{1}{u} (u-1)^2,$$

we obtain the following relation

$$\left( \left( \frac{T_N}{Nk} \right)^{-1} - 1 \right) \kappa_N(t) = - \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \kappa_N(t) + \left( \left( \frac{T_N}{Nk} \right) - 1 \right)^2 \frac{Nk}{T_N} \kappa_N(t). \tag{4.47}$$

We have the inequalities

$$\begin{aligned}
& \mathbb{P} \left( \sup_{0 \leq t \leq a} \left| \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \kappa_N(t) \right| > (A_{11} N^{-1/2} (\log aN)^{1/2}) \right. \\
& \quad \times \left. \left( \left( \frac{1}{2} \varepsilon (\log aN) \right)^{1/2} + A_1 N^{-1/4} (\log aN)^{3/4} \right) \right) \\
& \leq \mathbb{P} \left( \left| \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \right| > (A_{11} N^{-1/2} (\log aN)^{1/2}) \right) \\
& \quad + \mathbb{P} \left( \sup_{0 \leq t \leq a} |\kappa_N(t)| > \left( \left( \frac{1}{2} \varepsilon (\log aN) \right)^{1/2} + A_1 N^{-1/4} (\log aN)^{3/4} \right) \right) \\
& \leq \mathbb{P} \left( \left| \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \right| > (A_{11} N^{-1/2} (\log aN)^{1/2}) \right) \\
& \quad + \mathbb{P} \left( \sup_{0 \leq t \leq a} |\kappa_N(t) - B_N^1(t)| > (A_1 N^{-1/4} (\log aN)^{3/4}) \right) \\
& \quad + \mathbb{P} \left( \sup_{0 \leq t \leq a} |B_N^{(1)}(t)| > \left( \frac{1}{2} \varepsilon (\log aN) \right)^{1/2} \right) \\
& \leq B_{11} N^{-\varepsilon} + B_1 N^{-\varepsilon} + 2N^{-\varepsilon} \\
& \leq B_{14} N^{-\varepsilon}. \tag{4.48}
\end{aligned}$$

By the law of large numbers,  $T_N/N$  converges to  $k$ , as  $n$  tends to infinity. Then  $T_N/Nk$  tends to one when  $n$  tends to infinity. On the other hand, we remark, if  $T_N/Nk \geq 1/2$ , then  $Nk/T_N \leq 2$ . We can see that

$$\begin{aligned}
& \mathbb{P} \left( \sup_{0 \leq t \leq a} \left| \left( \left( \frac{T_N}{Nk} \right) - 1 \right)^2 \frac{Nk}{T_N} \kappa_N(t) \right| > (2A_{11}^2 N^{-1} (\log aN)) \right. \\
& \quad \times \left. \left( \left( \frac{1}{2} \varepsilon (\log aN) \right)^{1/2} + A_1 N^{-1/4} (\log aN)^{3/4} \right) \right) \\
& \leq B_{14} N^{-\varepsilon}. \tag{4.49}
\end{aligned}$$

Using (4.48) and (4.49), we obtain

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} \left| \left( \left( \frac{T_N}{Nk} \right)^{-1} - 1 \right) \kappa_N(t) \right| > A_{14} N^{-1/4} (\log aN)^{3/4} \right) \leq B_{14} N^{-\varepsilon}. \tag{4.50}$$

Moreover we have

$$\begin{aligned}
& \phi_k(t) N^{1/2} \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \left( \left( \frac{T_N}{Nk} \right)^{-1} - 1 \right) \\
& = -\phi_k(t) N^{1/2} \left( \left( \frac{T_N}{Nk} \right) - 1 \right)^2 + \phi_k(t) N^{1/2} \left( \left( \frac{T_N}{Nk} \right) - 1 \right)^3 \frac{T_N}{Nk}.
\end{aligned}$$

Now, on  $A_N(\delta)$ , we have

$$\sup_{0 \leq t \leq a} \phi_k(t) = M < \infty.$$

Taking  $A_{15} = A_{11}^2 M$  and applying similar techniques used in line 2 of (4.38) we get, by (4.33), that

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} \phi_k(t) N^{1/2} \left| \left( \left( \frac{T_N}{Nk} \right) - 1 \right)^2 \right| > A_{15} N^{-1/2} (\log aN) \right) \leq B_{11} N^{-\varepsilon}. \quad (4.51)$$

Let  $A_{16} = 2A_{11}^3 M$ . By using the same arguments, it follows

$$\mathbb{P} \left( \sup_{0 \leq t \leq a} \phi_k(t) N^{1/2} \left| \left( \left( \frac{T_N}{Nk} \right) - 1 \right)^3 \right| \frac{Nk}{T_N} > A_{16} N^{-1} (\log aN)^{3/2} \right) \leq B_{11} N^{-\varepsilon}. \quad (4.52)$$

From (4.51) and (4.52), we obtain

$$\begin{aligned} \mathbb{P} \left( \sup_{0 \leq t \leq a} \phi_k(t) N^{1/2} \left( \left( \frac{T_N}{Nk} \right) - 1 \right) \left( \left( \frac{T_N}{Nk} \right)^{-1} - 1 \right) \right. \\ \left. > A_{17} N^{-1/2} (\log aN) \right) \leq B_{17} N^{-\varepsilon}. \end{aligned} \quad (4.53)$$

Now, by combining (4.43), (4.44), (4.46), (4.50) and (4.53), we obtain

$$\begin{aligned} \mathbb{P} \left( \sup_{0 \leq t \leq a} \left| \gamma_N^1(t) - \left( B_N^{(1)}(t) - \frac{\phi_k(t)}{k} \int_0^\infty B_N^{(1)}(F_k(t)) dt \right) \right| \right. \\ \left. > A_2 N^{-1/4} (\log aN)^{3/4} \right) \leq B_2 N^{-\varepsilon}. \end{aligned} \quad (4.54)$$

By Lemma 4.4.4 of Csörgő and Révész (1981) and (1.17), we can define a sequence of Gaussian processes  $\{W_{Nk}(t) : 0 \leq t \leq 1\}$ ,  $N = 1, 2, \dots$  such that for each  $N$ , we have

$$\{\gamma_{Nk}(t), W_{Nk}(s) : 0 \leq t, s \leq 1\} \stackrel{d}{=} \{\gamma_N^1(t), W_N^*(t) : 0 \leq t, s \leq 1\}. \quad (4.55)$$

This completes the proof of Theorem 2.3. ■

## Proof of Theorem 2.4.

We are going to give the main steps of the proof. The details are the same as in Theorem 2.3. Assume first that  $n = Nk$ . Keep in mind the representation (1.16) for the empirical process of  $k$ -spacings. We are aimed to prove the following

$$\mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} |\alpha_N^1(x) - V_N^*(x)| \geq A_3 N^{-1/4} (\log aN) \right) \leq B_3 N^{-\varepsilon}, \quad (4.56)$$

where

$$V_N^*(x) = B_N^{(1)}(F_k(x)) - \frac{1}{k} x f_k(x) \int_0^\infty B_N^{(1)}(F_k(y)) dy. \quad (4.57)$$

By second order Taylor expansion in the second term of (1.16), we get

$$\begin{aligned} \alpha_N^1(x) - V_N^*(x) &= \beta_N \left( x \frac{T_N}{Nk} \right) - B_N^{(1)} \left( F_k \left( x \frac{T_N}{Nk} \right) \right) \\ &\quad + B_N^{(1)} \left( F_k \left( x \frac{T_N}{Nk} \right) \right) - B_N^{(1)}(F_k(x)) + N^{1/2} \left( \frac{T_N}{Nk} - 1 \right)^2 x^2 f^{(k)}(x_N) \\ &\quad + \frac{x f_k(x)}{k} \left( N^{1/2} \left( \frac{T_N}{Nk} - 1 \right) - \int_0^\infty t dB_N^{(1)}(F_k(t)) \right), \end{aligned}$$

where

$$|x_N - x| \leq x \left| \frac{T_N}{Nk} - 1 \right|.$$

Making use of Lemmas 4.2 and 4.3, together with Theorem 2.1 we obtain (4.56). Hence together with Lemma 4.4.4 of Csörgő and Révész (1981), we can define a sequence of Gaussian processes  $\{V_{Nk}(x) : 0 \leq x < \infty\}$ ,  $N = 1, 2, \dots$ , such that for each  $N$  we have

$$\{\alpha_{Nk}(x), V_{Nk}(y) : 0 \leq x, y < \infty\} \stackrel{d}{=} \{\alpha_N^1(x), V_N^*(y) : 0 \leq x, y < \infty\}. \quad (4.58)$$

This completes the proof Theorem (2.4) for the case where  $n = Nk$ . Now, we prove the general case where  $k(N-1) < n < Nk$ . It follows from (1.9) that

$$\begin{aligned} & \{\alpha_n(x) : 0 \leq x < \infty\} \\ & \stackrel{d}{=} \left\{ N^{1/2} \left( G_{N,k} \left( x \frac{S_n}{Nk} \right) - F_k(x) \right) : 0 \leq x < \infty \right\}, \end{aligned} \quad (4.59)$$

where

$$G_{N,k}(x) = \frac{1}{N} \sum_{i=1}^{N-1} \mathbb{1}_{\{Y_i < x\}} + \frac{1}{N} \mathbb{1}_{\{\sum_{\ell=(N-1)k+1}^n E_\ell < x\}}. \quad (4.60)$$

Notice that we have the following fact

$$\sup_{0 \leq x \leq Q_k(a)} \left| G_{N,k} \left( x \frac{S_n}{Nk} \right) - G_{N-1} \left( x \frac{S_n}{Nk} \right) \right| \leq \frac{1}{N} + \frac{1}{N(N-1)}, \quad (4.61)$$

and

$$\mathbb{P} \left( \left| \frac{S_n}{Nk} - \frac{T_{N-1}}{k(N-1)} \right| > A_{18} N^{-1} (\log aN) \right) \leq B_{18} N^{-\varepsilon}. \quad (4.62)$$

Set

$$\begin{aligned} \mathcal{P} = \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| N^{1/2} \left( G_{N,k} \left( x \frac{S_n}{Nk} \right) - F_k(x) \right) - V_{N-1}^*(x) \right| \right. \\ \left. > A_{19} N^{-1/4} (\log aN)^{3/4} \right). \end{aligned} \quad (4.63)$$

By the use of (4.56) in connection with (4.61), we infer that

$$\begin{aligned} \mathcal{P} & \leq \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} N^{1/2} \left| F_k \left( x \frac{S_n}{T_{N-1}} \right) - F_k(x) \right| > A_{20} N^{-1/2} (\log aN) \right) \\ & + \mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} N^{1/2} \left| V_{N-1}^* \left( x \frac{S_n}{T_{N-1}} \right) - V_{N-1}^*(x) \right| > A_{21} N^{-1/2} (\log aN) \right) \\ & + B_3 N^{-\varepsilon}. \end{aligned} \quad (4.64)$$

Once more, by a first order the Taylor expansion, we have

$$N^{1/2} \left| F_k \left( x \frac{S_n}{T_{N-1}} \right) - F_k(x) \right| = x f_k(x_N) \cdot N^{1/2} \left| \frac{S_n - T_{N-1}}{T_{N-1}} \right|, \quad (4.65)$$

where

$$|x_N - x| \leq x \left| \frac{S_n - T_{N-1}}{T_{N-1}} \right|.$$

By combining Lemma 4.2 with (4.62), it follows that

$$\mathbb{P} \left( \left| \frac{S_n}{T_{N-1}} - 1 \right| > A_{22} N^{-1} (\log aN) \right) \leq B_{22} N^{-\varepsilon}. \quad (4.66)$$

By arguing in a similar way as in the proof (4.38), we obtain that

$$\mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} N^{1/2} \left| F_k \left( x \frac{S_n}{T_{N-1}} \right) - F_k(x) \right| > A_{20} N^{-1/2} (\log aN) \right) \leq B_{20} N^{-\varepsilon}. \quad (4.67)$$

Now, by definitions (4.57), (4.67), and through a similar argument as that used in the end of the proof of Lemma 4.3, we get

$$\mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} N^{1/2} \left| V_{N-1}^* \left( x \frac{S_n}{T_{N-1}} \right) - V_{N-1}^*(x) \right| > A_{21} N^{-1/2} (\log aN) \right) \leq B_{21} N^{-\varepsilon}. \quad (4.68)$$

Then, making use of the equations (4.64), (4.67) and (4.68), we obtain

$$\mathbb{P} \left( \sup_{0 \leq x \leq Q_k(a)} \left| N^{1/2} \left( G_{N,k} \left( x \frac{S_n}{Nk} \right) - F_k(x) \right) - V_{N-1}^*(x) \right| \right) \quad (4.69)$$

$$> A_{23} N^{-1/4} (\log aN)^{3/4} \leq B_{23} N^{-\varepsilon}. \quad (4.70)$$

Again, by Lemma 4.4.4 of Csörgő and Révész (1981) and (4.59), we can get a sequence of Gaussian processes  $\{V_n(x) : 0 \leq x < \infty\}$ ,  $k(N-1) < n < Nk$ ,  $N = 1, 2, \dots$ , such that for each  $N$  we have

$$\begin{aligned} & \{\alpha_n(x), V_n(y) : 0 \leq x, y < \infty\} \\ & \stackrel{d}{=} \left\{ N^{1/2} \left( G_{N,k} \left( x \frac{S_n}{Nk} \right) - F_k(x) \right), V_{N-1}^*(y) : 0 \leq x, y < \infty \right\}. \end{aligned}$$

This completes the proof of Theorem 2.4. ■

## Proof of Corollary 2.5

The functional  $\Phi$  being Lipschitz, there exists a positive constant  $L$  such that, for any functions  $v, w$ ,

$$|\Phi(v) - \Phi(w)| \leq L \sup_{t \in \mathbb{R}} |v(t) - w(t)|. \quad (4.71)$$

Let us choose for  $v, w$  the processes

$$\gamma_n := \gamma_{Nk}(\cdot) \quad \text{and} \quad \mathbb{W}_n := W_{Nk}(\cdot).$$

Applying the elementary inequality

$$|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) = \mathbb{P}((A \setminus B) \cup (B \setminus A))$$



to the events  $A = \{\Phi(\gamma_n) \leq x\}$  and  $B = \{\Phi(\mathbb{W}_n) \leq x\}$  provides, for any  $x \in \mathbb{R}$  and any  $n \in \mathbb{N}^*$ ,

$$|\mathbb{P}\{\Phi(\gamma_n) \leq x\} - \mathbb{P}\{\Phi(\mathbb{W}_n) \leq x\}| \leq \mathbb{P}\{\Phi(\gamma_n) \leq x \leq \Phi(\mathbb{W}_n) \text{ or } \Phi(\mathbb{W}_n) \leq x \leq \Phi(\gamma_n)\}.$$

Now, applying to the elementary fact that  $[a \leq x \leq b \text{ or } b \leq x \leq a \text{ implies } |b-x| \leq |b-a|]$  to the numbers  $a = \Phi(\gamma_n)$  and  $b = \Phi(\mathbb{W}_n)$

$$\mathbb{P}\{\Phi(\gamma_n) \leq x \leq \Phi(\mathbb{W}_n)\} \leq \mathbb{P}\{|\Phi(\mathbb{W}_n) - x| \leq |\Phi(\mathbb{W}_n) - \Phi(\gamma_n)|\},$$

from which, due to (4.71), we deduce that

$$|\mathbb{P}\{\Phi(\gamma_n) \leq x\} - \mathbb{P}\{\Phi(\mathbb{W}_n) \leq x\}| \leq \mathbb{P}\left\{|\Phi(\mathbb{W}_n) - x| \leq L \sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)|\right\}. \quad (4.72)$$

We obtain the estimate below valid for large enough  $n$ , for  $\epsilon_n = c(N^{-1/4}(\log aN)^{3/4})$

$$\mathbb{P}\left\{\sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)| \geq \epsilon_n\right\} \leq o(N^{-1/4}(\log aN)^{3/4}) \quad (4.73)$$

Now, by (4.72), we write

$$\begin{aligned} & |\mathbb{P}\{\Phi(\gamma_n) \leq x\} - \mathbb{P}\{\Phi(\mathbb{W}_n) \leq x\}| \\ & \leq \mathbb{P}\left\{\sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)| < \epsilon_n, |\Phi(\mathbb{W}_n) - x| \leq L \sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)|\right\} \\ & \quad + \mathbb{P}\left\{\sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)| \geq \epsilon_n, |\Phi(\mathbb{W}_n) - x| \leq L \sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)|\right\} \\ & \leq \mathbb{P}\{|\Phi(\mathbb{W}_n) - x| \leq L\epsilon_n\} + \mathbb{P}\left\{\sup_{t \in \mathbb{R}} |\gamma_n(t) - \mathbb{W}_n(t)| \geq \epsilon_n\right\}. \end{aligned} \quad (4.74)$$

Noticing that the distribution of  $W_{Nk}$  does not depend on  $n$ , which entails the equality

$$\mathbb{P}\{|\Phi(\mathbb{W}_n) - x| \leq L\epsilon_n\} = \mathbb{P}\{|\Phi(\mathbb{W}) - x| \leq L\epsilon_n\}$$

where  $\mathbb{W}$  is a gaussian process with the same distribution as  $W_{Nk}$ . and recalling the assumption that the r.v.  $\Phi(\mathbb{W})$  admits a density function bounded by  $M$  say, we get that, for any  $x \in \mathbb{R}$  and any  $n \in \mathbb{N}^*$ ,

$$\mathbb{P}\{|\Phi(\mathbb{W}_n) - x| \leq L\epsilon_n\} \leq 2LM\epsilon_n. \quad (4.75)$$

Finally, putting (4.73) and (4.75) into (4.74) leads to (2.6), which completes the proof of Corollary 2.5. An alternative proof of a similar result may be found in [Shorack and Wellner \(1986\)](#) pp. 502–503. ■

## Proof of the Theorem 3.1

The proof of this result is based on the Theorem 2.8 and the Lemma 4.4 below. Let us consider the restriction on the interval  $[0, 1]$  of a standard Wiener process  $\{W(t) : t \geq 0\}$ .

**Lemma 4.4.** *Assume that  $\{h_N : N \geq 1\}$  verify **(H.1)** and **(H.2)**. Then*

$$\lim_{N \rightarrow \infty} \Delta \left( \{ \beta_N (W(t + h_N \cdot) - W(t)) : 0 \leq t \leq 1 - h_N \}, \mathcal{S}_{\frac{c}{c+1}} \right) \stackrel{\mathbb{P}}{=} 0. \quad (4.76)$$

**Proof of Lemma 4.4 .** For  $c < \infty$ , the proof of this lemma is due to the joint use of a result of [Deheuvels and Révész \(1993\)](#) and the scaling property. Notice that the result of [Deheuvels and Révész \(1993\)](#) is obtained with our the hypothesis of monotonically decreasing (resp. increasing) when  $h_N \rightarrow 0$  (resp.  $Nh_N \rightarrow \infty$ ), we may refer also to [Deheuvels and Révész \(1987\)](#). For the case of a non bounded arbitrary  $c$ , the proof of [Deheuvels and Révész \(1993\)](#) could still be used. We just notice that the quantity  $\delta_T$  defined in their article tends again to 0. Then, the replacement of their Theorem 1.1 initially used, by the Theorem 1.2 appearing in their article in connection with the use of the property of scaling leads to the expected result.  $\blacksquare$

Now, we are equipped to prove Theorem [3.1](#).

### Proof of Theorem [3.1](#).

For all  $0 \leq t \leq 1 - h_N$ , we define the quantity,

$$\omega_N(h_N, t; \cdot) = W(t + h_N \cdot) - W(t). \quad (4.77)$$

Through [\(4.76\)](#), the triangle inequality applied on the Hausdorff distance bring our proof to show that, as  $n \rightarrow \infty$ ,

$$\Delta \left( \left\{ \beta_N \eta_N(h_N, t; \cdot) : 0 \leq t \leq 1 - h_N \right\}, \left\{ \beta_N \omega_N(h_N, t; \cdot) : 0 \leq t \leq \Gamma'_1 - h_N \right\} \right) \stackrel{\mathbb{P}}{\rightarrow} 0,$$

By [\(3.1\)](#) and [\(2.11\)](#), it is easy to remark that, for all  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  such that the following inequality takes place for all  $n$  large enough, with probability greater than  $1 - \varepsilon$ ,

$$\begin{aligned} & \left\| \beta_N \eta_N(h_N, t; \cdot) - \beta_N \omega_N(h_N, t; \cdot) \right\| \\ & \leq \beta_N C_\varepsilon n^{-1/4} (\log n)^{3/4} + \left\| \beta_N (\Gamma'_N(t + h_N \cdot) - \Gamma'_N(t)) - \beta_N \omega_N(h_N, t; \cdot) \right\|. \end{aligned} \quad (4.78)$$

The hypotheses **(H.2)** and **(H.3)** implies that, as  $n \rightarrow \infty$ ,

$$\beta_N n^{-1/4} (\log n)^{3/4} \rightarrow 0,$$

Let us consider now the Brownian bridge

$$B(u) = W(u) - uW(1), 0 \leq u \leq 1.$$

Recall the definition [\(2.11\)](#) of  $\Gamma'_N$ , we infer that we have

$$\begin{aligned} & \left\| \beta_N (\Gamma'_N(t + h_N \cdot) - \Gamma'_N(t)) - \beta_N \omega_N(h_N, t; \cdot) \right\| \\ & \leq \beta_N h_N |W(1)| + \beta_N \left\| \{ \phi_k(t) - \phi_k(t + h_N \cdot) \} \int_0^1 B(u) dQ_k(u) \right\|, \end{aligned} \quad (4.79)$$

where we recall

$$\phi_k(t) = \frac{1}{k} Q_k(t) f_k(Q_k(t)),$$

and

$$\sup_{0 \leq t \leq 1} \phi_k(t) < \infty,$$

which implies that

$$\beta_N \left\| \{ \phi_k(t) - \phi_k(t + h_N \cdot) \} \right\| = o(1) \times O_{\mathbb{P}}(1) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty. \quad (4.80)$$

It follows from **(H.1)** that  $\beta_N h_N \rightarrow 0$  as  $n \rightarrow \infty$ . By (4.79) and (4.80) and making use of Tchebychev's inequality, we obtain readily that, as  $n \rightarrow \infty$ ,

$$\left\| \beta_N \eta_N(h_N, t; \cdot) - \beta_N \omega_N(h_N, t; \cdot) \right\| \xrightarrow{\mathbb{P}} 0, \quad (4.81)$$

Hence, we finally obtain, as  $n \rightarrow \infty$

$$\Delta \left( \left\{ \beta_N \eta_N(h_N, t; \cdot) : 0 \leq t \leq 1 - h_N \right\}, \left\{ \beta_N \omega_N(h_N, t; \cdot) : 0 \leq t \leq 1 - h_N \right\} \right) \xrightarrow{\mathbb{P}} 0,$$

The proof of Theorem 3.1 is therefore completed. ■

### Proof of Corollary 3.3.

As  $\mathcal{S}_{c/(c+1)}$  is a compact and connected set of  $B_0[0, 1]$  equipped with the uniform topology, the continuity of the functional  $\Phi$  implies that image set  $\Phi(\mathcal{S}_{c/(c+1)})$  is also a compact and connected set of  $\mathbb{R}$  of the form  $[l_\Phi, L_\Phi]$ , where

$$l_\Phi = \inf_{f \in \mathcal{S}_{c/(c+1)}} \Phi(f) \quad \text{and} \quad L_\Phi = \sup_{f \in \mathcal{S}_{c/(c+1)}} \Phi(f).$$

By making use of Theorem 3.1 and the continuity of the functional  $\Phi$ , we obtain

$$\lim_{n \rightarrow \infty} \Delta \left( \left\{ \beta_N \eta_N(h_N, t; \cdot) : 0 \leq t \leq 1 - h_N \right\}, [l_\Phi, L_\Phi] \right) \xrightarrow{\mathbb{P}} 0.$$

which readily imply that

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1 - h_N} \Phi(\beta_N \eta_N(h_N, t; \cdot)) \xrightarrow{\mathbb{P}} L_\Phi.$$

Hence the proof of Corollary 3.3 is complete. ■

### Proof of Corollary 3.4.

For any  $f(\cdot) \in \mathcal{S}_{c/(c+1)}$ , the Schwarz inequality shows that, for each  $0 \leq s \leq 1$ ,

$$|f(s)| = \left| \int_0^s \dot{f}(u) du \right| \leq \left( s \int_0^1 \dot{f}^2(u) du \right)^{1/2} \leq \left( \frac{c}{c+1} \right)^{1/2}.$$

Therefore, by choosing  $f(\cdot)$  in such a way that

$$f(s) = \left(\frac{c}{c+1}\right)^{1/2} s, \text{ for } 0 \leq s \leq 1,$$

gives, after a simple calculation, that

$$f(\cdot) \in \mathcal{S}_{c/(c+1)} \text{ and } \Phi_1(f) = \Phi_2(f) = \left(\frac{c}{c+1}\right)^{1/2}.$$

Thus, an application of Corollary 3.3 implies, in turn, that, (3.4) and (3.5) hold. Moreover, for all  $f(\cdot) \in \mathcal{S}_{c/(c+1)}$ , using Schwarz' inequality implies readily that

$$\mp \int_0^1 v(u) \dot{f}(u) du \leq \left(\frac{c}{c+1} \int_0^1 v^2(u) du\right)^{1/2}.$$

The function  $f(\cdot)$  of derivative,

$$\dot{f}(s) = \mp \left(\frac{c}{c+1}\right)^{1/2} \left(\int_0^1 v^2(u) du\right)^{-1/2} v(s),$$

belongs to  $\mathcal{S}_{c/(c+1)}$ . Furthermore,

$$\Phi_3(f) = \left(\frac{c}{c+1}\right)^{1/2} \left(\int_0^1 v^2(s) ds\right)^{1/2}.$$

An application of Corollary 3.3 implies that (3.6) holds. ■

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