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On uniqueness of large solutions of nonlinear parabolic equations in nonsmooth domains

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Abstract We study the existence and uniqueness of the positive solutions of the problem (P): $\partial_t u - \Delta u + u^q = 0$ ($q > 1$) in $\Omega \times (0, \infty)$, $u = \infty$ on $\partial\Omega \times (0, \infty)$ and $u(\cdot, 0) \in L^1(\Omega)$, when Ω is a bounded domain in \mathbb{R}^N . We construct a maximal solution, prove that this maximal solution is a large solution whenever $q < N/(N-2)$ and it is unique if $\partial\Omega = \partial\bar{\Omega}^c$. If $\partial\Omega$ has the local graph property, we prove that there exists at most one solution to problem (P).

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1 Introduction

Let $q > 1$ and let Ω be a bounded domain in \mathbb{R}^N with boundary $\partial\Omega := \Gamma$. It has been proved by Keller [5] and Osserman [11] that there exists a *maximal solution* \bar{u} to the stationary equation

$$-\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega. \quad (1.1)$$

When $1 < q < N/(N-2)$ this maximal solution is a *large solution* in the sense that

$$\lim_{\rho(x) \rightarrow 0} \bar{u}(x) = \infty \quad (1.2)$$

where $\rho(x) = \text{dist}(x, \partial\Omega)$. Furthermore Véron proves in [12] that \bar{u} is the unique large solution whenever $\partial\Omega = \partial\bar{\Omega}^c$. When $q \geq N/(N-2)$ his proof of uniqueness does not apply. Marcus and Véron prove in [7] that, there exists at most one large solution, provided $\partial\Omega$ is locally the graph of a continuous function. The aim of this article is to extend these questions to the parabolic equation

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty). \quad (1.3)$$

We are interested into positive solutions which satisfy

$$\lim_{t \rightarrow 0} u(\cdot, t) = f \quad \text{in } L^1_{loc}(\Omega), \quad (1.4)$$

where $f \in L^1_{loc+}(\Omega)$ and

$$\lim_{(x,t) \rightarrow (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0, \infty). \quad (1.5)$$

Notice that if the initial and boundary conditions are exchanged, i.e. $u(\cdot, t)$ blows-up when $t \rightarrow 0$ and coincides with a locally integrable function on $\Gamma \times (0, \infty)$, this problem is associated with the study of the initial trace, and much work has been done by Marcus and Véron [9] in the case of a smooth domain. In particular they obtain the existence and uniqueness when q is subcritical, i.e. $1 < q < 1 + 2/N$.

In this article we prove two series of results:

Theorem A *Assume $q > 1$ and Ω is a bounded domain. Then for any $f \in L^1_{loc+}(\Omega)$ there exists a maximal solution \bar{u}_f to problem (2.5) satisfying (1.4). If $1 < q < N/(N-2)$, \bar{u}_f satisfies (1.5). At end, if $1 < q < N/(N-2)$ and $\partial\Omega = \partial\bar{\Omega}^c$, \bar{u}_f is the unique solution of the problem which satisfies (1.5).*

The proof of uniqueness is based upon the construction of self-similar solutions of (2.5) in $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$, with a persistent strong singularity on the axis $\{0\} \times (0, \infty)$ and a zero initial trace on $\mathbb{R}^N \setminus \{0\}$. This solution, which is studied in Appendix, is reminiscent of the very singular solution of Brezis, Peletier and Terman [2], although the method of construction is far different. The uniqueness is a delicate adaptation to the parabolic framework of the proof by contradiction of [12].

Theorem B *Assume $q > 1$, Ω is a bounded domain and $\partial\Omega$, is locally a continuous graph. Then for any $f \in L^1_{loc+}(\Omega)$ there exists at most one solution to problem (2.5) satisfying (1.4) and (1.5).*

For proving this result, we adapt the idea which was introduced in [7] of constructing local super and subsolutions by small translations of the domain, but the non-uniformity of the boundary blow-up creates an extra-difficulty. In an appendix we study a self-similar equation which plays a key-role in our construction,

$$\begin{cases} H'' + \left(\frac{N-1}{r} + \frac{r}{2} \right) H' + \frac{1}{q-1} H - |H|^{q-1} = 0 \\ \lim_{r \rightarrow 0} H(r) = \infty \\ \lim_{r \rightarrow \infty} r^{2/(q-1)} H(r) = 0. \end{cases} \quad (1.6)$$

We prove the existence and the uniqueness of the positive solution of (1.6) when $1 < q < N/(N-2)$ and we give precise asymptotics when $r \rightarrow 0$ and $r \rightarrow \infty$.

This article is organised as follows: 1- Introduction. 2- The maximal solution 3- The case $1 < q < N/(N-2)$. 4- The local continuous graph property. 5- Appendix.

2 The maximal solution

In this section Ω is an open domain of \mathbb{R}^N , with a compact boundary $\Gamma := \partial\Omega$. If G is any open subset of \mathbb{R}^N and $0 < T \leq \infty$, we denote $Q_T^G := G \times (0, T)$. If $f \in L^1_{loc+}(\Omega)$, we

consider the problem

$$\begin{cases} \partial_t u - \Delta u + |u|^{q-1}u = 0 & \text{in } Q_\infty^\Omega \\ \lim_{t \rightarrow 0} u(\cdot, t) = f(\cdot) & \text{in } L_{loc}^1(\Omega) \\ \lim_{(x,t) \rightarrow (y,s)} u(x,t) = \infty & \forall (y,s) \in \Gamma \times (0, \infty). \end{cases} \quad (2.1)$$

By the next result, we reduce the lateral blow-up condition by a locally uniform one in which we set $\rho(x) = \text{dist}(x, \Gamma)$.

Lemma 2.1 *The following two conditions are equivalent*

$$\lim_{(x,t) \rightarrow (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0, \infty) \quad (2.2)$$

and

$$\lim_{\rho(x) \rightarrow 0} u(x,t) = \infty \quad \text{uniformly on } [\tau, T], \quad (2.3)$$

for any $0 < \tau < T < \infty$.

Proof. It is clear that (2.3) is equivalent to the fact that (2.2) holds uniformly on $\Gamma \times [\tau, T]$. By contradiction, we assume that (2.2) does not hold uniformly for some $T > \tau > 0$. Then there exists $\beta > 0$ such that for any $\delta > 0$, there exist two couples $(y_\delta, s_\delta) \in \Gamma \times [\tau, T]$ and $(x_\delta, t_\delta) \in \Omega \times [\tau, T]$ such that

$$|x_\delta - y_\delta| + |t_\delta - s_\delta| \leq \delta \quad \text{and} \quad u(x_\delta, t_\delta) \leq \beta. \quad (2.4)$$

Taking $\delta = 1/n$, $n \in \mathbb{N}^*$, we can assume that $\{\delta\}$ is discrete and that $y_\delta \rightarrow y \in \Gamma$ and $s_\delta \rightarrow s \in [\tau, T]$. Thus $x_\delta \rightarrow y$ and $t_\delta \rightarrow s$. Therefore (2.4) contradicts (2.2). \square

Theorem 2.2 *For any $q > 1$ and $f \in L_{loc}^1(\Omega)$, there exists a maximal solution $u := \bar{u}_f$ of*

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q_\infty^\Omega \quad (2.5)$$

which satisfies

$$\lim_{t \rightarrow 0} u(\cdot, t) = f(\cdot) \quad \text{in } L_{loc}^1(\Omega). \quad (2.6)$$

Proof. Let Ω_n be an increasing sequence of smooth bounded domains such that $\bar{\Omega}_n \subset \Omega_{n+1} \subset \Omega$ and $\cup \Omega_n = \Omega$. For each n let $u_{n,f}$ be the increasing limit when $k \rightarrow \infty$ of the $u_{n,k,f}$ solution of

$$\begin{cases} \partial_t u_{n,k,f} - \Delta u_{n,k,f} + u_{n,k,f}^q = 0 & \text{in } Q_\infty^{\Omega_n} \\ u_{n,k,f}(x,t) = k & \text{in } \partial\Omega_n \times (0, \infty) \\ u_{n,k,f}(x,0) = f\chi_{\Omega_n} & \text{in } \Omega_n. \end{cases} \quad (2.7)$$

By the maximum principle and a standard approximation argument $n \mapsto u_{n,k,f}$ is decreasing thus $n \mapsto u_{n,f}$ too. The limit \bar{u}_f of the $u_{n,f}$ satisfies (2.5) and (2.6). It is independent of the exhaustion $\{\Omega_n\}$ of Ω . Let u be a positive solution of (2.5) in Q_∞^Ω which satisfies (2.6). Since the initial trace of u is a locally integrable function, $u^q \in L_{loc}^1(\Omega \times [0, \infty))$. By

Fubini we can assume that, for any n , $u \in L^1_{loc}(\partial\Omega_n \times [0, \infty))$. Because $(u - u_{n,k,f})_+ \leq u$ and tends to 0 when $k \rightarrow \infty$, it follows by Lebesgue's theorem that

$$\lim_{k \rightarrow \infty} \|(u - u_{n,k,f})_+\|_{L^1(\partial\Omega_n \times (0, T))} = 0 \quad \forall T > 0.$$

Applying the maximum principle in $\Omega_n \times (0, \infty)$ yields to

$$u \leq \lim_{k \rightarrow \infty} u_{n,k,f} = u_{n,f} \implies u \leq \lim_{n \rightarrow \infty} u_{n,f} = \bar{u}_f.$$

□

Theorem 2.3 *For any $q > 1$ and $f \in L^1_{loc+}(\Omega)$, there exists a minimal nonnegative solution \underline{u}_f of (2.5) in Q_∞^Ω which satisfies (2.6).*

Proof. The scheme of the construction is similar to the one of \bar{u}_f : with the same exhaustion $\{\Omega_n\}$ of Ω , we consider the solution $u_{n,0,f}$ solution of

$$\begin{cases} \partial_t u_{n,0,f} - \Delta u_{n,0,f} + u_{n,0,f}^q = 0 & \text{in } Q_\infty^{\Omega_n} \\ u_{n,0,f}(x, t) = 0 & \text{in } \partial\Omega_n \times (0, \infty) \\ u_{n,0,f}(x, 0) = f\chi_{\Omega_n} & \text{in } \Omega_n. \end{cases} \quad (2.8)$$

By the maximum principle, $n \mapsto u_{n,0,f}$ is increasing and dominated by \bar{u}_f . Therefore it converges to some solution \underline{u}_f of (2.5), which satisfies (2.6) as $u_{n,0,f}$ and \bar{u}_f do it. Using the same argument as in the proof of Theorem 2.2, there holds $u_{n,0,f} \leq u$ in $Q_\infty^{\Omega_n}$ for a suitable exhaustion. Thus $\underline{u}_f \leq u$. □

Remark. Because of the lack of regularity of $\partial\Omega$, there is no reason for \bar{u}_f (resp \underline{u}_f) to tend to infinity (resp. zero) on $\partial\Omega \times (0, \infty)$.

The next statement will be very useful for proving uniqueness results.

Theorem 2.4 *Assume $q > 1$, $f \in L^1_{loc+}(\Omega)$ and u_f is a nonnegative solution of (2.5) satisfying (2.6). Then there exists a nonnegative solution u_0 of (2.5) satisfying*

$$\lim_{t \rightarrow 0} u_0(\cdot, t) = 0 \quad \text{in } L^1_{loc}(\Omega), \quad (2.9)$$

such that

$$0 \leq u_f - \underline{u}_f \leq u_0 \leq u_f, \quad (2.10)$$

and

$$0 \leq \bar{u}_f - u_f \leq \bar{u}_0 - u_0. \quad (2.11)$$

Proof. Step 1: construction of u_0 . The function $w = u_f - \underline{u}_f$ is a nonnegative subsolution of (2.5) which satisfies

$$\lim_{t \rightarrow 0} w(\cdot, t) = 0 \quad \text{in } L^1_{loc}(\Omega).$$

Using the above considered exhaustion of Ω , we denote by v_n the solution of

$$\begin{cases} \partial_t v_n - \Delta v_n + v_n^q = 0 & \text{in } Q_\infty^{\Omega_n} \\ v_n(x, t) = u_f - \underline{u}_f & \text{in } \partial\Omega_n \times (0, \infty) \\ v_n(x, 0) = 0 & \text{in } \Omega_n. \end{cases} \quad (2.12)$$

By the maximum principle

$$u_f - \underline{u}_f \leq v_n \leq u_f \quad \text{in } Q_\infty^{\Omega_n}.$$

Therefore $v_{n+1} \geq v_n$ on $\partial\Omega_n \times (0, \infty)$; this implies that the same inequality holds in $Q_\infty^{\Omega_n}$. If we denote by u_0 the limit of the $\{v_n\}$, it is a solution of (2.5) in Q_∞^Ω . For any compact $K \in \Omega$, there exists n_K and $\alpha > 0$ such that $\text{dist}(K, \Omega_n^c) \geq \alpha$ for $n \geq n_K$ therefore v_n remains uniformly bounded on K by Brezis-Friedman estimate [3]. Thus the local equicontinuity of the v_n (consequence of the regularity theory for parabolic equations) implies that u_0 satisfies (2.9).

Step 2: proof of (2.11). We follow a method introduced in [8] in a different context. For $n \in \mathbb{N}$ and $k > 0$ fixed, we set

$$Z_{f,n} = u_{f,n} - u_f \quad \text{and} \quad Z_{0,n} = u_{0,n} - u_0,$$

where we assume that the n are chosen such that $u_f, u_0 \in L_{loc}^1(\partial\Omega_n \times [0, \infty))$, and

$$\phi(r, s) = \begin{cases} \frac{r^q - s^q}{r - s} & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{cases}$$

By convexity,

$$\begin{cases} r_0 \geq s_0, r_1 \geq s_1 \\ r_1 \geq r_0, s_1 \geq s_0 \end{cases} \implies \phi(r_1, s_1) \geq \phi(r_0, s_0).$$

Therefore

$$\phi(u_{f,n}, u_f) \geq \phi(u_{0,n}, u_0) \quad \text{in } Q_T^{\Omega_n},$$

and

$$\begin{aligned} 0 &= \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + u_{f,n}^q - u_f^q - u_{0,n}^q + u_0^q \\ &= \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)Z_{f,n} - \phi(u_{0,n}, u_0)Z_{0,n}, \end{aligned}$$

which implies

$$\partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{0,n}) \leq 0.$$

But $Z_{f,n} - Z_{0,n} = 0$ in $\Omega_n \times \{0\}$ and

$$\int_0^\infty \int_{\partial\Omega_n} |Z_{f,n} - Z_{0,n}| dS dt = 0$$

by approximations. By the maximum principle $Z_{f,n,k} - Z_{0,n,k} \leq 0$. Letting $n \rightarrow \infty$ yields to

$$\bar{u}_f - u_f \leq \bar{u}_0 - u_0,$$

which ends the proof. \square

3 The case $1 < q < N/(N - 2)$

In this section we assume that Ω is a domain of \mathbb{R}^N with a compact boundary. We first prove that the maximal solution is a large solution

Theorem 3.1 *Assume $1 < q < N/(N - 2)$ and $f \in L^1_{loc+}(\Omega)$. Then the maximal solution \bar{u}_f of (2.5) in Q_T^Ω which satisfies (2.6) satisfies also (2.3).*

Proof. In Appendix we construct the self-similar solution $V := V_N$ of (2.5) in $Q_\infty^{\mathbb{R}^N \setminus \{0\}}$ which has initial trace zero in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$\lim_{|x| \rightarrow 0} V_N(x, t) = \infty,$$

locally uniformly on $[\tau, \infty)$, for any $\tau > 0$. Furthermore $V_N(x, t) = t^{-1/(q-1)} H_N(|x|/\sqrt{t})$. If $a \in \partial\Omega$, the restriction to Ω_n of the function $V_N(x - a, t)$ is bounded from above by $u_{n,f}$. Letting $n \rightarrow \infty$ yields to

$$V_N(x - a, t) \leq \bar{u}_f(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.1)$$

If we consider $x \in \Omega$ and denote by a_x a projection of x onto $\partial\Omega$, there holds

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) = V_N(x - a_x, t) \leq \bar{u}_f(x, t). \quad (3.2)$$

Using (5.2), we derive that \bar{u}_f satisfies (2.3). \square

Theorem 3.2 *Assume $1 < q < N/(N - 2)$, $f \in L^1_{loc+}(\Omega)$ and $\partial\Omega = \partial\bar{\Omega}^c$. Then \bar{u}_f is the unique solution of (2.5) in Q_T^Ω which satisfies (2.6) and (2.3).*

Proof. Assume that u_f is a solution of (2.5) in Q_T^Ω such that (2.6) and (2.3) hold. By Theorem 2.4 there exists a positive solution u_0 with zero initial trace such that

$$0 \leq u_f - u_0 \leq \underline{u}_f \quad (3.3)$$

and (2.11) are satisfied. Since $\underline{u}_f(x, t) \leq ((q-1)t)^{-1/(q-1)}$ (notice that this last expression is the maximal solution of (2.5) in $Q_\infty^{\mathbb{R}^N}$), the function u_0 satisfies also (2.3). Therefore, it is sufficient to prove that $\bar{u}_0 = u_0 := u$.

Step 1: bilateral estimates. Since $\partial\Omega = \partial\bar{\Omega}^c$, for any $a \in \partial\Omega$, there exists a sequence $\{a_n\} \subset \bar{\Omega}^c$ converging to a . If u is any solution of (2.5) in Q_T^Ω which satisfies (2.3) and (2.9), there holds

$$V_N(x - a_n, t) \leq u(x, t) \implies V_N(x - a, t) \leq u(x, t).$$

In particular, if $a = a_x$, we see that u satisfies (3.2). In order to obtain an estimate from above we consider for $r < \rho(x)$ the solution $(y, t) \mapsto u_{x,r}(y, t)$ of

$$\begin{cases} \partial_t u_{x,r} - \Delta u_{x,r} + u_{x,r}^q = 0 & \text{in } Q_\infty^{B_r(x)} \\ \lim_{(y,t) \rightarrow (z,0)} u_{x,r}(y, t) = 0 & \forall z \in B_r(x) \\ \lim_{|x| \uparrow r} u_{x,r}(x, t) = \infty & \text{locally uniformly on } [\tau, \infty), \text{ for any } \tau > 0 \end{cases} \quad (3.4)$$

Then

$$\bar{u}_0(y, t) \leq u_{x,r}(y, t) \implies \bar{u}_0(y, t) \leq u_{x,\rho(x)}(y, t) \quad \forall (y, t) \in Q_\infty^{B_{\rho(x)}(x)}.$$

In particular, with $u_{0,r} = u_r$,

$$\bar{u}_0(x, t) \leq u_{\rho(x)}(0, t) = (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2).$$

Therefore

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) \leq u(x, t) \leq \bar{u}_0(x, t) \leq (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2). \quad (3.5)$$

The function $s \mapsto u_1(0, s)$ is increasing by the same argument as the one of Corollary 4.3 and bounded from above by the unique solution P of

$$\begin{cases} -\Delta P + P^q = 0 & \text{in } B_1 \\ \lim_{|x| \rightarrow 1} P(x) = \infty. \end{cases} \quad (3.6)$$

Therefore it converges to P locally uniformly in B_1 and $\lim_{s \rightarrow \infty} u_1(0, s) = P(0)$. Thus

$$t/(\rho(x))^2 \rightarrow \infty \implies (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2) \approx P(0)(\rho(x))^{-2/(q-1)}. \quad (3.7)$$

On the other hand, if $t/(\rho(x))^2 \rightarrow \infty$, equivalently $\rho(x)/\sqrt{t} \rightarrow 0$,

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) \approx \lambda_{N,q} t^{-1/(q-1)} (\rho(x)/\sqrt{t})^{-2/(q-1)} = \lambda_{N,q} (\rho(x))^{-2/(q-1)}, \quad (3.8)$$

by (5.4).

Next, in order to obtain an estimate from above of $u_1(0, s)$ when $s \rightarrow 0$, we compare u_1 to a solution u_Θ of (2.5) in Q_∞^Θ , where Θ is a polyhedra inscribed in B_1 ; this polyhedra is a finite intersection of half spaces Γ_i containing Π . In each of the half space Γ_i , with boundary γ_i , we can consider the solution W_i of (2.5) in $Q_\infty^{\Gamma_i}$ which tends to infinity on $\gamma_i \times (0, \infty)$ and has value 0 on $\Gamma_i \times \{0\}$. This solution depends only on the distance to γ_i and t . Thus it is expressed by the function V_1 defined in Proposition 5.1 when $N = 1$. Moreover, since a sum of solutions is a super solution,

$$u_1 \leq u_\Theta \leq \sum_i W_i \implies u_1(0, s) \leq \sum_i H_1(\text{dist}(0, \gamma_i)/\sqrt{s}). \quad (3.9)$$

We can choose the hyperplanes γ_i such that for any $\delta \in (0, 1)$, there exists $C_\delta \in \mathbb{N}_*$ such that

$$u_1(0, s) \leq C_\delta H_1((1-\delta)/\sqrt{s}). \quad (3.10)$$

Using (5.3) we derive

$$u(x, t) \geq c_{N,q} (\rho(x))^{2/(q-1)-N} t^{N/2-1/(q-1)} e^{-(\rho(x))^2/4t},$$

when $\rho(x)/\sqrt{t} \rightarrow \infty$, and

$$\bar{u}_0(x, t) \leq C H_1((1-\delta)\rho(x)/\sqrt{t}) \leq C(1-\delta)^{2/(q-1)-1} (\rho(x))^{2/(q-1)-1} t^{1/2-1/(q-1)} e^{-((1-\delta)\rho(x))^2/4t}.$$

Therefore, there exists $\theta > 1$ such that

$$\bar{u}_0(x, t) \leq C(\rho(x))^{2/(q-1)-N} t^{N/2-1/(q-1)} e^{-(\rho(x))^2/4\theta t} \leq Cu(x, \theta t), \quad (3.11)$$

when $\rho(x)/\sqrt{t} \rightarrow \infty$. Finally, when $m^{-1} \leq \rho(x)/\sqrt{t} \leq m$ for some $m > 1$, (3.5) shows that $(\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2)$ and $t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t})$ are comparable. In conclusion, there exist constants $C > P(0)/\lambda_{N,q} > 1$ and $\theta > 1$ such that

$$u(x, t) \leq \bar{u}_0(x, t) \leq Cu(x, \theta t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.12)$$

Step 2: End of the proof. Let $\tau > 0$ and $C' > C$ be fixed. The function

$$t \mapsto u_\tau(x, t) := C' u(x, t + \theta\tau)$$

is a supersolution of (2.5) in $\Omega \times (0, \infty)$ which satisfies $u_\tau(x, 0) = C' u(x, \theta\tau) > \bar{u}_0(x, \tau)$ by (3.12). Furthermore,

$$C' u(x, t + \theta\tau) \geq C'(t + \theta\tau)^{-1/(q-1)} H_N(\rho(x)/\sqrt{t + \theta\tau}) = C' \lambda_{N,q} (1 + o(1)) (\rho(x))^{-2/(q-1)},$$

as $\rho(x) \rightarrow 0$, locally uniformly for $t \in [0, \infty)$. Similarly,

$$\bar{u}_0(x, t + \tau) \leq (\rho(x))^{-2/(q-1)} u_1(0, (t + \tau)/(\rho(x))^2) = P(0)(1 + o(1)) (\rho(x))^{-2/(q-1)},$$

as $\rho(x) \rightarrow 0$, and also locally uniformly for $t \in [0, \infty)$. Therefore $(\bar{u}_0(x, t) - u_\tau(x, t))_+$ vanishes in a neighborhood of $\partial\Omega \times [0, T]$ for any $T > 0$. By the maximum principle

$$u_\tau(x, t) \geq \bar{u}_0(x, t) \quad \forall (x, t) \in \Omega \times (0, \infty).$$

Letting $\tau \rightarrow 0$ and $C' \rightarrow C$ yields to

$$u(x, t) \leq \bar{u}_0(x, t) \leq Cu(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.13)$$

The conclusion of the proof is contradiction, following an idea introduced in [8] and developed by [12] in the elliptic case. We assume $u \neq \bar{u}_0$, thus $u < \bar{u}_0$. By convexity the function

$$w = u - \frac{1}{2C}(\bar{u}_0 - u)$$

is a supersolution and $w < u$. Moreover $w > w' := ((1 + C)/2C)u$ and w' is a subsolution. Consequently, there exists a solution u_1 of (2.5) which satisfies

$$w' < u_1 \leq w \implies \bar{u}_0 - u_1 \geq (1 + K^{-1})(\bar{u}_0 - u) \quad \text{in } Q_\infty^\Omega. \quad (3.14)$$

Notice that u_1 satisfies (2.9) and (2.3), therefore it satisfies (3.13) as u does it. Replacing u by u_1 and introducing the supersolution

$$w_1 = u_1 - \frac{1}{2C}(\bar{u}_0 - u_1)$$

and the subsolution $w'_1 := ((1 + C)/2C)u_1$ we see that there exists a solution u_2 of (2.5) such that

$$w'_1 < u_2 \leq w_1 \implies \bar{u}_0 - u_2 \geq (1 + K^{-1})^2 (\bar{u}_0 - u) \quad \text{in } Q_\infty^\Omega. \quad (3.15)$$

By induction, we construct a sequence of positive solutions u_k of (2.5), subject to (2.9) and (2.3) such that

$$\bar{u}_0 - u_k \geq (1 + K^{-1})^k (\bar{u}_0 - u) \quad \text{in } Q_\infty^\Omega. \quad (3.16)$$

This is clearly a contradiction since $(1 + K^{-1})^k \rightarrow \infty$ as $k \rightarrow \infty$ and \bar{u}_0 is locally bounded in Q_∞^Ω . \square

4 The local continuous graph property

In this section, we assume that $\partial\Omega$ is compact and is locally the graph of a continuous function, which means that there exists a finite number of open sets Ω_j ($j = 1, \dots, k$) such that $\Gamma \cap \Omega_j$ is the graph of a continuous function. Our main result is the following

Theorem 4.1 *Assume $q > 1$ and $f \in L_{loc+}^1(\Omega)$. Then there exists at most one positive solution of (2.5) in Q_∞^Ω satisfying (2.6) and (2.3).*

Suppose u_f satisfies (2.5) in Q_∞^Ω satisfying (2.6) and (2.3), then clearly the maximal solution \bar{u}_f endows the same properties. In order to prove that $u_f = \bar{u}_f$, we can assume that $f = 0$ by Theorem 2.4. We denote by u this large solution with zero initial trace. We consider some $j \in \{1, \dots, k\}$, perform a rotation, denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ the coordinates in \mathbb{R}^N and represent $\Gamma \cap \Omega_j$ as the graph of a continuous positive function ϕ defined in $C = \{x' \in \mathbb{R}^{N-1} : |x'| \leq R\}$. We identify C with $\{x = (x', 0) : |x'| \leq R\}$ and set

$$\Gamma_1 = \{x = (x', \phi(x')) : x' \in C\},$$

$$\Gamma_2 = \{x = (x', x_N) : x' \in \partial C, 0 \leq x_N < \phi(x'), \},$$

and

$$G_R = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi(x')\}.$$

We can assume that $\bar{G}_R \subset \Omega \cup \Gamma_1$,

$$\inf\{\phi(x') : x' \in C\} = R_0 > 0 \quad \text{and} \quad \sup\{\phi(x') : x' \in C\} = R_1 > R_0.$$

For $\sigma > 0$, small enough, we consider $\phi_\sigma \in C^\infty(C)$ satisfying

$$\phi(x') - \sigma/2 \leq \phi_\sigma(x') \leq \phi(x') + \sigma/2 \quad \forall x' \in C,$$

and set

$$G_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') - \sigma\}$$

and

$$G'_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') + \sigma\}.$$

The upper boundaries of G_σ and G'_σ are defined by

$$\Gamma_{1,\sigma} = \{x = (x', \phi_\sigma(x') - \sigma) : x' \in C\},$$

$$\Gamma'_{1,\sigma} = \{x = (x', \phi_\sigma(x') + \sigma) : x' \in C\},$$

and the remaining boundaries are

$$\begin{aligned}\Gamma_{2,\sigma} &= \{x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') - \sigma\}, \\ \Gamma'_{2,\sigma} &= \{x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') + \sigma\}.\end{aligned}$$

In order to have the monotonicity of the domains, we can also assume

$$\phi_\sigma(x') - \sigma < \phi_{\sigma'}(x') - \sigma' < \phi_{\sigma'}(x') + \sigma' < \phi_\sigma(x') + \sigma \quad \forall 0 < \sigma' < \sigma \quad \forall x' \in C, \quad (4.1)$$

thus, under the condition $0 < \sigma' < \sigma$,

$$G_{\sigma,R} \subset G_{\sigma',R} \subset G_R \subset G'_{\sigma',R} \subset G'_{\sigma,R}. \quad (4.2)$$

The localization procedure is to consider the restriction of u to $Q_\infty^{G_R} := G_R \times (0, \infty)$, thus u is regular in $G_R \cup \Gamma_2 \times [0, \infty)$ and satisfies

$$\lim_{x_N \rightarrow \phi(x')} u(x', x_N, t) = \infty, \quad (4.3)$$

uniformly with respect to $(x', t) \in C \times [\tau, T]$, for any $0 < \tau < T$. We construct v_σ as solution of

$$\partial_t v_\sigma - \Delta v_\sigma + v_\sigma^q = 0 \quad \text{in } Q_\infty^{G_{\sigma,R}} := G_{\sigma,R} \times (0, \infty), \quad (4.4)$$

subject to the initial condition

$$\lim_{t \rightarrow 0} v_\sigma(x, t) = 0 \quad \text{locally uniformly in } G_{\sigma,R}, \quad (4.5)$$

and the boundary conditions

$$\lim_{x_N \rightarrow \phi_\sigma(x') - \sigma} v_\sigma(x', x_N, t) = \infty \quad \forall (x', t) \in C \times (0, \infty], \quad (4.6)$$

uniformly on any set $K \times [\tau, T]$, where $T > \tau > 0$ and K is a compact subset of C and

$$v_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma_{2,\sigma} \times [0, \infty). \quad (4.7)$$

We also construct w_σ as solution of

$$\partial_t w_\sigma - \Delta w_\sigma + w_\sigma^q = 0 \quad \text{in } Q_T^{G'_{\sigma,R}} := G'_{\sigma,R} \times (0, \infty), \quad (4.8)$$

subject to the initial condition

$$\lim_{t \rightarrow 0} w_\sigma(x, t) = 0 \quad \text{locally uniformly in } G'_{\sigma,R}, \quad (4.9)$$

and the boundary conditions

$$\begin{cases} (i) & w_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma'_{1,\sigma} \times [0, T], \\ (i') & \lim_{(x,s) \rightarrow (y,t)} w_\sigma(x, t) = \infty \quad \forall (y, s) \in \Gamma'_{2,\sigma} \times [0, T]. \end{cases} \quad (4.10)$$

The functions v_σ and w_σ inherit the following properties in which the local graph property plays a fundamental role, allowing translations of the truncated domains in the x_N -direction.

Lemma 4.2 For $\sigma > \sigma' > 0$ there holds

$$v_{\sigma'} \leq v_{\sigma} \quad \text{in } Q_{\infty}^{G_{\sigma,R}}, \quad (4.11)$$

$$w_{\sigma'} \leq w_{\sigma} \quad \text{in } Q_{\infty}^{G'_{\sigma',R}}, \quad (4.12)$$

$$\begin{aligned} (i) \quad & v_{\sigma}(x', x_N - 2\sigma, t) \leq u(x', x_N, t) \quad \text{in } Q_{\infty}^{G_R} \\ (ii) \quad & u(x', x_N, t) \leq v_{\sigma}(x, t) + w_{\sigma}(x, t) \quad \text{in } Q_{\infty}^{G_{\sigma,R}}. \end{aligned} \quad (4.13)$$

Proof. The inequalities (4.11) and (4.12) are the direct consequence of the fact that the domains $G_{\sigma,R}$ and $G'_{\sigma',R}$ are Lipschitz and the functions v_{σ} and w_{σ} are constructed by approximations of solutions of (2.5) with bounded boundary data. For proving (4.13)-(i), we compare, for $\tau > 0$, $u(x, t - \tau)$ and $v_{\sigma}(x', x_N - 2\sigma, t)$ in $Q_{\infty}^{G_R}$. Because u satisfies (2.3), and $v_{\sigma}(x', x_N - 2\sigma, 0) = 0$ in G_R , (4.13)-(i) follows by the maximum principle. The proof of (4.13)-(ii) needs no translation, but the fact that the sum of two solutions is a supersolution. \square

Corollary 4.3 There exist $v_0 = \lim_{\sigma \rightarrow 0} v_{\sigma}$ and $w_0 = \lim_{\sigma \rightarrow 0} w_{\sigma}$ and there holds

$$v_0 \leq u \leq v_0 + w_0 \quad \text{in } Q_{\infty}^{G_R}. \quad (4.14)$$

Moreover, the functions $t \mapsto v_0(x, t)$ and $t \mapsto w_0(x, t)$ are increasing on $(0, \infty)$, $\forall x \in G_R$.

Proof. The first assertion follows from (4.11)-(4.12), and (4.14) from (4.13). Since v_0 is the limit, when $\sigma \rightarrow 0$ of v_{σ} which satisfy equation (4.4) in $Q_T^{G_{\sigma,R}}$, initial condition (4.5) and boundary conditions (4.6), (4.7), it is sufficient to prove the monotonicity of $t \mapsto v_{\sigma}(\cdot, t)$. Moreover v_{σ} is the limit, when k tends to infinity of the $v_{k,\sigma}$ solutions of (2.5) in $Q_T^{G_{\sigma,R}}$, which satisfy the same boundary conditions as v_{σ} on $\Gamma_{2,\sigma} \times [0, T]$, the same zero initial condition and

$$\lim_{x_N \rightarrow \phi(x') - \sigma} v_{k,\sigma}(x', x_N, t) = k.$$

For $\tau > 0$, we define V_{τ} by $V_{\tau}(x, t) = (v_{k,\sigma}(x, t) - v_{k,\sigma}(x, t + \tau))_+$. Because $\partial G_{\sigma,R}$ is Lipschitz and V_{τ} is a subsolution of (2.5) which vanishes on $\partial G_{\sigma,R} \times [0, T]$ and at $t = 0$, it is identically zero. This implies $v_{k,\sigma}(x, t) \leq v_{k,\sigma}(x, t + \tau)$, and the monotonicity property of v_0 , by strict maximum principle and letting $\sigma \rightarrow 0$. The proof of the monotonicity of w_0 is similar. \square

The key step of the proof is the following result.

Proposition 4.4 Let $\epsilon, \tau > 0$. Then there exists $\delta_{\epsilon} > 0$ such that, if we denote

$$G_{\delta,R'} = \{x = (x', x_N) : |x'| < R' \text{ and } \phi(x') - \delta \leq x_N < \phi(x')\},$$

there holds, for $R' < R/\sqrt{N-1}$,

$$w_0(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q_{\infty}^{G_{\delta,R'}}. \quad (4.15)$$

Proof. Using the result in Appendix, we recall that $V := V_1$ is the unique positive and self-similar solution of the problem

$$\begin{cases} \partial_t V - \partial_{zz} V + V^q = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ \lim_{t \rightarrow 0} V(z, t) = 0 & \forall z > 0 \\ \lim_{z \rightarrow 0} V(z, t) = \infty & \forall t > 0, \end{cases} \quad (4.16)$$

and it is expressed by $V_1(z, t) = t^{-1/(q-1)} H_1(x/\sqrt{t})$, where H_1 satisfies (5.2)-(5.3) with $N = 1$. We set $R_N = R/\sqrt{N-1}$ so that

$$C_\infty := \{x' = (x_1, \dots, x_{N-1}) : \sup_{j \leq N-1} |x_j| < R_N\} \subset C = \{x' : |x'| \leq R\}$$

and we define

$$\tilde{w}(x, t) = W(x_N, t) + \sum_{j=1}^{N-1} (W(x_j - R, t) + W(R - x_j, t)).$$

The function \tilde{w} a super solution in $\Theta \times \mathbb{R}^+$ where $\Theta := \{(x', x_N) : x' \in C_\infty, x_N > 0\}$ which blows up on

$$\{x : x_N = 0, \sup_{j \leq N-1} |x_j| \leq R\} \cup \{x : x_N \geq 0, x_j = \pm R\}.$$

Therefore $w_0 \leq \tilde{w}$ in $Q_T^{G_{R_N}}$. Moreover $\tilde{w}(x, t) \rightarrow 0$ when $t \rightarrow 0$, uniformly on

$$G_{\alpha, R'}^* := \{x = (x_1, x_2) : |x_1| \leq R', \alpha \leq x_2 \leq \phi(x_1)\},$$

for any $\alpha \in (0, R_0]$ and $R' \in (0, R_N)$. Since for any $\tau > 0$, $v_0(x, t + \tau) \rightarrow \infty$ when $\rho(x) \rightarrow 0$, locally uniformly on $[0, \infty)$, and $\tilde{w}(x, t)$ remains uniformly bounded on $Q_\infty^{G_{\delta, R'}}$, for any $\delta > R_0$, it follows that for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$w_0(x, t) \leq \tilde{w}(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q_\infty^{G_{\delta_\epsilon, R'}}.$$

□

Proof of Theorem 4.1. Assume u is a solution of (2.5) satisfying (2.6) and (2.3). Then there holds in $Q_\infty^{G_{\delta_\epsilon, R'}}$,

$$v_0(\cdot, t) \leq u(\cdot, t) \leq v_0(\cdot, t) + \epsilon v_0(\cdot, t + \tau). \quad (4.17)$$

Therefore

$$v_0(\cdot, t + \tau) \leq u(\cdot, t + \tau) \leq v_0(\cdot, t + \tau) + \epsilon v_0(\cdot, t + 2\tau),$$

from which follows

$$(1 + \epsilon)u(\cdot, t + \tau) \geq (1 + \epsilon)v_0(\cdot, t + \tau) \geq v_0(\cdot, t) + \epsilon v_0(\cdot, t + \tau)$$

since $t \mapsto v_0(\cdot, t)$ is increasing by Corollary 4.3. The maximal solution \bar{u}_0 satisfies (4.17) too; consequently the following inequality is verified in $Q_\infty^{G_{\delta\epsilon, R'}}$,

$$(1 + \epsilon)u(\cdot, t + \tau) \geq \bar{u}_0(\cdot, t). \quad (4.18)$$

Since $\partial\Omega$ is compact, there exists $\delta^* > 0$ such that (4.18) holds whenever $t \in [0, T]$ ($T > 0$ arbitrary) and $\rho(x) \leq \delta^*$. Furthermore

$$\lim_{t \rightarrow 0} \max\{(\bar{u}_0(x, t) - (1 + \epsilon)u(x, t + \tau))_+ : \rho(x) \geq \delta^*\} = 0$$

because of (2.6). Since $(\bar{u}_0(x, t) - (1 + \epsilon)u(x, t + \tau))_+$ is a subsolution, which vanishes at $t = 0$ and near $\partial\Omega \times [0, T]$, it follows that (4.18) holds in Q_T^Ω . Letting $\epsilon \rightarrow 0$ and $\tau \rightarrow 0$ yields to $u \geq \bar{u}_0$. \square

Remark. The existence of large solutions when $q \geq N/(N - 2)$ is a difficult problem as it is already in the elliptic case. We conjecture that the necessary and sufficient conditions, obtained by Dherzin-Le Gall when $q = 2$ [4] and Labutin [6] in the general case $q > 1$, and expressed by mean of a Wiener type criterion involving the $C_{2, q'}^{\mathbb{R}^N}$ -Bessel capacity, are still valid. As in [7], it is clear that if $\partial\Omega$ satisfies the exterior segment property and $1 < q < (N - 1)/(N - 3)$, then \bar{u}_0 is a large solution.

5 Appendix

The proof of this result is based upon the existence of solution of (2.5) in $Q_\infty^{\mathbb{R}^N \setminus \{0\}}$ with a persistent singularity on $\{0\} \times [0, \infty)$.

Proposition 5.1 *For any $q > 1$, there exists a unique positive function $V := V_N$ defined in $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying, for any $\tau > 0$*

$$\begin{cases} \partial_t V - \Delta V + V^q = 0 & \text{in } Q_\infty^{\mathbb{R}^N \setminus \{0\}} \\ \lim_{(x, t) \rightarrow (y, 0)} V(x, t) = 0 & \forall y \in \mathbb{R}^N \setminus \{0\} \\ \lim_{|x| \rightarrow 0} V(x, t) = \infty & \text{locally uniformly on } [\tau, \infty), \text{ for any } \tau > 0 \end{cases} \quad (5.1)$$

Then $V_N(x, t) = t^{-1/(q-1)} H_N(|x|/\sqrt{t})$, where $H := H_N$ is the unique positive function satisfying

$$\begin{cases} H'' + \left(\frac{N-1}{r} + \frac{r}{2}\right) H' + \frac{1}{q-1} H - H^q = 0 & \text{in } \mathbb{R}_+ \\ \lim_{r \rightarrow 0} H(r) = \infty \\ \lim_{r \rightarrow \infty} r^{2/(q-1)} H(r) = 0. \end{cases} \quad (5.2)$$

Furthermore there holds

$$H_N(r) = c_{N, q} r^{2/(q-1) - N} e^{-r^2/4} (1 + O(r^{-2})) \quad \text{as } r \rightarrow \infty, \quad (5.3)$$

and

$$H_N(r) = \lambda_{N, q} r^{-2/(q-1)} (1 + O(r)) \quad \text{as } r \rightarrow 0, \quad (5.4)$$

Proof. If we assume $1 < q < N/(N-2)$, the $C_{2,1,q'}$ parabolic capacity of the axis $\{0\} \times \mathbb{R} \subset \mathbb{R}^{N+1}$ is positive, therefore there exists a unique solution $u := u_\mu$ to the problem

$$\partial_t u - \Delta u + |u|^{q-1}u = \mu \quad \in \mathbb{R}^N \times \mathbb{R}, \quad (5.5)$$

(see [1]) where μ is the uniform measure on $\{0\} \times \mathbb{R}_+$ defined by

$$\int \zeta d\mu = \int_0^\infty \zeta(0, t) dt \quad \forall \zeta \in C_0^\infty(\mathbb{R}^{N+1}).$$

If we denote $T_\ell[u](x, t) = \ell^{2/(q-1)}u(\ell x, \ell^2 t)$ for $\ell > 0$, then T_ℓ leaves the equation (2.5) invariant, and $T_\ell[u_\mu] = u_{\ell^{2/(q-1)-N}\mu}$. If we replace μ by $k\mu$ ($k > 0$), we obtain

$$T_\ell[u_{k\mu}] = u_{\ell^{2/(q-1)-N}k\mu}. \quad (5.6)$$

Moreover, any solution of (2.5) in $\mathbb{R}^N \setminus \{0\} \times \mathbb{R}_+$ which vanishes on $\mathbb{R}^N \setminus \{0\} \times \{0\}$ is bounded from above by the maximum solution $u := U$ of

$$-\Delta u + u^q = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (5.7)$$

This is obtained by considering the solution U_ϵ of

$$\begin{cases} -\Delta u + u^q = 0 & \text{in } \mathbb{R}^N \setminus \overline{B_\epsilon} \\ \lim_{|x| \rightarrow \epsilon} u(x) = \infty. \end{cases} \quad (5.8)$$

Actually,

$$U(x) := \lim_{\epsilon \rightarrow 0} U_\epsilon(x) = \lambda_{N,q} |x|^{-2/(q-1)} \quad \text{with } \lambda_{N,q} := \left[\left(\frac{2}{q-1} \right) \left(\frac{2q}{q-1} - N \right) \right]^{1/(q-1)}, \quad (5.9)$$

an expression which exists since $1 < q < N/(N-2)$. If we let $k \rightarrow \infty$ in (5.6), using the monotonicity of $\mu \mapsto u_\mu$, we obtain that $u_{k\mu} \rightarrow u_{\infty\mu}$, $u_{\infty\mu} \leq U$ and

$$T_\ell[u_{\infty\mu}] = u_{\ell^{2/(q-1)-N}\infty\mu} = u_{\infty\mu} \quad \forall \ell > 0. \quad (5.10)$$

This implies that $u_{\infty\mu}$ is self-similar, that is

$$u_{\infty\mu}(x, t) = t^{-1/(q-1)}h(x/\sqrt{t}).$$

Furthermore, $h(\cdot)$ is positive and radial as $x \mapsto u_\mu(x, t)$ is, and it solves

$$h'' + \left(\frac{N-1}{r} + \frac{r}{2} \right) h' + \frac{1}{q-1}h - h^q = 0 \quad \text{in } \mathbb{R}_+. \quad (5.11)$$

Since $u_\mu(x, 0) = 0$ for $x \neq 0$, the a priori bounds $u_{k\mu} \leq U$, the equicontinuity of the $\{u_{k\mu}\}_{k>0}$ implies that $u_{\infty\mu}(x, 0) = 0$ for $x \neq 0$; therefore

$$\lim_{r \rightarrow \infty} r^{2/(q-1)}h(r) = 0. \quad (5.12)$$

The same argument as the one used in the proof of Corollary 4.3 implies that $t \mapsto u_\mu(x, t)$ is increasing, therefore $\lim_{x \rightarrow 0} u_\mu(x, t) = \infty$ for $t > 0$. This implies $\lim_{r \rightarrow 0} h(r) = \infty$. Then the proof of (5.3) follows from [10, Appendix]. When $r \rightarrow 0$, h could have two possible behaviours [13]:

(i) either

$$h(r) = \lambda_{N,q} r^{-2/(q-1)}(1 + O(r)), \quad (5.13)$$

(ii) or there exists $c \geq 0$ such that

$$h(r) = cm_N(r)(1 + O(r)), \quad (5.14)$$

where $m_N(r)$ is the Newtonian kernel if $N \geq 2$ and $m_1(r) = 1 + o(1)$.

If (ii) were true with $c > 0$ (the case $c = 0$ implying that $h = 0$ because of the behavior at ∞ and maximum principle), it would lead to

$$u_{\infty\mu}(x) = c|x|^{2-N}t^{N-2-1/(q-1)}(1 + o(1)) \quad \text{as } x \rightarrow 0, \quad (5.15)$$

for all $t > 0$. Therefore

$$\int_\epsilon^T \int_{B_1} u_{k\mu}^q dx dt < C(\epsilon), \quad (5.16)$$

for any $\epsilon > 0$ and $k \in (0, \infty]$. We write (5.5) under the form

$$\partial_t u_{k\mu} - \Delta u_{k\mu} = g_k + k\mu$$

where $g_k = -u_{k\mu}^q$, then $u_{k\mu} = u'_{k\mu} + u''_k$, where

$$\partial_t u'_{k\mu} - \Delta u'_{k\mu} = k\mu$$

and

$$\partial_t u''_k - \Delta u''_k = g_k.$$

By linearity $u'_{k\mu} = ku'_\mu$. Because of (5.16) u''_k remains uniformly bounded in $L^1(B_1 \times (\epsilon, T))$. This clearly contradicts $\lim_{k \rightarrow \infty} u'_{k\mu} = \infty$. Thus (5.4) holds. The proof of uniqueness is an easy adaptation of [7, Lemma 1.1]: the fact that the domain is not bounded being compensated by the strong decay estimate (5.3). This unique solution is denoted by V_N and $h = H_N$. \square

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