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Nonlinear Processes in Geophysics

Quasiadiabatic description of nonlinear particle dynamics in typical magnetotail configurations

D. L. Vainchtein[1,2], J. Büchner[3], A. I. Neishtadt[1], and L. M. Zelenyi[1]

[1]Space Research Institute, ul.Profsouznaya, 84/32, Moscow, Russia, GSP-7, 117997
[2]Department of Mechanical & Environmental Engineering, University of California, Santa Barbara, CA 93106, USA
[3]Max Planck Institut für Aeronomie, Max-Planck-Str. 2, D-37191 Katlenburg-Lindau, Germany

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Abstract. In the present paper we discuss the motion of charged particles in three different regions of the Earth magnetotail: in the region with magnetic field reversal and in the vicinities of neutral line of X- and O-types. The presence of small parameters (ratio of characteristic length scales in and perpendicular to the equatorial plane and the smallness of the electric field) allows us to introduce a hierarchy of motions and use methods of perturbation theory. We propose a parameter that plays the role of a measure of mixing in the system.

1 Introduction

In recent years there has been much interest in the nonlinear dynamics of charged particles in various regions of the earth magnetotail. This research started with the papers (Northrop, 1963; Speiser, 1965, 1967; Sonnerup, 1971; Lyons and Speiser, 1982). Since then, the motion of particles in the magnetic field reversal configurations was studied by different authors, both numerically and analytically for quite different approximations of the magnetic field geometry. The most basic, one-dimensional, approximation was discussed in many publications (see Chen (1992) and references therein). Longitudinal variations of the magnetic field were first taken into consideration analytically in Zelenyi et al. (1990) and numerically in Karimabadi et al. (1990), where the problem was considered for a variety of configurations of magnetic field.

The motion of particles in the vicinity of an X-type neutral line was studied theoretically and numerically (Martin, 1986; Burkhart et al., 1991; Moses et al., 1993), numerically by (Bruhwiler and Zweibel, 1992; Smets et al., 1993; Nocera et al., 1996). The impact of the electric field was studied analytically and numerically in Deeg et al. (1991) and numerically in Petkaki and Mackinnon (1994). The dynamics in the vicinity of an O-type neutral line was studied numerically in Martin et al. (1991) and analytically in Larson and Tracy (1993).

It is known that for a wide range of parameters the motion of charged particles in the magnetotail is chaotic (Chen and Palmadesso, 1986; Büchner and Zelenyi, 1989), see also Chen (1992) for a review. It was shown by Büchner and Zelenyi (1989), that in the domain where the Larmor radius of charged particles is much larger than the smallest radius of curvature of magnetic field lines the jumps of an adiabatic invariant lead to chaotic behaviour. In closed systems chaotic behaviour necessarily leads to complete mixing in the accessible part of the phase space. However, in the real magnetotail the electric field accelerates particles, thus limiting the time they spend in the magnetotail, which, in turn, defines how much mixing can happen before the particles leave the system.

One of the standard methods of studying the chaotization and mixing (in both analytical and numerical approaches) is to look at the behaviour of approximate invariants of a system. The action variable I_z (the action variable of the motion perpendicular to the equatorial plane), that is a standard action variable of the Hamiltonian dynamics and analogous to the the magnetic moment μ in the guiding center theory, was introduced by Speiser (1970). The important role played by I_z was first recognized by Sonnerup (1971). That single action variable is, naturally, not sufficient for a complete description of more realistic models, which include both variations of parameters of the magnetic field along the equatorial plane and the influence of the electric field. In Zelenyi et al. (1990) new results were obtained by introducing an additional, longitudinal, adiabatic invariant, that is analogous to $J_\parallel$ (the action associated with the motion along a magnetic field line) in the guiding center theory. This approach proved to be quite useful and allowed the authors to obtain important estimates of the particles' acceleration. The method was based on two consecutive averagings of the equations of motion, and the validity of the approach, that may be non-trivial (because of separatrix crossings, even the adiabatic invariant of

Correspondence to: D. L. Vainchtein
(dmitri@engineering.ucsb.edu)

the fastest motion, I_z, is not well-conserved) were not discussed. The situation here is quite different from the case when the guiding center theory works. In the latter case, two consecutive averagings of the equations of motions are clearly valid and the adiabatic invariants are preserved for timescales growing exponentially with the magnetic field intensity (see Benettin and Sempio, 1994).

In the present paper we consider the motion of charged particles in three distinct regions of the Earth magnetotail: in the region with magnetic field reversal (MFR), taking into consideration the longitudinal variations of the magnetic field, and in the vicinities of neutral lines of X- and O-type (XO). In all three regions the magnetic field was considered two-dimensional (no cross-tail component) and the presence of the constant electric field was taken into account. The are several objectives of the present publication. First, our goal is to define what results obtained for uniform normal component of the magnetic field, B_n, are applicable for x-dependent B_n and what phenomena are qualitatively different. Second, we would like to obtain the conditions under which the mixing in the phase space is not too strong. We propose a new parameter that describes the extent of mixing in the system (see Sect. 5). This parameter depends on the magnitude of the electric field and the smoothness and the curvature of the magnetic field. Depending on the value of this parameter, the jumps of the adiabatic invariants destroy original structures in the distribution function or particles accelerate and leave the tail quickly enough to keep the distribution function relatively intact. And finally, we use the technique developed in Büchner and Zelenyi (1986, 1989); Zelenyi et al. (1990) to study the motion of charged particles near neutral lines of X- and O-types in order to understand the impact of processes that occur in the vicinities of neutral lines on the overall picture of particles acceleration.

There are two basic phenomena that may affect the picture constructed in the present paper: the time-dependent perturbations (like electromagnetic waves) and the presence of the additional cross-tail component of the magnetic field. The impact of the electromagnetic waves on the dynamics of particles in the magnetotail was studied started in Cattell et al. (1995); Ma and Summers (1998). A range of parameters for which the structure obtained for stationary field configurations survive in the presence of the electromagnetic waves was obtained analytically in Vainchtein et al. (2004). The role of the cross-tail component of the magnetic field in the MFR region was first discussed in Karimabadi et al. (1990) and later studied, both analytically and numerically, in many papers, including Büchner and Zelenyi (1991); Zhu and Parks (1993); Chapman and Rowlands (1998); Ynnerman et al. (2000); Tsalas et al. (2001). It was shown that for a certain range of the ratios of the typical values of the components of the magnetic field the structures obtained for two-dimensional model stay qualitatively the same even in the presence of the third component.

The structure of the paper is as follows. In Sect. 2 we describe the basic models for all the configurations of electromagnetic fields that we use the rest of the paper. In Sect. 3 we discuss the properties of the fastest (perpendicular to the equatorial plane) motion. We show that dynamics is exactly the same for MFR and XO problems. In Sect. 4 we illustrate the averaging method and provide the description of the phase portraits of the averaged system. In Sect. 5 we describe the long-time properties of the particles' dynamics (on times of order of the lifetime of particles in the magnetotail) and introduce a parameter that can be used as a measure of mixing. In Sects. 6 and 7 we derive the acceleration rates in the adiabatic case in the MFR and XO problems. Section 8 contains conclusions.

2 Main equations

We start with a derivation of the Hamiltonian of a charged particle in different regions of the Earth magnetotail. We separate three distinct regions: the region with magnetic field reversal (referred as MFR below), and the vicinities of neutral lines of X- and O-types. The descriptions of the motion of charged particles near the neutral lines of two types are mathematically very similar. We denote by XO the equations applicable to the dynamics near either neutral line and provide the separate descriptions when necessary.

We use the following models of the magnetic field:

$$\mathbf{B} = -B\frac{z}{L_z}\mathbf{e}_x + B_n(x)\mathbf{e}_z \quad (MFR),$$
$$\mathbf{B} = sB\frac{z}{L_z}\mathbf{e}_x + B\frac{x}{L_x}\mathbf{e}_z \quad (XO). \tag{1}$$

In Eq. (1), $\mathbf{e}_i$ are the unit vectors of a Cartesian coordinate system in which the x-axis is directed towards the earth, the y-axis is in the equatorial plane and is directed from dusk to dawn and the z-axis is perpendicular to the equatorial plane and is directed from south to north; L_x and L_z are the characteristic length scales in x- and z-directions, respectively; s is a parameter that is equal to $+1$ for the problem for an X-line and is equal to -1 for the problem for an O-line.

The electric field is induced by the solar wind and has the form

$$\mathbf{E} = E\mathbf{e}_y.$$

In the magnetotail E can reach the values of order of 0.1 mV/m. These values of E correspond to a total potential drop of order of 25 kV. The latter quantity gives the upper boundary for the energy that a particle can gain in the magnetosphere due to the acceleration in a dawn-dusk electric field.

For the MFR problem the specified configuration of the electromagnetic field is schematically shown in Fig. 1a and can be described by a vector potential

$$\mathbf{A} = \left(0; \int_{x_0}^{x} B_n(\zeta)d\zeta - B\frac{z^2}{2L_z} - cEt; 0\right),$$

where x_0 is an arbitrary constant (it corresponds to an ambiguity in choosing a vector potential), t is the time and c is the

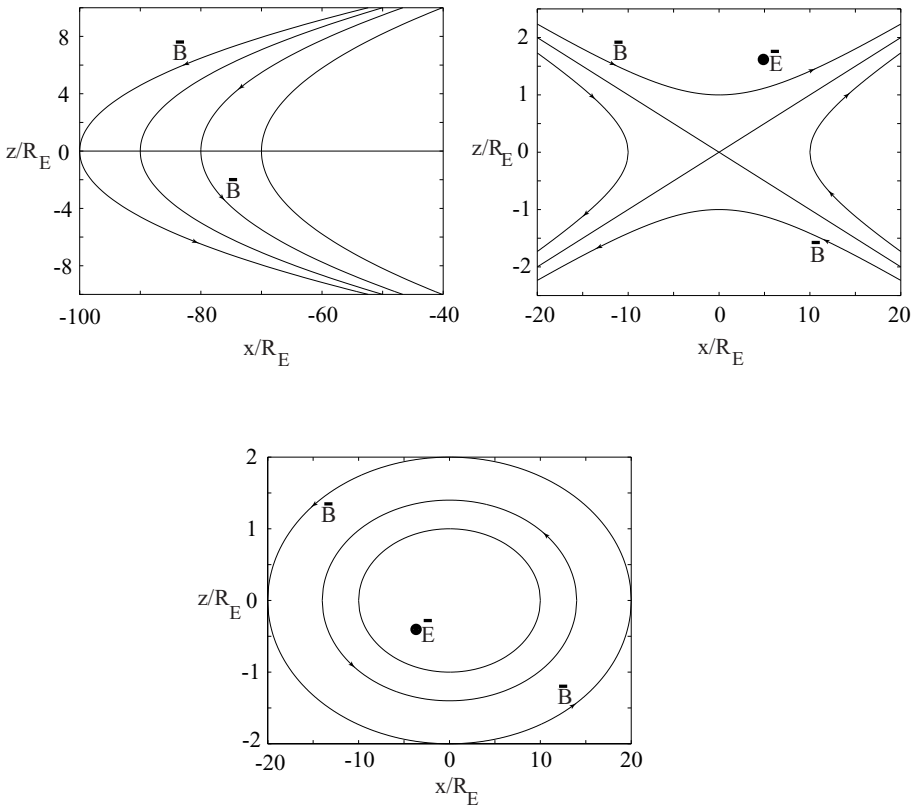

Fig. 1. Characteristic configuration of electromagnetic field: **(a)** In the MFR region, **(b)** in the vicinity of a neutral line of X type, and **(c)** in the vicinity of a neutral line of O type.

speed of light. In the specified gauge, the Hamiltonian of a charged particle is

$$H = \frac{1}{2m}\left[P_x^2 + \left(P_y - \frac{e}{c}\left(\overline{B}_n \int_{x_0}^{x} \tilde{\beta}(\zeta)d\zeta - B\frac{z^2}{2L_z} - cEt\right)\right)^2 + P_z^2\right], \tag{2}$$

where $\overline{B}_n$ is some typical value of $B_n(x)$ and $\tilde{\beta}(x)=B_n(x)/\overline{B}_n$.

Note, that the time dependence in (2) is just a standard pay off for using a vector potential and the Hamiltonian form of equations of motion (see e.g. Landau and Lifshitz, 1959) and is not related to time-dependent field reversal configurations similar to those considered in Chapman (1994); Chapman and Watkins (1996).

The configurations of electromagnetic fields in the vicinity of X- and O-lines are shown in Figs. 1b and c, respectively. The vector potential can be written as

$$\mathbf{A} = \left(0;\ sB\frac{x^2}{2L_x} - B\frac{z^2}{2L_z} - cEt;\ 0\right),$$

and the Hamiltonian of a charged particle is

$$H = \frac{1}{2m}\left[P_x^2 + \left(P_y - \frac{e}{c}\left(sB\frac{x^2}{2L_x} - B\frac{z^2}{2L_z} - cEt\right)\right)^2 + P_z^2\right]. \tag{3}$$

In Eqs. (2) and (3), $\mathbf{P} = \left(P_x, P_y, P_z\right)$, m and e are the generalized momentum, the mass and the charge of a particle respectively. As H does not depend on y explicitly, P_y is an integral of motion: $P_y = P_{y,0}$=const.

Introduce dimensionless variables

$$\mathbf{x_1} = \mathbf{x}/\sqrt{\rho_L L_z},\ \mathbf{P_1} = \mathbf{P}/mv,\ t_1 = tv/\sqrt{\rho_L L_z},\ h = H/mv^2, \tag{4}$$

where v is a typical velocity of a particle and $\rho_L = cmv/eB$ is the Larmor radius. Introduce a scaled time:

$$\tau = \varepsilon t_1 + P_{y,0},$$

where ε is the normalized electric field:

$$\varepsilon = \frac{cE}{v\overline{B}_n} \quad (MFR), \qquad \varepsilon = \frac{cE}{vB}\frac{L_z}{\sqrt{L_z\rho_L}} \quad (XO).$$

Estimates of the magnitude of ε (based on representative values of E=0.1 mV/m, B_n=1 nT, and v=700 km/sec, see e.g. (Kivelson and Russell, 1995)) yield $1/10<\varepsilon<1/3$. Note, that although the values of ε are not too small, it is the product of the normalized magnitude of the electric field, ε, and the parameter of the nonuniformity of the $x-$ component of the

magnetic field (ν for MFR problem, λ for the XO problem, see below) that defines the ratio of the characteristic time scales. And those products are sufficiently small. Therefore, the new time τ can be considered to be slow. In the new variables Hamiltonians (2) and (3) have the form

$$h = \frac{1}{2}\left[P_{x_1}^2 + \left(\kappa \int_{x_{1,0}}^{x_1} \tilde{\beta}(\zeta) d\zeta - \frac{1}{2} z_1^2 - \kappa\tau\right)^2 + P_{z_1}^2\right] \quad (MFR), \quad (5)$$

$$h = \frac{1}{2}\left[P_{x_1}^2 + \left(s\frac{1}{2}\lambda^2 x_1^2 - \frac{1}{2} z_1^2 - \tau\right)^2 + P_{z_1}^2\right] \quad (XO). \quad (6)$$

In (5) and (6),

$$\kappa = \frac{\overline{B}_n}{B}\sqrt{L_z/\rho_L} \quad (MFR), \quad (7)$$

and

$$\lambda^2 = \frac{L_z}{L_x} \quad (XO).$$

Parameters κ and λ, that is equivalent to b_n from Burkhart et al. (1991), play a crucial role in the description of the dynamics of charged particles.
Now let's consider two problems separately.

2.1 The Hamiltonian in MFR.

Let us start with the MFR problem. For constant B_n, the Hamiltonian is time-independent in the moving coordinate frame $\widehat{x}=x-tcE/B_n$ (de Hoffman – Teller substitution, de Hoffman and Teller, 1950). One of the main objectives of the present paper is to define what results obtained for uniform B_n (see Büchner and Zelenyi, 1989; Zelenyi et al., 1990) are applicable for x-dependent B_n and what phenomena are qualitatively different.

If B_n depends on x smoothly then the Hamiltonian depends smoothly on time. If B_n does not change the sign, we can introduce

$$x_2 = \int_{x_{1,0}}^{x_1} \tilde{\beta}(\zeta) d\zeta$$

as a new variable. Define a dimensionless parameter ν which characterizes the smoothness of $B_n(x)$ such that $\beta(\nu x_2)=\tilde{\beta}(x)$ and $(1/\beta)(\partial\beta(\nu x_2)/\partial(\nu x_2))\sim 1$. Hamiltonian (5) gets the form

$$h = \frac{1}{2}\left[P_x^2\beta^2(\nu x) + P_z^2 + \left(x' - \frac{z^2}{2}\right)^2\right], \quad (8)$$

where

$$x' = \kappa(x - \tau). \quad (9)$$

In Eq. (8) and below, we do not write subscripts of x, P_x and z, P_z.

For the parameter κ introduced in (7) we have $\kappa^2 \equiv \rho_c/\rho_L$, where is ρ_c is a characteristic minimal curvature radius of a magnetic field line. Recall, that to define κ we used some the quantity $\overline{B}_n$, which is typical value of the $B_n(x)$ for a region where the motion is localized. Therefore, κ is constant. It was shown by Wagner et al. (1979), that if $\kappa \gg 1$, the particles are magnetized and their motion can be described by the guiding center theory (**GCT**). In the magnetotail, electrons are always magnetized, except for the very close vicinity of the X-line. The value of κ for ions strongly depends on what part of the tail we study and can be anywhere from $\kappa \ll 1$ in the distant tail to $\kappa \gg 1$ near the earth with all the stops in between. Like electrons, ions are magnetized if $\kappa \gg 1$. Up to the best of our knowledge, there is no complete analytical picture of the dynamics if $\kappa \sim 1$ (see e.g. (Delcourt and Martin, 1999) and references therein for progress in research in that region). We are interested in the case $\kappa \ll 1$ with other parameters satisfying the following conditions:

$$\nu \le \kappa \ll 1, \quad \varepsilon \ll 1.$$

The physical meaning of $\nu \le \kappa$ is that $B_n(x)$ does not change too much over the amplitude of the x-motion. In particular, it excludes from the consideration all the trajectories that come too close to the earth. These inequalities are valid approximately further then 10 R_E from the earth, where R_E is the radius of the earth. In this approximation there is a hierarchy of motions. The fastest motion is on the (z, P_z) plane (i.e. the motion perpendicular to the equatorial plane), the motion on the (x, P_x) plane (the motion towards – away from the earth) is slower and on the (t, h) (acceleration) is the slowest.

Let us now comment on the approximations used in the present paper. First, we neglected the cross-tail component of the magnetic field, B_y, which was first discussed in Karimabadi et al. (1990) and later studied, both analytically and numerically, in many papers, including Büchner and Zelenyi (1991); Zhu and Parks (1993); Chapman and Rowlands (1998); Ynnerman et al. (2000); Tsalas et al. (2001). It was shown that for a certain range of the ratios of the typical values of the components of the magnetic field the structures obtained for two-dimensional model stay qualitatively the same even in the presence of the third component. One way to take B_y into account is to modify the definition of κ to

$$\kappa = \frac{\overline{B}_n}{B}\sqrt{L_z/\rho_L}\left(1 + \left(\frac{B_y}{\overline{B}_n}\right)^2\right)^{3/4}.$$

For realistic values of B_y (of order of B_n) the value of κ would increase approximately twice, remaining smaller than unity. However, further increase of B_y does result in chaotization of ion dynamics – the way it should be for κ of order 1. For even larger values of B_y, the ions are magnetized and we could use guiding center theory instead of the quasiadiabatic theory.

Another nuance is that definition (7), derived in Büchner and Zelenyi (1989) for thick sheets ($L_z > \rho_L$) should be used

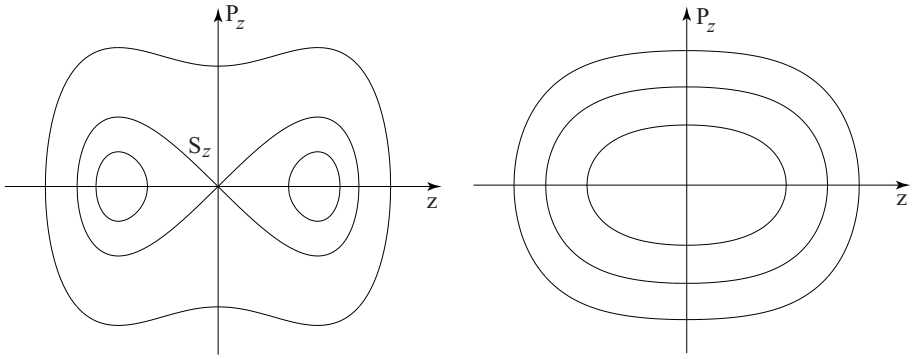

Fig. 2. Characteristic phase portrait of the z-motion for (a) $x'>0$ and (b) $x'<0$.

with care for thin sheets ($L_z>\rho_L$). As was shown in later papers (Savenkov et al., 1991, 1997) for thin sheets the definition of κ becomes $\kappa=\overline{B}_n/B$. Hence, as it was pointed by Chen (1992) the dynamics of ions depends on two parameters $\overline{B}_n/B$ and L_z/ρ_L. The latter parameter contains the energy of particles, H, and when $H\to\infty$, $L_z/\rho_L\to 0$, but the value of κ stays at a fixed value $\kappa=\overline{B}_n/B$. In this case one could use a bilinear model of tail field, where particle motion is easily integrable. However, the methods used in the present study are applicable for either definition of κ, in other words regardless of the relation between L_z and ρ_L.

2.2 The XO Hamiltonian

The motion of charged particles in the vicinity of the neutral lines of X- and O-types is described by Hamiltonian (6). The typical values of x are of order λ^{-1} and the typical values of z are of order 1. Following Martin (1986), we assume λ to be of order of 0.1−0.2, which does not contradict to available experimental data. Therefore, we have the same hierarchy of motions as in the MFR case. For XO we denote

$$x' = s\frac{1}{2}\lambda^2 x^2 - \tau,$$

and Hamiltonian (6) takes the form

$$h = \frac{1}{2}\left[P_x^2 + P_z^2 + \left(x' - \frac{z^2}{2}\right)^2\right]. \qquad (10)$$

For particles residing near X-line, a similar Hamiltonian was considered in Martin (1986); Moses et al. (1993) and (without the electric field) in Burkhart et al. (1991).

3 The motion on the (z, P_z) plane

The ratio of typical frequencies for z- and x-motion is of order $1/\kappa\gg 1$ (MFR) (in the part of the tail we are interested in) or of order $1/\lambda\gg 1$ (XO). Therefore, when studying the z-motion, we fix the values of x, P_x, τ and h. Therefore, the Hamiltonian of the z-motion,

$$h_z = \frac{1}{2}\left[P_z^2 + \left(x' - \frac{z^2}{2}\right)^2\right], \qquad (11)$$

is the same for MFR and XO and coincides with the Hamiltonian considered by Zelenyi et al. (1990); Burkhart et al. (1991).

Characteristic phase portraits of the z-motion are shown in Fig. 2a for $x'>0$ and in Fig. 2b for $x'<0$. The separatrix S_z in Fig. 2a is specified by $h_z=h_{z,sep}=(x')^2/2$. The action variable,

$$I_z = \frac{1}{2\pi}\oint P_z dz,$$

is equal to the normalized by 2π area inside a phase curve on the (z, P_z) plane. The value of I_z is given by Sonnerup (1971); Büchner and Zelenyi (1989); Burkhart et al. (1991):

$$I_z = \frac{8}{3\pi} I_z' = \frac{8}{3\pi}(2h_z)^{3/4} f(k), \qquad (12)$$

$$f(k) = \begin{cases} f_a(k) = \left(1-k^2\right)K(k) + \left(2k^2-1\right)E(k), & k<1, \\ f_b(k) = \frac{1}{2}\left(2\left(1-k^2\right)kK\left(k^{-1}\right) + \left(2k^2-1\right)kE\left(k^{-1}\right)\right), & k>1. \end{cases}$$

In (12), the subscript a corresponds to the motion in the domain with $h_z>h_{z,sep}$, and the subscript b corresponds to the motion in one of two domains with $h_z<h_{z,sep}$ (in what follows, we also use the notation a and b to denote the respective region);

$$k^2 = \frac{1}{2}\left(1 + x'(2h_z)^{-1/2}\right);$$

$E(k)$ and $K(k)$ are complete elliptic integrals. It is convenient to use I_z', which is just a constant multiple of I_z, in order to avoid repeating appearances of the factor $8/3\pi$.

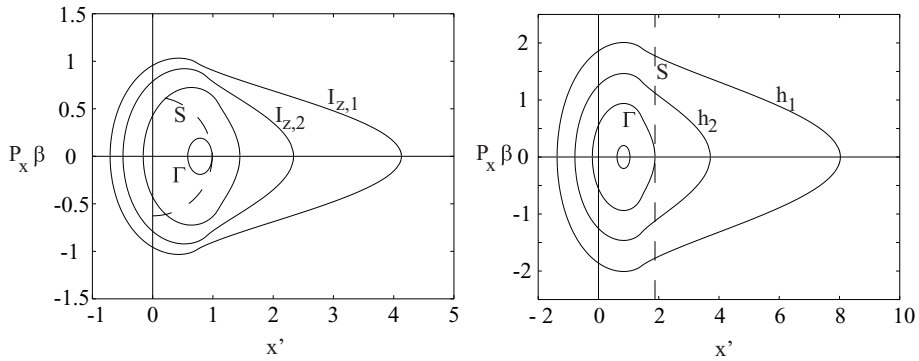

Fig. 3. Phase portraits on the $(x', P_x\beta)$ plane for the MFR problem: **(a)** $h{=}h_0$, phase curves are lines of constant I_z, the **UC** is a semi-circle $(x')^2 + (P_x\beta)^2 = 2h$, $x'>0$, of two marked trajectories $h_1>h_2$; (b) $I_z{=}I_{z,0}$, phase curves are lines of constant h, the **UC** is a straight line $x' = (I_z')^{2/3}$, of two marked trajectories $I_{z,1}<I_{z,2}$.

4 The motion on the (x, P_x) plane.

The action variable I_z is the adiabatic invariant of the x-motion. Substituting $h_z{=}h_z\left(I_z, x'\right)$ into (8) or (10), in other words, averaging the equations of motion over z-oscillations, we obtain the Hamiltonian of the x-motion possessing $1\frac{1}{2}$ degrees of freedom:

$$h = \frac{1}{2}P_x^2\beta^2(\nu x) + h_z\left(I_z, x'\right) \quad (MFR),$$
$$h = \frac{1}{2}P_x^2 + h_z\left(I_z, x'\right) \quad (XO). \tag{13}$$

The slow time, τ, appears in (13) via the dependence of x' on τ. Note, that in (13) the function h_z is the same for the MFR and XO problems. But as x' depends on x and τ differently for two problems, properties of the x-motion are also different.

We start our description of the x-motion with the construction of phase portraits.

4.1 Phase portraits for MFR.

There are two ways to plot a phase portrait on the $(x', P_x\beta)$ plane. One can either set $h{=}h_0$ and plot phase curves as lines of constant I_z (Fig. 3a), or set $I_z{=}I_{z,0}$ and plot phase curves as lines of constant h (Fig. 3b). The advantages of both of these methods will become quite clear later (see Sect. 5 below). Every point on the $(x', P_x\beta)$ plane corresponds to a unique closed phase curve on the (z, P_z) phase plane.

Figure 3a is very similar to what is shown in Büchner and Zelenyi (1989). The difference is that in Fig. 3a the vertical axis is $P_x\beta$ instead of P_x. The very similarity proves that by appropriate transformation of variables the problem with varying $B_n(x)$ can be reduced on some time scale to that of Büchner and Zelenyi (1989). This fact may not be immediately obvious from the original Hamiltonian structure (5).

The special curve S (an uncertainty curve, **UC**, see Wisdom, 1985) corresponds to the separatrix of the z-motion. When a phase point moving along a phase curve on the $(x', P_x\beta)$ plane arrives at S, the corresponding phase curve of the z-motion resides on S_z. In Fig. 3a the **UC** has the form of a semi-circle:

$$\left(x'\right)^2 + (P_x\beta)^2 = 2h, \quad x' > 0,$$

and it is a straight line in Fig. 3b:

$$x' = \left(I_z'\right)^{2/3}.$$

The fixed point C is the point of maximum of $f(k)$. At the maximum $k{=}k_c \approx 0.91$ and $f(k_c){=}f_c \approx 1.16$. Hence

$$x_c' = (2h)^{-1/2}\left(2k_c^2 - 1\right) \approx 0.47h^{-1/2} \text{ in Fig. 3a,} \tag{14}$$

$$x_c' = \left(I_z'/f_c\right)^{2/3}\left(2k_c^2 - 1\right) \approx 0.59\left(I_z'\right)^{2/3} \text{ in Fig. 3b.} \tag{15}$$

In Fig. 3a the outer orbits correspond to the smaller values of I_z', while the inner orbits correspond to the larger values of I_z', with I_z' reaching its maximum value of $I_{z,max}'{=}f_c\,(2h)^{3/4}$ at C. Similarly, in Fig. 3b the outer orbits correspond to the larger values of h, while the inner orbits correspond to the smaller values of h. On both phase portraits there is a unique curve Γ that is tangent to S. It divides the phase plane into two regions: the region A outside Γ and the region B within Γ. All the trajectories residing in the region B do not cross S, and all the trajectories residing in the region A cross S.

4.2 Phase portraits in the vicinity of an X-line

As Hamiltonian (6) depends on the parameter s, the phase portraits of the x-motion in the vicinities of neutral lines of X- and O-types are significantly different. The behaviour of particles in the vicinity of an X-line was studied in several papers (see e.g. Martin, 1986; Burkhart et al., 1991). In Burkhart et al. (1991) the phase portraits were numerically constructed as surface of sections of trajectories of the original system. Contrarly to that, the phase portraits presented below are constructed purely analytically using the method

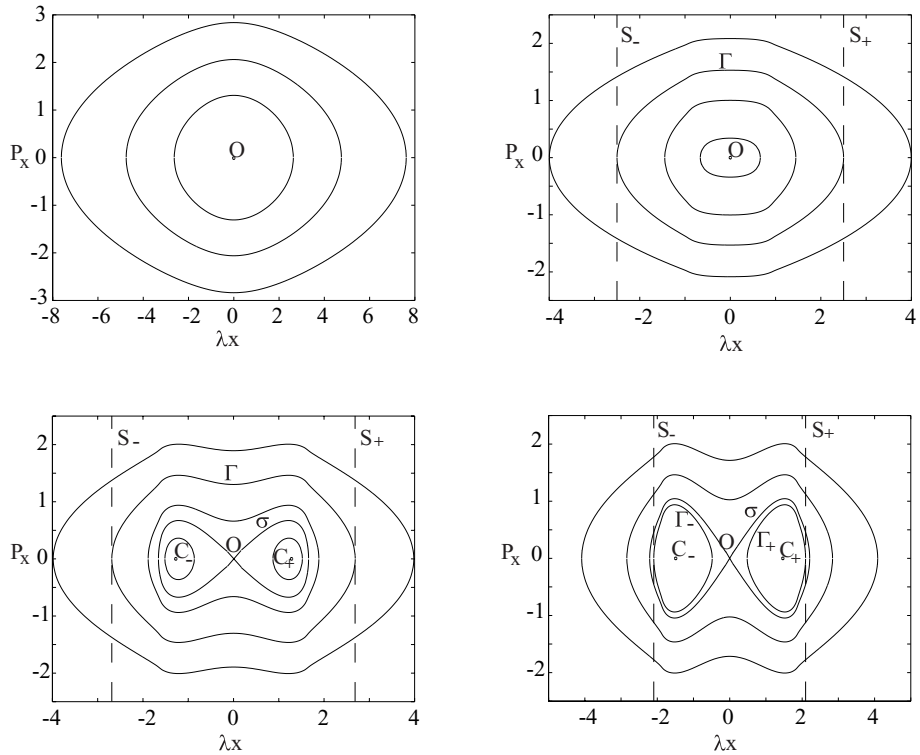

Fig. 4. Phase portraits on the $(\lambda x', P_x)$ plane for the X-line problem, $I_z = I_{z,0}$, phase curves are lines of constant h. **(a)** $\tau < -\left(I_z'\right)^{2/3}$, **(b)** $-\left(I_z'\right)^{2/3} < \tau < \tau_c$, **(c)** $\tau_c < \tau < 0$, **(d)** $\tau > 0$.

of averaging and the perturbation theory. As a result, we were able to describe the structure of phase portraits of x-motion in more details. In particular, we found that there were not three, but four qualitatively different types of the phase portraits depending on the values of the slow variables and I_z' (see below).

We construct phase portraits for the XO problem in the same way the phase portrait in Fig. 3b was constructed: on the whole phase plane $I_z = I_{z,0}$ and the phase curves are lines of constant h.

Depending on the value of τ, there are four qualitatively different types of phase portraits for the particles residing near an X-line.

1). $-\infty < \tau < -\left(I_z'\right)^{2/3}$

A typical phase portrait is shown in Fig. 4a. For these values of τ, there is no **UC**. Particles do not cross the equatorial (z=0) plane. Hence, they reside within the region B on the (z, P_z) phase plane . The only stationary point is at the origin.

2). $-\left(I_z'\right)^{2/3} < \tau < \tau_c$

A typical phase portrait is shown in Fig. 4b. The critical value τ_c is

$$\tau_c = -x_c', \tag{16}$$

where x_c' is given by (15). The **UC** consists of two straight lines, $S_\pm$, that are parallel to the P_x-axis and are separated from it by a distance

$$(\lambda x)_s = \sqrt{2\left(-\tau - \left(I_z'\right)^{2/3}\right)}.$$

The only stationary point is still at the origin. The phase curve Γ, that is tangent to **UC**, divides the phase plane into two regions: A and B.

3). $\tau_c < \tau < 0$.

A typical phase portrait is shown in Fig. 4c. At $\tau = \tau_c$ the elliptic stationary point O at the origin undergoes a bifurcation: the point O becomes hyperbolic and there are two new elliptic stationary points, $C_\pm$, located at

$$(\lambda x)_c = \pm\sqrt{2\left(\tau - \tau_c\right)}.$$

There is a new special curve on the phase portrait – the separatrix of the x-motion, σ. The curve σ separates the trajectories that are localized on one side of the X-line (those inside σ) from those that move from side to side of the X-line within one period of the x-motion.

4). $\tau > 0$.

A typical phase portrait is shown in Fig. 4d. At $\tau=0$, σ is tangent to $S_\pm$ and for larger values of τ, crosses it. The curve Γ now consists of two separate curves $\Gamma_\pm$ and, therefore, the region B consists of two separate domains, B_+ and B_-, that are bounded by Γ_+ and Γ_-, respectively.

4.3 Phase portraits in the vicinity of an O-line.

Depending on the value of τ there are three qualitatively different types of phase portraits.

1). $-\infty<\tau<-\left(I_z'\right)^{2/3}$.

A typical phase portrait is shown in Fig. 5a. The **UC** has the form of two straight lines $S_\pm$, that are parallel to the P_x-axis and are separated from it by a distance

$$(\lambda x)_s=\sqrt{2\left(-\tau-\left(I_z'\right)^{2/3}\right)}.$$

There are three fixed points: O (hyperbolic) at the origin and $C_\pm$ (elliptic) at

$$(\lambda x)_c=\pm\sqrt{2\left(-\tau+\tau_c\right)}.$$

2). $-\left(I_z'\right)^{2/3}<\tau<\tau_c$.

A typical phase portrait is shown in Fig. 5b. For these values of τ, there is no **UC**, hence, the motion of particles on the entire phase plane is regular outside of a small domain in the vicinity of σ.

3). $\tau_c<\tau$.

A typical phase portrait is shown in Fig. 5c. At $\tau=\tau_c$ the hyperbolic fixed point O and the elliptic fixed points $C_\pm$ merge giving rise to a single elliptic fixed point at the origin.

4.4 Motion in the region B

In both (MFR and XO) problems the behaviour of particles residing in the regions A and B is drastically different.

In the region B, both in MFR and XO problems, the phase space is filled by the invariant tori up to a residue of exponentially small measure (of order of $\sim\exp(-C/\sqrt{\kappa})$ where C is some constant) and the value of adiabatic invariant, I_z, is conserved eternally: on infinite time interval it has only oscillations of order κ (the Arnold theorem about perpetual adiabatic invariance, see e.g. Arnold et al., 1988). We can introduce the second action variable

$$I_x=\frac{1}{2\pi}\oint P_x dx, \tag{17}$$

where the integral is taken over a closed phase trajectory on the $(x,\,P_x)$ plane.

4.5 Motion in the region A

In the region A, the motion is more complicated. The value of I_z is conserved only far from S and undergoes a jump every time a phase point crosses S. The asymptotic formulas for the jump of adiabatic invariant on a separatrix were obtained by Timofeev (1978) for particular case of a pendulum in a slowly varying gravity field, by Neishtadt (1986) and Cary et al. (1986) for systems with one degree of freedom plus slowly varying parameter and by Neishtadt (1987) for systems with two degrees of freedom, one corresponding to the fast motion and the other corresponding to the slow motion. This theory was first applied to magnetospheric problems by Büchner and Zelenyi (1989). We have

$$\begin{aligned}\Delta I_z=I_{z,a}-2I_{z,b}&\approx\mp\frac{4}{\pi}\kappa\beta^2P_x\ln\left(2\sin\Theta\right)\quad(MFR),\\&\approx\mp\frac{1}{\pi}\lambda\left(\lambda x\right)P_x\ln\left(2\sin\Theta\right)\quad(XO).\end{aligned}\tag{18}$$

In (18), $I_{z,a}$ and $I_{z,b}$ are the values of I_z in the domains a and b, respectively, the minus and plus signs correspond to passages from b to a and from a to b, respectively (see the discussion below Eq. (12) in Sect. 3). The phase Θ characterizes a separatrix crossing and it depends not only on x and P_x but on z and P_z as well. The value of Θ is very sensitive to small perturbations of the initial conditions and can be treated as a random variable uniformly distributed on $(0,\pi)$. It is shown by Neishtadt (1987), that in the limit $\kappa\to0$, for multiple crossings ΔI_z can be treated as a random value with a zero mean and dispersion of order $\sim\kappa^2$ (MFR) or (due to normalizing conditions $\lambda x\sim1$) of order $\sim\lambda^2$ (XO). Because of these jumps, I_z was referred to as a quasi-adiabatic invariant by Büchner and Zelenyi (1989).

Because of the jumps in I_z no unique value of I_z can be assigned to a whole orbit of the x-motion. (Moreover, in the exact system projections of phase curves on the $(x,\,P_x)$ phase plane are not even closed curves – while in the averaged system the trajectories are closed.) Nevertheless, while a particle moves far from the uncertainty curve the value of I_z remains nearly constant (with accuracy of order κ, see the discussion in the previous subsection) and can be used to specify segments of trajectories between consecutive crossings of the uncertainty curve. Recall, that the characteristic values of the jumps' magnitude are small (of order $\sim\kappa$). Therefore, at every single crossing the value of I_z does not change much. It is the accumulation of the jumps (see Sect. 5 below) that leads to chaotic dynamics and mixing in the phase space. Despite the jumps, the notion of I_z was used to characterize orbits by many different authors (see e.g. Karimabadi et al., 1990; Burkhart et al., 1991; Moses et al., 1993). In particular, the value of I_z characterizes the motion in a sense that it foretells if a given trajectory (for frozen values of the slow variables) intersects the separatrix or not. There exists a critical value of I_z (see Büchner and Zelenyi, 1989, and Subsects. 4.1–4.3) such that if at any point along a trajectory I_z is larger (smaller) then that critical value, the trajectory does (does

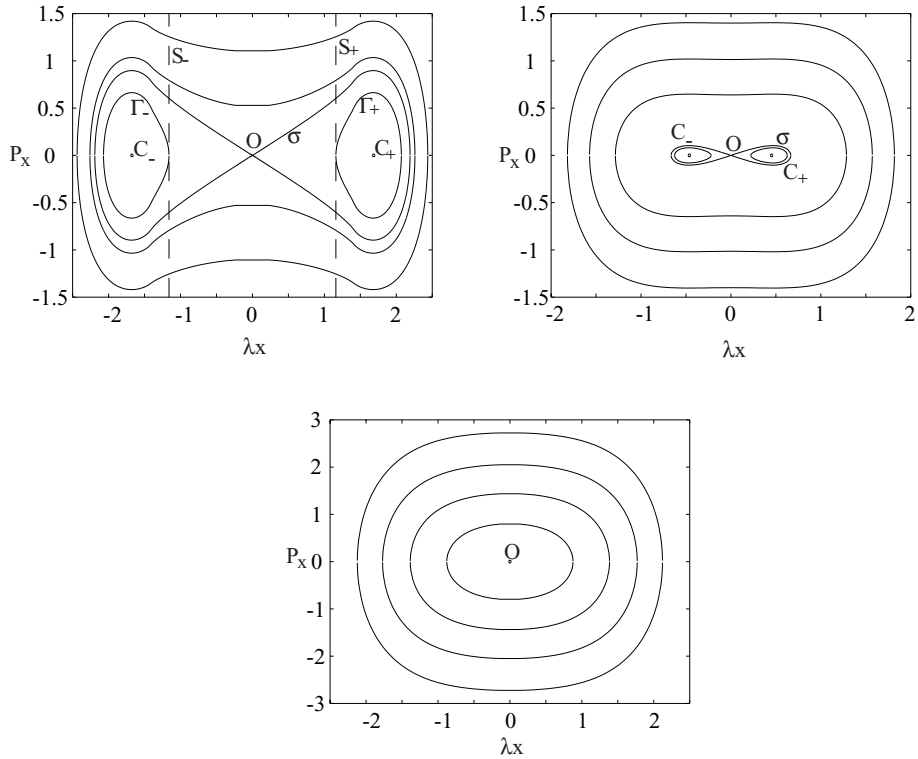

Fig. 5. Phase portraits on the $(\lambda x', P_x)$ plane for the O-line problem, $I_z = I_{z,0}$, phase curves are lines of constant h. **(a)** $\tau < -\left(I_z'\right)^{2/3}$, **(b)** $-\left(I_z'\right)^{2/3} < \tau < \tau_c$, **(c)** $\tau > \tau_c$.

not) intersect the separatrix. Jumps of I_z can not move I_z across the critical value. Besides that, the current value of I_z such important properties of the x-motion as the period and the distance of the closest approach to the earth (the location of the rightmost point in Fig. 3).

Despite the jumps of I_z in the region A, we can formally introduce the action I_x as the normalized by 2π area within a curve h=const, I_z=const on the (x, P_x) plane. I_x undergoes jumps, that are synchronized with the jumps of I_z:

$$\Delta I_x \sim \frac{\partial I_x}{\partial I_z} \Delta I_z.$$

As the jumps of I_z are small, $\Delta I_z \sim \kappa$, the corresponding jumps of I_x are also small. Hence, although even the original adiabatic invariant, I_z, is not exactly conserved, we can introduce and use the second adiabatic invariant in the case when the aggregate change in I_z is small (see Sect. 5 for more details). Therefore, I_x can also be called a quasi-adiabatic invariant and plays the role of the longitudinal invariant in **GCT**.

For the XO problem we can obtain the explicit formula for I_x, that is valid in the region A as well as in the region B. The similar formula for the MFR problem for B_n=const was derived by Zelenyi et al. (1990). It follows from (12) and (17), that

$$I_x = \alpha \frac{2\sqrt{2}}{\pi} I_z' \int_{k_1}^{k_2} \frac{Q(k)}{f^2(k)} \left[\frac{2h\left(I_z'\right)^{-4/3} f^{4/3}(k) - 1}{\left(2k^2 - 1\right) + \tau\left(I_z'\right)^{-2/3} f^{2/3}(k)} \right]^{1/2} dk, \tag{19}$$

where

$$Q(k) = \begin{cases} kK(k), & k < 1, \\ K\left(k^{-1}\right), & k > 1; \end{cases}$$

and $f(k)$ was defined in (12). The values of k_1, k_2 and α depend on whether a particle moves in the vicinity of X- or O-line and on the position of a phase curve on the phase plane.

For particles in the vicinity of an X-line k_2 is given by the conditions $k_2 > k_c$ and

$$2hf^{4/3}(k_2) = \left(I_z'\right)^{4/3}. \tag{20}$$

The values of k_1 and α depend on whether a phase curve intersects the line x=0 or not. If the whole curve is on one side of line x=0 then k_1 is given by (20) and the condition $k_1 < k_c$; in this case α=1. If a phase curve intersects line the x=0 then α=2 and k_1 is given by

$$-\tau = \left(I_z'/f(k_1)\right)^{2/3} \left(2k_1^2 - 1\right).$$

For a particle in the vicinity of an O-line k_1 is given by the conditions $k_1 < k_c$ and

$$2hf^{4/3}(k_1) = \left(I_z'\right)^{4/3}. \tag{21}$$

If the whole curve is on one side of the line $x=0$ then k_2 is given by (21) and the condition $k_2 > k_c$; in this case $\alpha=1$. If a phase curve intersects line $x=0$ then $\alpha=2$ and k_2 is given by

$$-\tau = \left(I_z'/f(k_2)\right)^{2/3}\left(2k_2^2 - 1\right).$$

5 Long-time dynamics: regular transport and mixing

The earth magnetotail is an open system. Charged particles from the solar wind enter in the far region and then accelerate and drift towards the earth. In the present section we discuss the properties of particles' dynamics on the time scales of the life time of particles in the magnetotail, which for a typical particle is much longer then the period of the x-motion.

After averaging over the x-motion, we can study the motion on the (τ, h) plane (i.e. the acceleration of the particles). For given I_z and I_x, every point on the (τ, h) plane corresponds to the entire phase curve on the (x, P_x) plane.

In the absence of the electric field the particles would stay in the region where they originally resided (A or B) forever. As it was discussed before, the jumps of adiabatic invariants cannot move particles from the region A to the region B (recall, the particle in the region B do not cross the separatrix). It is the presence of the electric field combined with the longitudinal variations of the magnetic field that causes the energy drift and the particles move from the region B to the region A, or from the region A to the region B. The direction of the drift depends on the location of a particle in the magnetotail.

5.1 Region B

While a particle resides in the region B, the values of both adiabatic invariants, I_z and I_x, are conserved up to the oscillations of order κ (see Subsect. 4.4). The motion of a particle on the (τ, h) plane is described by conservation laws

$$I_z = \text{const}, \qquad I_x = \text{const}. \tag{22}$$

Let us consider the drift in energy for the MFR problem. On the $(x', P_x\beta)$ phase plane in Fig. 3b, the shape of the phase curves does not change with time, each trajectory being specified by a value of energy. Consequently, as the energy drifts, particle drift from one phase curve to another. The direction of this drift depends on the sign of dh/dt averaged over the x-motion:

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} = \varepsilon\nu\left\langle P_x^2\beta\frac{\partial\beta}{\partial(\nu x)}\right\rangle, \tag{23}$$

where brackets denote the averaging over the x-motion. It follows from (23), that the particle's acceleration is affected only by large-scale variations of $\beta(\nu x)$ with a typical length scale comparable with the amplitude of x-oscillations.

The experimental observations show (see Tsyganenko (1987)) that, on average, $\partial\beta/\partial x > 0$ (i.e. $B_n(x)$ increases towards the earth). Hence, the energy increases with time and particles move from the region B to the region A. The time t_Γ that a particle spends in the region B (in other words, the time before a particle reaches Γ in Fig. 3b) is given by

$$I_x(h_0, t_0) = I_x(h_\Gamma, t_\Gamma),$$

where h_0 is the initial energy of the particle and

$$h_\Gamma = \frac{1}{2}\left(I_z'\right)^{4/3}$$

is the energy corresponding to Γ.

5.2 Region A

The motion of particles in the region A is affected by two phenomena: (1) the drift over energy under the influence of the electric field and (2) the jumps of adiabatic invariants. The behavior of particles depends on the relative importance of these phenomena.

Wiggins (1988) proved the presence of Smale horseshoes in domains with separatrix crossings, which, in turn, implies the existence of chaotic trajectories. The accumulation of the jumps of adiabatic invariants could produce chaotic dynamics, a phenomena called adiabatic chaos in Wiggins (1988). For the problem under consideration the presence of chaotic dynamics was demonstrated in (Büchner and Zelenyi, 1989; Zelenyi et al., 1990). The properties of the long-term evolution of adiabatic invariants depend on whether consecutive crossings are statistically dependent or independent.

In general case, the statistical independence follows from the divergence of phases along trajectories, which, in turn, depends on the relation between the values of the phase Θ (see Eq. 18) at two successive separatrix crossings. Denote the respective values of Θ as Θ_1 and Θ_2. A small variation $\delta\Theta_1$ of Θ_1 produces the variation of the jump value of I_z by a quantity of order $\kappa\ \delta\Theta_1$. As a result, the phase Θ_2 changes by the quantity $\delta\Theta_2 \sim C(1/\kappa)\kappa\ \delta\Theta_1 = C\delta\Theta_1$. The coefficient C, that is proportional to $\partial T_x/\partial I_z$, defines the divergence of trajectories. If C is large trajectories diverge fast: a small change in the resonance phase before one crossing generally results in a big change in the value of the resonance phase of the next crossing. In that case the jumps of adiabatic invariant at two successive crossings can be considered as independent. On the other hand, small values of C lead to nearly adiabatic motion.

With consecutive jumps being independent, evolution of the adiabatic invariant due to multiple separatrix crossings can be treated as a random walk with a step of order of κ without a preferred direction. Such a diffusion of adiabatic invariants is observed in (nearly all) numeric simulations of various systems. Diffusion time can be estimated as a characteristic slow period divided by a characteristic mean square value of the jump. Thus estimated, diffusion time was shown to be in a good agreement with numerical simulations (see Bruhwiler and Cary, 1989).

The assumption of statistical independence is crucial for chaotization: when phases on the successive crossings are related a regular motion can emerge. Cary and Skodje (1989) showed that for some initial conditions consecutive crossings are statistically dependent. It was shown by Elskens and Escande (1991) and Neishtadt et al. (1997) that the islands of stability, albeit being of a small measure, do exist inside large chaotic sea. A similar phenomenon of statistical dependence of consecutive jumps in a model of the roll convection was discussed in Itin et al. (2002). A somewhat different phenomenon appears if T_x is (approximately) independent of I_z. In this case $\partial T_x/\partial I_z$ approximately vanishes and consecutive phases become correlated. In particular, consecutive jumps may cancel each other producing a phenomenon called energy resonances or beamlets (see e.g. Chen, 1992; Ashour-Abdalla et al., 1993), that is well known in magnetospheric applications, but involves only a relatively small number of particles in the process of the convection from the distant tail to the earth. For nearby trajectories the net effect of jumps (averaged for all the crossings during one period of the x-motion), although not exactly vanishing, is much smaller than in a general case. Consequently, the beamlet particles behave as if they are in an adiabatic regime. It is important to note that the beamlets are occur only for not too small values of κ. In the formal limit of $\kappa \to 0$ the beamlets disappear. We return to the topic of the beamlets in the end of the current section where we discuss the rate of mixing.

In the pure parabolic field model the particles stay in the magnetotail forever. In that case, the mixing is complete and all the structures in the distribution functions of the chaotic region that were present when particles entered the magnetotail are washed out. On the other hand, in real magnespheric configurations the particles spend in the magnetotail only a finite time, and the number of jumps may not be sufficient for complete mixing.

Introduce the characteristic time of mixing, T_j, as a time over which the value of the adiabatic invariant changes by a quantity of order 1 (see below). As the jumps of the adiabatic invariant are localized near the uncertainty curve, **UC**, T_j can be written as

$$T_j \sim T_{per} \frac{1}{D},$$

where T_{per} is a typical time between consecutive crossings of **UC** and is of order of the period of the x-motion. D is the diffusion coefficient and is of order of the average square of the magnitude of a jump of the adiabatic invariant at one crossing:

$$D \sim \left\langle (\Delta I_z)^2 \right\rangle. \tag{24}$$

The extent of mixing in the system depends on the relation between T_j and T_L, that is a characteristic lifetime of particles in the magnetotail. If T_j is larger then T_L, then mixing is small everywhere in the phase space and distribution functions are transported towards the earth relatively intact. Contrarily, if T_j is less then T_L, then mixing is important and distribution functions are homogenized.

Introduce a parameter

$$\eta = \frac{T_L}{T_j}.$$

In MFR, the value of T_{per} is of order of $\sim 1/\kappa$. A characteristic size of jumps of the adiabatic invariant on the separatrix is of order $\sim\kappa$. Therefore, $T_j \sim \kappa^{-3}$.

The lifetime of particles is defined by the acceleration due to the electric field and depends on the magnitude of the constant solar wind-induced electric field and non-uniformity of magnetic field. A typical rate of change of energy is of order of $\varepsilon\nu$ (see Eq. 23). Hence, we have $T_L \sim (\varepsilon\nu)^{-1}$ and we arrive at

$$\eta = \frac{\kappa^3}{\nu\varepsilon}.$$

Similarly, in XO $T_L \sim \varepsilon^{-1}$, $T_j \sim \lambda^{-3}$ and

$$\eta = \frac{\lambda^3}{\varepsilon}.$$

If $\eta \ll 1$, drift dominates, jumps are small perturbations and the mixing is negligible. In the opposite case, $\eta \gg 1$, the mixing is strong: particles cross the separatrix sufficient number of times for adiabatic invariant cover the whole admissible domain. A similar observation was made (based on numerical simulations) by Karimabadi et al. (1990). Therefore, η is a measure of mixing in the system.

Now one can see the advantages of the two types of phase portraits on the (x', P_x) plane. If $\eta \ll 1$, I_z remains approximately constant and, like in the region B, particles drift from one phase curve to another in Fig. 3a. On the other hand, if $\eta \gg 1$, particles jump from one phase curve to another in Fig. 3b remaining on approximately the same energy level.

Note, that if D is defined by (24), η is an averaged characteristics, that is the same for all the phase space. In order to describe the evolution of coherent structures, we can consider η as a function of (say) the energy h. The most prominent consequence of the dependence of η on h is a possibility for consecutive jumps to be correlated. To include the correlation in the diffusion coefficient we can redefine D based on the average square of the magnitude of a total jump of the adiabatic invariant in one period of the x-motion:

$$D(h) \sim \left\langle (\Delta_2 I_z)^2 \right\rangle / 2,$$

If the consecutive crossings are statistically independent, $\Delta_2 I_z$, that is a sum of two changes of I_z during one period of the x-motion, is of order of the variation of ΔI_z at one crossing. However, if for a certain value of $h = h_b$ the jumps are correlated, like in the case of beamlets, when they cancel each other, $\Delta_2 I_z$ (and, correspondingly, $D(h_b)$) may go all the way to zero. In this case η goes to 0 and at the energy level $h = h_b$ the dynamics is adiabatic.

6 Orbits with small I_z in MFR

If the cumulative effect of the jumps of the adiabatic invariant is small ($\eta \ll 1$), then the mixing is weak in the whole domain and Eq. (22) specify the acceleration rate. In the general case, the exact solutions cannot be obtained explicitly.

These solutions can be simplified in a particular case $I_z \ll 1$, that corresponds to very long orbits with far reflecting points (a reflecting point being the rightmost point on a phase curve in Fig. 3a, see also Sect. 4). However, one must keep in mind that the separation of the x- and z-motions is valid only in the region where $\beta \sim 1$ and, therefore, we can not use the considerations below to describe the motion of the particles that come too close to the earth.

For large x' (see 9) we can linearize Hamiltonian (8) near $z_0 = \pm\sqrt{2x'}$. Introducing $z = z_0 + \bar{z}$, where $\bar{z} \ll z_0$, we get

$$h = \frac{1}{2}\left(P_x^2\beta^2 + P_z^2 + 2x'\bar{z}^2\right) = \frac{1}{2}P_x^2\beta^2 + h_z.$$

The result is a harmonic oscillator with a variable frequency $\omega(x') = \sqrt{2x'}$ and, therefore,

$$h_z = \frac{1}{2}\left(P_z^2 + \bar{z}^2\omega^2(x')\right) = I_z\omega(x'). \tag{25}$$

It follows from (17) and (25), that

$$I_x \approx \frac{1}{\pi}\int_{x'_{\min}}^{0} P_x\, dx + \int_0^{x'_{\max}} \frac{1}{\beta\,(x')}\frac{1}{\kappa}\sqrt{2h - I_z\sqrt{8x'}}\, dx', \tag{26}$$

where $x'_{\min}$ and $x'_{\max}$ are given by the condition $P_x(x = x_{\min,\max}) = 0$. One can see in Fig. 3a that the first term in (26) is much smaller then the second one and thus can be neglected. Introduce a new variable

$$\xi = \sqrt{2x'}I_z/h.$$

In terms of ξ, (26) can be written as

$$I_x \approx \frac{\sqrt{2}}{\pi}\frac{h^{5/2}}{\kappa I_z^2}\int_0^1 \frac{1}{\beta}\xi\sqrt{1-\xi}\, d\xi. \tag{27}$$

In (27) we substituted $\xi_{\max} = 1$. Integrating (27) by parts and taking into account that

$$\frac{d\beta}{d\xi} = \frac{1}{\kappa}\frac{d\beta}{dx}\frac{h^2}{I_z^2}\xi \quad \text{and} \quad \frac{d\beta}{dx}\frac{1}{\beta} \sim \nu.$$

we get

$$I_x \approx \frac{2}{3}\frac{\sqrt{2}}{\pi}\frac{h^{5/2}}{\kappa I_z^2}\left[\frac{2}{5}\frac{1}{\beta} - \frac{\nu}{\kappa}\frac{h^2}{I_z^2}\int_0^1\left((1-\xi)^{3/2} - \frac{3}{5}(1-\xi)^{5/2}\right)\frac{1}{\beta}\, d\xi\right]. \tag{28}$$

The first term in the square brackets in (28) depends on the average over an x-oscillation value of β. The second term describes variations of β over an x-orbit. The relative importance of the second term in (28) is given by a parameter

$$\gamma = \frac{5}{2}\frac{\nu}{\kappa}\frac{h^2}{I_z^2}.$$

If $\gamma \ll 1$ the second term is small and

$$I_x \approx \frac{4\sqrt{2}}{15\pi}\frac{h^{5/2}}{\kappa I_z^2}\frac{1}{\beta}.$$

Using (22) we obtain the acceleration rate,

$$h \sim \beta^{2/5}, \tag{29}$$

that confirms the result obtained by Zelenyi et al. (1990). Experimental results yield that the condition $\gamma \ll 1$ is satisfied for the particles with $I_z \ll 1$ beyond 50 R_E from the Earth. Note, that if $\gamma \geq 1$, (29) is not valid.

The period of x-oscillations is given by

$$T_x = 2\int_{x_{\min}}^{x_{\max}}\frac{dx}{\dot{x}} = 2\int_{x_{\min}}^{x_{\max}}\frac{dx}{P_x\beta^2} \approx \sqrt{2}\frac{h^{3/2}}{\kappa I_z^2}\int_0^1\frac{1}{\beta}\frac{\xi\, d\xi}{\sqrt{1-\xi}}.$$

Recall that the mixing parameter, η, is proportional to the period of the x-motion, T_x. Taking into account that the characteristic values of h are of order 1, we get that for small I_z, η has the form

$$\eta \sim I_z^2\frac{\kappa^3}{\nu\varepsilon}. \tag{30}$$

It follows from (30), that for the particles with small I_z the mixing is less prominent then for the particles with $I_z \sim 1$ as the particles with $I_z \ll 1$ cross the separatrix fewer times before being carried out of the magnetotail. The similar effect (that for shorter trajectories the rate of diffusion is larger) was observed by Karimabadi et al. (1990).

7 Acceleration of particles in XO

Again, consider the case $\eta \ll 1$ (small jumps of I_z). Conservation laws (22) are valid on the whole phase plane. The motion on the (time, energy) phase plane can be described using a single phase portrait. Apply a change of variables

$$x = \bar{x}\left(I_z'\right)^{1/3}, \quad h = \bar{h}\left(I_z'\right)^{4/3}, \quad \tau = \bar{\tau}\left(I_z'\right)^{2/3}.$$

$$p_x = \bar{p}_x\left(I_z'\right)^{-1/3}, \; h_z = \bar{h}_z\left(I_z'\right)^{4/3},$$

Note, that the value of k does not change under such a change of variables. Introduce a new adiabatic invariant

$$J = I_x/I_z'.$$

It follows from (19), that

$$J = \alpha\frac{2\sqrt{2}}{\pi}\int_{k_1}^{k_2}\frac{Q\,(k)}{f^2\,(k)}\left[s\frac{2\bar{h}f^{4/3}\,(k) - 1}{-\left(2k^2-1\right) - \bar{\tau}f^{2/3}\,(k)}\right]^{1/2} dk. \tag{31}$$

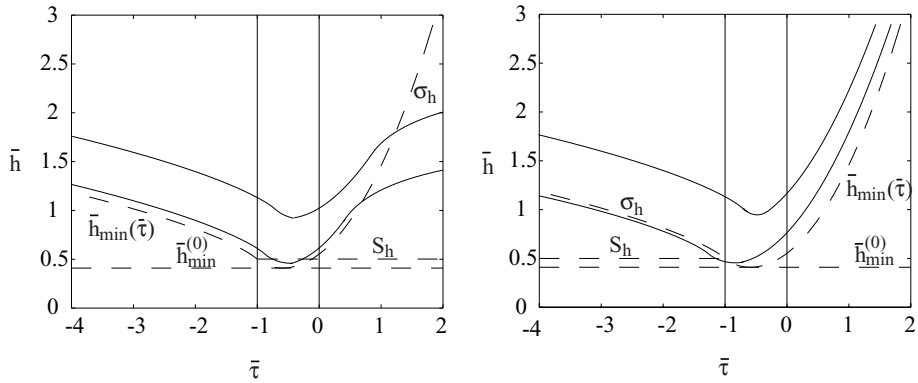

Fig. 6. Phase portraits on the $(\overline{\tau}, \overline{h})$ plane for (a) the X-line problem, (b) the O-line problem. Phase curves are lines of constant J.

On the $\left(\overline{\tau}, \overline{h}\right)$ plane phase curves are lines of constant J; the shape of curves does not depend on I_z.

It follows from (31), that the lowest possible value of h is $h^{(0)}_{\min} = (1/f_c)^{4/3}$, although phase curves do not fill the entire domain $\overline{h} > \overline{h}^{(0)}_{\min}$. The minimum possible energy $h_{\min}$ as a function of $\overline{\tau}$ is given by

$$\overline{h}_{\min} = \begin{cases} \overline{h}_{\min}(\overline{\tau}), & \overline{\tau} < \overline{\tau}_c, \\ \overline{h}^{(0)}_{\min}, & \overline{\tau} > \overline{\tau}_c, \end{cases}$$

for particles moving in the vicinity of an X-line and

$$\overline{h}_{\min} = \begin{cases} \overline{h}^{(0)}_{\min}, & \overline{\tau} < \overline{\tau}_c, \\ \overline{h}_{\min}(\tau), & \overline{\tau} > \overline{\tau}_c, \end{cases}$$

for particles moving in the vicinity of an O-line, where $\overline{h}_{\min}(\overline{\tau})$ and $\overline{\tau}_c$ (see Eqs. 15 and 16) are given by

$$\begin{aligned} &\left(2\overline{h}\right)^{4/3} f(k_{h\tau}) = 1, \\ &2k^2_{h\tau} - 1 = -\overline{\tau}\left(2\overline{h}\right)^{-1/2}, \\ &\overline{\tau}_c = -\frac{1}{f_c^{2/3}}\left(2k_c^2 - 1\right) \approx -0.59. \end{aligned} \tag{32}$$

In the vicinity of an X-line, Eqs. (32) define for $\overline{\tau} > \overline{\tau}_c$ on the $\left(\overline{\tau}, \overline{h}\right)$ plane curve σ_h, that corresponds to σ on the $\left(x', P_x\right)$ plane. In the vicinity of an O-line, Eqs. (32) define σ_h for $\overline{\tau} < \overline{\tau}_c$. In both cases, above σ_h particles cross the $x=0$ plane twice within each period of the x-motion. Below σ_h particles move in either $x>0$ or $x<0$ domain. The horizontal line S_h:$2\overline{h}_s=1$ corresponds to the **UC**.

It follows from J=const, that

$$\frac{\partial J}{\partial \overline{\tau}} d\overline{\tau} + \frac{\partial J}{\partial \overline{h}} d\overline{h} = 0,$$

and for the curves that do not cross the $x=0$ plane we obtain

$$\frac{d\overline{h}}{d\overline{\tau}} = -\frac{\partial J/\partial \overline{\tau}}{\partial J/\partial \overline{h}} = s\frac{1}{2}\frac{\int_{k_1}^{k_2}\left(2\overline{h} f^{4/3}(k) - 1\right)\Psi(k)dk}{\int_{k_1}^{k_2} f^{2/3}(k)\left(-\left(2k^2 - 1\right) - \overline{\tau} f^{2/3}(k)\right)\Psi(k)dk}, \tag{33}$$

where

$$\Psi(k) = \frac{Q(k)}{f^{4/3}(k)}\frac{\left(2\overline{h} f^{4/3}(k) - 1\right)^{-1/2}}{\left(-s\left(\left(2k^2 - 1\right) + \overline{\tau} f^{2/3}(k)\right)\right)^{3/2}} > 0.$$

It follows from (33), that in the vicinity of an X-line the energy of particles which do not cross the $x=0$ plane increases and the particles initially residing in the region B drift to the region A. Therefore, the motion of particles in the vicinity of an X-line becomes more and more chaotic with time. The phase portrait on the $\left(\overline{h}, \overline{\tau}\right)$ plane (in other words, solutions of J=const) is shown in Fig. 6a.

On the other hand, in the vicinity of an O-line the energy of particles which do not cross $x=0$ decreases and the particles drift from the region A to the region B. Therefore the motion of particles in the vicinity of an O-line becomes more and more regular. The phase portrait is shown in Fig. 6b.

The energy of the particles that cross the $x=0$ plane may increase as well as decrease. It can be shown numerically that for any J there is a value of $\overline{\tau}_J$ such that $d\overline{h}/d\overline{\tau} < 0$ for $\overline{\tau} < \overline{\tau}_J$ and $d\overline{h}/d\overline{\tau} > 0$ for $\overline{\tau} > \overline{\tau}_J$.

8 Conclusions

We considered the motion of charged particles in three different regions of the earth magnetotail: in the region with magnetic field reversal and in the vicinities of neutral line of X- and O-types. The presence of small parameters (ratio of characteristic length scales in and perpendicular to the equatorial plane and the smallness of the electric field) allows us to introduce the hierarchy of motions.

We introduced a parameter that plays the role of a measure of mixing in the system. This parameter describes the relative importance of the diffusion of adiabatic invariants on a time scale, defined by the acceleration of particles induced by the electric field. Depending on the value of this parameter, the jumps of the adiabatic invariants destroy original structures in the distribution function or particles accelerate and leave the tail quickly enough to keep the distribution function relatively intact. In the limit of small jumps of the adiabatic invariant we obtained the equations that govern the acceleration of particles. Using this parameter we showed that for particles on long elongated trajectories (small values of I_z) mixing is less prominent than for the rest of the particles.

Our approach illustrates the usefulness of such quantities as quasi-adiabatic invariants (even not perfectly conserved) for the description of particles' motion in the limit when usual guiding center theory breaks down. The developed approach may be useful not only for description of the dynamics of the particles in the magnetotail, but also in other open electromagnetic configurations.

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