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# HOCHSCHILD AND ORDINARY COHOMOLOGY RINGS OF SMALL CATEGORIES

FEI XU

ABSTRACT. Let  $\mathcal{C}$  be a small category and  $k$  a field. There are two interesting mathematical subjects: the category algebra  $k\mathcal{C}$  and the classifying space  $|\mathcal{C}| = B\mathcal{C}$ . We study the ring homomorphism  $\mathrm{HH}^*(k\mathcal{C}) \rightarrow \mathrm{H}^*(|\mathcal{C}|, k)$  and prove it is split surjective, using the factorization category of Quillen [18] and certain techniques from functor cohomology theory. This generalizes the well-known theorems for groups and posets. Based on this result, we construct a seven-dimensional category algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated, disproving a conjecture of Snashall and Solberg [20].

KEYWORDS. Hochschild cohomology ring, ordinary cohomology ring, category algebra, category of factorizations, left Kan extension, finite EI-categories, finite generation, nilpotent element.

## 1. INTRODUCTION

Let  $\mathcal{C}$  be a small category,  $k$  a field and  $Vect_k$  the category of  $k$ -vector spaces. We denote by  $\mathrm{Ob}\mathcal{C}$  and  $\mathrm{Mor}\mathcal{C}$  the sets of objects and morphisms in  $\mathcal{C}$ , respectively. The category algebra  $k\mathcal{C}$  [22, 23] of  $\mathcal{C}$  is a  $k$ -vector space with basis equal to  $\mathrm{Mor}\mathcal{C}$ , and the multiplication is given by the composition of base elements (if two morphisms are not composable then the product is zero). Suppose  $Vect_k^{\mathcal{C}}$  is the category of all covariant functors from  $\mathcal{C}$  to  $Vect_k$  and  $k\mathcal{C}\text{-mod}$  is the category of left  $k\mathcal{C}$ -modules. Mitchell [17, Theorem 7.1] showed that there exists a full faithful functor

$$R : Vect_k^{\mathcal{C}} \rightarrow k\mathcal{C}\text{-mod},$$

defined by  $R(F) = \bigoplus_{x \in \mathrm{Ob}\mathcal{C}} F(x)$ . The functor  $R$  has a left inverse  $L : k\mathcal{C}\text{-mod} \rightarrow Vect_k^{\mathcal{C}}$  defined by  $M \mapsto F_M$  such that  $F_M(x) = 1_x \cdot M$ , where  $1_x$  is the identity in  $\mathrm{End}_{\mathcal{C}}(x)$  for each  $x \in \mathrm{Ob}\mathcal{C}$ . When  $\mathrm{Ob}\mathcal{C}$  is finite, the category algebra  $k\mathcal{C}$  has an identity  $1_{k\mathcal{C}} = \sum_{x \in \mathrm{Ob}\mathcal{C}} 1_x$ , and the above two functors provide an equivalence between the two abelian categories. If  $\mathcal{C}$  is a group (regarded as a category with one object), the equivalence simply gives us the fundamental correspondence between group modules and group representations. In the present article we shall investigate  $\mathrm{Ext}_{Vect_k^{\mathcal{C}}}^*(M, N) = \bigoplus_{i \geq 0} \mathrm{Ext}_{Vect_k^{\mathcal{C}}}^i(M, N)$  for various  $\mathcal{C}$  and functors  $M, N \in Vect_k^{\mathcal{C}}$ . Due to the existence of the above faithful functor  $R$ , every functor is a  $k\mathcal{C}$ -module. For simplicity, throughout this article we shall write the above Ext as  $\mathrm{Ext}_{k\mathcal{C}}^*(M, N)$ . Whenever we need to emphasize that a  $k\mathcal{C}$ -module  $M$  is indeed an object in  $Vect_k^{\mathcal{C}}$ , we say  $M$  is a functor in  $k\mathcal{C}\text{-mod}$ . Let  $\theta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  be a covariant functor between small categories. We use frequently the functor  $\mathrm{Res}_{\theta} : Vect_k^{\mathcal{C}_2} \rightarrow Vect_k^{\mathcal{C}_1}$ , which is

called the restriction along  $\theta$  (precomposition with  $\theta$ ). The functor  $\theta$  does not always induce an algebra homomorphism from  $k\mathcal{C}_1$  to  $k\mathcal{C}_2$  [23]. Hence it does not give rise to a functor  $k\mathcal{C}_2\text{-mod} \rightarrow k\mathcal{C}_1\text{-mod}$ . Despite this potential hole, in Section 2 we often write  $\text{Res}_\theta : k\mathcal{C}_2\text{-mod} \rightarrow k\mathcal{C}_1\text{-mod}$ , again for simplicity and consistency. As almost all modules we consider are functors, it will not cause any real problem.

Let  $\underline{k} \in k\mathcal{C}\text{-mod}$  be the constant functor, sending every object to  $k$  and every morphism to the identity. When  $\mathcal{C}$  is a group,  $\underline{k} = k$  becomes the trivial group module. For this reason, the functor  $\underline{k}$  is often called the trivial  $k\mathcal{C}$ -module, and it plays the role of trivial module for a group algebra. The ordinary cohomology ring of  $\mathcal{C}$  with coefficients in  $k$  can be defined as  $\text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k})$ , which is isomorphic to  $H^*(|\mathcal{C}|, k)$  [22, 23] and hence is graded commutative. Such an ordinary cohomology ring modulo nilpotents is not finitely generated in general, see for example [24].

Let  $\mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op}$ , where  $\mathcal{C}^{op}$  is the opposite category. The enveloping algebra of  $k\mathcal{C}$ ,  $(k\mathcal{C})^e = k\mathcal{C} \otimes_k (k\mathcal{C})^{op}$ , is naturally isomorphic to  $k\mathcal{C}^e$  as  $k$ -algebras. Hence in the present article we shall not distinguish the two algebras  $(k\mathcal{C})^e$  and  $k\mathcal{C}^e$ . By introducing  $\mathcal{C}^e$  and  $k\mathcal{C}^e$ , one can use functor cohomology theory to investigate Hochschild cohomology. We want to consider  $\text{Ext}_{k\mathcal{C}^e}^*(M, N)$ , where  $M, N \in k\mathcal{C}^e\text{-mod}$ . When  $M = N = k\mathcal{C}$ ,  $\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C})$  becomes a graded commutative ring [20]. If  $\text{Ob } \mathcal{C}$  is finite (thus  $k\mathcal{C}$  has an identity), one can identify the above ring with the Hochschild cohomology ring  $\text{HH}^*(k\mathcal{C})$  (see [6, Section 7] and [14, Chapter 1]). For this reason, we shall call  $\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C})$  the Hochschild cohomology ring of  $\mathcal{C}$  in the present article. We note that the module  $k\mathcal{C} \in k\mathcal{C}^e\text{-mod}$  comes from a functor  $\mathcal{C}^e \rightarrow \text{Vect}_k$  such that  $k\mathcal{C}(x, y) = k \text{Hom}_{\mathcal{C}}(y, x)$  for each  $(x, y) \in \text{Ob } \mathcal{C}^e$  (if  $\text{Hom}_{\mathcal{C}}(y, x) = \emptyset$  then we assume  $k\mathcal{C}(x, y) = 0$ ).

Suppose  $A$  is an associative  $k$ -algebra and  $A^e$  is its enveloping algebra. Let  $M$  be an  $A^e$ -module. Then one has a ring homomorphism induced by the tensor product  $- \otimes_A M$

$$\phi_M : \text{Ext}_{A^e}^*(A, A) \rightarrow \text{Ext}_A^*(M, M).$$

If we take  $A = k\mathcal{C}$  for a small category  $\mathcal{C}$  and  $M = \underline{k}$ , we get a ring homomorphism

$$\phi_{\mathcal{C}} : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}).$$

In this situation,  $\phi_{\mathcal{C}}$  is really induced by the projection functor  $pr : \mathcal{C}^e \rightarrow \mathcal{C}$  (see Section 2.3). The structures of these two cohomology rings and the homomorphism are the main subjects of our investigation. Note that we name the ring homomorphism  $\phi_{\mathcal{C}}$ , not  $\phi_{\underline{k}}$ , since we need to deal with various categories and  $\phi_{\underline{k}}$  can cause confusion. It is well-known that when  $\mathcal{C}$  is a group,  $\phi_{\mathcal{C}}$  is a split surjection (see for instance [2] or [12, 2.9]), whilst, when  $\mathcal{C}$  is a poset,  $\phi_{\mathcal{C}}$  is an isomorphism [7]. The two results are proved in completely different ways in the literature. In our article, we use functor cohomology theory to establish a general statement on the ring homomorphism  $\phi_{\mathcal{C}}$ , including the above two results as special cases. In order to deal with the general situation, we need to consider the category of factorizations in a category  $\mathcal{C}$ , introduced by Quillen [18]. The category of factorizations in  $\mathcal{C}$ ,  $F(\mathcal{C})$ , has all the morphisms in  $\mathcal{C}$  as its objects. If we write the objects in  $F(\mathcal{C})$  as  $[\alpha]$ , for any  $\alpha \in \text{Mor } \mathcal{C}$ , then there exists morphisms from  $[\alpha]$  to  $[\alpha']$  if  $\alpha$  factors through  $\alpha'$  in  $\text{Mor } \mathcal{C}$ . The category

$F(\mathcal{C})$  admits natural functors  $t$  and  $s$  into  $\mathcal{C}$  and  $\mathcal{C}^{op}$ , respectively, inducing homotopy equivalences of classifying spaces. One can assemble these two functors together to form a new functor  $\tau = (t, s) : F(\mathcal{C}) \rightarrow \mathcal{C}^e$ . Quillen observed that  $F(\mathcal{C})$  is cofibred over  $\mathcal{C}^e$  and described the fibres. Based on these, we prove the following statements (Theorem 2.3.1 and Proposition 2.3.5). We comment that Mac Lane [16] discussed the question for monoids in Section X.5 of his book and obtained part of the result (stated for homology).

**Theorem A** *Let  $\mathcal{C}$  be a small category and  $k$  a field. For any functor  $M \in k\mathcal{C}^e\text{-mod}$ , we have*

$$\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, M) \cong \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \text{Res}_\tau M),$$

where  $\text{Res}_\tau$  is the restriction along  $\tau : F(\mathcal{C}) \rightarrow \mathcal{C}^e$  (precomposition with  $\tau$ ). In particular we have

$$\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k}) \cong \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \cong \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}),$$

and  $\phi_{\mathcal{C}} : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k})$  is a split surjection, induced by the following decompositions  $\text{Res}_\tau(k\mathcal{C}) \cong \underline{k} \oplus N_{\mathcal{C}}$  and

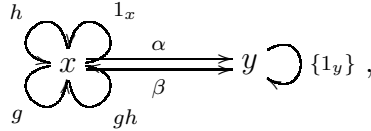
$$\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \cong \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}) \oplus \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, N_{\mathcal{C}}),$$

where  $N_{\mathcal{C}} \in kF(\mathcal{C})\text{-mod}$  as a functor takes the following value

$$N_{\mathcal{C}}([\alpha]) = k\{\beta - \gamma \mid \beta, \gamma \in \text{Hom}_{\mathcal{C}}(y, x)\},$$

if  $[\alpha] \in \text{Ob } F(\mathcal{C})$  and  $\alpha \in \text{Hom}_{\mathcal{C}}(y, x)$ .

Especially, the existence of a surjective homomorphism implies that if the ordinary cohomology ring, modulo nilpotents, is not finitely generated, neither is the Hochschild cohomology ring, modulo nilpotents. In [24] we computed the mod-2 ordinary cohomology ring of the following category  $\mathcal{E}_0$



where  $g^2 = h^2 = 1_x$ ,  $gh = hg$ ,  $\alpha h = \beta g = \alpha$ , and  $\alpha g = \beta h = \beta$ . It was shown there that its ordinary cohomology ring doesn't have any nilpotents and is not finitely generated. Thus its Hochschild cohomology ring modulo nilpotents is not finitely generated, providing a counterexample against the conjecture in [20]. We note that the category algebra  $k\mathcal{C}$  is not a self-injective algebra, in contrast to the fact that the Hochschild cohomology ring of a finite-dimensional cocommutative Hopf algebra, or of a finite-dimensional self-injective algebra of finite representation type, is finitely generated [5, 8] (a finite-dimensional Hopf algebra is always self-injective [11]). In this particular case, the category algebra is graded and is Koszul, which was brought attention to the author by Nicole Snashall.

A small category is called EI if every endomorphism is an isomorphism. A category is finite if the morphism set is finite. Typical examples of finite EI-categories

are posets and groups. The above category  $\mathcal{E}_0$  is finite EI as well. Some sophisticated finite EI-categories have been heavily used in, for example, the  $p$ -local finite group theory [1] and modular representation theory [21, Chapter 7]. Let  $\mathcal{C}$  be a finite EI-category. We can define a full subcategory  $\mathcal{A}_{\mathcal{C}} = \mathcal{A}$  such that  $\text{Ob } \mathcal{A} = \text{Ob } \mathcal{C}$  and  $\text{Mor } \mathcal{A}$  contains exactly all the isomorphisms in  $\text{Mor } \mathcal{C}$ . The category  $\mathcal{A}$  can be considered as the disjoint union of all finite groups in  $\mathcal{C}$ . The following is Theorem 2.4.2.

**Theorem B** Let  $\mathcal{C}$  be a finite EI-category and  $k$  a field. Then we have the following commutative diagram

$$\begin{array}{ccc} \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) & \xrightarrow{\phi_{k\mathcal{A}}} & \text{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A}) \\ \phi_{\mathcal{C}} \downarrow & & \downarrow \phi_{\mathcal{A}} \\ \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}) & \xrightarrow{\text{Res}_{\mathcal{C}, \mathcal{A}}} & \text{Ext}_{k\mathcal{A}}^*(\underline{k}, \underline{k}). \end{array}$$

Here  $\text{Res}_{\mathcal{C}, \mathcal{A}}$  is induced by the inclusion  $\iota : \mathcal{A} \hookrightarrow \mathcal{C}$ . In this theorem the category  $\mathcal{A}$  may be replaced by any full subcategory of it.

Our paper begins with a brief introduction to the ring homomorphisms from the Hochschild cohomology of an associative algebra to some relevant rings. Afterwards, we introduce the concept of an enveloping category and reinterpret the ring homomorphism using functor cohomology theoretic methods. Based on Quillen's work, we continue to prove  $\phi_{\mathcal{C}} : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k})$  is split surjective for any small category  $\mathcal{C}$ . Some consequences of this splitting surjection and further properties will be given. Finally, we end this paper with four examples. The first example provides a counter-example to a conjecture of Snashall and Solberg.

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## 2. HOCHSCHILD AND ORDINARY COHOMOLOGY RINGS OF CATEGORIES

We first describe the ring homomorphism from the Hochschild cohomology ring, of an associative algebra, to some relevant cohomology rings, induced by tensor products with modules. When the associative algebra is a category algebra and the target is the ordinary cohomology ring, we reconstruct the ring homomorphism, using a different method. Based on the alternative description, we show the ring homomorphism  $\phi_{\mathcal{C}}$  is split surjective.

### 2.1. The ring homomorphisms from the Hochschild cohomology ring.

**Definition 2.1.1.** *Let  $A$  be an associative  $k$ -algebra and  $M, N$  two  $A$ -modules. We write  $\text{Ext}_A^*(M, N) = \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N)$ .*

In general, if  $\Lambda$  and  $\Gamma$  are two associative  $k$ -algebras and  $M$  is a  $\Lambda \otimes_k \Gamma^{op}$ -module, or equivalently a  $\Lambda$ - $\Gamma$ -bimodule, we can define a ring homomorphism induced by the tensor product  $-\otimes_\Lambda M$

$$\phi_M : \text{Ext}_{\Lambda^e}^*(\Lambda, \Lambda) \rightarrow \text{Ext}_{\Lambda \otimes_k \Gamma^{op}}^*(M, M).$$

Let  $\mathcal{R}_* \rightarrow \Lambda \rightarrow 0$  be a projective resolution of the  $\Lambda^e$ -module  $\Lambda$ . The exact sequence is split if we regard it as a complex of right  $\Lambda$ -modules. Thus by tensoring  $M$  over  $\Lambda$  from the right, we obtain an exact sequence ending at the  $\Lambda \otimes_k \Gamma^{op}$ -module  $M$

$$\mathcal{R}_* \otimes_\Lambda M \rightarrow \Lambda \otimes_\Lambda M \cong M \rightarrow 0.$$

Now one can build a projective resolution of  $M$ ,  $\mathcal{R}'_* \rightarrow M \rightarrow 0$ , along with a chain map

$$\begin{array}{ccccc} \mathcal{R}'_* & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow = & & \\ \mathcal{R}_* \otimes_\Lambda M & \longrightarrow & \Lambda \otimes_\Lambda M & \longrightarrow & 0. \end{array}$$

This induces an algebra homomorphism  $\phi_M : \text{Ext}_{\Lambda^e}^*(\Lambda, \Lambda) \rightarrow \text{Ext}_{\Lambda \otimes_k \Gamma^{op}}^*(M, M)$ . If  $N$  is another  $\Lambda \otimes_k \Gamma^{op}$ -module, we see  $\text{Ext}_{\Lambda \otimes_k \Gamma^{op}}^*(M, N)$  has an  $\text{Ext}_{\Lambda^e}^*(\Lambda, \Lambda)$ -module structure via the ring homomorphisms  $\phi_M$  and  $\phi_N$  together with the Yoneda splice. We quote the following theorem of Snashall and Solberg [20].

**Theorem 2.1.2.** *Let  $\Lambda$  and  $\Gamma$  be two associative  $k$ -algebras. Let  $\eta$  be an element in  $\text{Ext}_{\Lambda^e}^n(\Lambda, \Lambda)$  and  $\theta$  an element in  $\text{Ext}_{\Lambda \otimes_k \Gamma^{op}}^m(M, N)$  for two  $\Lambda$ - $\Gamma$ -bimodules  $M$  and  $N$ . Then  $\phi_N(\eta)\theta = (-1)^{mn}\theta\phi_M(\eta)$ .*

When  $\Lambda$  has an identity, it means  $\text{Ext}_{\Lambda^e}^*(\Lambda, \Lambda) \cong \text{HH}^*(\Lambda)$  is a graded commutative ring, which was first proved by Gerstenhaber [6].

**2.2. Enveloping category of a small category.** Let  $\mathcal{C}$  be a small category. Quillen [18, page 94 Example] considered the category  $\mathcal{C}^{op} \times \mathcal{C}$ . We slightly modify it and give it a name, in order to be consistent with our investigation of the Hochschild cohomology.

**Definition 2.2.1.** *We call  $\mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op}$  the enveloping category of a small category  $\mathcal{C}$ .*

The following result is just a simple observation. It implies the enveloping algebra of a category algebra of  $\mathcal{C}$  is the category algebra of its enveloping category, so later on we will just use the terminology  $k\mathcal{C}^e$  when dealing with Hochschild cohomology. This identification enables us to apply functor cohomology theory to the investigation of the Hochschild cohomology theory of category algebras.

**Lemma 2.2.2.** *Let  $\mathcal{C}$  be a small category. There is a natural isomorphism  $k\mathcal{C}^e \cong (k\mathcal{C})^e$ . As a functor,  $k\mathcal{C}(x, y) = k \text{Hom}_{\mathcal{C}}(y, x)$  if  $\text{Hom}_{\mathcal{C}}(y, x) \neq \emptyset$  and  $k\mathcal{C}(x, y) = 0$  otherwise. Here  $(x, y) \in \text{Ob } \mathcal{C}^e$ .*

*Proof.* We define a map  $k\mathcal{C}^e \rightarrow (k\mathcal{C})^e$  on the natural base elements of  $k\mathcal{C}^e$  by  $(\alpha, \beta^{op}) \mapsto \alpha \otimes \beta^{op}$ ,  $\alpha, \beta \in \text{Mor } \mathcal{C}$ . It extends linearly to an algebra isomorphism.

If  $M$  is a  $k\mathcal{C}^e$ -module and  $m \in M$ , then  $(\alpha, \beta^{op}) \cdot m = \alpha \cdot m \cdot \beta$  and as a functor  $M : \mathcal{C}^e \rightarrow \text{Vect}_k$

$$M(x, y) = 1_{(x, y)} \cdot M = (1_x, 1_y^{op}) \cdot M = 1_x \cdot M \cdot 1_y,$$

on each object  $(x, y) \in \mathcal{C}^e$ . In particular,

$$k\mathcal{C}(x, y) = (1_x, 1_y^{op}) \cdot k\mathcal{C} = 1_x \cdot k\mathcal{C} \cdot 1_y = k\text{Hom}_{\mathcal{C}}(y, x)$$

if  $\text{Hom}_{\mathcal{C}}(y, x) \neq \emptyset$ , and  $k\mathcal{C}(x, y) = 0$  otherwise.  $\square$

Let  $\mathcal{C}$  be a small category. We recall Quillen's category  $F(\mathcal{C})$  of factorizations in  $\mathcal{C}$ . In his article [18], Quillen named this category  $S(\mathcal{C})$ . However since  $S(\mathcal{C})$  has been used to denote the subdivision of a small category  $\mathcal{C}$  [19, 13], we adopt Baues and Wirsching's terminology [3] which we believe is suitable. The category  $F(\mathcal{C})$  has the morphisms in  $\mathcal{C}$  as its objects. In order to avoid confusion, we write an object in  $F(\mathcal{C})$  as  $[\alpha]$ , whenever  $\alpha \in \text{Mor } \mathcal{C}$ . A morphism from  $[\alpha] \in \text{Ob } F(\mathcal{C})$  to  $[\alpha'] \in \text{Ob } F(\mathcal{C})$  is given by a pair of  $u, v \in \text{Mor } \mathcal{C}$ , making the following diagram commutative

$$\begin{array}{ccc} x & \xleftarrow{\alpha} & y \\ u \downarrow & & \downarrow v^{op} \\ x' & \xleftarrow{\alpha'} & y' \end{array}$$

In other words, there is a morphism from  $[\alpha]$  to  $[\alpha']$  if and only if  $\alpha' = u\alpha v$  for some  $u, v \in \text{Mor } \mathcal{C}$ , or equivalently  $\alpha$  is a factor of  $\alpha'$  in  $\text{Mor } \mathcal{C}$ . The category  $F(\mathcal{C})$  admits two natural covariant functors to  $\mathcal{C}$  and  $\mathcal{C}^{op}$

$$\mathcal{C} \xleftarrow{t} F(\mathcal{C}) \xrightarrow{s} \mathcal{C}^{op},$$

where  $t$  and  $s$  send an object  $[\alpha]$  to its target and source, respectively. Using his Theorem A and its corollary, Quillen showed these two functors induce homotopy equivalences of the classifying spaces. We will be interested in the functor

$$\tau = (t, s) : F(\mathcal{C}) \rightarrow \mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op},$$

sending an  $[\alpha] \in \text{Ob } F(\mathcal{C})$  to  $(x, y) \in \text{Ob } \mathcal{C}^e$  if  $\alpha \in \text{Hom}_{\mathcal{C}}(y, x)$  and a morphism  $(u, v^{op}) \in \text{Mor } F(\mathcal{C})$  to  $(u, v^{op}) \in \text{Mor } (\mathcal{C}^e)$ .

The importance of the functor  $\tau : F(\mathcal{C}) \rightarrow \mathcal{C}^e$  lies in the fact that its target category gives rise to the Hochschild cohomology ring of  $\mathcal{C}$ , while its source category determines the ordinary cohomology ring of  $\mathcal{C} \simeq F(\mathcal{C})$ . In the situation of (finite) posets and groups, the functor is well-understood and in the group case it has been implicitly used to establish the homomorphism from the Hochschild cohomology ring to the ordinary cohomology ring.

**Example 2.2.3.** (1) When  $\mathcal{C}$  is a poset,  $\tau : F(\mathcal{C}) \rightarrow \mathcal{C}^e$  sends  $F(\mathcal{C})$  isomorphically onto a full category  $\mathcal{C}_{\Delta}^e \subset \mathcal{C}^e$ , where

$$\text{Ob } \mathcal{C}_{\Delta}^e = \{(x, y) \in \text{Ob } \mathcal{C}^e \mid \text{Hom}_{\mathcal{C}}(y, x) \neq \emptyset\}$$

(the full subcategory  $\mathcal{C}_\Delta^e$  is well-defined whenever  $\mathcal{C}$  is EI, see Section 2.4). One can easily see that  $k\mathcal{C}$  as a functor only takes non-zero values at objects in  $\text{Ob } \mathcal{C}_\Delta^e$ . Furthermore as a  $k\mathcal{C}_\Delta^e$ -module,  $k\mathcal{C} \cong \underline{k}$  is the trivial module by Lemma 2.2.2. Since  $\mathcal{C}_\Delta^e \cong F(\mathcal{C})$  is a co-ideal in the poset  $\mathcal{C}^e$ , we obtain  $\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \cong \text{Ext}_{k\mathcal{C}_\Delta^e}^*(k\mathcal{C}, k\mathcal{C}) \cong \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \cong \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k})$ , where the last isomorphism comes from the fact that  $|F(\mathcal{C})| \simeq |\mathcal{C}|$ . This isomorphism between the two cohomology rings was first established in [7];

- (2) When  $\mathcal{C}$  is a group, the category  $F(\mathcal{C})$  is a groupoid and is equivalent to a subcategory of the one object category  $\mathcal{C}^e$  with morphism set

$$\{(g, g^{-1\text{op}}) \mid g \in \text{Mor } \mathcal{C}\} \subset \text{Mor } \mathcal{C}^e.$$

Based on this description, one can prove the existence of the surjective homomorphism from the Hochschild cohomology ring to the ordinary cohomology ring of a group, which is basically the same as the classical approach. See for example [2].

**2.3. The main theorem.** In order to deal with the general situation, we need to recall the definition of an overcategory. It is used to define and understand the left Kan extension, which generalizes the concept of an induction.

Let  $\theta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  be a covariant functor between small categories. For each  $z \in \text{Ob } \mathcal{C}_2$ , the overcategory  $\theta/z$  consists of objects  $(x, \alpha)$ , where  $x \in \text{Ob } \mathcal{C}_1$  and  $\alpha \in \text{Hom}_{\mathcal{C}_2}(\theta(x), z)$ . A morphism from  $(x, \alpha)$  to  $(x', \alpha')$  is a morphism  $\beta \in \text{Hom}_{\mathcal{C}_1}(x, x')$  such that  $\alpha = \alpha'\theta(\beta)$ . Let  $\text{Res}_\theta : k\mathcal{C}_2\text{-mod} \rightarrow k\mathcal{C}_1\text{-mod}$  be the restriction on functors along  $\theta$  (precomposition with  $\theta$ ). The left adjoint of  $\text{Res}_\theta$  is called the left Kan extension  $LK_\theta : k\mathcal{C}_1\text{-mod} \rightarrow k\mathcal{C}_2\text{-mod}$  and is defined by

$$LK_\theta(M)(z) = \varinjlim_{\theta/z} M \circ \pi,$$

where  $z \in \text{Ob } \mathcal{C}_2$ ,  $\pi : \theta/z \rightarrow \mathcal{C}_1$  is the projection functor  $(x, \alpha) \mapsto x$  and  $M$  is a functor in  $k\mathcal{C}_1\text{-mod}$ . When  $\mathcal{C}_2$  is a subgroup of a group  $\mathcal{C}_1$  and  $\theta$  is the inclusion, the left Kan extension is the usual induction, i.e.  $LK_\theta(M) \cong k\mathcal{C}_2 \otimes_{k\mathcal{C}_1} M$ .

With the definition of an overcategory, one can continue to define two functors  $\theta/? : \mathcal{C}_2 \rightarrow \text{sCat}$  (the category of small categories), and  $C_*(\theta/?) : \mathcal{C}_2 \rightarrow k\mathcal{C}_2\text{-Cplx}$  (the category of complexes of  $k\mathcal{C}_2$ -modules). For each  $x \in \text{Ob } \mathcal{C}_2$ ,  $C_*(\theta/x)$  is the simplicial complex coming from the nerve of the small category  $\theta/x$ . When  $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$  and  $\theta = \text{Id}_\mathcal{C}$ , we have functors  $\text{Id}_\mathcal{C}/?$  and  $C_*(\text{Id}_\mathcal{C}/?)$ . It is well-known that the latter can be used to define a projective resolution of the  $k\mathcal{C}$ -module  $\underline{k} : C_*(\text{Id}_\mathcal{C}/?) \rightarrow \underline{k} \rightarrow 0$ . For each  $n \geq 0$ ,  $C_n(\text{Id}_\mathcal{C}/?) : \mathcal{C} \rightarrow k\mathcal{C}\text{-Cplx}$  is the functor sending each  $x \in \text{Ob } \mathcal{C}$  to the vector space whose basis is the set of all  $n$ -chains of morphisms in  $\text{Id}_\mathcal{C}/x$ . The differential, a  $k\mathcal{C}$ -map,  $\sigma^n : C_n(\text{Id}_\mathcal{C}/?) \rightarrow C_{n-1}(\text{Id}_\mathcal{C}/?)$  is defined as follows. For each  $x \in \text{Ob } \mathcal{C}$ ,

$$\begin{aligned} & \sigma_x^n((x_0, \alpha_0) \rightarrow \cdots \rightarrow (x_i, \alpha_i) \rightarrow \cdots \rightarrow (x_n, \alpha_n)) \\ &= \sum_{i=0}^n (-1)^i [(x_0, \alpha_0) \rightarrow \cdots \rightarrow \widehat{(x_i, \alpha_i)} \rightarrow \cdots \rightarrow (x_n, \alpha_n)], \end{aligned}$$

where  $\alpha_i \in \text{Hom}_\mathcal{C}(x_i, x)$ . Let  $\theta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  be a covariant functor. There is an isomorphism of complexes of projective  $k\mathcal{C}_2$ -modules (a left Kan extension always



preserves projectives)

$$LK_\theta(C_*(\text{Id}_{\mathcal{C}_1}/?)) \cong C_*(\theta/?),$$

which can be found for example in Hollender-Vogt [10, 4.3]. Under certain conditions, the above complex may be a projective resolution of the  $k\mathcal{C}_2$ -module  $LK_\theta(\underline{k})$ . This is the key to our future investigation.

We want to discuss the left Kan extensions of the functors  $\tau$ ,  $t$  and  $pr$  in the following commutative diagram of small categories

$$\begin{array}{ccc} F(\mathcal{C}) & \xrightarrow{\tau} & \mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op} \\ & \searrow t & \swarrow pr \\ & \mathcal{C} & \end{array},$$

where  $pr$  is the projection onto the first component. Since  $t = pr \circ \tau$ , we have

$$LK_t \cong LK_{pr} \circ LK_\tau.$$

In the rest of this section, we will establish and describe the following ring homomorphisms, induced by the three left Kan extensions  $LK_t$ ,  $LK_{pr}$  and  $LK_\tau$  respectively,

$$\begin{aligned} t^* & : \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) & \rightarrow & \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}), \\ pr^* & : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) & \rightarrow & \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}) \\ \tau^* & : \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) & \rightarrow & \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}). \end{aligned}$$

The first two homomorphisms are not difficult to describe and we do it now. The homomorphism  $t^*$  is an isomorphism since  $t$  induces a homotopy equivalence of  $F(\mathcal{C})$  and  $\mathcal{C}$  by [18]. More explicitly, let  $C_*(\text{Id}_{F(\mathcal{C})}/?) \rightarrow \underline{k} \rightarrow 0$  be the projective resolution of the  $kF(\mathcal{C})$ -module  $\underline{k}$ . The left Kan extension of  $t$ ,  $LK_t$ , sends it to a projective resolution of the  $k\mathcal{C}$ -module  $\underline{k}$

$$LK_t(C_*(\text{Id}_{F(\mathcal{C})}/?)) \cong C_*(t/?) \rightarrow LK_t(\underline{k}) \cong \underline{k} \rightarrow 0.$$

The reason is that first of all,  $C_*(t/?)$  is a complex of projective  $k\mathcal{C}$ -modules, and second of all, for each  $x \in \text{Ob}\mathcal{C}$ ,  $t/x$  is contractible [18] and thus  $C_*(t/x)$  is exact except having homology  $k$  at the end.

The homomorphism  $pr^*$ , induced by  $pr$ , is exactly  $\phi_{\mathcal{C}}$ , defined earlier, which is induced by tensoring over  $k\mathcal{C}$  with  $\underline{k}$  from the right. We see this from the fact that  $LK_{pr}$  is exactly the tensor product  $- \otimes_{k\mathcal{C}} \underline{k}$  on a projective resolution of the  $k\mathcal{C}^e$ -module  $k\mathcal{C}$ . In fact for each  $x \in \text{Ob}\mathcal{C}$  since  $pr/x \cong (\text{Id}_{\mathcal{C}}/x) \times \mathcal{C}^{op}$ ,

$$LK_{pr}(k\mathcal{C}^e)(x) = \varinjlim_{pr/x} k\mathcal{C}^e \cong \varinjlim_{\text{Id}_{\mathcal{C}}/x} (k\mathcal{C}) \otimes_k \varinjlim_{\mathcal{C}^{op}} (k\mathcal{C}^{op}) \cong 1_x \cdot k\mathcal{C} \otimes_k \underline{k}.$$

It implies  $LK_{pr}(k\mathcal{C}^e) \cong k\mathcal{C} \otimes_k \underline{k} \cong k\mathcal{C}^e \otimes_{k\mathcal{C}} \underline{k}$ . Also we have

$$LK_{pr}(k\mathcal{C}) \cong LK_{pr}(LK_\tau(\underline{k})) \cong LK_t(\underline{k}) \cong \underline{k}.$$

Now we turn to investigate  $LK_\tau$  and  $\tau^*$ . Our goal is to use  $\tau^*$  and  $t^*$  to interpret  $pr^* = \phi_{\mathcal{C}}$ . The main result in this section is as follows.

**Theorem 2.3.1.** *Let  $\mathcal{C}$  be a small category and  $k$  a field. There exists a ring homomorphism*

$$\epsilon^* : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k})$$

*such that  $\epsilon^*\tau^* \cong 1$ . Moreover the following composition  $t^*\epsilon^*$  is a split surjection*

$$\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \xrightarrow{\epsilon^*} \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \xrightarrow{t^*} \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}),$$

*with the property that  $t^*\epsilon^* \cong pr^* \cong \phi_{\mathcal{C}}$ .*

The proof of this theorem will be divided into three lemmas. We first discuss the action of  $LK_{\tau}$  on a certain projective resolution of the  $kF(\mathcal{C})$ -module  $\underline{k}$ . In his example on page 94 of [18], Quillen asserted that the category  $F(\mathcal{C})$  is a cofibred category over  $\mathcal{C}^e$ , via  $\tau$ , with discrete fibres defined by the functor  $(x, y) \mapsto \text{Hom}_{\mathcal{C}}(y, x)$ , where  $(x, y) \in \text{Ob } \mathcal{C}^e$ . As a consequence of the assertion Quillen indicated that each overcategory  $\tau/(x, y)$  is homotopy equivalent to the fibre  $\tau^{-1}(x, y)$ , which is the discrete category  $\text{Hom}_{\mathcal{C}}(y, x)$ . Hence the left Kan extension of  $\underline{k}$  takes the following value at each object  $(x, y)$

$$LK_{\tau}(\underline{k})(x, y) = \varinjlim_{\tau/(x, y)} \underline{k} \cong \text{H}_0(|\tau/(x, y)|, k) \cong \text{H}_0(|\tau^{-1}(x, y)|, k),$$

which equals  $k\text{Hom}_{\mathcal{C}}(y, x)$  if  $\text{Hom}_{\mathcal{C}}(y, x) \neq \emptyset$  and zero otherwise. It implies  $LK_{\tau}(\underline{k}) \cong k\mathcal{C}$  as  $k\mathcal{C}^e$ -modules. Further more, the following lemma implies  $LK_{\tau}(C_*(\text{Id}_{F(\mathcal{C})}/?)) \rightarrow LK_{\tau}(\underline{k}) \cong k\mathcal{C} \rightarrow 0$  is indeed a projective resolution.

**Lemma 2.3.2.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two small categories and  $\theta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  a covariant functor. If  $\theta/w$  is a discrete category for every  $w \in \text{Ob } \mathcal{C}_2$ , then we obtain a projective resolution of the  $k\mathcal{C}_2$ -module  $LK_{\theta}(\underline{k}) \cong \text{H}_0(|\theta/?|, k)$*

$$LK_{\theta}(C_*(\text{Id}_{\mathcal{C}_1}/?)) \cong C_*(\theta/?)) \rightarrow LK_{\theta}(\underline{k}) \rightarrow 0.$$

*Proof.* Evaluating  $C_*(\theta/?)$  at an object  $w \in \text{Ob } \mathcal{C}_2$ , one gets a complex  $C_*(\theta/w)$  that computes the homology of  $|\theta/w|$  with coefficients in  $k$ . Thus if  $\theta/w$  is a discrete category, we get an exact sequence

$$LK_{\theta}(C_*(\text{Id}_{\mathcal{C}_1}/w)) \cong C_*(\theta/w) \rightarrow LK_{\theta}(\underline{k})(w) \cong \text{H}_0(|\theta/w|, k) \rightarrow 0.$$

If  $\theta/w$  is a discrete category for every  $w \in \text{Ob } \mathcal{C}_2$ , then we obtain a projective resolution of the  $k\mathcal{C}_2$ -module  $LK_{\theta}(\underline{k})$

$$LK_{\theta}(C_*(\text{Id}_{\mathcal{C}_1}/?)) \cong C_*(\theta/?)) \rightarrow LK_{\theta}(\underline{k}) \rightarrow 0,$$

because it's exact and meanwhile the left Kan extension preserves projectives.  $\square$

Since  $LK_{\theta}$  is the left adjoint of  $\text{Res}_{\theta}$ , there are natural transformations  $\text{Id} \rightarrow \text{Res}_{\theta} LK_{\theta}$  and  $LK_{\theta} \text{Res}_{\theta} \rightarrow \text{Id}$ . We pay attention to the case of  $\tau : F(\mathcal{C}) \rightarrow \mathcal{C}^e$ . There exists a  $kF(\mathcal{C})$ -homomorphism  $\underline{k} \rightarrow \text{Res}_{\tau} LK_{\tau}(\underline{k}) = \text{Res}_{\tau}(k\mathcal{C})$  as well as a  $k\mathcal{C}^e$ -homomorphism  $k\mathcal{C} = LK_{\tau} \text{Res}_{\tau}(\underline{k}) \rightarrow \underline{k}$ . The latter gives rise to a  $kF(\mathcal{C})$ -homomorphism  $\text{Res}_{\tau}(k\mathcal{C}) = \text{Res}_{\tau} LK_{\tau} \text{Res}_{\tau}(\underline{k}) \rightarrow \underline{k} = \text{Res}_{\tau} \underline{k}$ . In case  $\mathcal{C}$  is a poset, one has  $\underline{k} = \text{Res}_{\tau}(k\mathcal{C})$ . When  $\mathcal{C}$  is a group,  $F(\mathcal{C})$  is a groupoid, equivalent to the automorphism group of  $[1_{\mathcal{C}}] \in \text{Ob } F(\mathcal{C})$ , that is,  $\{(g, g^{-1op}) \mid g \in \text{Mor } \mathcal{C}\}$ . If we name the full subcategory of  $F(\mathcal{C})$ , consisting of one object  $[1_{\mathcal{C}}]$ , by  $\tilde{\Delta}\mathcal{C}$  and the inclusion (an

equivalence) by  $i : \tilde{\Delta}\mathcal{C} \hookrightarrow F(\mathcal{C})$ . Then  $\text{Res}_{\tau_i}(k\mathcal{C}) = \text{Res}_{\tau}(k\mathcal{C})([1_{\mathcal{C}}])$  is a  $k\tilde{\Delta}\mathcal{C}$ -module with the action  $(g, g^{-1\text{op}}) \cdot a = gag^{-1}$ ,  $a \in \text{Res}_{\tau_i}(k\mathcal{C})$ . Thus  $\text{Res}_{\tau_i}(k\mathcal{C}) = \bigoplus kc_g$ , where  $c_g$  is the conjugacy class of  $g \in \text{Mor } \mathcal{C}$ . In particular  $\underline{k} = kc_{1_{\mathcal{C}}}$  is a direct summand of  $\text{Res}_{\tau_i}(k\mathcal{C})$  and it implies  $\underline{k} \mid \text{Res}_{\tau}(k\mathcal{C})$  as  $kF(\mathcal{C})$ -modules because  $i$  is an equivalence of categories.

**Lemma 2.3.3.** *Let  $\mathcal{C}$  be a small category. Then  $\underline{k} \mid \text{Res}_{\tau}(k\mathcal{C})$  as  $kF(\mathcal{C})$ -modules.*

*Proof.* One needs to keep in mind that the restriction of a module usually has a large  $k$ -dimension than the module itself since  $\tau$  is not injective on objects. We define a  $kF(\mathcal{C})$ -homomorphism (a natural transformation)  $\iota : \underline{k} \rightarrow \text{Res}_{\tau}(k\mathcal{C})$  by the assignments  $\iota_{[\alpha]}(1_k) = \alpha \in \text{Res}_{\tau}(k\mathcal{C})([\alpha])$  for each  $[\alpha] \in \text{Ob } F(\mathcal{C})$ . If  $[\beta]$  is another object in  $\text{Ob } F(\mathcal{C})$  and  $(u, v^{\text{op}}) \in \text{Hom}_{F(\mathcal{C})}([\alpha], [\beta])$  is an arbitrary morphism, then by the definition of an  $F(\mathcal{C})$ -morphism,  $(u, v^{\text{op}}) \cdot \alpha = u\alpha v = \beta$ . Hence  $\iota$  maps  $\underline{k}$  isomorphically onto a submodule of  $\text{Res}_{\tau}(k\mathcal{C})$ . On the other hand, we may define a  $kF(\mathcal{C})$ -homomorphism  $\epsilon : \text{Res}_{\tau}(k\mathcal{C}) \rightarrow \underline{k}$  such that, for any  $[\alpha] \in \text{Ob } F(\mathcal{C})$ ,  $\epsilon_{[\alpha]} : \text{Res}_{\tau}(k\mathcal{C})([\alpha]) \rightarrow \underline{k}([\alpha]) = k$  sends each base element in  $\text{Res}_{\tau}(k\mathcal{C})([\alpha]) = k \text{Hom}_{\mathcal{C}}(y, x)$  to  $1_k$ . One can readily check the composite of these two maps is the identity

$$\underline{k} \xrightarrow{\iota} \text{Res}_{\tau}(k\mathcal{C}) \xrightarrow{\epsilon} \underline{k},$$

and this means  $\underline{k} \mid \text{Res}_{\tau}(k\mathcal{C})$  or  $\text{Res}_{\tau}(k\mathcal{C}) = \underline{k} \oplus N_{\mathcal{C}}$  for some  $kF(\mathcal{C})$ -module  $N_{\mathcal{C}}$ .  $\square$

The module  $N_{\mathcal{C}}$  as a functor can be described by

$$N_{\mathcal{C}}([\alpha]) = k\{\beta - \gamma \mid \beta, \gamma \in \text{Hom}_{\mathcal{C}}(y, x)\},$$

if  $[\alpha] \in \text{Ob } F(\mathcal{C})$  and  $\alpha \in \text{Hom}_{\mathcal{C}}(y, x)$ . It will be useful to our computation since it determines the “difference” between the Hochschild and ordinary cohomology rings of a category. The next lemma finishes off our proof of the main theorem.

**Lemma 2.3.4.** *Let  $\mathcal{C}$  be a small category. There is a surjective ring homomorphism  $\epsilon^*$*

$$\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}),$$

such that  $\epsilon^*\tau^* \cong 1$  and  $pr^* \cong t^*\epsilon^*$ .

*Proof.* By Quillen’s observation [18], we know every overcategory  $\tau/(x, y)$  has the homotopy type of  $\text{Hom}_{\mathcal{C}}(y, x)$ . Applying Lemma 2.3.2 to  $\tau : F(\mathcal{C}) \rightarrow \mathcal{C}^e$ , we know the left Kan extension  $LK_{\tau}$  sends a certain projective resolution  $\mathcal{P}_*$  of the  $kF(\mathcal{C})$ -module  $\underline{k}$  to a projective resolution  $LK_{\tau}(\mathcal{P}_*)$  of the  $k\mathcal{C}^e$ -module  $k\mathcal{C}$ . Then on the cochain level we see  $\tau^*$  is determined by the following composition.

$$\text{Hom}_{kF(\mathcal{C})}(\mathcal{P}_*, \underline{k}) \rightarrow \text{Hom}_{k\mathcal{C}^e}(LK_{\tau}(\mathcal{P}_*), LK_{\tau}(\underline{k})) \cong \text{Hom}_{kF(\mathcal{C})}(\mathcal{P}_*, \text{Res}_{\tau}LK_{\tau}(\underline{k})).$$

Lemma 2.3.3 says  $\text{Res}_{\tau}LK_{\tau}(\underline{k}) = \underline{k} \oplus N_{\mathcal{C}}$  for some  $kF(\mathcal{C})$ -module  $N_{\mathcal{C}}$ . As a consequence, we have a split exact sequence of  $k$ -vector spaces

$$0 \rightarrow \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \hookrightarrow \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \cong \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k} \oplus N_{\mathcal{C}}) \rightarrow \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \rightarrow 0.$$

The leftmost map is  $\tau^*$  and the rightmost map is named  $\epsilon^*$ , induced by  $\epsilon$  in Lemma 2.3.3, and is given by

$$\mathrm{Hom}_{k\mathcal{C}^e}(LK_\tau(\mathcal{P}_*), LK_\tau(\underline{k})) \cong \mathrm{Hom}_{kF(\mathcal{C})}(\mathcal{P}_*, \mathrm{Res}_\tau LK_\tau(\underline{k})) \rightarrow \mathrm{Hom}_{kF(\mathcal{C})}(\mathcal{P}_*, \underline{k}).$$

From here, we can see  $pr^* \cong t^*\epsilon^*$  because of the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{k\mathcal{C}^e}(LK_\tau(\mathcal{P}_*), LK_\tau(\underline{k})) & \xrightarrow{\epsilon^*} & \mathrm{Hom}_{kF(\mathcal{C})}(\mathcal{P}_*, \underline{k}) \\ pr^* \downarrow & & \downarrow t^* \\ \mathrm{Hom}_{k\mathcal{C}}(LK_{pr}LK_\tau(\mathcal{P}_*), LK_{pr}LK_\tau(\underline{k})) & \xrightarrow{\cong} & \mathrm{Hom}_{k\mathcal{C}}(LK_t(\mathcal{P}_*), LK_t(\underline{k})). \end{array}$$

Finally we show  $\epsilon^*$  is a ring homomorphism. Since  $\underline{k} = \mathrm{Res}_\tau \underline{k}$ , we get

$$\mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \cong \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \mathrm{Res}_\tau \underline{k}) \cong \mathrm{Ext}_{k\mathcal{C}^e}^*(LK_\tau(\underline{k}), \underline{k}) \cong \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k}).$$

It implies the cup product in the Hochschild cohomology ring

$$\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \otimes_k \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \xrightarrow{\smile} \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) = \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C} \otimes_{k\mathcal{C}} k\mathcal{C})$$

is compatible with the cup product in the ordinary cohomology ring since we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \otimes_k \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) & \xrightarrow{\smile} & \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C} \otimes_{k\mathcal{C}} k\mathcal{C}) = \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \\ \epsilon^* \otimes_k \epsilon^* \downarrow & & \downarrow \epsilon^* \\ \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k}) \otimes_k \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k}) & \xrightarrow{\smile} & \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k} \otimes_{k\mathcal{C}} \underline{k}) = \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k}). \end{array}$$

Thus

$$\epsilon^* : \mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k})$$

is a left inverse of  $\tau^*$ . □

From the proof of last lemma, we have

$$\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, M) \cong \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \mathrm{Res}_\tau M)$$

for any functor  $M \in k\mathcal{C}^e\text{-mod}$ . This is not necessarily true for any  $M \in k\mathcal{C}^e\text{-mod}$  as  $\tau : F(\mathcal{C}) \rightarrow \mathcal{C}^e$  does not always induce an algebra homomorphism hence the restriction on  $M$  may not make sense. Together with our earlier discussion, we have the following formula for computation. Since we showed  $\mathrm{Res}_\tau(k\mathcal{C}) = T \oplus N_{\mathcal{C}}$  with  $T \cong \underline{k}$ , we may use the decomposition to compute the Hochschild cohomology ring when the structure of  $N_{\mathcal{C}}$  is understood.

**Proposition 2.3.5.** *Let  $\mathcal{C}$  be a small category and  $k$  a field. For any functor  $M \in k\mathcal{C}^e\text{-mod}$ , we have*

$$\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, M) \cong \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \mathrm{Res}_\tau M).$$

*In particular we have*

$$\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, \underline{k}) \cong \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \cong \mathrm{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}),$$

*and*

$$\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \cong \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \underline{k}) \oplus \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, N_{\mathcal{C}}) \cong \mathrm{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}) \oplus \mathrm{Ext}_{kF(\mathcal{C})}^*(\underline{k}, N_{\mathcal{C}}),$$

where  $N_{\mathcal{C}}$  is the submodule of  $\text{Res}_{\tau}(k\mathcal{C}) \in kF(\mathcal{C})\text{-mod}$  which as a functor takes the following value

$$N_{\mathcal{C}}([\alpha]) = k\{\beta - \gamma \mid \beta, \gamma \in \text{Hom}_{\mathcal{C}}(y, x)\},$$

if  $[\alpha] \in \text{Ob } F(\mathcal{C})$  and  $\alpha \in \text{Hom}_{\mathcal{C}}(y, x)$ .

Note that when  $\mathcal{C}$  is a finite abelian group, we obtain Holm's isomorphism [9, 4]

$$\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \cong \text{Ext}_{kF(\mathcal{C})}^*(\underline{k}, \text{Res}_{\tau}(k\mathcal{C})) \cong k\mathcal{C} \otimes_k \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}).$$

In Section 3 we will compute some further examples of Hochschild cohomology rings, using the above formula.

**2.4. EI-categories.** A small category is EI if every endomorphism is an isomorphism, and is finite if the morphism set is finite. The reader is referred to [22, 23] for a general description of the representation and ordinary cohomology theory of finite EI-categories. In this subsection we always assume  $\mathcal{C}$  is a finite EI-category. The finiteness condition implies all  $k\mathcal{C}$ -modules are functors, while the EI-condition implies that  $x \cong x'$  in  $\text{Ob } \mathcal{C}$  if both  $\text{Hom}_{\mathcal{C}}(x, x')$  and  $\text{Hom}_{\mathcal{C}}(x', x)$  are non-empty. The EI-condition allows us to give a partial order on the set of isomorphism classes of objects in  $\text{Ob } \mathcal{C}$  and hence a natural filtration to each functor in  $k\mathcal{C}\text{-mod}$  with respect to the partial order. The simple and (finitely generated) projective  $k\mathcal{C}$ -modules have been classified by Lück [15].

For future reference, we quote the following result [23]: let  $\mathcal{C}$  be a finite EI-category and  $M, N \in k\mathcal{C}\text{-mod}$ . An object  $x \in \text{Ob } \mathcal{C}$  is called  $M$ -minimal if  $M(x) \neq 0$  and there is no object  $y \in \text{Ob } \mathcal{C}$  such that  $\text{Hom}_{\mathcal{C}}(y, x) \neq \emptyset$  and  $M(y) \neq 0$ . If the  $M$ -minimal objects are  $x_1, \dots, x_n \in \text{Ob } \mathcal{C}$ , and  $X_M$  is the full subcategory of  $\mathcal{C}$  consisting of all  $M$ -minimal objects, then

$$\text{Ext}_{k\mathcal{C}}^*(M, N) \cong \text{Ext}_{kX_M}^*(M, N),$$

given that  $N$  as a functor takes non-zero values only at objects in  $X_M$ . This isomorphism will be used in this subsection as well as in the next section where we compute some Hochschild cohomology rings.

Suppose  $\mathcal{A}$  is the full subcategory of  $\mathcal{C}$  which consists of all objects and all isomorphisms in  $\mathcal{C}$ . The category  $\mathcal{A}$  is a disjoint union of finitely many finite groups. Its category algebra  $k\mathcal{A} = \bigoplus_{x \in \text{Ob } \mathcal{C}} k \text{Aut}_{\mathcal{C}}(x)$  is a  $k\mathcal{C}^e$ -module, and is a quotient of  $k\mathcal{C}$ , with kernel written as  $\ker$ . Considered as a functor  $\ker \subset k\mathcal{C}$  takes non-zero values at  $(x, y)$  for which there exists a  $\mathcal{C}$ -morphism from  $y$  to  $x$  and  $x \not\cong y$ .

The short exact sequence of  $k\mathcal{C}^e$ -modules

$$0 \rightarrow \ker \rightarrow k\mathcal{C} \xrightarrow{\pi} k\mathcal{A} \rightarrow 0$$

induces a long exact sequence

$$\dots \rightarrow \text{Ext}_{k\mathcal{C}^e}^n(k\mathcal{C}, \ker) \rightarrow \text{Ext}_{k\mathcal{C}^e}^n(k\mathcal{C}, k\mathcal{C}) \xrightarrow{\tilde{\pi}} \text{Ext}_{k\mathcal{C}^e}^n(k\mathcal{C}, k\mathcal{A}) \xrightarrow{\eta} \text{Ext}_{k\mathcal{C}^e}^{n+1}(k\mathcal{C}, \ker) \rightarrow \dots$$

By the previously quoted result from [23], one can see  $\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{A})$  is naturally isomorphic to

$$\text{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A}),$$

which is isomorphic to the direct sum of the Hochschild cohomology rings of the automorphism groups of objects in  $\mathcal{C}$ :  $\bigoplus_{x \in \text{Ob } \mathcal{C}} \text{Ext}_{k \text{Aut}_{\mathcal{C}}(x)}^*(k \text{Aut}_{\mathcal{C}}(x), k \text{Aut}_{\mathcal{C}}(x))$ . The following map will still be written as  $\tilde{\pi}$

$$\tilde{\pi} : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A}).$$

We show  $\tilde{\pi}$  can be identified with the algebra homomorphism induced by  $- \otimes_{k\mathcal{C}} k\mathcal{A}$

$$\phi_{k\mathcal{A}} : \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) \rightarrow \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{A}, k\mathcal{A}) \cong \text{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A}).$$

Hence we do not need to distinguish the maps  $\phi_{k\mathcal{A}}$  and  $\tilde{\pi}$ .

**Lemma 2.4.1.** *The following diagram is commutative*

$$\begin{array}{ccc} \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) & \xrightarrow{\tilde{\pi}} & \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{A}) \\ \phi_{k\mathcal{A}} \downarrow & & \downarrow \cong \\ \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{A}, k\mathcal{A}) & \xrightarrow{\cong} & \text{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A}). \end{array}$$

*Proof.* This can be seen on the cochain level. Suppose  $\mathcal{R}_* \rightarrow k\mathcal{C} \rightarrow 0$  is the minimal projective resolution of the  $k\mathcal{C}^e$ -module  $k\mathcal{C}$ . Then  $\text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C})$  is the homology of the cochain complex  $\text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_*, k\mathcal{C})$ . The tensor product  $- \otimes_{k\mathcal{C}} k\mathcal{A}$  induces a map

$$\text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_*, k\mathcal{C}) \rightarrow \text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_* \otimes_{k\mathcal{C}} k\mathcal{A}, k\mathcal{C} \otimes_{k\mathcal{C}} k\mathcal{A}) \cong \text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_* \otimes_{k\mathcal{C}} k\mathcal{A}, k\mathcal{A}),$$

which gives rise to  $\phi_{k\mathcal{A}}$ . On the other hand  $\tilde{\pi}$  is given by

$$\text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_*, k\mathcal{C}) \rightarrow \text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_*, k\mathcal{A}) \cong \text{Hom}_{k\mathcal{A}^e}(\text{Res}_{\mathcal{C}, \mathcal{A}}(\mathcal{R}_*), k\mathcal{A}),$$

where  $\text{Res}_{\mathcal{C}, \mathcal{A}}(\mathcal{R}_*)$  is the restriction of  $\mathcal{R}_*$  along the inclusion  $\mathcal{A} \hookrightarrow \mathcal{C}$  and is the minimal projective resolution of the  $k\mathcal{A}^e$ -module  $k\mathcal{A}$ . But

$$\text{Hom}_{k\mathcal{C}^e}(\mathcal{R}_* \otimes_{k\mathcal{C}} k\mathcal{A}, k\mathcal{A}) \cong \text{Hom}_{k\mathcal{A}^e}(\mathcal{R}_* \otimes_{k\mathcal{C}} k\mathcal{A}, k\mathcal{A}) \cong \text{Hom}_{k\mathcal{A}^e}(\text{Res}_{\mathcal{C}, \mathcal{A}}(\mathcal{R}_*), k\mathcal{A}).$$

□

We have the following commutative diagram, involving four cohomology rings.

**Theorem 2.4.2.** *Let  $\mathcal{C}$  be a finite EI-category and  $k$  a field. Then we have the following commutative diagram*

$$\begin{array}{ccc} \text{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) & \xrightarrow{\phi_{k\mathcal{A}} = \tilde{\pi}} & \text{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A}) \\ \phi_{\mathcal{C}} \downarrow & & \downarrow \phi_{\mathcal{A}} \\ \text{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}) & \xrightarrow{\text{Res}_{\mathcal{C}, \mathcal{A}}} & \text{Ext}_{k\mathcal{A}}^*(\underline{k}, \underline{k}). \end{array}$$

*Proof.* As usual, we prove it on the cochain level. Let  $\mathcal{R}_* \rightarrow k\mathcal{C} \rightarrow 0$  be the minimal projective resolution of the  $k\mathcal{C}^e$ -module  $k\mathcal{C}$ . Then we have the following commutative

diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{k\mathcal{C}^e}(\mathcal{R}_*, k\mathcal{C}) & \longrightarrow & \mathrm{Hom}_{k\mathcal{C}^e}(\mathcal{R}_* \otimes_{k\mathcal{C}} k\mathcal{A}, k\mathcal{C} \otimes_{k\mathcal{C}} k\mathcal{A}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{k\mathcal{C}}(\mathcal{R}_* \otimes_{k\mathcal{C}} \underline{k}, k\mathcal{C} \otimes_{k\mathcal{C}} \underline{k}) & \longrightarrow & \mathrm{Hom}_{k\mathcal{A}}(\mathcal{R}_* \otimes_{k\mathcal{C}} k\mathcal{A} \otimes_{k\mathcal{A}} \underline{k}, k\mathcal{C} \otimes_{k\mathcal{C}} k\mathcal{A} \otimes_{k\mathcal{A}} \underline{k}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{k\mathcal{C}}(\mathcal{R}'_*, \underline{k}) & \longrightarrow & \mathrm{Hom}_{k\mathcal{A}}(\mathcal{R}'_*, \underline{k}) \\
& & \downarrow \\
& & \mathrm{Hom}_{k\mathcal{A}}(\mathcal{R}''_*, \underline{k}),
\end{array}$$

in which  $\mathcal{R}'_* \rightarrow \underline{k} \rightarrow 0$  and  $\mathcal{R}''_* \rightarrow \underline{k} \rightarrow 0$  are the projective resolutions of  $k\mathcal{C}$ - and  $k\mathcal{A}$ -modules satisfying the following commutative diagrams of  $k\mathcal{C}$ -modules and  $k\mathcal{A}$ -modules, respectively,

$$\begin{array}{ccc}
\mathcal{R}'_* & \longrightarrow & \underline{k} \longrightarrow 0 \\
\downarrow & & \downarrow \cong \\
\mathcal{R}_* \otimes_{k\mathcal{C}} \underline{k} & \longrightarrow & k\mathcal{C} \otimes_{k\mathcal{C}} \underline{k} \longrightarrow 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{R}''_* & \longrightarrow & \underline{k} \longrightarrow 0 \\
\downarrow & & \downarrow = \\
\mathcal{R}'_* & \longrightarrow & \underline{k} \longrightarrow 0.
\end{array}$$

In the main diagram, upper left cochain complex computes  $\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C})$ , upper right corner computes  $\mathrm{Ext}_{k\mathcal{A}^e}^*(k\mathcal{A}, k\mathcal{A})$ , lower left corner computes  $\mathrm{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k})$  and lower right corner computes  $\mathrm{Ext}_{k\mathcal{A}}^*(\underline{k}, \underline{k})$ . Hence our statement follows.  $\square$

We note that in the theorem the category  $\mathcal{A}$  may be replaced by any full subcategory of it. Especially, we have a commutative diagram for each  $\mathrm{Aut}_{\mathcal{C}}(x) \subset \mathcal{A}$

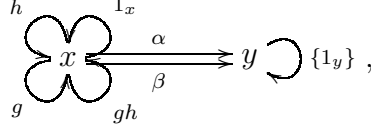
$$\begin{array}{ccc}
\mathrm{Ext}_{k\mathcal{C}^e}^*(k\mathcal{C}, k\mathcal{C}) & \xrightarrow{\phi_{k \mathrm{Aut}_{\mathcal{C}}(x)}} & \mathrm{Ext}_{k \mathrm{Aut}_{\mathcal{C}}(x)^e}^*(k \mathrm{Aut}_{\mathcal{C}}(x), k \mathrm{Aut}_{\mathcal{C}}(x)) \\
\downarrow \phi_{\mathcal{C}} & & \downarrow \phi_{\mathrm{Aut}_{\mathcal{C}}(x)} \\
\mathrm{Ext}_{k\mathcal{C}}^*(\underline{k}, \underline{k}) & \xrightarrow{\mathrm{Res}_{\mathcal{C}, \mathrm{Aut}_{\mathcal{C}}(x)}} & \mathrm{Ext}_{k \mathrm{Aut}_{\mathcal{C}}(x)}^*(\underline{k}, \underline{k}).
\end{array}$$

### 3. EXAMPLES OF THE HOCHSCHILD COHOMOLOGY RINGS OF CATEGORIES

In this section we calculate the Hochschild cohomology rings for four finite EI-categories, with base field  $k$  of characteristic 2. In particular the first category gives rise to a counterexample against the finite generation conjecture of the Hochschild cohomology rings in [20].

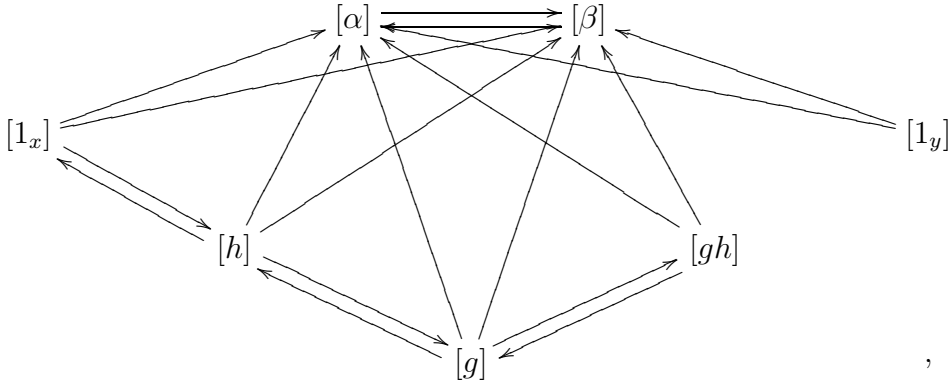
Since all of our four categories are finite EI-categories, for the reader's convenience we give a description of the simple  $k\mathcal{C}$ -modules for a finite EI-category  $\mathcal{C}$ . By [15], any simple  $k\mathcal{C}$ -module  $S_{x,V}$  is indexed by the isomorphism class of an object  $x \in \mathrm{Ob} \mathcal{C}$  and a simple module  $V$  of the automorphism group  $\mathrm{Aut}_{\mathcal{C}}(x)$  of  $x$ . As a functor,  $S_{x,V}(y) \cong V$  if  $y \cong x$  in  $\mathrm{Ob} \mathcal{C}$  and  $S_{x,V}(y) = 0$  otherwise.

3.1. **The category  $\mathcal{E}_0$ .** In [24] we presented an example, by Aurélien Djament, Laurent Piriou and the author, of the mod-2 ordinary cohomology ring of the following category  $\mathcal{E}_0$

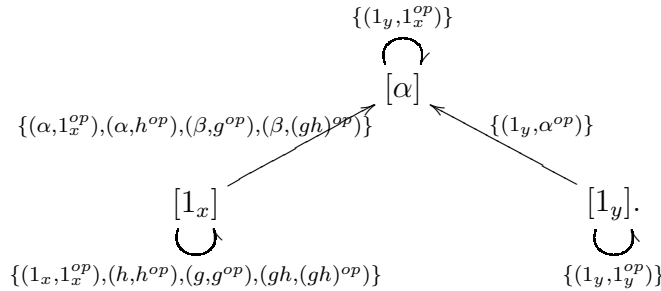


where  $g^2 = h^2 = 1_x, gh = hg, \alpha h = \beta g = \alpha$ , and  $\alpha g = \beta h = \beta$ . The ordinary cohomology ring  $\text{Ext}_{k\mathcal{E}_0}^*(\underline{k}, \underline{k})$  is a subring of the polynomial ring  $H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, k) \cong k[u, v]$ , removing all  $u^n, n \geq 1$ , and their scalar multiples. It has no nilpotents and is not finitely generated. By Theorem 2.3.4, it implies that the Hochschild cohomology ring  $\text{Ext}_{k\mathcal{E}_0}^*(k\mathcal{E}_0, k\mathcal{E}_0)$  is not finitely generated either, which gives a counterexample against the conjecture in [20]. We compute its Hochschild cohomology ring using Proposition 2.3.5.

The category of factorizations in  $\mathcal{E}_0$ ,  $F(\mathcal{E}_0)$ , has the following shape



in which  $[1_x] \cong [h] \cong [g] \cong [gh]$  and  $[\alpha] \cong [\beta]$ . For the purpose of computation, we use the skeleton  $F'(\mathcal{E}_0)$  of  $F(\mathcal{E}_0)$  (which is equivalent to  $F(\mathcal{E}_0)$  hence the two category algebras and their module categories are Morita equivalent)



In the above category, next to each arrow is the set of homomorphisms in  $F'(\mathcal{E}_0)$  from one object to another. The module  $N_{\mathcal{E}_0} \in kF'(\mathcal{E}_0)\text{-mod}$  (see Proposition 2.3.5) takes



the following values

$$\begin{aligned} N_{\mathcal{C}}([1_x]) &= k\{1_x + h, g + gh, 1_x + g\} & , & \quad N_{\mathcal{C}}([h]) &= k\{1_x + h, g + gh, 1_x + g\}, \\ N_{\mathcal{C}}([g]) &= k\{1_x + h, g + gh, 1_x + g\} & , & \quad N_{\mathcal{C}}([gh]) &= k\{1_x + h, g + gh, 1_x + g\}, \\ N_{\mathcal{C}}([\alpha]) &= k\{\alpha + \beta\} & , & \quad N_{\mathcal{C}}([\beta]) &= k\{\alpha + \beta\}, \\ N_{\mathcal{C}}([1_y]) &= 0. \end{aligned}$$

Thus  $N_{\mathcal{E}_0} = S_{[1_x], k(1_x+h)} \oplus S_{[1_x], k(g+gh)} \oplus \underline{k}'_{1_x+g}$ , where  $S_{[1_x], k(1_x+h)}$  and  $S_{[1_x], k(g+gh)}$  are simple  $kF'(\mathcal{E}_0)$ -modules such that  $S_{[1_x], k(1_x+h)}([1_x]) = k(1_x+h)$  and  $S_{[1_x], k(g+gh)}([1_x]) = k(g+gh)$ , and  $\underline{k}'_{1_x+g}$  is a  $kF'(\mathcal{E}_0)$ -module such that  $\underline{k}'_{1_x+g}([1_x]) = k(1_x+g)$ ,  $\underline{k}'_{1_x+g}([\alpha]) = k(\alpha+\beta)$  and  $\underline{k}'_{1_x+g}([1_y]) = 0$ . Note that  $S_{[1_x], k(1_x+h)}([1_x]) = k(1_x+h)$ ,  $S_{[1_x], k(g+gh)}([1_x]) = k(g+gh)$  and  $\underline{k}'_{1_x+g}([1_x]) = k(1_x+g)$  are all isomorphic to the trivial  $k \operatorname{Aut}_{F'(\mathcal{E}_0)}([1_x])$ -module, and have the same trivial ring structure in the sense that the product of any two elements is zero. Hence we have (along with the result quoted in Section 2.4, paragraph two)

$$\operatorname{Ext}_{kF'(\mathcal{E}_0)}^*(\underline{k}, S_{[1_x], k(1_x+h)}) \cong k(1_x+h) \otimes_k \operatorname{Ext}_{k \operatorname{Aut}_{F'(\mathcal{E}_0)}([1_x])}^*(k, k)$$

and

$$\operatorname{Ext}_{kF'(\mathcal{E}_0)}^*(\underline{k}, S_{[1_x], k(g+gh)}) \cong k(g+gh) \otimes_k \operatorname{Ext}_{k \operatorname{Aut}_{F'(\mathcal{E}_0)}([1_x])}^*(k, k)$$

as rings, in which  $k(1_x+h)$  and  $k(g+gh)$  are concentrated in degree zero in each ring. From the structure of  $F(\mathcal{E}_0)$ , one has  $\operatorname{Aut}_{F'(\mathcal{E}_0)}([1_x]) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .

For computing  $\operatorname{Ext}_{kF'(\mathcal{E}_0)}^*(\underline{k}, \underline{k}'_{1_x+g})$ , we use the following short exact sequence of  $kF(\mathcal{E}_0)$ -modules

$$0 \rightarrow \underline{k}'_{1_x+g} \rightarrow \underline{k} \rightarrow S_{[1_y], k} \rightarrow 0.$$

It induces a long exact sequence in which one can find  $\operatorname{Ext}_{kF'(\mathcal{E}_0)}^0(\underline{k}, S_{[1_y], k}) = k$  and  $\operatorname{Ext}_{kF'(\mathcal{E}_0)}^n(\underline{k}, S_{[1_y], k}) = 0$  if  $n \geq 1$ . Thus  $\operatorname{Ext}_{kF'(\mathcal{E}_0)}^0(\underline{k}, \underline{k}'_{1_x+g}) = 0$  while  $\operatorname{Ext}_{kF'(\mathcal{E}_0)}^n(\underline{k}, \underline{k}) \cong \operatorname{Ext}_{kF'(\mathcal{E}_0)}^n(\underline{k}, \underline{k}'_{1_x+g})$  for each  $n \geq 1$ . Hence as a ring

$$\operatorname{Ext}_{kF'(\mathcal{E}_0)}^*(\underline{k}, \underline{k}'_{1_x+g}) \cong k(1_x+g) \otimes_k \operatorname{Ext}_{kF'(\mathcal{E}_0)}^{*>0}(\underline{k}, \underline{k}) \cong k(1_x+g) \otimes_k \operatorname{Ext}_{k\mathcal{E}_0}^{*>0}(\underline{k}, \underline{k}).$$

All in all, we have

$$\operatorname{Ext}_{k\mathcal{E}_0}^0(k\mathcal{E}_0, k\mathcal{E}_0) \cong \operatorname{Ext}_{k\mathcal{E}_0}^0(\underline{k}, \underline{k}) \oplus k(1_x+h) \oplus k(g+gh),$$

and if  $n \geq 1$

$$\begin{aligned} & \operatorname{Ext}_{k\mathcal{E}_0}^n(k\mathcal{E}_0, k\mathcal{E}_0) \\ & \cong \operatorname{Ext}_{k\mathcal{E}_0}^n(\underline{k}, \underline{k}) \oplus \{k(1_x+g) \otimes_k \operatorname{Ext}_{k\mathcal{E}_0}^n(\underline{k}, \underline{k})\} \\ & \quad \oplus \{k(1_x+h) \otimes_k \operatorname{Ext}_{k(\mathbb{Z}_2 \times \mathbb{Z}_2)}^*(k, k)\} \oplus \{k(g+gh) \otimes_k \operatorname{Ext}_{k(\mathbb{Z}_2 \times \mathbb{Z}_2)}^n(k, k)\}. \end{aligned}$$

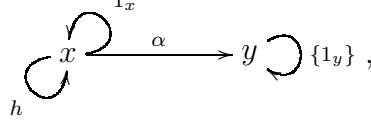
Combining all the information we obtained, the surjective ring homomorphism

$$\phi_{\mathcal{E}_0} : \operatorname{Ext}_{k\mathcal{E}_0}^*(k\mathcal{E}_0, k\mathcal{E}_0) \rightarrow \operatorname{Ext}_{k\mathcal{E}_0}^*(\underline{k}, \underline{k})$$

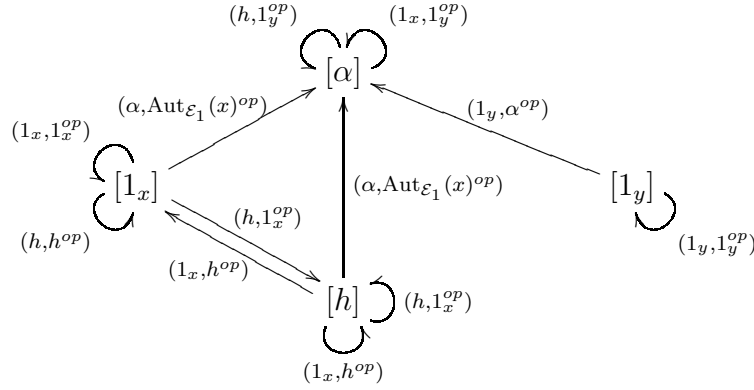
has its kernel consisting of all nilpotents. Consequently this Hochschild cohomology ring modulo nilpotents is not finitely generated, against the finite generation conjecture in [20]. We comment that the category algebra  $k\mathcal{E}_0$  is not a self-injective algebra

(hence is not Hopf, by [11]). Nicole Snashall points out to the author that this algebra is Koszul since both  $k\mathcal{E}_0$  and  $\text{Ext}_{k\mathcal{E}_0}^*(k\overline{\mathcal{E}_0}, k\overline{\mathcal{E}_0})$  as graded algebras are generated in degrees zero and one, where  $k\overline{\mathcal{E}_0} = k\mathcal{E}_0 / \text{Rad}(k\mathcal{E}_0) \cong S_{x,k} \oplus S_{y,k}$ .

**3.2. The category  $\mathcal{E}_1$ .** The following category  $\mathcal{E}_1$  has a terminal object and hence is contractible:



where  $h^2 = 1_x$  and  $\alpha h = \alpha$ . The contractibility implies the ordinary cohomology ring is simply the base field  $k$ . In this case  $F(\mathcal{E}_1)$  is the following category



We calculate its Hochschild cohomology ring. By proposition 2.3.5, we only need to compute  $\text{Ext}_{kF(\mathcal{E}_1)}^*(\underline{k}, N_{\mathcal{E}_1})$ , where  $N_{\mathcal{E}_1}$  has the following value at objects of  $F(\mathcal{E}_1)$

$$\begin{aligned} N_{\mathcal{E}_1}([1_x]) &= k\{1_x + h\} & , & & N_{\mathcal{E}_1}([h]) &= k\{1_x + h\}, \\ N_{\mathcal{E}_1}([1_y]) &= 0 & , & & N_{\mathcal{E}_1}([\alpha]) &= 0. \end{aligned}$$

One can easily see that  $N_{\mathcal{E}_1} = S_{[1_x], k(1_x+h)}$  is a simple module of dimension one with a specified value  $k(1_x + h)$  at  $[1_x]$ . Since  $[1_x] \cong [h] \in \text{Ob } F(\mathcal{E}_1)$  are minimal objects, using quoted result in Section 2.4 paragraph two, we get

$$\text{Ext}_{kF(\mathcal{E}_1)}^*(\underline{k}, N_{\mathcal{E}_1}) \cong \text{Ext}_{k\text{Aut}_{F(\mathcal{E}_1)}([1_x])}^*(k, k(1_x + h)) \cong k(1_x + h) \otimes_k \text{Ext}_{k\mathbb{Z}_2}^*(k, k),$$

which is isomorphic to  $k(1_x + h) \otimes_k k[u]$ . Here  $k[u]$  is a polynomial algebra with an indeterminate  $u$  at degree one and  $k(1_x + h)$  is at degree zero. Thus

$$\text{Ext}_{k\mathcal{E}_1}^*(k\mathcal{E}_1, k\mathcal{E}_1) \cong \text{Ext}_{k\mathcal{E}_1}^*(\underline{k}, \underline{k}) \oplus \text{Ext}_{kF(\mathcal{E}_1)}^*(\underline{k}, N_{\mathcal{E}_1}) \cong k \oplus \{k(1_x + h) \otimes_k k[u]\}.$$

The kernel of  $\phi_{\mathcal{E}_1}$  consists of all nilpotents in the Hochschild cohomology ring.

**3.3. The category  $\mathcal{E}_2$ .** The following category has its classifying space homotopy equivalent to the join,  $B\mathbb{Z}_2 * B\mathbb{Z}_2 = \Sigma(B\mathbb{Z}_2 \wedge B\mathbb{Z}_2) = \Sigma[B(\mathbb{Z}_2 \times \mathbb{Z}_2) / (B\mathbb{Z}_2 \vee B\mathbb{Z}_2)]$ ,

of the classifying spaces of the two automorphism groups:

$$\begin{array}{c} \begin{array}{ccc} & 1_x & \\ & \curvearrowright & \\ x & \xrightarrow{\alpha} & y \\ & \curvearrowleft & \\ & g & \\ & 1_y & \end{array} , \end{array}$$

where  $h^2 = 1_x$ ,  $\alpha h = \alpha = g\alpha$  and  $g^2 = 1_y$ . As direct consequences, its ordinary cohomology groups are equal to  $k, 0, 0$  at degrees zero, one and two, and  $k^{n-2}$  at each degree  $n \geq 3$ , and furthermore the cup product in this ring is trivial [24]. We compute its Hochschild cohomology ring. The category  $F(\mathcal{E}_2)$  is as follows

$$\begin{array}{ccccc} & & (\text{Aut}_{\mathcal{E}_2}(x), \text{Aut}_{\mathcal{E}_2}(y)^{op}) & & \\ & & \curvearrowright & & \\ & & [\alpha] & & \\ & (\alpha, \text{Aut}_{\mathcal{E}_2}(x)^{op}) & \nearrow & \nwarrow & (\text{Aut}_{\mathcal{E}_2}(y), \alpha^{op}) \\ (1_x, 1_x^{op}) & [1_x] & & & [g] & (1_y, g^{op}) \\ & \curvearrowleft & & & \curvearrowright & \\ & (h, h^{op}) & & & (g, 1_y^{op}) & \\ & & \searrow & \swarrow & & \\ & & [h] & & [1_y] & \\ & (1_x, h^{op}) & \curvearrowright & \curvearrowleft & (1_y, 1_y^{op}) & \curvearrowright \\ & & (h, 1_x^{op}) & & (g, g^{op}) & \end{array}$$

By Proposition 2.3.5, we need to compute  $\text{Ext}_{k\mathcal{E}_2}^*(\underline{k}, N_{\mathcal{E}_2})$ . In this case we have

$$\begin{aligned} N_{\mathcal{E}_2}([1_x]) &= k\{1_x + h\} \quad , \quad N_{\mathcal{E}_2}([h]) = k\{1_x + h\}, \\ N_{\mathcal{E}_2}([1_y]) &= k\{1_y + g\} \quad , \quad N_{\mathcal{E}_2}([g]) = k\{1_y + g\}, \\ N_{\mathcal{E}_2}([\alpha]) &= 0. \end{aligned}$$

It means  $N_{\mathcal{E}_2} = S_{[1_x], k(1_x+h)} \oplus S_{[1_y], k(1_y+g)}$  and thus by Proposition 2.2.5

$$\begin{aligned} \text{Ext}_{k\mathcal{E}_2}^*(\underline{k}, N_{\mathcal{E}_2}) &\cong \text{Ext}_{k \text{Aut}_{F(\mathcal{E}_2)}([1_x])}^*(k, k(1_x + h)) \oplus \text{Ext}_{k \text{Aut}_{F(\mathcal{E}_2)}([1_y])}^*(k, k(1_y + g)) \\ &\cong \{k(1_x + h) \otimes_k \text{Ext}_{k\mathbb{Z}_2}^*(k, k)\} \oplus \{k(1_y + g) \otimes_k \text{Ext}_{k\mathbb{Z}_2}^*(k, k)\}. \end{aligned}$$

Hence

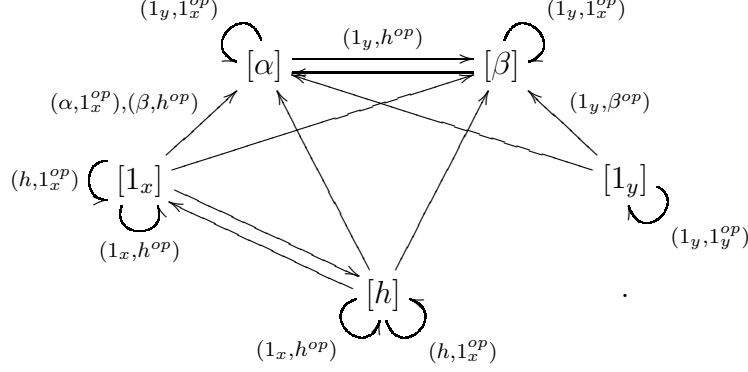
$$\text{Ext}_{k\mathcal{E}_2}^*(k\mathcal{E}_2, k\mathcal{E}_2) \cong \text{Ext}_{k\mathcal{E}_2}^*(\underline{k}, \underline{k}) \oplus \{k(1_x + h) \otimes_k k[u]\} \oplus \{k(1_y + g) \otimes_k k[v]\},$$

where  $k[u]$  and  $k[v]$  are two polynomial algebras with indeterminants in degree one. Both the Hochschild and ordinary cohomology rings modulo nilpotents are isomorphic to the base field  $k$ .

**3.4. The category  $\mathcal{E}_3$ .** The following category has a classifying space homotopy equivalent to that of  $\text{Aut}_{\mathcal{E}_3}(x) \cong \mathbb{Z}_2$  (by Quillen's Theorem A [18], or see [23])

$$\begin{array}{c} \begin{array}{ccc} & 1_x & \\ & \curvearrowright & \\ x & \xrightarrow{\alpha} & y \\ & \curvearrowleft & \\ & \beta & \\ & \{1_y\} & \end{array} , \end{array}$$

where  $h^2 = 1_x$  and  $\alpha h = \beta$ . We compute its Hochschild cohomology ring. The category  $F(\mathcal{E}_3)$  is as follows (not all morphisms are presented since only its skeleton is needed)



The module  $N_{\mathcal{E}_3}$  takes the following values

$$\begin{aligned} N_{\mathcal{E}_2}([1_x]) &= k\{1_x + h\} \quad , \quad N_{\mathcal{E}_2}([h]) = k\{1_x + h\}, \\ N_{\mathcal{E}_2}([1_y]) &= 0 \quad , \quad N_{\mathcal{E}_2}([\alpha]) = k\{\alpha + \beta\}, \\ N_{\mathcal{E}_2}([\alpha]) &= k\{\alpha + \beta\}. \end{aligned}$$

Thus  $N_{\mathcal{E}_3}$  fits into the following short exact sequence of  $kF(\mathcal{E}_3)$ -modules

$$0 \rightarrow N_{\mathcal{E}_3} \rightarrow \underline{k} \rightarrow S_{[1_y],k} \rightarrow 0.$$

Just like in our first example, using the long exact sequence coming from it, we know  $\text{Ext}_{k\mathcal{E}_2}^0(\underline{k}, N_{\mathcal{E}_3}) = 0$  and  $\text{Ext}_{k\mathcal{E}_2}^{*>0}(\underline{k}, N_{\mathcal{E}_3}) \cong k(1_x + h) \otimes_k \text{Ext}_{kF(\mathcal{E}_3)}^{*>0}(\underline{k}, \underline{k}) \cong k(1_x + h) \otimes_k \text{Ext}_{k\mathcal{E}_3}^{*>0}(\underline{k}, \underline{k})$ . Hence

$$\text{Ext}_{k\mathcal{E}_3}^*(k\mathcal{E}_3, k\mathcal{E}_3) \cong \text{Ext}_{k\mathcal{E}_3}^*(\underline{k}, \underline{k}) \oplus \{k(1_x + h) \otimes_k \text{Ext}_{k\mathcal{E}_3}^{*>0}(\underline{k}, \underline{k})\}.$$

The kernel of  $\phi_{\mathcal{E}_3}$  contains all nilpotents in the Hochschild cohomology ring.

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