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Nonlinear Processes in Geophysics (1998) 5: 181–186

Nonlinear Processes in Geophysics

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Received: 27 May 1997 – Accepted: 13 October 1998

Abstract. The paper concerns the temporal evolution of the Rayleigh-Taylor instability of two superimposed fluids in a vertical channel. At large times the instability results in the formation of a wide nearly-steady bubble of the lighter fluid rising through the channel, and thin long unsteady jets of the heavier fluid flowing down the channel walls. The jet flow appears to be tractable asymptotically by the method of matched asymptotic expansions. The solution has been obtained with the planar tips of jets characterized by the jump of the interface slope.

1 Introduction

The Rayleigh-Taylor instability occurring when the heavier fluid is superimposed over the lighter one has been extensively studied experimentally, and the general features of the system are well known (Davies and Taylor, 1950; Lewis, 1950; Zukosky, 1966). After an initial transient period, the system enters the asymptotic stage where the wide bubble of the lighter fluid acquires a fixed shape and rises through the channel at a constant speed. Simultaneously thin long jets of the heavier fluid are observed that slide down along the channel walls at a constant acceleration (Fig. 1). The problem has also been explored theoretically both in the time - independent and unsteady formulations (Birkhoff and Carter, 1957; Garabedian, 1957; Vanden-Broeck, 1984a,b; Gertsenshtein et al. , 1989; Cherniavski and Shtemler, 1994; Baker et al. , 1980; Gertsenshtein and Cherniavski, 1985).In the former case, as was found by Birkhoff and Carter (1957), the bubble rises at the dimensionless speed $w = 0.23$, where $w = w^d/\sqrt{h^d g^d}$, with w^d, h^d and g^d as the dimensional bubble speed, channel width and gravity acceleration.

It is well established by now that the solution of the steady - state problem exists for any value of the bubble speed w smaller than some critical value w_*. Garabedian (1957) showed that $w_* = 0.24$, while Vanden-Broeck (1984a) , in his more accurate analysis obtained $w_* = 0.36$. Note that in the absence of viscosity and surface tension effects, the solution with the smooth tip of the bubble occurs only for $w = 0.23$ (Gertsenshtein et al. , 1989; Cherniavski and Shtemler, 1994). This is also in compliance with the results of Vanden-Broeck (1984b) where $w = 0.23$ was obtained as a limit at vanishing values of surface tension.

The solution of the steady-state problem is singular since the falling jets are of infinite length. In the at-

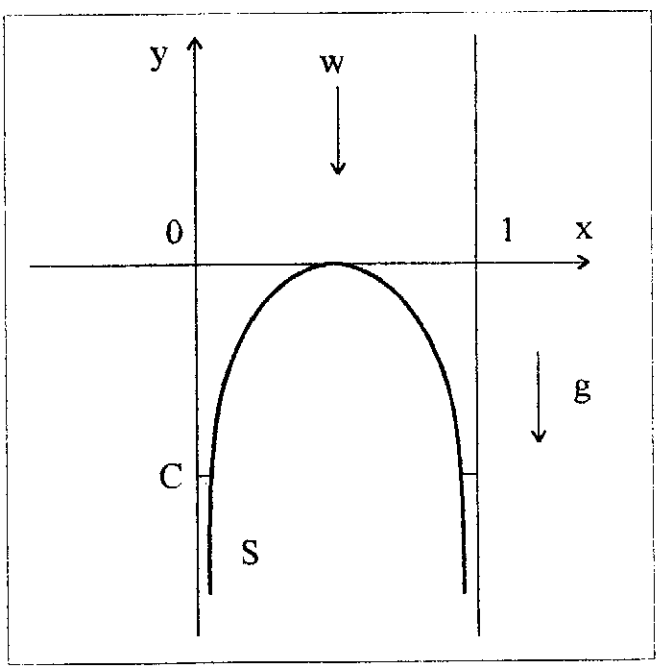

Fig. 1. Diagram for the advanced stage of the Rayleigh-Taylor instability. The bold line (S) corresponds to the steady-state interface, the thin line (C) to the interface of the associated unsteady interface.

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tempt to simulate the pertinent dynamical system numerically, the presence of unsteady continuously shrinking jets leads to computational complications in the unsteady problem (Baker et al. , 1980; Gertsenshtein and Cherniavski, 1985). The flow within the jets has been approximately resolved analytically by the integral relations technique (Gertsenshtein et al. , 1989; Cherniavski and Shtemler, 1994). The integral method, however, fails to yield the jet interface shape.

The main goal of the present work is to describe the unsteady flow and the interface shape in the jet at long times, the numerical calculations of which met with computational complications, and to match them with the steady-state solution. The well-settled downward-sliding unsteady jets are described asymptotically, where the time reciprocal is regarded as a parameter of expansions. The problem is solved by means of the method of matched asymptotic expansions and for comparison by a somewhat modified integral approach developed in the previous studies (Gertsenshtein et al. , 1989; Cherniavski and Shtemler, 1994).

The explicit description for the unsteady jets obtained by the method of matched asymptotic expansions in addition to the steady-state solution for the bubble permits to describe the solution in the whole region of flow. The method developed here can be employed to study a wide number of systems. Thus, a similar procedure has recently been employed in the context of a physically related problem concerning the upward propagating flames (Shtemler and Sivashinsky, 1994).

2 Basic equations

The present study is limited to the case of a negligibly small density of the light fluid, when the interface is actually the free boundary of the heavy fluid. In the two-dimensional formulation discussed in this paper, the potential flow of the inviscid incompressible fluid entails the validity of the Cauchy-Riemann relations for the potential and stream function, as well as of the Bernoulli integral for the pressure. In the frame of reference xOy moving upward at a constant speed w, the appropriately scaled (see below) set of the governing equations and boundary conditions reads

(i) equations for the flow-field

$$\begin{aligned} &\partial_x \phi = \partial_y \psi, \ \partial_y \phi = -\partial_x \psi, \\ &\tfrac{p}{\rho} + \tfrac{1}{2}(\nabla\phi)^2 + gy + \partial_t \phi = 0. \end{aligned} \tag{1}$$

(ii) conditions at the interface $\gamma(x, y, t) = 0$

$$\begin{aligned} &\partial_t \gamma + \partial_x \phi \, \partial_x \gamma + \partial_y \phi \, \partial_y \gamma = 0, \\ &p = 0. \end{aligned} \tag{2}$$

(iii) impermeability conditions at the walls $x = 0$ and $x = h$

$$\partial_y \psi = 0, \tag{3}$$

(iv) condition at infinity

$$\phi \to -wy + \phi_\infty(t) \qquad \text{as} \quad y \to \infty. \tag{4}$$

Here the vertical axis is chosen in such a way that it coincides with the wall $x = 0$; h is the channel width; g is the gravity acceleration; ρ is the heavy fluid density; (x, y) are the spatial coordinates; t is the time; ϕ and ψ are the potential and stream function; the additive constant in the stream function ψ is set to be zero at the wall $x = 0$; p is the pressure; $\gamma(x, y, t) = 0$ is the interface profile. Since we shall be concerned with the long-time assumption, $t \to \infty$, the initial data are assumed to be irrelevant and not specified.

Dimensional values of the channel width, h^d, of the gravity acceleration, g^d, and of the fluid density, ρ^d, are taken as the basic scales of the system. Hence, everywhere below

$$h = 1, \ g = 1, \ \rho = 1, \ w = \frac{w^d}{\sqrt{h^d g^d}}, \tag{5}$$

where w^d is the dimensional speed of the rising bubble. Here the superscript ^{d} denotes corresponding dimensional variables, the dimensionless speed w of the rising bubble may be regarded as the Froude number of the problem. To fix the arbitrary constants by the choice of the horizontal axis and the additive constant in the potential function, the following additional conditions are introduced:

$$\begin{aligned} &\nabla\phi = 0 \qquad \text{at} \quad x = \tfrac{1}{2}, \ y = 0, \\ &y_b(t) \to 0, \ \ \phi_{\infty}(t) \to 0 \qquad \text{as } t \to \infty. \end{aligned} \tag{6}$$

Here $y_b(t)$ is the height of the bubble tip in the reference frame xOy (Fig. 1). The tip of the settled bubble therefore belongs to the axis Ox.

Setting time derivatives in Eqs. (1) (4) to zero and employing the condition of the mass-flux conservation, one ends up with a steady-state formulation of the problem

(i) equations for the flow-field

$$\begin{aligned} &\partial_x \phi^s = \partial_y \psi^s, \quad \partial_y \phi^s = -\partial_x \psi^s, \\ &p^s + y + \tfrac{1}{2}(\nabla\phi^s)^2 = 0. \end{aligned} \tag{7}$$

(ii) conditions at the interface $\gamma^s(x, y) = 0$

$$\psi^s = \frac{w}{2}, \ \ p^s = 0, \tag{8}$$

(iii) impermeability conditions at the walls

$$\begin{aligned} &\psi^s = 0 \qquad \text{as } x = 0, \\ &\psi^s = w \qquad \text{as } x = 1, \end{aligned} \tag{9}$$

(iv) condition at infinity

$$\phi^s \to -wy \qquad \text{as} \quad y \to \infty. \tag{10}$$

Here the superscript ^{s} denotes the steady-state solution. Due to the symmetry of the problem relative to the channel midline $x = 1/2$, our discussion is limited

to the left half of the channel. According to the mass conservation law and the Bernoulli integral the fluid entering the channel inflow crossection ($y \to +\infty$) at the velocity w forms an infinitely long thin jet near the wall ($x = 0$ and $y \to -\infty$). The asymptotic structure of the steady-state solution near $x = 0$ is given by the relations (Birkhoff and Carter, 1957)

$$\begin{aligned} &\phi^s(x,y) = \tilde{\phi}^s(x,y) + O(x^3),\\ &\psi^s(x,y) = \tilde{\psi}^s(x,y) + O(x^6),\\ &p^s(x,y) = \tilde{p}^s(x,y) + O(x^4),\\ &\gamma^s(x,y) = \tilde{\gamma}^s(x,y) + O(x^4) = 0\\ &\qquad \text{as } x \to 0 \ (y \to -\infty). \end{aligned} \tag{11}$$

Here

$$\begin{aligned} &\tilde{\phi}^s(x,y) = \tfrac{1}{3}(-2y)^{\frac{3}{2}},\\ &\tilde{\psi}^s(x,y) = (-2y)^{\frac{1}{2}}x,\\ &\tilde{p}^s(x,y) = 0,\\ &\tilde{\gamma}^s(x,y) = x - \tfrac{w}{2}(-2y)^{-\frac{1}{2}}. \end{aligned} \tag{12}$$

3 Integral description of the falling jets

The steady-state jet has an infinite length. However, in the unsteady case its length is clearly finite for any fixed instant of time and increases as time grows. For further analysis it is useful to write out the integral conservation laws (Benjamin and Olver, 1982). Presuming non-zero values of the pressure and the normal speed at the surface bounding the flow region, the pertinent relations may be written as

(i) energy conservation

$$d_t[i_1 + \frac{w}{2}\phi_\infty(t)] = b_1, \tag{13}$$

(ii) mass conservation

$$d_t i_2 = w + b_2, \tag{14}$$

(iii) vertical momentum conservation

$$d_t[i_3 - \phi_\infty(t) + w y_b(t)] = -\frac{1}{2}w^2 - i_2 + b_3, \tag{15}$$

(iv) conservation of height of mass centroid times mass

$$d_t i_4 = i_3 + w y_b(t) + b_4. \tag{16}$$

Here

$$\begin{aligned} &i_1 = \tfrac{1}{2}\int_L \phi\partial_n\phi ds + \tfrac{1}{2}\int_L y^2 dx,\\ &i_2 = \int_L y dx\ ,\\ &i_3 = \int_L \psi dy\ ,\\ &i_4 = \tfrac{1}{2}\int_L y^2 dx,\\ &b_1 = -\int_L [p v_n\ + (v_n - \partial_n\phi)\partial_t\phi] ds,\\ &b_2 = \int_L (v_n - \partial_n\phi) ds,\\ &b_3 = \int_L [-p dx + (v_n - \partial_n\phi)\partial_y\phi ds],\\ &b_4 = \int_L (v_n - \partial_n\phi) y ds, \end{aligned} \tag{17}$$

where L ($\tilde{\gamma}(x,y,t) = 0$) is the free boundary or a curve approximating it, both with positive clockwise direction of traversal; ∂_n is the normal derivative; $ds = \sqrt{(dx)^2 + (dy)^2}$; v_n is the normal velocity of L:

$$\begin{aligned} &\partial_n\phi ds = -\partial_x\psi dx - \partial_y\psi dy;\\ &v_n = -\tfrac{\partial_t\tilde{\gamma}(x,y,t)}{|\nabla\tilde{\gamma}(x,y,t)|}. \end{aligned} \tag{18}$$

When deriving Eqs. (13) – (17) the boundary conditions at the walls, infinity and the problem symmetry were employed (Benjamin and Olver, 1982). The values i_k and b_k ($k = 1, ..., 4$) are the interface parts of energy, mass, vertical momentum, height of mass centroid times mass and their fluxes, respectively. Relations (13) – (16) are actually the integral identities for the exact unsteady solution of the problem for arbitrary L and t ($0 \le t < \infty$). In particular, if L is the unsteady free boundary, due to the kinematic ($v_n = \partial_n\phi$) and dynamic ($p = 0$) conditions, some terms in Eqs. (13) – (17) can be dropped.

Note that the solution of the steady-state problem (7) – (10) is not a uniformly valid approximation of the unsteady problem (1) – (4) at $t \to \infty$. This can be easily shown by substituting the steady-state solution of order t^0 (as $t \to \infty$) into the conservation relations (13) – (16) where L is regarded as the interface. For instance, the steady-state solution violates the mass conservation relation (14). Indeed, the left and right hand parts of Eq. (14) are equal to zero and w ($b_2 \equiv 0$), respectively, and therefore cannot balance each other. The source of the nonuniformity is the formation of infinitely long and thin jets.

Within the jet, the region of the steady-state solution nonuniformity, the unsteady solution may be described integrally. Assume that at $t \to \infty$ the solution of the unsteady problem becomes time-independent. Approximate the interface boundary by a portion of the steady-state interface S and by the moving control-surface C (Fig. 1)

$$\tilde{\gamma}(x,y,t) \equiv \begin{cases} \gamma^s(x,y) & \text{at S:} \quad x > x_*(t),\\ y - y_* & \text{at C:} \quad 0 \le x \le x_*(t). \end{cases} \tag{19}$$

Here $\gamma^s(x,y)$ is given by Eq. (11) , and the unknown functions $x_*(t)$ and $y_*(t)$ are to be found.

Set the mean velocity of the control surface C relative to the fluid, q_*, as zero, i.e. in such a way that the kinematic condition on C is met integrally

$$q_*(t) = \frac{\int_C (v_n - \partial_n\varphi) ds}{\int_C ds} = 0. \tag{20}$$

The flow parameters at S are found by solving the steady-state problem (7) – (10), while at C they are given by the asymptotic expressions (11) and

definitions (18), (20) of v_n and q_*

$$\begin{aligned}
&\phi = \phi_*(t) + O(t^{-3}),\ \ \psi = \tfrac{w}{2x_*(t)}x + O(t^{-6}),\\
&p = O(t^{-4}),\ v_n = -d_t y_*(t),\\
&\partial_n\phi = -\partial_y\phi = \tfrac{w}{2x_*(t)} + O(t^{-5}),\\
&q_*(t) = \tfrac{w^2}{4x_*(t)}d_t[\tfrac{1}{x_*(t)}] - \tfrac{w}{2x_*(t)} + O(t^{-5})\\
&\varphi_*(t) = \tfrac{w^3}{72x_*^3(t)} + \tfrac{2\kappa}{w} + O(t^{-3}),\\
&y_*(t) = -\tfrac{w^2}{8x_*^2(t)} + O(t^{-4}),\\
&x_*(t) = \tfrac{w}{2t} + O(t^{-7}) \qquad \text{as} \quad t\to\infty.
\end{aligned} \tag{21}$$

where κ is a constant calculated in Gertsenshtein et al. (1989); Cherniavski and Shtemler (1994). According to (21) $p = 0$ at C and such an approach also guarantees satisfaction in the integral sense of the dynamic boundary condition at C. Moreover, since $p = 0$ and $q_* = 0$ at C in the present case, the fluxes through the control surface C are equal to zero. The expressions for functions $x_*(t)$ and $y_*(t)$ in (21) determine the control surface C (in the expression for x_* the additive constant of integration is set to zero corresponding to arbitrary choice of the time origin). Employing the relations (21), one observes that within the jet all variables are of the following t-orders:

$$\begin{aligned}
&x\sim t^{-1},\ \ y\sim t^2,\ \ \varphi\sim t^3,\\
&v\sim t^0,\ \ p\sim t^2 \qquad \text{as } t\to\infty.
\end{aligned} \tag{22}$$

Suppose that at $t\to\infty$ all the integrals (17) may be approximated by the sums of integrals over S and C

$$\begin{aligned}
&i_k \to i_k^s + i_k^c,\\
&b_k \to b_k^s + b_k^c\ \ (k = 1,2,3,4) \qquad \text{as } t\to\infty.
\end{aligned} \tag{23}$$

In compliance with (20), (21) at $t\to\infty$ for integrals over C one obtains,

$$\begin{aligned}
&i_1^c = \tfrac{w}{2}\varphi_*(t) - y_*^2(t)x_*(t) + O(t^{-3}),\\
&i_2^c = -2y_*(t)x_*(t) + O(t^{-5}),\\
&i_3^c = O(t^{-4}),\\
&i_4^c = -y_*^2(t)x_*(t) + O(t^{-3}),\\
&b_1^c = O(t^{-4}),\ \ b_2^c = O(t^{-6}),\\
&b_3^c = O(t^{-5}),\ \ b_4^c = O(t^{-4}).
\end{aligned} \tag{24}$$

Arrange the expressions for i_k^s and b_k^s $(k = 1,2,3,4)$ as the sums of the asymptotic parts (12) and the corresponding remainders. The contributions of the asymptotic terms at $t\to\infty$ are given by

$$\begin{aligned}
&i_1^s = i_4^s = -\tfrac{w^4}{192x_*^3(t)} + v + O(t^{-3}),\\
&i_2^s = \tfrac{w^2}{4x_*(t)} + m + O(t^{-5}),\ \ i_3^s = wy_*(t),\\
&b_1^s = b_2^s = b_3^s = b_4^s = 0.
\end{aligned} \tag{25}$$

When deriving Eq. (25) the following identities were used:

$$\begin{aligned}
&v_n = 0,\ \partial_n\phi = 0,\ p = 0,\\
&v = w/2 \qquad \text{at S.}
\end{aligned} \tag{26}$$

The values of b_k^s $(k = 1,\ldots 4)$ and i_3^s are evaluated exactly; integrals m and v can be calculated by solving the steady-state problem (7) – (10)

$$\begin{aligned}
&v = -\int_0^{\frac{1}{2}}[\eta^2(x) - \tilde\eta^2(x)]dx + \tfrac{w^3}{24},\\
&m = -2\int_0^{\frac{1}{2}}[\eta(x) - \tilde\eta(x)]dx \quad - \tfrac{w^2}{2}.
\end{aligned} \tag{27}$$

Here $y = \eta(x)$ and $y = \tilde\eta(x)$ are the exact $(\gamma^s(x,\eta(x)) = 0)$ and asymptotic $(\tilde\gamma^s(x,\tilde\eta(x)) = 0)$ profiles for the steady-state interface (see Eq. (12)). From Eqs. (22) – (25) at $t\to\infty$ one obtains,

$$\begin{aligned}
&i_1 = \kappa + v + O(t^{-3}),\\
&i_2 = \tfrac{w^2}{2x_*(t)} + m + O(t^{-5}),\\
&i_3 = -\tfrac{w^3}{8x_*^2(t)} + O(t^{-4}),\\
&i_4 = -\tfrac{w^4}{48x_*^3(t)} + v + O(t^{-3}),\\
&b_1 = O(t^{-4}),\ \ b_2 = O(t^{-6}).\\
&b_3 = O(t^{-5}),\ \ b_4 = O(t^{-4}).
\end{aligned} \tag{28}$$

Substituting Eq. (27) into the conservation relations (13) – (16) and employing Eq. (5), the conservation of vertical momentum (15) at $t\to\infty$ yields,

$$m + \frac{w^2}{2} = 0. \tag{29}$$

For the exact solution of the steady-state problem the relation (29) is an integral identity (Gertsenshtein et al., 1989; Cherniavski and Shtemler, 1994). Thus at $t\to\infty$ the unsteady solution may be described integrally by the steady-state solution (discussed above in Sect. 1) and the moving control surface, in the sense of satisfying the associated dynamical conservation laws. For example, taking into account Eq. (6) and the choice of the dimensional variables, one concludes from Eq. (21) that at $t\to\infty$ the jet falls with the gravity acceleration. The relations (21) provide the asymptotic expressions of the jet length and width. However, in the general case, it fails to yield the interface shape.

4 Asymptotic shape of the falling jets

To describe the shape of the interface in the fluid jet at long times, the problem (1) – (4) may also be tackled by the method of matched asymptotic expansions, where the time reciprocal t^{-1} is employed as a parameter of expansion. To ensure that the outer problem is reduced to that of the steady-state as $t\to\infty$, the outer expansion may be written as

$$\begin{aligned}
&\phi(x,y,t) = \phi_0^{(o)}(x,y) + t^{-1}\phi_1^{(o)}(x,y) + \ldots,\\
&\psi(x,y,t) = \psi_0^{(o)}(x,y) + t^{-1}\psi_1^{(o)}(x,y) + \ldots,\\
&p(x,y,t) = p_0^{(o)}(x,y) + t^{-1}p_1^{(o)}(x,y) + \ldots,\\
&\gamma(x,y,t) = \gamma_0^{(o)}(x,y) + t^{-1}\gamma_1^{(o)}(x,y) + \ldots\ .
\end{aligned} \tag{30}$$

Here the superscript $^{(o)}$ corresponds to the outer solution. For the leading order approximation the

original unsteady problem (1) - (4) is reduced to the steady-state problem (7) - (10). As has been already mentioned, the steady solution is not a uniformly valid approximation to the unsteady one due to the singularity developing near the walls as $t \to \infty$. Following the estimates (22), introduce the inner variables

$$\begin{aligned} &t^{(i)} = t, \quad x^{(i)} = tx, \\ &y^{(i)} = t^{-2}y, \quad \phi^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = t^{-3}\phi(x, y, t), \\ &v^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = \psi(x, y, t), \\ &p^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = t^{-2}p(x, y, t), \\ &\gamma^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = \gamma(x, y, t). \end{aligned} \tag{31}$$

Here the superscript $^{(i)}$ corresponds to the inner solution. From Eq. (31) one obtains

$$\begin{aligned} &\partial_t = \partial_{t^{(i)}} + t^{(i)-1}x^{(i)}\partial_{x^{(i)}} - 2t^{(i)-1}y^{(i)}\partial_{y^{(i)}}, \\ &\partial_x = t^{(i)}\partial_{x^{(i)}}, \ \partial_y = t^{(i)-2}\partial_{y^{(i)}}, \\ &\partial_t\phi = t^{(i)3}\partial_{t^{(i)}}\phi^{(i)} + 3t^{(i)2}\phi^{(i)} \\ &\quad + t^{(i)2}x^{(i)}\partial_{x^{(i)}}\phi^{(i)} - 2t^{(i)2}y^{(i)}\partial_{y^{(i)}}\phi^{(i)}, \\ &\partial_t\gamma = \partial_{t^{(i)}}\gamma^{(i)} + t^{(i)-1}x^{(i)}\partial_{x^{(i)}}\gamma^{(i)} \\ &\quad - 2t^{(i)-1}y^{(i)}\partial_{y^{(i)}}\gamma^{(i)}. \end{aligned} \tag{32}$$

The inner expansion for the solution of the unsteady problem may be written as

$$\begin{aligned} &\phi^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = \phi_0^{(i)}(x^{(i)}, y^{(i)}) \\ &\quad + t^{(i)-1}\phi_1^{(i)}(x^{(i)}, y^{(i)}) + \dots, \\ &v^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = \psi_0^{(i)}(x^{(i)}, y^{(i)}) \\ &\quad + t^{(i)-1}\psi_1^{(i)}(x^{(i)}, y^{(i)}) + \dots, \\ &p^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = p_0^{(i)}(x^{(i)}, y^{(i)}) \\ &\quad + t^{(i)-1}p_1^{(i)}(x^{(i)}, y^{(i)}) + \dots, \\ &\gamma^{(i)}(x^{(i)}, y^{(i)}, t^{(i)}) = \gamma_0^{(i)}(x^{(i)}, y^{(i)}) \\ &\quad + t^{(i)-1}\gamma_1^{(i)}(x^{(i)}, y^{(i)}) + \dots. \end{aligned} \tag{33}$$

Substituting (31) (33) into the system (1) - (4) for the leading order approximation one obtains,

(i) equations for the flow-field

$$\begin{aligned} &\partial_{x^{(i)}}\psi^{(i)} = -\partial_{y^{(i)}}\phi^{(i)}, \\ &\partial_{x^{(i)}}\phi^{(i)} = 0, \\ &p^{(i)} + 3\phi^{(i)} - 2y^{(i)}\partial_{y^{(i)}}\phi^{(i)} + y^{(i)} \\ &\quad + \tfrac{1}{2}(\partial_{y^{(i)}}\phi^{(i)})^2 = 0, \end{aligned} \tag{34}$$

(ii) conditions at the interface $\gamma^{(i)}(x^{(i)}, y^{(i)}) = 0$

$$\begin{aligned} &x^{(i)}\partial_{x^{(i)}}\gamma^{(i)} - 2y^{(i)}\partial_{y^{(i)}}\gamma^{(i)} \\ &\quad + \partial_{x^{(i)}}\gamma^{(i)}\partial_{y^{(i)}}\psi^{(i)} + \partial_{y^{(i)}}\gamma^{(i)}\partial_{y^{(i)}}\phi^{(i)} = 0, \\ &p^{(i)} = 0, \end{aligned} \tag{35}$$

(iii) impermeability condition at the wall $x^{(i)} = 0$

$$v^{(i)} = 0. \tag{36}$$

Here the subscript zero is omitted. From (34) one obtains

$$\begin{aligned} &\phi^{(i)}(x^{(i)}, y^{(i)}) = \phi^{(i)}(y^{(i)}), \\ &v^{(i)}(x^{(i)}, y^{(i)}) = -x^{(i)}d_{y^{(i)}}\phi^{(i)}(y^{(i)}), \\ &p^{(i)}(x^{(i)}, y^{(i)}) = -3\phi^{(i)} + 2y^{(i)}d_{y^{(i)}}\phi^{(i)} \\ &\quad - \tfrac{1}{2}(d_{y^{(i)}}\phi^{(i)})^2 - y^{(i)}. \end{aligned} \tag{37}$$

Substitution of Eq. (37) into Eq. (35) yields the potential $\phi^{(i)}(y^{(i)})$ and the interface equation $\gamma^{(i)}(x^{(i)}, y^{(i)}) = 0$. Indeed, from the second relation in Eq. (35) one obtains

$$d_{y^{(i)}}\phi^{(i)} = -1 + C_1 \pm (-2C_1y^{(i)} + C_1^2 - C_1)^{\frac{1}{2}}. \tag{38}$$

The outer and inner solutions should be matchable at the interface. The inner expansion of the outer solution is supposed to be equal to the outer expansion of the inner solution (Van Dyke, 1964). The outer expansion (30) written in terms of the inner variables (31) up to order t^0 yields, due to Eq. (11),

$$d_{y^{(o)}}\phi^{(oi)} = -(-2y^{(i)})^{\frac{1}{2}}. \tag{39}$$

In view of Eqs. (38), (39) the matching $d_{y^{(o)}}\phi^{(i)}$ and $d_{y^{(o)}}\phi^{(oi)}$ is possible only provided $C_1 = 1$. Hence,

$$d_{y^{(i)}}\phi^{(i)} = -(-2y^{(i)})^{\frac{1}{2}}. \tag{40}$$

Substituting Eq. (40) into the first condition of (35), and employing Eq. (37) one obtains

$$\begin{aligned} &\left[1 - (-2y^{(i)})^{\frac{1}{2}}\right](x^{(i)}\partial_{x^{(i)}}\gamma^{(i)} \\ &\quad - 2y^{(i)}\partial_{y^{(i)}}\gamma^{(i)}) = 0. \end{aligned} \tag{41}$$

Equation (41) has two families of solutions

$$\begin{aligned} &x^{(i)} - C_2(-2y^{(i)})^{-\frac{1}{2}} = 0 \qquad \text{or} \\ &y^{(i)} + \tfrac{1}{2} = 0. \end{aligned} \tag{42}$$

Equations (42) determine the interface with the jump of the shape slope at the point $y^{(i)} = y_*^{(i)}$, $x^{(i)} = x_*^{(i)}$

$$\gamma^{(i)} = \begin{cases} x^{(i)} - C_2(-2y^{(i)})^{-\frac{1}{2}} & \text{if } x^{(i)} \geq x_*^{(i)} \\ y^{(i)} - y_*^{(i)} & \text{if } x^{(i)} \leq x_*^{(i)} \end{cases} \tag{43}$$

with

$$\begin{aligned} &y_*^{(i)} = -\tfrac{1}{2}, \\ &x_*^{(i)} = C_2. \end{aligned} \tag{44}$$

The matching of $\gamma^{(o)}(x, y)$ and $\gamma^{(oi)}(x^{(i)}, y^{(i)})$ in Eqs. (30) and (43) up to order t^0 yields, due to Eq. (11)

$$C_2 = \frac{w}{2}. \tag{45}$$

Finally from Eqs. (37), (40) and (43), (45) one obtains

$$\begin{aligned} &\phi^{(i)}(x^{(i)}, y^{(i)}) = \tfrac{1}{3}(-2y^{(i)})^{\frac{3}{2}}, \\ &\psi^{(i)}(x^{(i)}, y^{(i)}) = x^{(i)}(-2y^{(i)})^{\frac{1}{2}}, \\ &p^{(i)}(x^{(i)}, y^{(i)}) = 0, \\ &\gamma^{(i)} = \begin{cases} x^{(i)} - \frac{1}{2}w(-2y^{(i)})^{-\frac{1}{2}} & \text{if } x^{(i)} \geq x_*^{(i)} \\ y^{(i)} - y_*^{(i)} & \text{if } x^{(i)} \leq x_*^{(i)} \end{cases} \end{aligned} \tag{46}$$

where

$$y_*^{(i)} = -\tfrac{1}{2}, \ x_*^{(i)} = \tfrac{w}{2}. \tag{47}$$

Hence, returning to the outer variables, for the jet length and width, one obtains

$$y_* \equiv y_*^{(i)} t^2 = -\tfrac{1}{2} t^2, \qquad x_* \equiv x_*^{(i)} t^{-1} = \tfrac{w}{2t}. \tag{48}$$

The comparison of Eqs. (11) and (46) shows that the outer solution is invariant to the change of variables (32). Thus, the unsteady flow field at $t \to \infty$ is determined by the steady-state one. The surface of the jet is made of the steady-state one (outer solution) and the plane segment that is orthogonal to the wall and descends with the gravity acceleration along the steady-state interface. Note that the flow parameters in Eq. (48) are identical to those obtained by the integral method.

5 Concluding remarks and discussion

The method of matched asymptotic expansions described in Sect. 4, where the time reciprocal t^{-1} is employed as a parameter of expansions, provides an effective means to capture the description of the unsteady jet developing at the advanced stage of the Rayleigh-Taylor instability. This description is corroborated via a more simple perturbative approach of Sect. 3, where the integral approach provides an effective means to capture some basic features of the unsteady jet. In particular, it was found that the interface shape in the jet coincides with the control surface used in the integral approach, the width and the length of the heavy fluid jet evolve in time as $t^{-1}$and t^2, and the jet falls down with the gravity acceleration of the jet tip. Thus, the overall configuration of the well-settled rising bubble and the unsteady falling jet appears to involve two planar gradually shrinking segments in the falling jets. These segments produce the corner points on the descending jet interface. The jump of the jet interface slope stems from neglect of the viscosity effects. When the jet becomes thin, the effects of viscosity can become predominant. Direct numerical simulations (Baker et al. , 1980; Gertsenshtein and Cherniavski, 1985) shows that , the asymptotic stage of the Rayleigh-Taylor instability occurs already at values of dimensionless time $t(g^d/h^d) \sim 3$. The model adopted is assumed as an intermediate asymptotics, which is valid for moderately large times, i.e. for times large enough to form the well-settled jets, but small enough for neglecting the viscosity effects. Such an intermediate asymptotics permits us to avoid employment of a more complicated boundary-layer model.

Acknowledgements. These studies have been supported in part by the U.S. Department of Energy under Grant No. DE-FG02-88ER13822, by the National Science Foundation under Grant No. CTS-95-21084 and by the Russian Foundation for Basic Sciences under Grant No. 96-05-65412.

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