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## ► To cite this version:

Henri Gouin, Jean-François Debieve. Variational Principle Involving the Stress Tensor in Elastodynamics. International Journal of Engineering Science, 1986, 24 (7), pp.1057-1066. <10.1016/0020-7225(86)90001-7>. <hal-00303656>

**HAL Id: hal-00303656**

**<https://hal.science/hal-00303656v1>**

Submitted on 22 Jul 2008

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# Variational Principle Involving the Stress Tensor in Elastodynamics

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## Abstract

In the mechanics of inviscid conservative fluids, it is classical to generate the equations of dynamics by formulating with adequate variables, that the pressure integral calculated in the time-space domain corresponding to the motion of the continuous medium is stationary. The present study extends this principle to the dynamics of large deformations for isentropic motions in thermo-elastic bodies: we use a new way of writing the equations of motion in terms of *potentials* and we substitute the trace of the stress tensor for the pressure term.

*Key words:* Elastodynamics; Variational principle; Stress tensor canonical decomposition.

*PACS:* 46.05.+b; 46.15.Cc; 62.20.D-; 81.40.Jj.

*1991 MSC:* 73C50; 73V25; 73B27

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## 1 Introduction

HAMILTON'S PRINCIPLE holds for all conservative mechanical systems with holonomic side conditions. This is the case with perfect fluids in adiabatic motion. If  $(F)$  represents the set of the virtual motions which assign to the

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system two given positions at times  $t_1$  and  $t_2$ , the principle can be stated as follows (*principle of least action*):

*Among the (F)-motions, the motion of the system is that which stationarizes the integral of the difference between kinetic energy and potential energy over the time-space domain occupied by the continuous medium.*

Several authors have observed that in conventional fluid mechanics, many variational principles can be reduced to Hamilton's principle, although the procedures are diverse, according to the choice of the unknown functions. In the Eulerian description of fluids, several authors [1-5] have shown that the equations of motion may be derived from a variational principle through the introduction of Lagrange multipliers corresponding to the side conditions that the variations of the mass density, the entropy and the Lagrangian coordinates are coupled with some conservation conditions. Minor details apart, and with  $\mathbf{V}, \rho, s, \mathbf{X}, \phi, \psi, \varepsilon$  as independent variables, these methods consist in stationarizing

$$\int_W \left\{ \frac{1}{2} \rho \mathbf{V}^2 - \rho \varepsilon - \rho \Omega + \phi \left( \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} \right) + \psi \left( \frac{\partial \rho s}{\partial t} + \operatorname{div} \rho s \mathbf{V} \right) + \left( \frac{\partial \rho \mathbf{X}^\tau}{\partial t} + \operatorname{div}(\rho \mathbf{V} \mathbf{X}^\tau) \right) \boldsymbol{\Xi} \right\} d\mathbf{x} dt, \quad (1)$$

where  $\mathbf{X} = \begin{vmatrix} X^1 \\ X^2 \\ X^3 \end{vmatrix}$  are the Lagrangian coordinates,  $\mathbf{x} = \begin{vmatrix} x^1 \\ x^2 \\ x^3 \end{vmatrix}$  the Eulerian coordinates,

$\rho$  is the mass density,  $\mathbf{V}$  the velocity,  $\varepsilon$  the internal energy density,  $s$  the entropy density,  $\Omega$  the extraneous force potential;  $\phi$  and  $\psi$  are scalar Lagrange multipliers,  $\boldsymbol{\Xi}$  is a vector Lagrange multiplier and  $^\tau$  stands for the transposition. The variations with respect to  $\phi, \psi, \boldsymbol{\Xi}$  give the constraints:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0, \quad \text{density equation} \quad (2)$$

$$\frac{\partial \rho s}{\partial t} + \operatorname{div} \rho s \mathbf{V} = 0, \quad \text{entropy equation} \quad (3)$$

$$\frac{\partial(\rho \mathbf{X}^\tau)}{\partial t} + \operatorname{div}(\rho \mathbf{V} \mathbf{X}^\tau) = 0, \quad \text{Lin's constraint [5,6]}. \quad (4)$$

The last equation expresses that, in the eulerian description, the initial coordinates  $\mathbf{X}$  do not change along a particle path [2]. The connection of this principle with Hamilton's principle is somewhat intricate for any direct use. Casal [7], Seliger & Whitham [8] have introduced considerable simplification. By integrating by parts, and supposing that the integral corresponding to the

boundary of the time-space domain has a zero-variation, these authors reduce the integrand in (1) to:

$$\frac{1}{2}\rho\mathbf{V}^2 - \rho\varepsilon - \rho\Omega - \rho\dot{\phi} - \rho s\dot{\psi} - \rho\mathbf{X}^\tau\dot{\Xi}$$

where  $\dot{\phantom{x}}$  is the material derivative. By using *Clebsch representation* [9] for the motion and thermodynamic relations, they reduce the Lagrangian density to be just the pressure  $p$ . Conversely they have to find a suitable variational principle which generates the dynamical equations of a fluid and auxiliary conditions by expressing that

$$\int_W p \, d\mathbf{x} dt \quad (5)$$

is stationary.

On this purpose, Seliger and Whitham express the pressure as a function of the enthalpy and the entropy. They give a Clesch representation of the velocity field by means of Lagrange multipliers  $\phi, \psi, \Xi$ . This does not restrict the form of the velocity vector and gives a redundant number of unknowns. Under the constraint

$$p = \frac{1}{2}\rho\mathbf{V}^2 - \rho\varepsilon - \rho\Omega - \rho\dot{\phi} - \rho s\dot{\psi} - \rho\mathbf{X}^\tau\dot{\Xi},$$

the variational principle which states that (5) is stationary with respect to the variables  $\phi, s, \psi, \mathbf{X}$  and  $\Xi$ , yields the equations of the fluid motion in terms of potentials, and the density equation.

Obviously, this principle is far from Hamilton's principle. Seliger and Whitham noticed that Clesch's representation of the velocity is a crucial step for the final form (5) and said *this seems to be an especially simple form limited to fluids*.

The purpose of the present study is to generalize this principle to isentropic motions in thermo-elastic bodies with thermodynamic potentials [10]. The medium may be inhomogeneous.

We analyze the article by Seliger & Whitham and the limitation caused by Clesch representation. By using an appropriate representation for the internal energy, we overcome the difficulties associated with the fact that the stress tensor of an elastic medium is no more spherical.

In the second section, a decomposition of the stress tensor into spherical and deviatoric parts is associated to partial derivations of the internal energy. *The decomposition we obtained in 1986 is now largely used in the literature.*

In the third section, an Eulerian approach of the variational principle similar to the method used by Seliger and Whitham leads to a new Clesch representation and to the equations of motion in terms of potentials.

In the fourth section, we observe that the Lagrangian density can be reduced to some expression of the trace of the stress tensor.

We extend the results of Seliger and Whitham by writing that  $\int_W tr \sigma \, d\mathbf{x} dt$  is stationary, where  $tr \sigma$  stands for the trace of the stress tensor. Now, we consider  $tr \sigma$  as a function of the enthalpy  $h$ , the entropy  $s$ , the material variable  $\mathbf{X}$  and of the tensor  $C$  where  $C$  is the right Cauchy-Green tensor.

The variations are taken with respect to variables  $s, \mathbf{X}, \phi, \psi, \Xi$  introduced in the same way as in relation (5) and under the constraint:

$$h - \frac{1}{2} \mathbf{V}^2 + \dot{\phi} + \psi \dot{s} - \mathbf{X}^\tau \dot{\Xi} + \Omega = 0.$$

Three appendices present auxiliary calculations and, by means of the convective derivation associated with the velocity field, show the equivalence between Euler equations and our new motion equations in terms of potentials for isentropic processes.

## 2 A decomposition of the stress tensor in a hyperelastic medium

Each particle of the continuous medium is labelled by a material variable  $\mathbf{X}$ , ranging over a reference configuration  $\mathcal{D}_0$  in an Euclidian space [10]. The reference density  $\rho_0$  is given as a function on  $\mathcal{D}_0$  [11].

The expression  $\mathbf{x} = \phi(\mathbf{X}, t)$  of the spatial position describes the motion of the continuous medium. Generally,  $\phi(\cdot, t)$  is a twice continuously differentiable diffeomorphism of  $\mathcal{D}_0$  onto a compact orientable manifold  $\mathcal{D}_t$  constituting the image of the material at time  $t$  (see Appendix 1). As usual, we denote by  $F$  the deformation gradient; then,  $C = F^\tau F$  is the right Cauchy-Green tensor. Recall that:

$$\rho (\det C)^{\frac{1}{2}} = \rho_0(\mathbf{X}). \quad (6)$$

The internal energy density is supposed to be a function of the tensor  $C$ , the specific entropy  $s$  and the material variable  $\mathbf{X}$  (the hyperelastic medium is not necessary materially homogeneous):

$$\varepsilon = e(C, s, \mathbf{X}).$$

In the case of an isentropic process, this leads us back to a medium constituted with hyperelastic points (see Ref.10, p.19). Now, let us put:

$$C' = \frac{1}{(\det C)^{\frac{1}{3}}} C.$$

Then,  $C = \left( \frac{\rho_0(\mathbf{X})}{\rho} \right)^{\frac{2}{3}} C'$  and the internal energy density can be expressed in the form:

$$\varepsilon = f(\rho, C', s, \mathbf{X}). \quad (7)$$

When  $f$  is independent of both  $C'$  and  $\mathbf{X}$ , we retrieve the case of an elastic materially homogeneous fluid.

Let us observe that the independent variables  $\rho$  and  $C'$  are substituted to  $C$ . Since essentially  $\det C' = 1$ , the variable  $\rho$  corresponds to the change of volume while the tensorial variable  $C'$  represents the distortion of the medium. This point is fundamental for the decomposition of the stress tensor and will be the key of the representation of the equations of motion in terms of *potentials*. But  $f$  is defined on the manifold  $\det C' = 1$  and it is more convenient to introduce the function  $g$  such that

$$\varepsilon = g(\rho, C, s, \mathbf{X}) = f\left(\rho, \frac{C}{(\det C)^{\frac{1}{3}}}, s, \mathbf{X}\right) \quad (8)$$

where  $g$  is a homogeneous function of degree zero with respect to  $C$ . We assume  $g$  to be a differentiable function of  $\rho, C, s, \mathbf{X}$ . The stress tensor can be written [11,12]:

$$\sigma = 2\rho F \frac{\partial e}{\partial C} F^T. \quad (9)$$

From (6) and (8), it follows:

$$\sigma = 2\rho F \left( \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial C} + \frac{\partial g}{\partial C} \right) F^T.$$

Using Eq. (27) proved in Appendix 2, we deduce:

$$\sigma = -\rho^2 \frac{\partial g}{\partial \rho} I + 2\rho F \frac{\partial g}{\partial C} F^T, \quad (10)$$

where  $I$  represents the identity tensor. Hence, with (10) and (29), we deduce:

$$\sigma = -pI + \sigma_1$$

with:

$$p = \rho^2 \frac{\partial g}{\partial \rho}, \quad \sigma_1 = 2\rho F \frac{\partial g}{\partial C} F^T \quad \text{and} \quad \text{tr } \sigma_1 = 0. \quad (11)$$

### 3 A new transformation of the Hamilton principle for isentropic processes in elastodynamics

The Lagrangian density in the classical principle of Hamilton has the form [2]:

$$L = \rho \left( \frac{1}{2} \mathbf{V}^2 - \varepsilon - \Omega \right).$$

We introduce the variables  $C$  and  $s$  by means of (8) and the various quantities  $\rho, \mathbf{V}, C, s$  and  $\Omega$  involved in the Lagrangian density become functions of  $\mathbf{X}$  and of the motion  $\phi$  of the continuous medium [2].

Let us denote by  $\delta$ , the virtual displacement as defined by Serrin (see Ref. 2, p. 145). Under the Eulerian form, the variational principle of Hamilton reads:

For all variations  $\delta$ , vanishing on the boundary of  $W$ , one has

$$\delta \int_W L \, d\mathbf{x} dt = 0.$$

Consequently, equations of the motion of isentropic processes in elastodynamics are obtained.

In another way, it is convenient to consider the quantities  $\rho, \mathbf{V}, s$  and  $\mathbf{X}$  as variables. To be able to take into account the constraints imposed on  $\rho, s$  and  $\mathbf{X}$ , it is necessary to introduce appropriate Lagrange multipliers [2,7,8] ( $C$  being a function of  $\mathbf{X}$  by its derivative with respect to  $\mathbf{X}$ ). We are led to the variational principle:

**Theorem 1** *Let  $\phi, \psi$  and  $\Xi$  be three Lagrange multipliers where  $\phi, \psi$  are scalars and  $\Xi$  is a vector, the conditions for*

$$\int_W \left\{ \rho \left( \frac{1}{2} \mathbf{V}^2 - \varepsilon - \Omega \right) + \phi \left( \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} \right) - \rho \psi \dot{s} - \rho \Xi^T \dot{\mathbf{X}} \right\} d\mathbf{x} dt = 0 \quad (12)$$

*to be stationary for every variation of  $\rho, \mathbf{V}, s, \mathbf{X}, \phi, \psi, \Xi$  vanishing on the boundary of  $W$  yield the equations of isentropic motion and the constraints (2)-(4).*

The Lagrange multiplier  $\Xi$  is associated with Lin's constraint:

$$\frac{\partial}{\partial t} (\rho \mathbf{X}^T) + \operatorname{div} (\rho \mathbf{V} \mathbf{X}^T) = 0.$$

The latter arises from the fact that the Lagrangian coordinates no longer need to be given by explicit expressions; only the velocity field  $\mathbf{V}$  must be such that it should be possible to obtain such a coordinate system by integration [5,7,8].

By integrating by parts the two expressions in integral (12)

$$\phi \left( \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} \right) \quad \text{and} \quad -\rho \boldsymbol{\Xi}^\tau \dot{\mathbf{X}},$$

with zero-variation for the terms on the boundary of  $W$ , we are reduced to an equivalent theorem:

**Theorem 2** *With  $\rho, \mathbf{V}, s, \mathbf{X}, \phi, \psi, \boldsymbol{\Xi}$  as independent unknown functions with variations vanishing on the boundary of  $W$ , and*

$$\Lambda = \rho \left( \frac{1}{2} \mathbf{V}^2 - \varepsilon - \Omega - \dot{\phi} - \psi \dot{s} + \mathbf{X}^\tau \dot{\boldsymbol{\Xi}} \right) \quad (13)$$

*the conditions for*

$$\int_W \Lambda \, d\mathbf{x} dt \quad (14)$$

*to be stationary yield the equations of isentropic motion in elastodynamics and the constraints (2)-(4).*

Proof of the theorem:

In the last part of this paper,  $\frac{\partial}{\partial \mathbf{x}}$  will be used to denote the gradient associated with a tensor quantity defined on the motion space  $\mathcal{D}_t$  and  $(\ )'$  will be used to denote the partial derivatives. Let us write  $\theta = g'_s$ , the Kelvin temperature, and

$$h = \varepsilon + \rho g'_\rho.$$

We call  $h$ , the scalar specific enthalpy density and shortly enthalpy in this paper. The variations corresponding to  $\mathbf{V}, \rho, s$  and  $\mathbf{X}$  in expression (14) lead to the following equations: (see Appendix 3)

$$\delta \mathbf{V} : \mathbf{V}^\tau = \frac{\partial \phi}{\partial \mathbf{x}} + \psi \frac{\partial s}{\partial \mathbf{x}} - \mathbf{X}^\tau \frac{\partial \boldsymbol{\Xi}}{\partial \mathbf{x}} \quad (15)$$

$$\delta \rho : \Lambda'_\rho = 0, \quad \text{hence:} \quad \frac{1}{2} \mathbf{V}^2 - h - \Omega - \dot{\phi} - \psi \dot{s} + \mathbf{X}^\tau \dot{\boldsymbol{\Xi}} = 0 \quad (15^2)$$

$$\delta s : -\rho \theta + \rho \dot{\psi} = 0 \quad (15^3)$$

$$\delta \mathbf{X} : \dot{\boldsymbol{\Xi}} = g'_\mathbf{X}^\tau + \frac{1}{\rho} [\operatorname{div}(\sigma_1 F)]^\tau. \quad (15^4)$$

Then, the variations of  $\phi, \psi$  and  $\boldsymbol{\Xi}$  give the constraints:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0, \quad \dot{s} = 0 \quad \text{and} \quad \dot{\mathbf{X}} = 0.$$



By denoting  $\lambda = \phi - \Xi^\tau \mathbf{X}$ , we deduce the equations in terms of potentials for the motion:

$$\begin{aligned} \mathbf{V}^\tau &= \frac{\partial \lambda}{\partial \mathbf{x}} + \psi \frac{\partial s}{\partial \mathbf{x}} + \Xi^\tau F^{-1} \\ \dot{\lambda} &= \frac{1}{2} \mathbf{V}^2 - h - \Omega, \quad \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0, \\ \dot{\psi} &= \theta, \quad \dot{s} = 0, \\ \dot{\Xi}^\tau &= g'_\mathbf{x} + \frac{1}{\rho} [\operatorname{div}(\sigma_1 F)], \quad \dot{\mathbf{X}} = 0. \end{aligned} \tag{16}$$

In Appendix 1, we prove directly the equivalence between system (16) and the Euler equations joined with constraint requirements.

#### 4 A simplified form of variational principle

In the same way as Selinger and whitham, in the case of perfect compressible fluid [8], we shall show that the form of the variational principle given in Section 3 may be considerably simplified. We obtain a similar result in the case of isentropic motions for elastodynamics with large deformations.

Equation (13) may be written  $\Lambda = \rho \Lambda'_\rho + \rho^2 g'_\rho$  hence  $\Lambda = \rho \Lambda'_\rho + p$ , where  $p = -\frac{1}{3} \operatorname{tr} \sigma$ . From (15<sup>2</sup>), we obtain for the actual motions of the medium:

$$\Lambda = p.$$

The Legendre transformation of  $\rho g$  with respect to  $\rho$  gives  $-p$  with  $p$  being a function of the variables  $h, s, C, \mathbf{X}$ ,

$$p = l(h, s, C, \mathbf{X}),$$

where  $l$  is a homogeneous function of degree zero with respect to  $C$ .

*Let us examine a converse.* First, we notice that, for the thermo-elastic points of the medium, the knowledge of  $p$  as a function of the enthalpy  $h$ , the entropy  $s$ ,  $C$  and  $\mathbf{X}$ , homogeneous of degree zero with respect to  $C$ , permits the deduction by means of conventional thermodynamic relations, of the values of mass density, temperature and internal energy density. Then,

$$\rho = p'_h, \quad \theta = -\frac{p'_s}{p'_h} \tag{17}$$

and

$$\varepsilon = h - \frac{p}{p'_h}. \tag{18}$$

By differentiating (18), we obtain,

$$d\varepsilon = dh - \frac{dp}{\rho} + \frac{p}{\rho^2} d\rho.$$

Or

$$d\varepsilon = dh - \frac{1}{\rho} p'_h dh - \frac{1}{\rho} p'_s ds - \frac{1}{\rho} p'_{\mathbf{X}} d\mathbf{X} - \frac{1}{\rho} \text{tr}(p'_C dC) + \frac{p}{\rho^2} d\rho.$$

From (17), we deduce:

$$d\varepsilon = \theta ds + \frac{p}{\rho^2} d\rho - \frac{1}{\rho} p'_{\mathbf{X}} d\mathbf{X} - \frac{1}{\rho} \text{tr}(p'_C dC).$$

Recall that the internal energy density  $\varepsilon$  is in the form:

$$\varepsilon = g(\rho, C, s, \mathbf{X}),$$

where  $g$  is a homogeneous function of degree zero with respect to  $C$ . If we choose  $d\mathbf{X} = 0$ ,  $ds = 0$  and  $dC = C d\tau$ , from relations  $\text{tr}(p'_C C) = 0$ ,  $\text{tr}(g'_C C) = 0$ , and (28) we obtain:

$$\rho^2 g'_\rho = p$$

(see Appendix 2).

We can write the thermodynamic relations

$$\rho^2 g'_\rho = p, \quad \rho g'_C = -p'_C, \quad \theta = g'_s, \quad \rho g'_{\mathbf{X}} = -p'_{\mathbf{X}}.$$

The velocity vector may be written under a Clebsch representation:

$$\mathbf{V}^\tau = \frac{\partial \phi}{\partial \mathbf{x}} + \psi \frac{\partial s}{\partial \mathbf{x}} - \mathbf{X}^\tau \frac{\partial \Xi}{\partial \mathbf{x}}, \quad (19)$$

where  $\phi, \psi$  are scalar and  $\Xi$  is a vector, constituting evidently a redundant set of unknown functions.

As stated in the last principle, where the equation  $\frac{\partial \rho}{\partial t} + \text{div } \rho \mathbf{V} = 0$  specifies  $\rho$ ,  $\dot{s} = 0$  specifies  $s$  and  $\dot{\mathbf{X}} = 0$  specifies  $X$ , we shall require  $h$  to satisfy

$$h = \frac{1}{2} \mathbf{V}^2 - \dot{\phi} - \psi \dot{s} + \mathbf{X}^\tau \dot{\Xi} - \Omega. \quad (20)$$

(This constraint is connected with partial result (15<sup>2</sup>)). We may write:

**Theorem 3** *For every variation of  $s, \mathbf{X}, \psi, \phi, \Xi$  submitted to*

$$h - \frac{1}{2} \mathbf{V}^2 + \dot{\phi} + \psi \dot{s} - \mathbf{X}^\tau \dot{\Xi} + \Omega = 0$$

and vanishing on the boundary of  $W$ , the conditions for  $\int_W \text{tr } \sigma \, d\mathbf{x} dt$  to be stationary yield the equations of isentropic motions in elastodynamics and the relations (2)-(4) (the variations of  $C$  are deduced from the variations of  $\mathbf{X}$  by means of (30)). In the same way as in paragraph 3, we obtain System (21):

$$\begin{aligned} \delta\phi : \quad & \frac{\partial\rho}{\partial t} + \text{div } \rho\mathbf{V} = 0, \\ \delta\psi : \quad & \dot{s} = 0, \\ \delta s : \quad & \dot{\psi} = \theta, \\ \delta\Xi : \quad & \dot{\mathbf{X}} = 0, \\ \delta\mathbf{X} : \quad & \dot{\Xi}^\tau = g'_\mathbf{X} + \frac{1}{\rho} \text{div}(\sigma_1 F). \end{aligned} \tag{21}$$

The set Eqs. (19-20) and System (21) constitutes a system of equations of the motion and side conditions.

In this new form of variational principle, constraints are different from those of Hamilton's principle. The result is that Lagrangian and independent parameters are completely changed.

Let us notice that with the new parameter  $\lambda = \phi - \Xi^\tau \mathbf{X}$  and with a Clebsch representation for the velocity vector in the form

$$\mathbf{V}^\tau = \frac{\partial\lambda}{\partial\mathbf{x}} + \psi \frac{\partial s}{\partial\mathbf{x}} + \Xi^\tau F^{-1},$$

we may write:

**Theorem 4** *For every variation of  $s, \mathbf{X}, \psi, \lambda, \Xi$  submitted to*

$$h - \frac{1}{2}\mathbf{V}^2 + \dot{\lambda} + \psi \dot{s} + \Xi^\tau \dot{\mathbf{X}} + \Omega = 0$$

*and vanishing on the boundary of  $W$ , the conditions for  $\int_W \text{tr } \sigma \, d\mathbf{x} dt$  to be stationary yield the equations of isentropic motions and the relations (2)-(4).*

## 5 Conclusion

The variational statement in Theorem 4 is far from Hamilton's principle. The resulting motion equations involve thermodynamic variables like temperature and entropy.

The constraint condition  $h - \frac{1}{2}\mathbf{V}^2 + \dot{\lambda} + \psi \dot{s} + \Xi^\tau \dot{\mathbf{X}} + \Omega = 0$  involves the

specific enthalpy. Although this constraint explicitly uses entropy and the non uniquely defined parameters  $\lambda$  and  $\psi$ , the variational principle in this form is interesting in view of the fact that the stress tensor  $\sigma$  has an experimental meaning.

## Acknowledgements

We are grateful to Professor P. Casal for his incisive and illuminating criticism. Partial support of this research (*H.G.*) was provided by DGA/DRET-France under contract 82-455.

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## 6 Appendix 1

### Equations of isentropic motions in elastodynamics

*Preliminaries* [13,14]

The motion of the medium consists in the  $t$ -dependent  $C^2$ -diffeomorphism

$$\mathbf{X} = \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} \in \mathcal{D}_0 \xrightarrow{\phi_t} \mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathcal{D}_t, \quad [11, 12].$$

We immediately deduce  $\dot{F} = \frac{\partial \mathbf{V}}{\partial \mathbf{x}} F$  and  $\widehat{\dot{F}^{-1}} = -F^{-1} \frac{\partial \mathbf{V}}{\partial \mathbf{x}}$ . Let us write  $T^*(\mathcal{D}_t)$  for the cotangent fiber bundle of  $\mathcal{D}_t$  and  $T_{\mathbf{x}}^*(\mathcal{D}_t)$  the cotangent linear space to  $\mathcal{D}_t$  at  $\mathbf{x}$ ; then,

$$\mathbf{x} \in \mathcal{D}_t \longrightarrow L(t, \mathbf{x}) \in T_{\mathbf{x}}^*(\mathcal{D}_t)$$

represents a differential form field on  $\mathcal{D}_t$ .

Let us write  $T^*(\mathcal{D}_0)$  the cotangent fiber bundle of  $\mathcal{D}_0$  and  $T_{\mathbf{X}}^*(\mathcal{D}_0)$  the cotangent linear space to  $\mathcal{D}_0$  at  $\mathbf{X}$ . The mapping  $\phi_t^*$  is induced by  $\phi_t$  for the form fields. The convective derivation  $d_c$  of a form field  $L$  is deduced from the diagram:

$$\begin{array}{ccc} L \in T^*(\mathcal{D}_t) & \xrightarrow{\phi_t^{-1*}} & LF \in T^*(\mathcal{D}_0) \\ \downarrow d_c & & \downarrow \frac{d}{dt} \\ \dot{L} + L \frac{\partial \mathbf{V}}{\partial \mathbf{x}} \in T^*(\mathcal{D}_t) & \xleftarrow{\phi_t^*} & \dot{L}F + L \frac{\partial \mathbf{V}}{\partial \mathbf{x}} F \in T^*(\mathcal{D}_0) \end{array} \quad (22)$$

where  $\dot{L} + L \frac{\partial \mathbf{V}}{\partial \mathbf{x}}$  is the Lie derivative of  $L$  with respect to the velocity field  $\mathbf{V}$  that is the infinitesimal transformation of the one-parameter group of transformations  $\phi_t$  [15].

#### *Consequences*

Let  $b$  be a scalar field on  $\mathcal{D}_t$  assumed to be an Euclidian space;  $\text{grad}$  represents the gradient operator on  $\mathcal{D}_t$ . We define two form fields by their values  $\mathbf{V}^\tau$  and  $(\text{grad } b)^\tau$ . From (22) we deduce:

$$d_c(\mathbf{V}^\tau) = \mathbf{\Gamma}^\tau + \frac{\partial}{\partial \mathbf{x}} \left( \frac{1}{2} \mathbf{V}^2 \right), \quad (23)$$

(where  $\mathbf{\Gamma}$  is the acceleration), and

$$d_c(\text{grad } b)^\tau = (\text{grad } \dot{b})^\tau \quad (24)$$

$$d_c(LF^{-1}) = \dot{L}F^{-1}. \quad (25)$$

### *Potential equations*

With the notations of Section 2, the motion equation is:

$$\rho \mathbf{\Gamma}^\tau + \frac{\partial p}{\partial \mathbf{x}} - \text{div } \sigma_1 + \rho \frac{\partial \Omega}{\partial \mathbf{x}} = 0. \quad (26)$$

Relation (8) gives:

$$d\varepsilon = \theta ds + \frac{p}{\rho^2} d\rho + g'_\mathbf{x} d\mathbf{X} + \text{tr}(g'_C dC).$$

By symmetry property of the tensor  $g'_C$ , we obtain:

$$\text{tr}(g'_C dC) = 2\text{tr}(g'_C F^\tau dF),$$

and Eq. (11) implies

$$\text{tr}(g'_C dC) = \frac{1}{\rho} \text{tr}(\sigma_1 dF F^{-1}).$$

From the definition of the specific enthalpy, we deduce:

$$\text{grad } h = \text{grad } \varepsilon + \frac{1}{\rho} \text{grad } p - \frac{p}{\rho^2} \text{grad } \rho,$$

$$\frac{\partial h}{\partial \mathbf{x}} = \theta \frac{\partial s}{\partial \mathbf{x}} + \frac{1}{\rho} \frac{\partial p}{\partial \mathbf{x}} + g'_\mathbf{x} F^{-1} + \frac{1}{\rho} \text{div}(\sigma_1 F) F^{-1} - \frac{1}{\rho} \text{div } \sigma_1.$$

Equation (26) becomes:

$$\rho \mathbf{\Gamma}^\tau + \rho \frac{\partial}{\partial x}(h + \Omega) - \text{div}(\sigma_1 F) F^{-1} - \rho \theta \frac{\partial s}{\partial x} - g'_\mathbf{x} F^{-1} = 0.$$

Equation (23) implies:

$$d_c(\mathbf{V}^\tau) = \frac{\partial}{\partial \mathbf{x}} \left( \frac{1}{2} \mathbf{V}^2 - h - \Omega \right) + \theta \frac{\partial s}{\partial \mathbf{x}} + \frac{1}{\rho} \{ \text{div}(\sigma_1 F) + \rho g'_\mathbf{x} \} F^{-1}.$$

Let  $\Xi^\tau$  be a form field such that  $\dot{\Xi}^\tau = \frac{1}{\rho} \text{div}(\sigma_1 F) + g'_\mathbf{x}$ . With substituting from Eq. (25),

$$d_c(\mathbf{V}^\tau - \Xi^\tau F^{-1}) = \frac{\partial}{\partial \mathbf{x}} \left( \frac{1}{2} \mathbf{V}^2 - h - \Omega \right) + \theta \frac{\partial s}{\partial \mathbf{x}}.$$

Let  $\lambda$  and  $\psi$  be two scalar fields verifying  $\dot{\lambda} = \frac{1}{2}\mathbf{V}^2 - h - \Omega$  and  $\dot{\psi} = \theta$ . By adding to  $\Xi^\tau$ , if necessary, an appropriate one-form with a zero convective derivation, Eq. (24) and  $\dot{s} = 0$  (isentropic process), give:

$$\mathbf{V}^\tau = \frac{\partial \lambda}{\partial \mathbf{x}} + \psi \frac{\partial s}{\partial \mathbf{x}} + \Xi^\tau F^{-1},$$

where

$$\dot{\lambda} = \frac{1}{2}\mathbf{V}^2 - h - \Omega, \quad \dot{\psi} = \theta \quad \text{and} \quad \dot{\Xi}^\tau = \frac{1}{\rho} \operatorname{div}(\sigma_1 F) + g'_\mathbf{x}.$$

These equations along with the relations (2), (3), (4) are potential equations of the motion such as have been obtained in paragraph 3.

## 7 Appendix 2

### Some calculus

#### *Mass density*

By differentiating Eq. (6) and using Jacobi's identity, we obtain:

$$d\rho = -\frac{1}{2}\rho \operatorname{tr}(C^{-1}dC) + (\det C)^{-\frac{1}{2}} \frac{\partial \rho}{\partial \mathbf{X}} d\mathbf{X}.$$

Hence,

$$\rho'_C = -\frac{1}{2}\rho C^{-1}. \quad (27)$$

If we choose  $dC = C d\tau$  and  $d\mathbf{X} = 0$ , this reduces to

$$d\rho = -\frac{3}{2}\rho d\tau. \quad (28)$$

With respect with  $C$ ,  $g$  is a homogeneous function of degree zero. We deduce immediately (Euler identity):

$$\operatorname{tr}(g'_C C) = 0. \quad (29)$$

#### *Cauchy-Green tensor*

Let us calculate the Cauchy-Green tensor variations:

$$dC = dF^\tau F + F^\tau dF, \quad \delta F = -F \frac{\partial \delta \mathbf{X}}{\partial \mathbf{x}} F. \quad (30)$$

Hence,

$$\delta C = - \left[ C \frac{\partial \delta \mathbf{X}}{\partial \mathbf{x}} F + (C \frac{\partial \delta \mathbf{X}}{\partial \mathbf{x}} F)^\tau \right]. \quad (31)$$

## 8 Appendix 3

### Proof of Eqs.(15)

Let us write (14) under the form:

$$\begin{aligned} \int_W \Lambda \, d\mathbf{x} dt = \int_W \rho \left\{ \frac{1}{2} \mathbf{V}^2 - \varepsilon - \Omega - \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \mathbf{x}} \mathbf{V} - \psi \frac{\partial s}{\partial t} \right. \\ \left. - \psi \frac{\partial s}{\partial \mathbf{x}} \mathbf{V} + \mathbf{X}^\tau \frac{\partial \Xi}{\partial t} + \mathbf{X}^\tau \frac{\partial \Xi}{\partial \mathbf{x}} \mathbf{V} \right\} d\mathbf{x} dt. \end{aligned}$$

For every  $\delta \mathbf{V}$ -variation, we deduce:

$$\int_W \rho \left\{ \mathbf{V} - \frac{\partial \phi}{\partial \mathbf{x}} - \psi \frac{\partial s}{\partial \mathbf{x}} + \mathbf{X}^\tau \frac{\partial \Xi}{\partial \mathbf{x}} \right\} \delta \mathbf{V} d\mathbf{x} dt = 0,$$

and we obtain immediately Eq. (15). An integration by parts, on the boundary of  $W$ , leads with the same calculation to Eqs. (15<sup>2</sup>), (15<sup>3</sup>) and the constraint relations.

For every  $\delta \mathbf{X}$ -variation, vanishing on the boundary of  $W$ , we obtain:

$$\int_W \left\{ -\rho g'_\mathbf{X} \delta \mathbf{X} + \rho \dot{\Xi}^\tau \delta \mathbf{X} - \rho \operatorname{tr}(g'_C \delta C) \right\} d\mathbf{x} dt = 0,$$

where  $\delta C$  is a function of  $\delta \mathbf{X}$  (see Appendix 2, Eq.(30)).

By using Eqs.(11), (30), we obtain:

$$-\rho \operatorname{tr}(g'_C \delta C) = \operatorname{tr} \left( \sigma_1 F \frac{\partial \delta \mathbf{X}}{\partial \mathbf{x}} \right).$$

Moreover, we can write:

$$\int_W \left[ \rho (\dot{\Xi}^\tau - g'_\mathbf{X}) \delta \mathbf{X} + \operatorname{tr} \left( \sigma_1 F \frac{\partial \delta \mathbf{X}}{\partial \mathbf{x}} \right) \right] d\mathbf{x} dt = 0.$$

Hence

$$\int_W \left\{ \rho \left[ \dot{\Xi}^\tau - g'_\mathbf{X} - \frac{1}{\rho} \operatorname{div} (\sigma_1 F) \right] \right\} \delta \mathbf{X} \, d\mathbf{x} dt = 0.$$



We thus obtain

$$\dot{\Xi}^\tau = g'_{\mathbf{x}} + \frac{1}{\rho} \operatorname{div}(\sigma_1 F).$$

Let us notice that the divergence operator is applied to a tensor defined on  $\mathcal{D}_0$  with values in  $\mathcal{D}_t$ ; the partial derivatives are calculated on  $\mathcal{D}_t$ .

□