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Conditionally Independent Generalized Competing Risks for Maintenance Analysis

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Abstract. A complex repairable system is subjected to corrective maintenance (CM) and condition-based preventive maintenance (PM) actions. In order to take into account both the dependency between PM and CM and the possibility of imperfect maintenances, a generalized competing risks model have been introduced in [5]. In this paper, we study the particular case for which the potential times to next PM and CM are independent conditionally to the past of the maintenance process. We address the identifiability issue and find a result similar to that of [2] for usual competing risks. We propose a realistic model with exponential risks and derive the maximum likelihood estimators of its parameters.

Keywords. Reliability, imperfect maintenance, competing risks, point processes

Introduction

Complex repairable systems are submitted to two kinds of maintenance actions. *Corrective maintenance (CM)*, also called *repair*, is carried out after a failure and intends to put the system into a state in which it can perform its function again. *Preventive maintenance (PM)* is carried out when the system is operating and intends to slow down the wear process and reduce the frequency of occurrence of system failures. *Planned PM* occur at predetermined times. *Condition-based PM* occur at times which are determined according to the results of inspections and degradation or operation controls. In this study, we focus on condition-based PM. Then CM and PM times are both random and the sequence of maintenance times is a random point process.

In [5], we introduced the Generalized Competing Risks (GCR) models. It is a modelling framework for the maintenance process which takes both into account the possibility of imperfect maintenance and the dependency between CM and condition-based PM. The aim of this paper is to study the particular case of Conditionally Independent Generalized Competing Risks (CIGCR).

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1. Modelling of the maintenance process

The PM-CM process is the sequence of PM times and CM times. Maintenance durations are assumed to be negligible or not taken into account. Then, we introduce the following notations.

- $\{C_k\}_{k \geq 1}$ the maintenance times (CM and PM), with $C_0 = 0$.
- $\{W_k\}_{k \geq 1}$ the times between maintenances, $W_k = C_k - C_{k-1}$.
- $K = \{K_t\}_{t \geq 0}$ the counting maintenance (CM and PM) process.
- $N = \{N_t\}_{t \geq 0}$ the counting CM process.
- $M = \{M_t\}_{t \geq 0}$ the counting PM process.
- $\{U_k\}_{k \geq 1}$ the indicators of maintenance types: $U_k = 0$ if the k^{th} maintenance is a CM and $U_k = 1$ if the k^{th} maintenance is a PM.

In the following, bold characters denote vectors, for instance $\mathbf{W}_k = (W_1, \dots, W_k)$.

The PM-CM process can either be written as a bivariate counting process $\{N_t, M_t\}_{t \geq 0}$ or as a colored counting process: $\{K_t, U_{K_t}\}_{t \geq 0}$. The color associated to an event of the global maintenance process specifies whether the maintenance is preventive or corrective.

2. Characterization of the PM-CM process

Let $\mathcal{H}_t = \sigma(\{N_s, M_s\}_{0 \leq s \leq t}) = \sigma(\{K_s, U_{K_s}\}_{0 \leq s \leq t})$ be the natural filtration generated by the past of the processes N and M at time t . It is well known [1] that the maintenance process is characterized by three stochastic intensities. The CM intensity is:

$$\lambda_t^N = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_{t-} = 1 | \mathcal{H}_{t-}) \quad (1)$$

The PM intensity is:

$$\lambda_t^M = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(M_{t+\Delta t} - M_{t-} = 1 | \mathcal{H}_{t-}) \quad (2)$$

The (global) maintenance intensity is:

$$\lambda_t^K = \lambda_t^N + \lambda_t^M = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(K_{t+\Delta t} - K_{t-} = 1 | \mathcal{H}_{t-}) \quad (3)$$

In a parametric approach, the parameters θ of the PM and CM intensities can be estimated by the maximum likelihood method. The likelihood function associated to a single observation of the PM-CM process on $[0, t]$ is :

$$L_t(\theta) = \left[\prod_{i=1}^{K_t} \lambda_{C_i}^N {}^{1-U_i} \lambda_{C_i}^M {}^{U_i} \right] \exp \left(- \sum_{i=1}^{K_t+1} \int_{C_{i-1}}^{C_i} \lambda_s^K ds \right) \quad (4)$$

In order to build a model of the maintenance process, it is necessary to express the probability of instantaneous PM and CM given all the past of the maintenance process. A realistic model has to take into account both the efficiency of maintenance (which is not necessarily perfect or minimal) and the possible dependency between both kinds of maintenance, due to the fact that CM and PM are linked to the degradation process.

3. Usual competing risks models

A simple way of modelling the PM-CM process is the competing risks (CR) approach, developed in the context of maintenance e.g. in [3]. After the k^{th} maintenance, the latent time to the next failure (or the next CM) is a random variable Z_{k+1} . But the failure can be avoided by a potential PM that can take place at a random time Y_{k+1} after the k^{th} maintenance. Z_{k+1} and Y_{k+1} are not observed. The observations are the time to next maintenance $W_{k+1} = \min(Y_{k+1}, Z_{k+1})$ and the type of next maintenance $U_{k+1} = \mathbb{I}_{\{Y_{k+1} \leq Z_{k+1}\}}$. Y_{k+1} and Z_{k+1} are called the *risk variables*.

In the usual competing risks problem, it is assumed that the couples $\{(Y_k, Z_k)\}_{k \geq 1}$ are independent and identically distributed (iid), so the $\{(W_k, U_k)\}_{k \geq 1}$ are also iid. This means that the effect of every PM and CM is supposed to be perfect.

The dependency between each type of maintenance is expressed by the joint distribution of (Y_1, Z_1) , characterized by the *joint survival function*:

$$S_1(y, z) = P(Y_1 > y, Z_1 > z) \quad (5)$$

A well known problem of usual competing risks models is that the distribution of (Y_1, Z_1) is not identifiable. In fact, the distribution of the observations $\{(W_k, U_k)\}_{k \geq 1}$ depends only on the *sub-survival functions* [4] :

$$S_{Z_1}^*(z) = P(Z_1 > z, Z_1 < Y_1) = P(W_1 > z, U_1 = 0) \quad (6)$$

$$S_{Y_1}^*(y) = P(Y_1 > y, Y_1 \leq Z_1) = P(W_1 > y, U_1 = 1) \quad (7)$$

The assumption that the $\{(Y_k, Z_k)\}_{k \geq 1}$ are iid is not realistic because the effects of all CM and PM are not perfect. Moreover, PM and CM should be dependent because:

- PM and CM are linked to the degradation process.
- The aim of PM is to reduce the frequency of failures, so PM should delay CM.
- CM can have an influence on the future PM policy.

Then, it is interesting to generalize the usual competing risks models in order to take into account any kind of imperfect maintenance effect and any kind of dependency between CM and PM.

4. Generalized competing risks models

By a *generalized competing risks model* (GCR [5]), we mean a competing risks model for which the couples $\{(Y_k, Z_k)\}_{k \geq 1}$ are not assumed to be iid. The couples $\{(W_k, U_k)\}_{k \geq 1}$ are therefore also not iid. Thus, the effect of every PM and CM can be imperfect. The usual competing risks objects are naturally generalized by introducing a conditioning on the past of the PM-CM process.

The *CM-PM conditional generalized survival function* is:

$$S_{k+1}(y, z; \mathbf{W}_k, \mathbf{U}_k) = P(Y_{k+1} > y, Z_{k+1} > z \mid \mathbf{W}_k, \mathbf{U}_k) \quad (8)$$

The *generalized sub-survival functions* are:

$$S_{Z_{k+1}}^*(z; \mathbf{W}_k, \mathbf{U}_k) = P(Z_{k+1} > z, Z_{k+1} < Y_{k+1} \mid \mathbf{W}_k, \mathbf{U}_k) \quad (9)$$

$$S_{Y_{k+1}}^*(y; \mathbf{W}_k, \mathbf{U}_k) = P(Y_{k+1} > y, Y_{k+1} \leq Z_{k+1} | \mathbf{W}_k, \mathbf{U}_k) \quad (10)$$

The *conditional survival functions* of the risk variables are:

$$S_{Z_{k+1}}(z; \mathbf{W}_k, \mathbf{U}_k) = P(Z_{k+1} > z | \mathbf{W}_k, \mathbf{U}_k) \quad (11)$$

$$S_{Y_{k+1}}(y; \mathbf{W}_k, \mathbf{U}_k) = P(Y_{k+1} > y | \mathbf{W}_k, \mathbf{U}_k) \quad (12)$$

The maintenance intensities can be written in terms of the PM-CM survival functions:

$$\lambda_t^N = \frac{\left[-\frac{\partial}{\partial z} S_{K_{i-}+1}(y - C_{K_{i-}}, z - C_{K_{i-}}; \mathbf{W}_{K_{i-}}, \mathbf{U}_{K_{i-}}) \right]_{(t,t)}}{S_{K_{i-}+1}(t - C_{K_{i-}}, t - C_{K_{i-}}; \mathbf{W}_{K_{i-}}, \mathbf{U}_{K_{i-}})} \quad (13)$$

$$\lambda_t^M = \frac{\left[-\frac{\partial}{\partial y} S_{K_{i-}+1}(y - C_{K_{i-}}, z - C_{K_{i-}}; \mathbf{W}_{K_{i-}}, \mathbf{U}_{K_{i-}}) \right]_{(t,t)}}{S_{K_{i-}+1}(t - C_{K_{i-}}, t - C_{K_{i-}}; \mathbf{W}_{K_{i-}}, \mathbf{U}_{K_{i-}})} \quad (14)$$

$$\lambda_t^K = -\frac{d}{dt} \ln S_{K_{i-}+1}(t - C_{K_{i-}}, t - C_{K_{i-}}; \mathbf{W}_{K_{i-}}, \mathbf{U}_{K_{i-}}) \quad (15)$$

Finally, the likelihood (4) can be rewritten.

$$L_t(\theta) = S_{K_{i-}+1}(t - C_{K_{i-}}, t - C_{K_{i-}}; \mathbf{W}_{K_{i-}}, \mathbf{U}_{K_{i-}}) \prod_{i=1}^{K_t} \left[-\frac{\partial}{\partial y} S_i(y, z; \mathbf{W}_{i-1}, \mathbf{U}_{i-1}) \right]_{(W_i, W_i)}^{U_i} \left[-\frac{\partial}{\partial z} S_i(y, z; \mathbf{W}_{i-1}, \mathbf{U}_{i-1}) \right]_{(W_i, W_i)}^{1-U_i} \quad (16)$$

It can be seen that the PM-CM intensities and the likelihood depend only on the values of the PM-CM survival functions on the first diagonal. Then, there will be here the same identifiability problem as in classical competing risks models.

5. Conditionally independent generalized competing risks models

The most simple way of building a GCR model is to make a conditional independence assumption. The risks variables $\{(Y_k, Z_k)\}_{k \geq 1}$ are said to be *conditionally independent* if they are independent conditionally to the past of the maintenance process: $\forall k \geq 0, \forall y \geq 0, \forall z \geq 0,$

$$S_{k+1}(y, z; \mathbf{W}_k, \mathbf{U}_k) = S_{Y_{k+1}}(y; \mathbf{W}_k, \mathbf{U}_k) S_{Z_{k+1}}(z; \mathbf{W}_k, \mathbf{U}_k) \quad (17)$$

The corresponding GCR models are called the *conditionally independent generalized competing risks models (CIGCR)*. Note that PM and CM are dependent through the past of the maintenance process.

The maintenance intensities are:

$$\lambda_t^N = \lambda_{Z_{K_{t-}+1}}(t - C_{K_{t-}}; \mathbf{W}_{K_{t-}}, \mathbf{U}_{K_{t-}}) \quad (18)$$

$$\lambda_t^M = \lambda_{Y_{K_{t-}+1}}(t - C_{K_{t-}}; \mathbf{W}_{K_{t-}}, \mathbf{U}_{K_{t-}}) \quad (19)$$

$$\lambda_t^K = \lambda_{W_{K_{t-}+1}}(t - C_{K_{t-}}; \mathbf{W}_{K_{t-}}, \mathbf{U}_{K_{t-}}) \quad (20)$$

where λ_X denotes the hazard rate of the random variable X .

The conditional survival functions can be expressed as functions of the maintenance intensities:

$$S_{Z_{k+1}}(z; \mathbf{W}_k, \mathbf{U}_k) = \exp\left(-\int_0^z \lambda_{c_k+u}^N(k; \mathbf{W}_k, \mathbf{U}_k) du\right) \quad (21)$$

$$S_{Y_{k+1}}(y; \mathbf{W}_k, \mathbf{U}_k) = \exp\left(-\int_0^y \lambda_{c_k+u}^M(k; \mathbf{W}_k, \mathbf{U}_k) du\right) \quad (22)$$

Then, a CIGCR model is identifiable. Now we have an identifiability result, equivalent to that of [2].

1. Two CIGCR models with the same CM and PM intensities have the same generalized joint survival function.
2. For every GCR model, there exists a CIGCR model with the same CM and PM intensities.

The first result confirms that, for a CIGCR model, S_{k+1} is identifiable for all k . The second one proves that it is not true for all GCR models. Then, in order to predict the future of the maintenance process, it is possible to use a CIGCR model. But in order to obtain information on the failure process without PM, additional assumptions are needed on the joint distribution of (Y_{k+1}, Z_{k+1}) given $(\mathbf{W}_k, \mathbf{U}_k)$.

6. Exponential CIGCR models

An exponential CIGCR model is such that the conditional distributions of Y_{k+1} and Z_{k+1} given $(\mathbf{W}_k, \mathbf{U}_k)$ are exponential, with respective parameters $\lambda^Y(\mathbf{W}_k, \mathbf{U}_k)$ and $\lambda^Z(\mathbf{W}_k, \mathbf{U}_k)$. Then, the conditional survival functions are:

$$S_{Z_{k+1}}(z; \mathbf{W}_k, \mathbf{U}_k) = e^{-\lambda^Z(\mathbf{W}_k, \mathbf{U}_k) z} \quad (23)$$

$$S_{Y_{k+1}}(y; \mathbf{W}_k, \mathbf{U}_k) = e^{-\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) y} \quad (24)$$

The joint survival function is:

$$S_{k+1}(y, z; \mathbf{W}_k, \mathbf{U}_k) = e^{-\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) y - \lambda^Z(\mathbf{W}_k, \mathbf{U}_k) z} \quad (25)$$

and the conditional distribution of W_{k+1} is also exponential:

$$S_{W_{k+1}}(w; \mathbf{W}_k, \mathbf{U}_k) = e^{-\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) + \lambda^Z(\mathbf{W}_k, \mathbf{U}_k) w} \quad (26)$$

The maintenance intensities and the conditional subsrvival functions can easily be derived:

$$\lambda_t^N = \lambda^Z(\mathbf{W}_{K_t^-}, \mathbf{U}_{K_t^-}) \quad (27)$$

$$\lambda_t^M = \lambda^Y(\mathbf{W}_{K_t^-}, \mathbf{U}_{K_t^-}) \quad (28)$$

$$S_{Z_{k+1}}^*(z; \mathbf{W}_k, \mathbf{U}_k) = \frac{\lambda^Z(\mathbf{W}_k, \mathbf{U}_k)}{\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) + \lambda^Z(\mathbf{W}_k, \mathbf{U}_k)} e^{-\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) + \lambda^Z(\mathbf{W}_k, \mathbf{U}_k) z} \quad (29)$$

$$S_{Y_{k+1}}^*(y; \mathbf{W}_k, \mathbf{U}_k) = \frac{\lambda^Y(\mathbf{W}_k, \mathbf{U}_k)}{\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) + \lambda^Z(\mathbf{W}_k, \mathbf{U}_k)} e^{-\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) + \lambda^Z(\mathbf{W}_k, \mathbf{U}_k) y} \quad (30)$$

Finally, the likelihood function associated to the observation of the maintenance process on $[0, t]$ is:

$$L_t(\theta) = \prod_{i=1}^{K_t} \lambda^Z(\mathbf{W}_{i-1}, \mathbf{U}_{i-1})^{1-U_i} \lambda^Y(\mathbf{W}_{i-1}, \mathbf{U}_{i-1})^{U_i} \times \exp\left(-\sum_{i=1}^{K_t+1} \left[\lambda^Y(\mathbf{W}_{i-1}, \mathbf{U}_{i-1}) + \lambda^Z(\mathbf{W}_{i-1}, \mathbf{U}_{i-1})\right] W_i\right) \quad (31)$$

In order to build an exponential CIGCR model, it is necessary to define how λ_Y and λ_Z depend on $(\mathbf{W}_k, \mathbf{U}_k)$. In other words, we have to find a model of the influence of past CM and PM to next CM and PM.

7. A tractable exponential CIGCR model

The dependency between PM and CM can be expressed on the following way. If there have been lots of failures (CM) in the past, the system is not reliable enough. To improve it, the PM have to be performed sooner than expected. In other words, CM accelerate PM. Conversely, if there have been lots of PM, the PM should delay the occurrence of failures. In other words, PM delay CM. We will build a model which reflects these assumptions.

We first assume that Z_1 and Y_1 are independent and exponentially distributed with respective parameters λ_c and λ_p . We consider here that delaying a maintenance is multiplying the concerned rate by a constant $\alpha < 1$. Similarly, accelerating a maintenance is multiplying the concerned rate by a constant $\beta > 1$.

Then, if the first maintenance is a PM ($U_1 = 1$), we assume that :

- $\lambda^Y(W_1, 1) = \lambda_p$ (PM frequency is unchanged).
- $\lambda^Z(W_1, 1) = \alpha \lambda_c$ (CM frequency is decreased : CM is delayed).

If the first maintenance is a CM ($U_1 = 0$), we assume that :

- $\lambda^Y(W_1, 0) = \beta \lambda_p$ (PM frequency is increased : PM is accelerated).
- $\lambda^Z(W_1, 0) = \lambda_c$ (CM frequency is unchanged).

Both cases lead to :

- $\lambda^Y(W_1, U_1) = \lambda_p \beta^{1-U_1}$.

- $\lambda^Z(W_1, U_1) = \lambda_c \alpha^{U_1}$.

With the same assumptions on next maintenances, we obtain:

$$\lambda^Y(\mathbf{W}_k, \mathbf{U}_k) = \lambda_p \beta^{\sum_{i=1}^k (1-U_i)} = \lambda_p \beta^{N_{C_k}} \quad (32)$$

$$\lambda^Z(\mathbf{W}_k, \mathbf{U}_k) = \lambda_c \alpha^{\sum_{i=1}^k U_i} = \lambda_c \alpha^{M_{C_k}} \quad (33)$$

where N_{C_k} and M_{C_k} are respectively the numbers of CM and PM occurred before k^{th} maintenance. Note that $N_{C_k} + M_{C_k} = k$.

The maintenance intensities of this model are:

$$\lambda_t^N = \lambda_c \alpha^{M_t} \quad (34)$$

$$\lambda_t^M = \lambda_p \beta^{N_t} \quad (35)$$

$$\lambda_t^K = \lambda_p \beta^{N_t} + \lambda_c \alpha^{M_t} \quad (36)$$

The model parameters have a simple practical interpretation.

- λ_c characterizes the *initial reliability*: it is the failure rate of the system if it is not maintained.
- λ_p characterizes the *initial preventive maintenance policy*: it is the PM rate if the system is replaced by a new one at each failure.
- α characterizes the *PM efficiency* : the smaller α is, the more PM will manage to delay failures.
- β characterizes the *reactivity of the maintenance team* : the larger β is, the more PM will be anticipated in case of failure.

The likelihood function associated to the observation of k maintenances between 0 and t is:

$$L_t(\lambda_p, \lambda_c, \alpha, \beta; \mathbf{W}_k, \mathbf{U}_k) = \prod_{i=1}^k (\lambda_c \alpha^{M_{C_{i-1}}})^{1-U_i} (\lambda_p \beta^{N_{C_{i-1}}})^{U_i} \times \exp\left(-\sum_{i=1}^{k+1} [\lambda_p \beta^{N_{C_{i-1}}} + \lambda_c \alpha^{M_{C_{i-1}}}] w_i\right) \quad (37)$$

with $W_{k+1} = t - C_k$.

Then, it is easy to prove that the maximum likelihood estimators $\hat{\lambda}_p$, $\hat{\lambda}_c$, $\hat{\alpha}$ and $\hat{\beta}$ are such that:

$$\hat{\lambda}_p = \frac{M_{C_k}}{\sum_{i=1}^{k+1} \hat{\beta}^{N_{C_{i-1}}} W_i} \quad \hat{\lambda}_c = \frac{N_{C_k}}{\sum_{i=1}^{k+1} \hat{\alpha}^{M_{C_{i-1}}} W_i} \quad (38)$$

$\hat{\alpha}$ and $\hat{\beta}$ are solution of two implicit equations :

$$\left[\sum_{i=1}^k (1 - U_i) M_{C_{i-1}} \right] \left[\sum_{i=1}^{k+1} \hat{\alpha}^{M_{C_{i-1}}} W_i \right] = N_{C_k} \sum_{i=1}^{k+1} \hat{\alpha}^{M_{C_{i-1}}} M_{C_{i-1}} W_i \quad (39)$$

$$\left[\sum_{i=1}^k U_i N_{C_{i-1}} \right] \left[\sum_{i=1}^{k+1} \hat{\beta}^{N_{C_{i-1}}} W_i \right] = M_{C_k} \sum_{i=1}^{k+1} \hat{\beta}^{N_{C_{i-1}}} N_{C_{i-1}} W_i \quad (40)$$

With these estimates, it is possible to assess the system reliability and the efficiency of both types of maintenance.

8. Discussion

The generalized competing risks provide a general framework for the modelling of the maintenance process, with possibly dependent CM-PM and imperfect maintenance. The conditional independence assumption allows to build simple models with a practical interpretation. The identifiability property shows that, for each kind of data set, a CIGCR model can be adapted.

The properties of the exponential CIGCR model have to be studied. The model should be applied to real data. Finally it is possible to build other models, for instance with different CM-PM dependency assumptions (random sign, delay), or with Weibull distribution instead of exponential.

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