Boundaries for Banach spaces determine weak compactness

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BOUNDARIES FOR BANACH SPACES DETERMINE WEAK COMPACTNESS

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Abstract. A boundary for a Banach space is a subset of the dual unit sphere with the property that each element of the Banach space attains its norm on an element of that subset. Trivially, the pointwise convergence with respect to such a boundary is coarser than the weak topology on the Banach space. Godefroy's Boundary Problem asks whether nevertheless both topologies have the same bounded compact sets. This paper contains the answer in the positive.

This paper deals with boundaries for Banach spaces in the sense of

Definition 1. *Let X be a real Banach space. A subset B of $\mathcal{S}(X^*)$ is called a boundary for X if for each $x \in X$ there is $b \in B$ such that $b(x) = \|x\|$.*

The set of extremal points of the dual unit ball of a Banach space is a well-known example of a boundary. In 1980 Bourgain and Talagrand [3] showed that a norm bounded subset of a Banach space is weakly compact if it is compact in the pointwise topology on the set of extremal points of the dual unit ball. Some years later Godefroy [9] asked whether the result still holds if the extremal points are replaced by an arbitrary boundary.

Theorem 6 shows that the answer is "yes". The proof can be described vaguely as an amalgam of Behrends' quantitative version of Rosenthal's l^1-Theorem (cf. Lemma 2), Simons' equality (cf. Lemma 3), a variant of Hagler-Johnson's construction (cf. Lemma 4) and James' distortion theorem (everywhere).

Besides the result of Bourgain and Talagrand a positive answer to Godefroy's question has been known in the important case when the set in question is convex [10, p. 44]. Bourgain's and Talagrand's proof relies on the results of [2] which are of topological nature and do not seem applicable here because general boundaries seem to lack sufficient topological structures. As a - technically quite different - substitute we use Simons' equality [15] (or, if the reader prefers, Simons' inequality [14], see [8, Th. 3.48]) which has been advocated at several instances by Godefroy (e. g. [4, 11]). The idea to look for the key Lemma 4 of the present proof was inspired by the result of Cascales et al. [5, 6] on the existence of an independent sequence.

Throughout this article X denotes a real Banach space, X^* its dual, $\mathcal{B}(X)$ its unit ball and $\mathcal{S}(X)$ its unit sphere. The norm closed linear span of a subset A of X is written $[A]$. $\mathbb{N}$ starts at 1. Our references for unexplained Banach space notions are [12] (for boundaries in particular see Ch. 15, Infinite Dimensional Convexity by Fonf, Lindenstrauss and Phelps), [7] and [13].

Acknowledgement It is a pleasure to thank G. Godefroy for having introduced me to the subject and for encouraging conversations.

To each bounded sequence (x_n) in X we associate its James constant

$$\varepsilon_J(x_n) = \sup_m \inf_{\sum_{n\geq m} |\alpha_n|=1} \|\sum_{n\geq m} \alpha_n x_n\|.$$

1991 *Mathematics Subject Classification.* 46B20.

If a sequence is equivalent to the canonical basis of l^1 we call it simply an l^1-sequence. Clearly, $\varepsilon_J \geq 0$ with $\varepsilon_J(x_n) > 0$ if and only if there is an integer m such that $(x_n)_{n\geq m}$ is an l^1-sequence.

Two more moduli will be of importance. Let D be a subset of X^*. We define

$$\delta_D(x_n) = \sup_{x^*\in D} \big(\overline{\lim}\, x^*(x_n) - \underline{\lim}\, x^*(x_n)\big), \quad \delta_{\mathrm{HJ},\, D}(x_n) = \sup_{x^*\in D} x^*(x_n).$$

for bounded sequences (x_n) and write $\delta = \delta_{\mathcal{B}(X^*)}$ and $\delta_{\mathrm{HJ}} = \delta_{\mathrm{HJ},\, \mathcal{B}(X^*)}$ for short. Clearly $\delta_D \geq 0$ for all D and $\delta_{\mathrm{HJ}} \geq 0$ and equality $\delta(x_n) = 0$ (respectively $\delta_{\mathrm{HJ}}(x_n) = 0$) holds if and only if (x_n) is weakly Cauchy (respectively converges weakly to zero). Both δ_D and $\delta_{\mathrm{HJ},\, D}$ are non-increasing and similarly ε_J is non-decreasing if one passes to subsequences. We introduce the notation

$$\tilde{\delta}_D(x_n) = \inf_{n_k} \delta_D(x_{n_k}) \quad \text{and} \quad \tilde{\varepsilon}_J(x_n) = \sup_{n_k} \varepsilon_J(x_{n_k})$$

and say that (x_n) is δ_D-stable if $\tilde{\delta}_D(x_n) = \delta_D(x_n)$. Likewise (x_n) is ε_J-stable if $\tilde{\varepsilon}_J(x_n) = \varepsilon_J(x_n)$.

If one takes a norm preserving Hahn-Banach extension of the functional defined on $[x_n]_{n\geq m}$ by $x_n \mapsto (-1)^n \varepsilon_J((x_k)_{k\geq m})$ for all $n \geq m$ and m big enough then it is completely elementary but important to deduce that $\delta_{\mathrm{HJ}} \geq \varepsilon_J$ and $\delta \geq 2\varepsilon_J$, even $\tilde{\delta} \geq 2\varepsilon_J$. While in general strict inequality may occur for δ_{HJ} this cannot happen in $\tilde{\delta} \geq 2\varepsilon_J$ as soon as (x_n) is ε_J-stable. This follows from Behrends' quantitative version of Rosenthal's l^1-Theorem.

Lemma 2. *Let X be a Banach space.*

(1) *Each bounded sequence in X contains an ε_J-stable subsequence.*
(2) *Each bounded sequence in X contains a δ-stable subsequence.*
(3) *If (x_n) is ε_J-stable then*

$$\tilde{\delta}(x_n) = 2\tilde{\varepsilon}_J(x_n). \tag{1}$$

If moreover (x_n) is δ-stable then

$$\tilde{\delta}(z_n) = 2\tilde{\varepsilon}_J(x_n). \tag{2}$$

whenever

$$z_n = \frac{1}{2}\sum_{l=1}^{q} \lambda_l (x_{p_{2l}(n)} - x_{p_{2l-1}(n)})$$

where $q \in \mathbb{N}$, $\sum_{l=1}^{q} \lambda_l = 1$, $\lambda_l \geq 0$ and where $(\{p_1(n), \ldots, p_{2q}(n)\})_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets (of cardinality $2q$) of $\mathbb{N}$.

Proof: (1) The proof consists in a routine diagonal argument. Choose a subsequence $(y_n^{(1)})$ of (x_n) such that $\varepsilon_J(y_n^{(1)}) \geq \tilde{\varepsilon}_J(x_n) - 2^{-1}$. Successiveley, choose a subsequence $(y_n^{(k+1)})$ of $(y_n^{(k)})$ for each $k \in \mathbb{N}$ such that $\varepsilon_J(y_n^{(k+1)}) \geq \tilde{\varepsilon}_J(y_n^{(k)}) - 2^{-(k+1)}$. Let $y_n = y_n^{(n)}$ and let (z_n) be a subsequence of (y_n). Then $\varepsilon_J(y_n) \geq \varepsilon_J(y_n^{(k+1)}) \geq \tilde{\varepsilon}_J(y_n^{(k)}) - 2^{-(k+1)} \geq \varepsilon_J(z_n) - 2^{-(k+1)}$ whence $\varepsilon_J(y_n) \geq \varepsilon_J(z_n)$ and $\varepsilon_J(y_n) \geq \tilde{\varepsilon}_J(y_n)$ that is $\varepsilon_J(y_n) = \tilde{\varepsilon}_J(y_n)$. The proof of (2) works alike.

(3) Before the statement of the lemma we have already noticed that $\tilde{\delta} \geq 2\varepsilon_J$ whence $\tilde{\delta} \geq 2\tilde{\varepsilon}_J$ because (x_n) is assumed to be ε_J-stable. In order to prove the other inequality of (1) we may suppose that $\tilde{\delta}(x_n) > 0$ because otherwise $\tilde{\delta}(x_n) = 0$ and (1) holds trivially. In our notation Behrends' main result [1, Th. 3.2] reads: If $\tilde{\delta}(x_n) > 0$ then, given $\eta > 0$, there is a subsequence whose ε_J-value is greater than $-\eta + (\tilde{\delta}(x_n))/2$. Now let η run through a sequence tending to zero and pass successively to according subsequences; the process results

in a diagonal sequence (x_{n_k}) such that $\varepsilon_J(x_{n_k}) \geq (\tilde{\delta}(x_n))/2$. Hence $\tilde{\varepsilon}_J(x_n) \geq (\tilde{\delta}(x_n))/2$ which proves (1).

For "$\geq$" of (2) proceed as before the statement of the lemma by using functionals of the kind $x_{p_{2l-i}(n)} \mapsto (-1)^i \varepsilon_J((x_k)_{k\geq m})$, $i \in \{0,1\}$, $1 \leq l \leq q$, with n and m big enough. For the other inequality of (2) note that δ-stability and (1) reduce it to $\tilde{\delta}(z_n) \leq \delta(x_n)$ which in turn by subadditivity of δ is reduced to $\delta(x_{n_k} - x_{m_k}) \leq 2\delta(x_n)$ where (x_{n_k}) and (x_{m_k}) are two disjoint subsequences of (x_n). But the latter inequality follows from the fact that for each $x^* \in \mathcal{B}(X^*)$ both $\overline{\lim}\, x^*(x_{n_k} - x_{m_k})$ and $-\underline{\lim}\, x^*(x_{n_k} - x_{m_k})$ are each the difference of two cluster points of $(x^*(x_n))$ and hence majorized by $\delta(x_n)$. ■

If B is a boundary for X, topological notions that refer to the $\sigma(X,B)$-topology are preceded by "B-", for example B-closed, B-compact.
The following lemma is a consequence of Simons' equality [15] which in our notation reads $\delta_{\mathrm{HJ},B} = \delta_{\mathrm{HJ}}$.

Lemma 3. *Let B be a boundary for X. Then*

$$\delta_B = \delta \quad \text{and} \quad \tilde{\delta}_B = \tilde{\delta}. \tag{3}$$

Moreover, if (x_n) is an ε_J-stable bounded sequence and z_n is as in Lemma 2 then

$$\tilde{\delta}_B(z_n) = 2\tilde{\varepsilon}(x_n). \tag{4}$$

Proof: (4) follows from (2) and (3) and the second half of (3) is immediate from the first one which, in turn, is a routine consequence of (2) and Simons' equality: Fix $x^* \in \mathcal{B}(X^*)$ and choose subsequences $(u_k) = (x_{n_k})$ and $(v_k) = (x_{m_k})$ such that

$$\begin{aligned} &\overline{\lim}\, x^*(x_n) - \underline{\lim}\, x^*(x_n) \\ &= \lim x^*(u_k - v_k) \leq \delta_{\mathrm{HJ}}(u_k - v_k) = \delta_{\mathrm{HJ},B}(u_k - v_k) \\ &\leq \sup_{b\in B}\big(\overline{\lim}\, b(u_k) - \underline{\lim}\, b(v_k)\big) \leq \sup_{b\in B}\big(\overline{\lim}\, b(x_n) - \underline{\lim}\, b(x_n)\big). \end{aligned}$$

This shows "$\geq$" of $\delta_B = \delta$ whereas "$\leq$" is trivial. ■

Let $S = \bigcup_n S_n$ where $S_n = \{0,1\}^n$ for each $n \in \mathbb{N}$. If $\sigma \in S_n$ we write $\sigma = (\sigma_1, \ldots, \sigma_n)$ (with, of course, $\sigma_k \in \{0,1\}$) and for $i \in \{0,1\}$ we write (σ, i) for the element $(\sigma_1, \ldots, \sigma_n, i) \in S_{n+1}$. Recall that a tree of non empty subsets of $\mathbb{N}$ is a sequence $(\Omega_\sigma)_{\sigma\in S}$ such that $\Omega_{(\sigma,0)}$ and $\Omega_{(\sigma,1)}$ are two disjoint non empty (hence infinite) subsets of $\Omega_\sigma \subset \mathbb{N}$ for all $\sigma \in S$.

Lemma 4. *Let B be a boundary for X, (x_n) be an l^1-sequence and $\eta_k > 0$ decrease to zero. Then there are a sequence (b_k) in B, a tree $(\Omega_\sigma)_{\sigma\in S}$ and $\varepsilon \geq \varepsilon_J(x_n)$ such that for all k*

$$b_k(x_n - x_{n'}) > 2(1-\eta_k)\varepsilon \quad \text{if } n \in \Omega_\sigma, n' \in \Omega_{\sigma'}, \ \sigma, \sigma' \in S_k \text{ and } \sigma_k = 0, \sigma'_k = 1. \tag{5}$$

Furthermore, if the set $\{x_n \,|n \in \mathbb{N}\}$ is relatively B-compact in X, there is a sequence (y_m) of B-cluster points of the x_n such that

$$b_k(y_m - y_{m'}) \geq 2(1-\eta_k)\varepsilon \quad \text{if} \quad m \leq k < m', \ k, m, m' \in \mathbb{N}. \tag{6}$$

Proof: Choose $\Omega_\emptyset \subset \mathbb{N}$ such that the x_n with indices in $\Omega_\emptyset$ form an ε_J-stable and δ-stable l^1-sequence which exists by Lemma 2 and set $\varepsilon = \varepsilon_J((x_n)_{n\in\Omega_\emptyset})$. Then $\varepsilon \geq \varepsilon_J(x_n)$.
The b_k and Ω_σ for $\sigma \in S_k$ will be constructed by induction over $k \in \mathbb{N}$.

For the first induction step $k = 1$, equalities (3) and (1) allow to find $b_1 \in B$ and a subsequence (n_l) of $\Omega_\emptyset$ such that

$$b_1(x_{n_{2l}} - x_{n_{2l'+1}}) > 2(1-\eta_1)\varepsilon$$

for all $l, l' \in \mathbb{N}$. It remains to set $\Omega_{(i)} = \{n_{2l+i} \,|l \in \mathbb{N}\}$ for $i = 0$ and $i = 1$ in order to settle the first induction step.

Suppose that $b_1, \dots, b_k$ and $\Omega_\sigma = \{\omega_\sigma(n) \mid n \in \mathbb{N}\}$ for $\sigma \in S_k$ have been constructed according to (5). (Of course, we suppose the $\omega_\sigma : \mathbb{N} \to \mathbb{N}$ to be strictly increasing.) Set $\eta = \eta_{k+1}/2^{k+1}$. Apply (4) to

$$z_n = 2^{-k} \sum_{\sigma \in S_k} (-1)^{\sigma_k}\, x_{\omega_\sigma(n)}$$

(with $q = 2^{k-1}$ and $\lambda_l = 1/q$ for $l \le q$) in order to get $b_{k+1} \in B$ and a sequence (n_l) of integers such that

$$b_{k+1}(z_{n_{2l}} - z_{n_{2l+1}}) > 2(1-\eta)\varepsilon \quad \text{for all} \quad l \in \mathbb{N}. \tag{7}$$

Note that (by omitting at most finitely many members of the n_k) one has furthermore that

$$b_{k+1}(x_{\omega_\sigma(n_{2l})} - x_{\omega_{\sigma'}(n_{2l+1})}) < 2(1+\eta)\varepsilon \quad \text{for all} \quad l \in \mathbb{N}, \sigma, \sigma' \in \mathrm{S_k} \tag{8}$$

because by (1) the difference of two cluster points of $b_{k+1}(x_n)$ cannot exceed $\delta(x_n) = \tilde{\delta}(x_n) \le 2\varepsilon$.

Define, for all $\sigma \in S_k$,

$$\left.\begin{array}{ll} \omega_{(\sigma,0)}(l) & = \omega_\sigma(n_{2l}) \\ \omega_{(\sigma,1)}(l) & = \omega_\sigma(n_{2l+1}) \end{array}\right\} \text{if } \sigma_k = 0 \quad \text{and} \quad \left.\begin{array}{ll} \omega_{(\sigma,0)}(l) & = \omega_\sigma(n_{2l+1}) \\ \omega_{(\sigma,1)}(l) & = \omega_\sigma(n_{2l}) \end{array}\right\} \text{if } \sigma_k = 1.$$

With this notation we have

$$\begin{aligned} z_{n_{2l}} - z_{n_{2l+1}} &= 2^{-k} \sum_{\sigma \in S_k} (-1)^{\sigma_k} (x_{\omega_\sigma(n_{2l})} - x_{\omega_\sigma(n_{2l+1})}) \\ &= 2^{-k} \sum_{\sigma \in S_k} (x_{\omega_{(\sigma,0)}(l)} - x_{\omega_{(\sigma,1)}(l)}). \end{aligned}$$

Consider $\sigma, \sigma' \in S_k$ and distinguish the two cases $\sigma = \sigma'$ and $\sigma \ne \sigma'$. In the first case we have

$$x_{\omega_{(\sigma,0)}(l)} - x_{\omega_{(\sigma,1)}(l)} = 2^k (z_{n_{2l}} - z_{n_{2l+1}}) - \sum_{\tau \in S, \tau \ne \sigma} (x_{\omega_{(\tau,0)}(l)} - x_{\omega_{(\tau,1)}(l)}),$$

and in the second case

$$x_{\omega_{(\sigma,0)}(l)} - x_{\omega_{(\sigma',1)}(l)} = 2^k (z_{n_{2l}} - z_{n_{2l+1}}) - \Big[(x_{\omega_{(\sigma',0)}(l)} - x_{\omega_{(\sigma,1)}(l)}) + \sum{}' (x_{\omega_{(\tau,0)}(l)} - x_{\omega_{(\tau,1)}(l)}) \Big]$$

where the sum $\sum'$ runs over all $\tau \in S_k$ such that $\tau \ne \sigma$ and $\tau \ne \sigma'$. In both cases we obtain

$$b_{k+1}(x_{\omega_{(\sigma,0)}(l)} - x_{\omega_{(\sigma',1)}(l)}) > 2^{k+1}(1-\eta)\varepsilon - 2(2^k - 1)(1+\eta)\varepsilon = 2(1 - \eta_{k+1} + \eta)\varepsilon$$

for all $l \in \mathbb{N}$ by (7) and (8). Finally, since all $b_{k+1}(x_n)$ are contained in a compact subset of $\mathbb{R}$ there is an infinite set $N \subset \mathbb{N}$ such that

$$b_{k+1}(x_{\omega_{(\sigma,0)}(l)} - x_{\omega_{(\sigma',1)}(l')}) > 2(1 - \eta_{k+1})\varepsilon$$

for all $l, l' \in N$. It remains to set $\Omega_{(\sigma,i)} = \omega_{(\sigma,i)}(N)$ for $i \in \{0,1\}$ and all $\sigma \in S_k$. This shows (5) for $k+1$ and ends the induction.

For the last part of the lemma fix m, take, for all $k \ge m$,

$$\sigma^{(k)} = (\underbrace{1, \dots, 1}_{m-1}, \underbrace{0, \dots, 0}_{k-m+1}) \in S_k,$$

take n_k to be the k-th element of $\Omega_{\sigma^{(k)}}$ and define y_m to be a B-cluster point of the x_{n_k}. Then, whenever $1 \le m \le k < m'$, there are $\sigma, \sigma' \in S_k$ with $\sigma_k = 0$, $\sigma'_k = 1$ such that y_m(respectivlely $y_{m'}$) is a B-cluster point of the x_n with indices in Ω_σ (respectively in $\Omega_{\sigma'}$)) and (6) follows from (5). ■

Theorem 5. *In a real Banach space with a boundary B, a bounded relatively B-compact set cannot contain an l^1-sequence.*

Proof: Suppose to the contrary that there is a relatively B-compact l^1-sequence in a real Banach space X. By Lemma 2 this l^1-sequence contains a δ-stable subsequence which we denote by (x_n). Let (η_k) be a decreasing sequence of positive numbers with limit zero. Take $\varepsilon \geq \varepsilon_J(x_n) > 0$ and two sequences (b_k) and (y_m) that fulfill (6) of Lemma 4 and set

$$x = \big(\sum 2^{-m} y_m\big) - y = \sum 2^{-m}(y_m - y)$$

where y is a B-cluster point of the y_m. Inequality (6) extends to y in the sense that $b_k(y_m - y) \geq 2(1-\eta_k)\varepsilon$ for all $m \leq k$. Note that y is also a B-cluster point of the x_n. Therefore the difference $b(y_m) - b(y)$ is majorized by $\delta(x_n) = \tilde{\delta}(x_n) \leq 2\varepsilon$ for all $b \in B$ hence $\|x\| \leq 2\varepsilon$. But actually $\|x\| = 2\varepsilon$: Let $\eta > 0$, let m_0 such that $\|\sum_{m_0+1}^{\infty} 2^{-m}(y_m - y)\| < \eta$ and let $k > m_0$. Then

$$\begin{aligned} b_k(x) &\geq \big(\sum_{m=1}^{m_0} 2^{-m} b_k(y_m - y)\big) - \eta \\ &\geq \big(\sum_{m=1}^{m_0} 2^{-m}\, 2(1-\eta_k)\varepsilon\big) - \eta \\ &= 2(1-\eta_k)(1-2^{-m_0})\varepsilon - \eta \end{aligned}$$

which shows that $\sup_k b_k(x) \geq 2\varepsilon$ and thus

$$\|x\| = \sup_k b_k(x) = 2\varepsilon.$$

Since B is a boundary it contains a b_0 such that $b_0(x) = 2\varepsilon$. So $b_0(y_m - y) = 2\varepsilon$ for all m and $b_0(y_m) = 2\varepsilon + b_0(y)$. But y is a B-cluster point of the y_m thus $b_0(y) = 2\varepsilon + b_0(y)$, a contradiction which ends the proof. ■

Theorem 6. *Let B be a boundary for the real Banach space X. Then a B-compact bounded subset of X is weakly compact.*

Proof: Let A be a bounded B-compact subset of X. By the theorem of Eberlein-Šmulyan, in order to show that A is weakly compact it is enough to prove that each sequence in A admits a weakly convergent subsequence. Take a bounded sequence in A and denote by (x_n) the ε_J-stable subsequence of it which exists by Lemma 2. Theorem 5 entails that $\tilde{\varepsilon}_J(x_n) = 0$ hence $\tilde{\delta}(x_n) = 0$ by 1. That is, (x_n) is weakly Cauchy hence B-Cauchy. Since moreover A is B-compact, (x_n) B-converges (as a sequence) to a limit, say $x \in X$. Now a final application of Simons' equality ends the proof because by

$$\sup_{x^* \in \mathcal{B}(X^*)} \overline{\lim}\, x^*(x_n - x) = \sup_{b \in B} \overline{\lim}\, b(x_n - x) = 0$$

we see that (x_n) converges weakly to x. ■

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