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# Distinguished representations and exceptional poles of the Asai-L-function

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## Abstract

Let  $K/F$  be a quadratic extension of  $p$ -adic fields. We show that a generic irreducible representation of  $GL(n, K)$  is distinguished if and only if its Rankin-Selberg Asai  $L$ -function has an exceptional pole at zero. We use this result to compute Asai  $L$ -functions of ordinary irreducible representations of  $GL(2, K)$ . In the appendix, we describe supercuspidal dihedral representations of  $GL(2, K)$  in terms of Langlands parameter.

## 1 Introduction

For  $K/F$  a quadratic extension of local fields, let  $\sigma$  be the conjugation relative to this extension, and  $\eta_{K/F}$  be the character of  $F^*$  with kernel norms of  $K^*$ . The conjugation  $\sigma$  extends naturally to an automorphism of  $GL(n, K)$ , and we still note this automorphism  $\sigma$ . If  $\pi$  is a representation of  $GL(n, K)$ , we note  $\pi^\sigma$  the representation  $g \mapsto \pi(\sigma(g))$ .

If  $\pi$  is a smooth irreducible representation of  $GL(n, K)$ , and  $\chi$  a character of  $F^*$ , the dimension of the space of linear forms on its space, which transform by  $\chi$  under  $GL(n, F)$  (with respect to the action  $[(L, g) \mapsto L \circ \pi(g)]$ ), is known to be at most one (Proposition 11, [F1]). One says that  $\pi$  is  $\chi$ -distinguished if this dimension is one, and says that  $\pi$  is distinguished if it is 1-distinguished. Jacquet conjectured two results about distinguished representations of  $GL(n, K)$ . Let  $\pi$  be a smooth irreducible representation of  $GL(n, K)$  and  $\pi^\vee$  its contragredient. The first conjecture states that it is equivalent for  $\pi$  with central character trivial on  $F^*$  to be isomorphic to  $\pi^{\vee\sigma}$  and for  $\pi$  to be distinguished or  $\eta_{K/F}$ -distinguished. In [K], Kable proved it for discrete series representations, using Asai  $L$ -functions.

The second conjecture, which is proved in [K], states that if  $\pi$  is a discrete series representation, then it cannot be distinguished and  $\eta_{K/F}$ -distinguished at the same time.

One of the key points in Kable's proof is that if a discrete series representation of  $GL(n, K)$  is such that its Asai  $L$ -function has a pole at zero, then it is distinguished, Theorem 1.4 of [A-K-T] shows that it is actually an equivalence. This theorem actually shows that Asai  $L$ -functions of tempered distinguished representations admit a pole at zero.

In this article, using a result of Youngbin Ok which states that for a distinguished representation, linear forms invariant under the affine subgroup of  $GL(n, F)$  are actually  $GL(n, F)$ -invariant (which generalises Corollary 1.2 of [A-K-T]), we prove in Theorem 3.1 that a generic representation is distinguished if and only if its Asai  $L$ -function admits an exceptional pole at zero. A pole at zero is always exceptional for Asai  $L$ -functions of discrete series representations (see explanation before Proposition 3.4). As a first application, we give in Proposition 3.6 a formula for Asai  $L$ -functions of supercuspidal representations of  $GL(n, K)$ .

There are actually three different ways to define Asai  $L$ -functions: one via the local Langlands correspondence and in terms of Langlands parameters denoted by  $L_W(\pi, s)$ , the one we use via the theory of Rankin-Selberg integrals denoted by  $L_{As}(\pi, s)$ , and the Langlands-Shahidi method applied to a suitable unitary group, denoted by  $L_{As,2}(\pi, s)$  (see [A-R]). It is expected that the above three  $L$ -functions are equal.

For a discrete series representation  $\pi$ , it is shown in [He] that  $L_W(\pi, s) = L_{As,2}(\pi, s)$ , and in

[A-R] that  $L_{As}(\pi, s) = L_{As,2}(\pi, s)$ , both proofs using global methods.

As a second application of our principal result, we show (by local methods) in Theorem 4.2 of section 4 that for an ordinary representation (i.e. corresponding through Langlands correspondence to an imprimitive 2 dimensional representation of the Weil-Deligne group)  $\pi$  of  $GL(2, K)$ , we have  $L_W(\pi, s) = L_{As}(\pi, s)$ . We recall that for odd residual characteristic, every smooth irreducible infinite dimensional representation of  $GL(2, K)$  is ordinary.

In the appendix (section 5), we describe in Theorem 5.4 distinguished dihedral supercuspidal representations, this description is used in section 4 for the computation of  $L_{As}$  for such representations.

## 2 Preliminaries

Let  $E_1$  be a field, and  $E_2$  a finite galois extension of  $E_1$ , we note  $Gal(E_2/E_1)$  the Galois group of  $E_2$  over  $E_1$ , and we note  $Tr_{E_2/E_1}$  (respectively  $N_{E_2/E_1}$ ) the trace (respectively the norm) application from  $E_2$  to  $E_1$ . If  $E_2$  is quadratic over  $E_1$ , we note  $\sigma_{E_2/E_1}$  the non trivial element of  $Gal(E_2/E_1)$ .

In the rest of this paper, the letter  $F$  will always designate a non archimedean local field of characteristic zero in a fixed algebraic closure  $\bar{F}$ , and the letter  $K$  a quadratic extension of  $F$  in  $\bar{F}$ . We note  $q_F$  and  $q_K$  the cardinality of their residual fields,  $R_K$  and  $R_F$  their integer rings,  $P_K$  and  $P_F$  the maximal ideals of  $R_K$  and  $R_F$ , and  $U_K$  and  $U_F$  their unit groups. We also note  $v_K$  and  $v_F$  the respective normalized valuations, and  $|\cdot|_K$  and  $|\cdot|_F$  the respective absolute values. We fix an element  $\delta$  of  $K - F$  such that  $\delta^2 \in F$ , hence  $K = F(\delta)$ .

Let  $\psi$  be a non trivial character of  $K$  trivial on  $F$ , it is of the form  $x \mapsto \psi' \circ Tr_{K/F}(\delta x)$  for some non trivial character  $\psi'$  of  $F$ .

Whenever  $G$  is an algebraic group defined over  $F$ , we note  $G(K)$  its  $K$ -points and  $G(F)$  its  $F$ -points. The group  $GL(n)$  is noted  $G_n$ , its standard maximal unipotent subgroup is noted  $N_n$ . If  $\pi$  is a representation of a group, we also note  $\pi$  its isomorphism class. Let  $\mu$  be a character of  $F^*$ , we say that a representation  $\pi$  of  $G_n(K)$  is  $\mu$ -distinguished if it admits on its space  $V_\pi$  a linear form  $L$ , which verifies the following: for  $v$  in  $V$  and  $h$  in  $G_n(K)$ , then  $L(\pi(h)v) = \mu(det(h))L(v)$ . If  $\mu = 1$ , we say that  $\pi$  is distinguished.

We note  $K_n(F)$  the maximal compact subgroup  $G_n(R_F)$  of  $G_n(F)$ , and for  $r \geq 1$ , we note  $K_{n,r}(F)$ , the congruence subgroup  $I_n + M_n(P_F^r)$ .

The character  $\psi$  defines a character of  $N_n(K)$  that we still note  $\psi$ , given by  $\psi(n) = \psi(\sum_{i=1}^{n-1} n_{i,i+1})$ .

We now recall standard results from [F2].

Let  $\pi$  be a generic smooth irreducible representation of  $G_n(K)$ , we note  $\pi^\vee$  its smooth contra-gradient, and  $c_\pi$  its central character.

We note  $D(F^n)$  the space of smooth functions with compact support on  $F^n$ , and  $D_0(F^n)$  the subspace of  $D(F^n)$  of functions vanishing at zero. We note  $\rho$  the natural action of  $G_n(F)$  on  $D(F^n)$ , given by  $\rho(g)\phi(x_1, \dots, x_n) = \phi((x_1, \dots, x_n)g)$ , and we note  $\eta$  the line vector  $(0, \dots, 0, 1)$  of length  $n$ .

If  $W$  belongs to the Whittaker model  $W(\pi, \psi)$  of  $\pi$ , and  $\phi$  belongs to  $D(F^n)$ , the following integral converges for  $s$  of real part large enough:

$$\int_{N_n(F) \backslash G_n(F)} W(g)\phi(\eta g)|det(g)|_F^s dg.$$

This integral as a function of  $s$  has a meromorphic extension to  $\mathbb{C}$  which we note  $\Psi(W, \phi, s)$ . For  $s$  of real part large enough, the function  $\Psi(W, \phi, s)$  is a rational function in  $q_F^{-s}$ , which actually has a Laurent series development.

The  $\mathbb{C}$ -vector space generated by these functions is in fact a fractional ideal  $I(\pi)$  of  $\mathbb{C}[q_F^{-s}, q_F^s]$ . This ideal  $I(\pi)$  is principal, and has a unique generator of the form  $1/P(q_F^{-s})$ , where  $P$  is a polynomial with  $P(0) = 1$ .

**Definition 2.1.** We note  $L_{As}(\pi, s)$  the generator of  $I(\pi)$  defined just above, and call it the Asai  $L$ -function of  $\pi$ .

**Remark 2.1.** If  $P$  belongs to  $\mathbb{C}[X]$  and has constant term equal to one, then the function of the complex variable  $L_P : s \mapsto 1/P(q_F^{-s})$  is called an Euler factor. It is a meromorphic function on  $\mathbb{C}$  and admits  $(2i\pi/Ln(q_F))\mathbb{Z}$  as a period subgroup. Hence if  $s_0$  is a pole of  $L_P$ , the elements  $s_0 + (2i\pi/Ln(q_F))\mathbb{Z}$  are also poles of  $L_P$ , with same multiplicities, we identify  $s_0$  and  $s_0 + (2i\pi/Ln(q_F))\mathbb{Z}$  when we talk about poles. A pole  $s_0$  then corresponds to a root  $\alpha_0$  of  $P$  by the formula  $q^{-s_0} = \alpha_0$ , its multiplicity in  $L_P$  equal to the multiplicity of  $\alpha_0$  in  $P$ .

Let  $w_n$  be the matrix of  $G_n(\mathbb{Z})$  with ones on the antidiagonal, and zeroes elsewhere. For  $W$  in  $W(\pi, \psi)$ , we note  $\tilde{W}$  the function  $g \mapsto W(w_n^t g^{-1})$  which belongs to  $W(\pi^\vee, \psi^{-1})$ , and we note  $\hat{\phi}$  the Fourier transform (with respect to  $\psi'$  and its associate autodual Haar measure) of  $\phi$  in  $D(F^n)$ .

**Theorem 2.1. (Functional equation)** (Th. of [F2])

There exists an epsilon factor  $\epsilon_{As}(\pi, s, \psi)$  which is, up to scalar, a (maybe negative) power of  $q^s$ , such that the following functional equation is satisfied for any  $W$  in  $W(\pi, \psi)$  and any  $\phi$  in  $D(F^n)$ :

$$\Psi(\tilde{W}, \hat{\phi}, 1-s)/L_{As}(\pi^\vee, 1-s) = c_\pi(-1)^{n-1} \epsilon_{As}(\pi, s, \psi) \Psi(W, \phi, s)/L_{As}(\pi, s).$$

### 3 Poles of the Asai $L$ -function and distinguishedness

Now suppose  $L_{As}(\pi, s)$  has a pole at  $s_0$ , its order  $d$  is the highest order pole of the family of functions of  $I(\pi)$ .

Then we have the following Laurent expansion at  $s_0$ :

$$\Psi(W, \phi, s) = B_{s_0}(W, \phi)/(q_F^s - q_F^{s_0})^d + \text{smaller order terms.} \quad (1)$$

The residue  $B_{s_0}(W, \phi)$  defines a non zero bilinear form on  $W(\pi, \psi) \times D(F^n)$ , satisfying the quasi-invariance:

$$B_{s_0}(\pi(g)W, \rho(g)\phi) = |\det(g)|_F^{-s_0} B_{s_0}(W, \phi).$$

Following [C-P] for the split case  $K = F \times F$ , we state the following definition:

**Definition 3.1.** A pole of the Asai  $L$ -function  $L_{As}(\pi, s)$  at  $s_0$  is called exceptional if the associated bilinear form  $B_{s_0}$  vanishes on  $W(\pi, \psi) \times D_0(F^n)$ .

As an immediate consequence, if  $s_0$  is an exceptional pole of  $L_{As}(\pi, s)$ , then  $B_{s_0}$  is of the form  $B_{s_0}(W, \phi) = \lambda_{s_0}(W)\phi(0)$ , where  $\lambda_{s_0}$  is a non zero  $|\det(\cdot)|_F^{-s_0}$  invariant linear form on  $W(\pi, \psi)$ . Hence we have:

**Proposition 3.1.** Let  $\pi$  be a generic irreducible representation of  $G_n(K)$ , and suppose its Asai  $L$ -function has an exceptional pole at zero, then  $\pi$  is distinguished.

We note  $P_n(F)$  the affine subgroup of  $G_n(F)$ , given by matrices with last row equal to  $\eta$ . For more convenience, we introduce a second  $L$ -function: for  $W$  in  $W(\pi, \psi)$ , by standard arguments, the following integral is convergent for  $Re(s)$  large, and defines a rational function in  $q^{-s}$ , which has a Laurent series development:

$$\int_{N_n(F) \backslash P_n(F)} W(p) |\det(p)|_F^s dp.$$

We note  $\Psi_1(W, s)$  the corresponding Laurent series. By standard arguments again, the vector space generated by the functions  $\Psi_1(W, s-1)$  is a fractional ideal  $I_1(\pi)$  of  $\mathbb{C}[q_F^{-s}, q_F^s]$ , which has a

unique generator of the form  $1/Q(q_F^{-s})$ , where  $Q$  is a polynomial with  $Q(0) = 1$ . We note  $L_1(\pi, s)$  this generator.

**Lemma 3.1.** (*[J-P-S] p. 393*)

Let  $W$  be in  $W(\pi, \psi)$ , one can choose  $\phi$  with support small enough around  $(0, \dots, 0, 1)$  such that  $\Psi(W, \phi, s) = \Psi_1(W, s - 1)$ .

*Proof.* As we gave a reference, we only sketch the proof. We first recall the following integration formula (cf. proof of the proposition in paragraph 4 of [F]), for  $\text{Re}(s) > 0$ :

$$\Psi(W, \phi, s) = \int_{K_n(F)} \int_{N_n(F) \setminus P_n(F)} W(pk) |det(p)|_F^{s-1} dp \int_{F^*} \phi(\eta ak) c_\pi(a) |a|_F^{ns} d^* adk. \quad (2)$$

Choosing  $r$  large enough for  $W$  to be right invariant under  $K_{n,r}(F)$ , we take  $\phi$  a positive multiple of the characteristic function of  $\eta K_{n,r}(F)$ , and conclude from equation (2).  $\square$

Hence we have the inclusion  $I_1(\pi) \subset I(\pi)$ , which implies that  $L_1(\pi, s) = L_{As}(\pi, s) R(q_F^s, q_F^{-s})$  for some  $R$  in  $\mathbb{C}[q_F^{-s}, q_F^s]$ . But because  $L_1$  and  $L_{As}$  are both Euler factors,  $R$  is actually just a polynomial in  $q_F^{-s}$ , with constant term equal to one. Noting  $L_{rad(ex)}(\pi, s)$  its inverse (which is an Euler factor), we have  $L_{As}(\pi, s) = L_1(\pi, s) L_{rad(ex)}(\pi, s)$ , we will say that  $L_1$  divides  $L_{As}$ . The explanation for the notation  $L_{rad(ex)}$  is given in Remark 3.1.

We now give a characterisation of exceptional poles:

**Proposition 3.2.** *A pole of  $L_{As}(\pi, s)$  is exceptional if and only if it is a pole of the function  $L_{rad(ex)}(\pi, s)$  defined just above.*

*Proof.* From equation (2), it becomes clear that the vector space generated by the integrals  $\Psi(W, \phi, s)$  with  $W$  in  $W(\pi, \psi)$  and  $\phi$  in  $D_0(F^n)$ , is contained in  $I_1(\pi)$ , but because of Lemma 3.1, those two vector spaces are equal. Hence  $L_1(\pi, s)$  is a generator of the ideal generated as a vector space by the functions  $\Psi(W, \phi, s)$  with  $W$  in  $W(\pi, \psi)$  and  $\phi$  in  $D_0(F^n)$ .

From equation (1), if  $s_0$  is an exceptional pole, a function  $\Psi(W, \phi, s)$ , with  $\phi$  in  $D_0(F^n)$ , cannot have a pole of highest order at  $s_0$ , hence we have one implication.

Now if the order of the pole  $s_0$  for  $L_{As}(\pi, s)$  is strictly greater than the one of  $L_1(\pi, s)$ , then the first residual term corresponding to a pole of highest order of the Laurent development of any function  $\Psi(W, \phi, s)$  with  $\phi(0) = 0$  must be zero, and zero is exceptional.  $\square$

Lemma 3.1 also implies:

**Proposition 3.3.** *The functional  $\Lambda_{\pi, s} : W \mapsto \Psi_1(W, s - 1)/L_{As}(\pi, s)$  defines a (maybe null) linear form on  $W(\pi, \psi)$  which transforms by  $|det(\cdot)|_F^{1-s}$  under the affine subgroup  $P_n(F)$ . For fixed  $W$  in  $W(\pi, \psi)$ , then  $s \mapsto \Lambda_{\pi, s}(W)$  is a polynomial of  $q^{-s}$ .*

Now we are able to prove the converse of Proposition 3.1:

**Theorem 3.1.** *A generic irreducible representation  $\pi$  of  $G_n(K)$  is distinguished if and only if  $L_{As}(s, \pi)$  admits an exceptional pole at zero.*

*Proof.* We only need to prove that if  $\pi$  is distinguished, then  $L_{As}(s, \pi)$  admits an exceptional pole at zero, so we suppose  $\pi$  distinguished.

From equation (2), for  $\text{Re}(s) < 0$ , and  $\pi$  distinguished (so that  $c_\pi$  has trivial restriction to  $F^*$ ), one has:

$$\Psi(\tilde{W}, \hat{\phi}, 1 - s) = \int_{K_n(F)} \int_{N_n(F) \setminus P_n(F)} \tilde{W}(pk) |det(p)|_F^{-s} dp \int_{F^*} \hat{\phi}(\eta ak) |a|_F^{n(1-s)} d^* adk. \quad (3)$$

This implies that:

$$\Psi(\tilde{W}, \hat{\phi}, 1-s)/L_{As}(\pi^\vee, 1-s) = \int_{K_n(F)} \Lambda_{\pi^\vee, 1-s}(\pi^\vee(k)\tilde{W}) \int_{F^*} \hat{\phi}(\eta ak) |a|_F^{n(1-s)} d^* adk. \quad (4)$$

The second member of the equality is actually a finite sum:  $\sum_i \lambda_i \Lambda_{\pi^\vee, 1-s}(\pi^\vee(k_i)\tilde{W}) \int_{F^*} \hat{\phi}(\eta ak_i) |a|_F^{n(1-s)} d^* a$ , where the  $\lambda_i$ 's are positive constants and the  $k_i$ 's are elements of  $K_n(F)$  independant of  $s$ .

Note that there exists a positive constant  $\epsilon$ , such that for  $Re(s) < \epsilon$ , the integral  $\int_{F^*} \hat{\phi}(\eta ak_i) |a|_F^{n(1-s)} d^* a$  is absolutely convergent, and defines a holomorphic function. So we have an equality (equality 4) of analytic functions (actually of polynomials in  $q_F^{-s}$ ), hence it is true for all  $s$  such that  $Re(s) < \epsilon$ . For  $s = 0$ , we get:

$$\Psi(\tilde{W}, \hat{\phi}, 1)/L_{As}(\pi^\vee, 1) = \int_{K_n(F)} \Lambda_{\pi^\vee, 1}(\pi^\vee(k)\tilde{W}) \int_{F^*} \hat{\phi}(\eta ak) |a|_F^n d^* adk.$$

But as  $\pi$  is distinguished, so is  $\pi^\vee$ , and as  $\Lambda_{\pi^\vee, 1}$  is a  $P_n(F)$ -invariant linear form on  $W(\pi^\vee, \psi^{-1})$ , it follows from theorem 3.1.2 of [Ok] that it is actually  $G_n(F)$ -invariant. Finally

$$\Psi(\tilde{W}, \hat{\phi}, 1)/L_{As}(\pi^\vee, 1) = \Lambda_{\pi^\vee, 1}(\tilde{W}) \int_{K_n(F)} \int_{F^*} \hat{\phi}(\eta ak) |a|_F^n d^* adk$$

which is equal to:

$$\Lambda_{\pi^\vee, 1}(\tilde{W}) \int_{P_n(F) \backslash G_n(F)} \hat{\phi}(\eta g) d_\mu g$$

where  $d_\mu$  is up to scalar the unique  $|det(\cdot)|^{-1}$  invariant measure on  $P_n(F) \backslash G_n(F)$ . But as

$$\int_{P_n(G) \backslash G_n(F)} \hat{\phi}(\eta g) d_\mu g = \int_{F^n} \hat{\phi}(x) dx = \phi(0),$$

we deduce from the functional equation that  $\Psi(W, \phi, 0)/L_{As}(\pi, 0) = 0$  whenever  $\phi(0) = 0$ .

As one can choose  $W$ , and  $\phi$  vanishing at zero, such that  $\Psi(W, \phi, s)$  is the constant function equal to 1 (see the proof of Theorem 1.4 in [A-K-T]), hence  $L_{As}(\pi, s)$  has a pole at zero, which must be exceptional.  $\square$

For a discrete series representation  $\pi$ , it follows from Lemma 2 of [K], that the integrals of the form

$$\int_{N_n(F) \backslash P_n(F)} W(p) |det(p)|_F^{s-1} dp.$$

converge absolutely for  $Re(s) > -\epsilon$  for some positive  $\epsilon$ , hence as functions of  $s$ , they cannot have a pole at zero.

This implies that  $L_1(\pi, s)$  has no pole at zero, hence Theorem 3.1 in this case gives:

**Proposition 3.4.** (*[K], Theorem 4*)

*A discrete series representation  $\pi$  of  $G_n(K)$  is distinguished if and only if  $L_{As}(s, \pi)$  admits a pole at zero.*

Let  $s_0$  be in  $\mathbb{C}$ . We notice that if  $\pi$  is a generic irreducible representation of  $G_n(K)$ , it is  $|\cdot|_F^{s_0}$ -distinguished if and only if  $\pi \otimes |\cdot|_K^{s_0/2}$  is distinguished, but as  $L_{As}(s, \pi \otimes |\cdot|_K^{s_0/2})$  is equal to  $L_{As}(s + s_0, \pi)$ , Theorem 3.1 becomes:

**Theorem 3.2.** *A generic irreducible representation  $\pi$  of  $G_n(K)$  is  $|\cdot|_F^{s_0}$ -distinguished if and only if  $L_{As}(s, \pi)$  admits an exceptional pole at  $s_0$ .*

**Remark 3.1.** Let  $P$  and  $Q$  be two polynomials in  $\mathbb{C}[X]$  with constant term 1, we say that the Euler factor  $L_P(s) = 1/P(q_F^{-s})$  divides  $L_Q(s) = 1/Q(q_F^{-s})$  if and only  $P$  divides  $Q$ . We note  $L_P \vee L_Q$  the Euler factor  $1/(P \vee Q)(q_F^{-s})$ , where the l.c.m  $P \vee Q$  is chosen such that  $(P \vee Q)(0) = 1$ . We define the g.c.d  $L_P \wedge L_Q$  the same way.

It follows from equation (2) that if  $c_{\pi|F^*}$  is ramified, then  $L_{As}(\pi, s) = L_1(\pi, s)$ . It also follows from the same equation that if  $c_{\pi|F^*} = | \cdot |_F^{-s_1}$  for some  $s_1$  in  $\mathbb{C}$ , then  $L_{rad(ex)}(\pi, s)$  divides  $1/(1 - q_F^{s_1 - ns})$ . Anyway,  $L_{rad(ex)}(\pi, s)$  has simple poles.

Now we can explain the notation  $L_{rad(ex)}$ . We refer to [C-P] where the case  $K = F \times F$  is treated. In fact, in the latter,  $L_{ex}(\pi, s)$  is the function  $1/P_{ex}(\pi, q_F^{-s})$ , with  $P_{ex}(\pi, q_F^{-s}) = \prod_{s_i} (1 - q_F^{s_i - s})^{d_i}$ , where the  $s_i$ 's are the exceptional poles of  $L_{As}(\pi, s)$  and the  $d_i$ 's their order in  $L_{As}(\pi, s)$ . Hence  $L_{rad(ex)}(\pi, s) = 1/P_{rad(ex)}(\pi, q_F^{-s})$ , where  $P_{rad(ex)}(\pi, X)$  is the unique generator with constant term equal to one, of the radical of the ideal generated by  $P_{ex}(\pi, X)$  in  $\mathbb{C}[X]$ .

We proved:

**Proposition 3.5.** *Let  $\pi$  be an irreducible generic representation of  $G_n(K)$ , the Euler factor  $L_{rad(ex)}(\pi, s)$  has simple poles, it is therefore equal to  $\prod 1/(1 - q_F^{s_0 - s})$  where the product is taken over the  $q_F^{s_0}$ 's such that  $\pi$  is  $| \cdot |_F^{-s_0}$ -distinguished.*

Suppose now that  $\pi$  is supercuspidal, then the restriction to  $P_n(K)$  of any  $W$  in  $W(\pi, \psi)$  has compact support modulo  $N_n(K)$ , hence  $\Psi_1(W, s - 1)$  is a polynomial in  $q^{-s}$ , and  $L_1(\pi, s)$  is equal to 1. Hence Proposition 3.5 becomes:

**Proposition 3.6.** *Let  $\pi$  be an irreducible supercuspidal representation of  $G_n(K)$ , then  $L_{As}(\pi, s) = \prod 1/(1 - q^{s_0 - s})$  where the product is taken over the  $q^{s_0}$ 's such that  $\pi$  is  $| \cdot |_F^{-s_0}$ -distinguished.*

## 4 Asai $L$ -functions of $GL(2)$

### 4.1 Asai $L$ -functions for imprimitive Weil-Deligne representations of dimension 2

The aim of this paragraph is to compute  $L_W(\rho, s)$  (see the introduction) when  $\rho$  is an imprimitive two dimensional representation of the Weil-Deligne group of  $K$ .

We note  $W_K$  (resp.  $W_F$ ) the Weil group of  $K$  (resp.  $F$ ),  $I_K$  (resp.  $I_F$ ) the inertia subgroup of  $W_K$  (resp.  $W_F$ ),  $W'_K$  (resp.  $W'_F$ ) the group  $W_K \times SL(2, \mathbb{C})$  (resp.  $W_F \times SL(2, \mathbb{C})$ ) and  $I'_K$  (resp.  $I'_F$ ) the group  $I_K \times SL(2, \mathbb{C})$  (resp.  $I_F \times SL(2, \mathbb{C})$ ). We note  $\phi_F$  a Frobenius element of  $W_F$ , and we also note  $\phi'_F$  the element  $(\phi_F, I_2)$  of  $W'_F$ .

We note  $sp(n)$  the unique (up to isomorphism) complex irreducible representation of  $SL(2, \mathbb{C})$  of dimension  $n$ .

If  $\rho$  is a finite dimensional representation of  $W'_K$ , we note  $M_{W'_K}^{W'_F}(\rho)$  the representation of  $W'_F$  induced multiplicatively from  $\rho$ . We recall its definition:

If  $V$  is the space of  $\rho$ , then the space of  $M_{W'_K}^{W'_F}(\rho)$  is  $V \otimes V$ . Noting  $\tau$  an element of  $W_F - W_K$ , and  $\sigma$  the element  $(\tau, I)$  of  $W'_F$ , we have:

$$M_{W'_K}^{W'_F}(\rho)(h)(v_1 \otimes v_2) = \rho(h)v_1 \otimes \rho^\sigma(h)v_2$$

for  $h$  in  $W'_K$ ,  $v_1$  and  $v_2$  in  $V$ .

$$M_{W'_K}^{W'_F}(\rho)(\sigma)(v_1 \otimes v_2) = \rho(\sigma^2)v_2 \otimes v_1$$

for  $v_1$  and  $v_2$  in  $V$ .

We refer to paragraph 7 of [P] for definition and basic propositionerties of multiplicative induction in the general case.

**Definition 4.1.** The function  $L_W(\rho, s)$  is by definition the usual  $L$ -function of the representation  $M_{W'_K}^{W'_F}(\rho)$ , i.e.  $L_W(\rho, s) = L(M_{W'_K}^{W'_F}(\rho), s)$ .

- i) If  $\rho$  is of the form  $\text{Ind}_{W'_B}^{W'_K}(\omega)$  for some multiplicative character  $\omega$  of a biquadratic extension  $B$  of  $F$ , we note  $K'$  and  $K''$  the two other extensions between  $F$  and  $B$ . If we call  $\sigma_1$  an element of  $W'_K$  which is not in  $W'_{K'} \cup W'_{K''}$  and  $\sigma_3$  an element of  $W'_{K''}$  which is not in  $W'_K \cup W'_{K'}$ , then  $\sigma_2 = \sigma_3 \sigma_1$  is an element of  $W'_{K'}$  which is not in  $W'_K \cup W'_{K''}$ . The elements  $(1, \sigma_1, \sigma_2, \sigma_3)$  are representatives of  $W'_F/W'_B$ , and 1 and  $\sigma_3$  are representatives of  $W'_F/W'_K$ .

If one identifies  $\omega$  with a character (still called  $\omega$ ) of  $B^*$ , then  $\omega^{\sigma_1}$  identifies with  $\omega \circ \sigma_{B/K}$ ,  $\omega^{\sigma_2}$  with  $\omega \circ \sigma_{B/K'}$  and  $\omega^{\sigma_3}$  with  $\omega \circ \sigma_{B/K''}$ . One then verifies that if  $a$  belongs to  $W_B$ , one has:

- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(a)] = \omega\omega^{\sigma_2} + \omega\omega^{\sigma_3} + \omega^{\sigma_1}\omega^{\sigma_2} + \omega^{\sigma_1}\omega^{\sigma_3}.$
- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(\sigma_1 a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_1 a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_1 a)] = 0.$
- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(\sigma_2 a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_2 a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_2 a)] = \omega(\sigma_2 a \sigma_2 a) + \omega^{\sigma_1}(\sigma_2 a \sigma_2 a).$
- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(\sigma_3 a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_3 a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_3 a)] = \omega(\sigma_3 a \sigma_3 a) + \omega^{\sigma_1}(\sigma_3 a \sigma_3 a).$

Hence we have the isomorphism

$$M_{W'_K}^{W'_F}(\rho) \simeq \text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega)) \oplus \text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega)).$$

From this we deduce that

$$L(M_{W'_K}^{W'_F}(\rho), s) = L(\omega|_{K'^*}, s) L(\omega|_{K''^*}, s).$$

- ii) Let  $L$  be a quadratic extension of  $F$ , such that  $\rho = \text{Ind}_{W'_L}^{W'_K}(\chi)$ , with  $\chi$  regular, is not isomorphic to a representation of the form  $\text{Ind}_{W'_B}^{W'_K}(\omega)$  as in i), then

$$L(M_{W'_K}^{W'_F}(\rho), s) = 1.$$

Indeed, we show that  $M_{W'_K}^{W'_F}(\rho)^{I'_F} = \{0\}$ . If it wasn't the case, the representation  $(M_{W'_K}^{W'_F}(\rho), V)$  would admit a  $I'_F$ -fixed vector, and so would its contragredient  $V^*$ . Now in the subspace of  $I'_F$ -fixed vectors of  $V^*$ , choosing an eigenvector of  $M_{W'_K}^{W'_F}(\rho)(\phi_F)$ , we would deduce the existence of a linear form  $L$  on  $(M_{W'_K}^{W'_F}(\rho), V)$  which transforms under  $W'_F$  by an unramified character  $\mu$  of  $W'_F$ . If we identify  $\mu$  with a character  $\mu'$  of  $F^*$ , the restriction of  $\mu$  to  $W'_K$  corresponds to  $\mu' \circ N_{K/F}$  of  $K^*$ , so we can write it as  $\theta\theta^\sigma$ , where  $\theta$  is a character of  $W'_K$  corresponding to an extension of  $\mu'$  to  $K^*$ . As the restriction of  $M_{W'_K}^{W'_F}$  to  $W'_K$  is isomorphic to  $\rho \otimes \rho^\sigma$ , we deduce that  $\theta^{-1}\rho \otimes (\theta^{-1}\rho)^\sigma$  is  $W'_K$  distinguished, that is  $\theta\rho^{vee} \simeq (\theta^{-1}\rho)^\sigma$ . But from the proof of Theorem 5.2, this would imply that  $\theta^{-1}\rho$  hence  $\rho$ , could be induced from a character of a biquadratic extension of  $F$ , which we supposed is not the case.

- iii) Suppose  $\rho = sp(2)$  acts on the space  $\mathbb{C}^2$  with canonical basis  $(e_1, e_2)$  by the natural action  $\rho[h, M](v) = M(v)$  for  $h$  in  $W_K$ ,  $M$  in  $SL(2, \mathbb{C})$  and  $v$  in  $\mathbb{C}^2$ . Then the space of  $M_{W'_K}^{W'_F}(\rho)$  is  $V \otimes V$  and  $SL(2, \mathbb{C})$  acts on it as  $sp(2) \otimes sp(2)$ . Decomposing  $V \otimes V$  as the direct sum  $\text{Alt}(V) \oplus$



$Sym(V)$ , we see that  $SL(2, \mathbb{C})$  acts as 1 on  $Alt(V)$ , and  $M_{W'_K}^{W'_F}(\rho) \left[ 1, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \right] (e_1 \otimes e_1) = x^2 e_1 \otimes e_1$ . Hence the representation of  $SL(2, \mathbb{C})$  on  $Sym(V)$  must be  $sp(3)$ . The Weil group  $W_F$  acts as  $\eta_{K/F}$  on  $Alt(V)$  and trivially on  $Sym(V)$ , finally  $M_{W'_K}^{W'_F}(\rho)$  is isomorphic to  $sp(3) \oplus \eta_{K/F}$ . Tensoring with a character  $\chi$ , we have  $M_{W'_K}^{W'_F}(\chi sp(2)) = \chi|_{F^*} M_{W'_K}^{W'_F}(sp(2)) = \chi|_{F^*} \eta_{K/F} \oplus \chi|_{F^*} sp(3)$ . Hence one has the following equality:

$$L(M_{W'_K}^{W'_F}(\chi sp(2)), s) = L(\chi|_{F^*} \eta_{K/F}, s) L(\chi|_{F^*}, s+1).$$

iv) If  $\rho = \lambda \oplus \mu$ , with  $\lambda$  and  $\mu$  two characters of  $W'_K$ , then from [P], Lemma 7.1, we have  $M_{W'_K}^{W'_F}(\rho) = \lambda|_{F^*} \oplus \mu|_{F^*} \oplus Ind_{W'_K}^{W'_F}(\lambda\mu^\sigma)$ . Hence we have

$$L(M_{W'_K}^{W'_F}(\rho)) = L(\lambda|_{F^*}, s) L(\mu|_{F^*}, s) L(\lambda\mu^\sigma, s).$$

## 4.2 Asai $L$ -functions for ordinary representations of $GL(2)$

In this subsection, we compute Asai  $L$ -functions for ordinary (i.e. non exceptional) representations of  $G_2(K)$ , and prove (Theorem 4.2) that they are equal to the corresponding functions  $L_W$  of imprimitive representations of  $W'_K$ .

In order to compute  $L_{As}$ , we first compute  $L_1$ , but this latter computation is easy because Kirillov models of infinite dimensional irreducible representations of  $G_2(K)$  are well-known (see [B], Th. 4.7.2 and 4.7.3).

Let  $\pi$  be an irreducible infinite dimensional (hence generic) representation of  $G_2(K)$ , we have the following situations for the computation of  $L_1(\pi, s)$ .

i) and ii) If  $\pi$  is supercuspidal, its Kirillov model consists of functions with compact support on  $K^*$ , hence

$$L_1(\pi, s) = 1.$$

iii) If  $\pi = \sigma(\chi)$  ( $\sigma(\chi|_K^{1/2}, \chi|_K^{-1/2})$  in [B]) is a special series representation of  $G_2(K)$ , twist of the Steinberg representation by the character  $\chi$  of  $K^*$ , the Kirillov model of  $\pi$  consists of functions of  $D(K)$  multiplied by  $\chi|_K$ . Hence their restrictions to  $F$  are functions of  $D(F)$  multiplied by  $\chi|_F^2$ , and the ideal  $I_1(\pi)$  is generated by functions of  $s$  of the form

$$\int_{F^*} \phi(t) \chi(t) |t|_F^{s-1} |t|_F^2 d^*t = \int_{F^*} \phi(t) \chi(t) |t|_F^{s+1} d^*t,$$

for  $\phi$  in  $D(F)$ , hence we have

$$L_1(\pi, s) = L(\chi|_{F^*}, s+1).$$

iv) If  $\pi = \pi(\lambda, \mu)$  is the principal series representation ( $\lambda$  and  $\mu$  being two characters of  $K^*$ , with  $\lambda\mu^{-1}$  different from  $| \cdot |$  and  $| \cdot |^{-1}$ ) corresponding to the representation  $\lambda \oplus \mu$  of  $W'_K$ .

If  $\lambda \neq \mu$ , the Kirillov model of  $\pi$  is given by functions of the form  $| \cdot |_K^{1/2} \chi \phi_1 + | \cdot |_K^{1/2} \mu \phi_2$ , for  $\phi_1$  and  $\phi_2$  in  $D(K)$ , and

$$L_1(\pi, s) = L(\lambda|_{F^*}, s) \vee L(\mu|_{F^*}, s).$$

If  $\lambda = \mu$ , the Kirillov model of  $\pi$  is given by functions of the form  $| \cdot |_K^{1/2} \lambda \phi_1 + | \cdot |_K^{1/2} \lambda v_K(t) \phi_2$ , for  $\phi_1$  and  $\phi_2$  in  $D(K)$ , and

$$L_1(\pi, s) = L(\lambda|_{F^*}, s)^2.$$

In order to compute  $L_{rad(ex)}$  for ordinary representations, we need to know when they are distinguished by a character  $|\cdot|_F^{-s_0}$  for some  $s_0 \in \mathbb{C}$ , we will then use Theorem 3.2. The answer is given by the following, which is a mix of Theorem 5.4 and proposition B.17 of [F-H]:

- Theorem 4.1.** a) *A dihedral supercuspidal representation  $\pi$  of  $G_2(K)$  is  $|\cdot|_F^{-s_0}$ -distinguished if and only if there exists a quadratic extension  $B$  of  $K$ , biquadratic over  $F$  (hence there are two other extensions between  $F$  and  $B$  that we call  $K'$  and  $K''$ ), and a character of  $B^*$  regular with respect to  $N_{B/K}$  which restricts either to  $K'$  as  $|\cdot|_{K'}^{-s_0}$  or to  $K''$  as  $|\cdot|_{K''}^{-s_0}$ , such that  $\pi$  is equal to  $\pi(\omega)$ .*
- b) *Let  $\mu$  be a character of  $K^*$ , then the sepcial series representation  $\sigma(\mu)$  is  $|\cdot|_F^{-s_0}$ -distinguished if and only if  $\mu$  restricts to  $F^*$  as  $\eta_{K/F}|\cdot|_F^{-s_0}$ .*
- c) *Let  $\lambda$  and  $\mu$  be two characters of  $K^*$ , with  $\lambda\mu^{-1}$  and  $\lambda^{-1}\mu$  different from  $|\cdot|_K$ , then the principal series representation  $\pi(\lambda, \mu)$  is  $|\cdot|_F^{-s_0}$ -distinguished if and only if either  $\lambda$  and  $\mu$  restrict as  $|\cdot|_F^{-s_0}$  to  $F^*$  or  $\lambda\mu^\sigma$  is equal to  $|\cdot|_K^{-s_0}$ .*

*Proof.* Let  $\pi$  be a representation, it is  $|\cdot|_F^{-s_0}$ -distinguished if and only if  $\pi \otimes |\cdot|_K^{s_0/2}$  is distinguished because  $|\cdot|_K^{-s_0/2}$  extends  $|\cdot|_F^{-s_0}$ , it then suffices to apply Theorem 5.4 and proposition B.17 of [F-H]. We give the full proof for case a). Suppose  $\pi$  is dihedral supercuspidal and  $\pi \otimes |\cdot|_K^{s_0/2}$  is distinguished. From Theorem 5.4, the representation  $\pi \otimes |\cdot|_K^{s_0/2}$  must be of the form  $\pi(\omega)$ , for  $\omega$  a character of quadratic extension  $B$  of  $K$ , biquadratic over  $F$ , such that if we call  $K'$  and  $K''$  two other extensions between  $F$  and  $B$ ,  $\omega$  doesn't factorize through  $N_{B/K}$  and restricts either trivially on  $K'^*$ , or trivially on  $K''^*$ . But  $\pi$  is equal to  $\pi(\omega) \otimes |\cdot|_K^{-s_0/2} = \pi(\omega|_B^{-s_0/2})$  because  $|\cdot|_B = |\cdot|_K \circ N_{B/K}$ . As  $|\cdot|_B^{-s_0/2}$  restricts to  $K'$  (resp.  $K''$ ) as  $|\cdot|_{K'}^{-s_0}$  (resp.  $|\cdot|_{K''}^{-s_0}$ ), case a) follows.  $\square$

We are now able to compute  $L_{rad(ex)}$ , hence  $L_{As}$  for ordinary representations.

- i) Suppose that  $\pi = \pi(Ind_{W_B'}^{W_K'}(\omega)) = \pi(\omega)$  is supercuspidal, with Langlands parameter  $Ind_{W_B'}^{W_K'}(\omega)$ , where  $\omega$  is a multiplicative character of a biquadratic extension  $B$  over  $F$  that doesn't factorize through  $N_{B/K}$ . We note  $K'$  and  $K''$  the two other extensions between  $B$  and  $F$ . Here  $L_1(\pi, s)$  is equal to one. We have the following series of equivalences:

$$\begin{aligned} s_0 \text{ is a pole of } L_{As}(\pi(\omega), s) &\iff \pi(\omega) \text{ is } |\cdot|_F^{-s_0} \text{-distinguished} \\ &\iff \omega|_{K'^*} = |\cdot|_{K'}^{-s_0} \text{ or } \omega|_{K''^*} = |\cdot|_{K''}^{-s_0} \\ &\iff s_0 \text{ is a pole of } L(\omega|_{K'^*}, s) \text{ or of } L(\omega|_{K''^*}, s) \\ &\iff s_0 \text{ is a pole of } L(\omega|_{K'^*}, s) \vee L(\omega|_{K''^*}, s) \end{aligned}$$

As both functions  $L_{As}(\pi(\omega), s)$  and  $L(\omega|_{K'^*}, s) \vee L(\omega|_{K''^*}, s)$  have simple poles and are Euler factors, they are equal. Now suppose that  $L(\omega|_{K'^*}, s)$  and  $L(\omega|_{K''^*}, s)$  have a common pole  $s_0$ , this would imply that  $\omega|_{K'^*} = |\cdot|_{K'}^{-s_0}$  and  $\omega|_{K''^*} = |\cdot|_{K''}^{-s_0}$ , which would mean that  $\omega|_B^{-s_0/2}$  is trivial on  $K'^*K''^*$ . According to Lemma 5.2, this would contradict the fact that  $\omega$  does not factorize through  $N_{B/K}$ , hence  $L(\omega|_{K'^*}, s) \vee L(\omega|_{K''^*}, s) = L(\omega|_{K'^*}, s)L(\omega|_{K''^*}, s)$ . Finally we proved:

$$\boxed{L_{As}(\pi(\omega), s) = L(\omega|_{K'^*}, s)L(\omega|_{K''^*}, s).}$$

- ii) Suppose that  $\pi$  is a supercuspidal representation, corresponding to an imprimitive representation of  $W_K'$  that cannot be induced from a character of the Weil-Deligne group of a biquadratic extension of  $F$ . Then necessarily  $\pi$  cannot be  $|\cdot|_F^{-s_0}$ -distinguished, for any complex number  $s_0$  of  $\mathbb{C}$ .

If it was the case, from Theorem 4.1, it would correspond to a Weil representation  $\pi(\omega)$  for some multiplicative character of a biquadratic extension of  $F$ , which cannot be. Hence  $L_{rad(ex)}(\pi, s)$  has no pole and is equal to one because it is an Euler factor, so we proved that:

$$\boxed{L_{As}(\pi, s) = 1.}$$

- iii) If  $\pi$  is equal to  $\sigma(\chi)$ , then  $L_1(\pi, s) = L(\chi|_{F^*}, s+1)$ . We want to compute  $L_{rad(ex)}(\pi, s)$ , we have the following series of equivalences:

$$\begin{aligned} s_0 \text{ is an exceptional pole of } L_{As}(\sigma(\chi), s) &\iff \sigma(\chi) \text{ is } | \cdot |_F^{-s_0} - \text{distinguished} \\ &\iff \chi|_{F^*} = \eta_{K/F} | \cdot |_F^{-s_0} \\ &\iff s_0 \text{ is a pole of } L(\chi|_{F^*} \eta_{K/F}, s) \end{aligned}$$

As both functions  $L_{rad(ex)}(\pi, s)$  and  $L(\chi|_{F^*} \eta_{K/F}, s)$  have simple poles and are Euler factors, they are equal, we thus have:

$$\boxed{L_{As}(\sigma(\chi)) = L(\chi|_{F^*}, s+1)L(\chi|_{F^*} \eta_{K/F}, s).}$$

- iv) If  $\pi = \pi(\lambda, \mu)$ , we first compute  $L_{rad(ex)}(\pi, s)$ . We have the following series of equivalences:

$$\begin{aligned} s_0 \text{ is an exceptional pole of } L_{As}(\pi(\lambda, \mu), s) &\iff \pi(\lambda, \mu) \text{ is } | \cdot |_F^{-s_0} - \text{distinguished} \\ &\iff \lambda\mu^\sigma = | \cdot |_K^{-s_0} \text{ or, } \lambda|_{F^*} = | \cdot |_F^{-s_0} \text{ and } \mu|_{F^*} = | \cdot |_F^{-s_0} \\ &\iff s_0 \text{ is a pole of } L(\lambda\mu^\sigma, s) \text{ or of } L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s) \\ &\iff s_0 \text{ is a pole of } L(\lambda\mu^\sigma, s) \vee [L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)] \end{aligned}$$

As both functions  $L_{rad(ex)}(\pi(\lambda, \mu), s)$  and  $L(\lambda\mu^\sigma, s) \vee [L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)]$  have simple poles and are Euler factors, they are equal.

If  $\lambda \neq \mu$ , then  $L_1(\pi, s) = L(\lambda|_{F^*}, s) \vee L(\mu|_{F^*}, s)$ . But  $L(\lambda\mu^\sigma, s)$  and  $L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)$  have no common pole. If there was a common pole  $s_0$ , one would have  $\lambda\mu^\sigma = | \cdot |_K^{-s_0}$ ,  $\lambda|_{F^*} = | \cdot |_F^{-s_0}$  and  $\mu|_{F^*} = | \cdot |_F^{-s_0}$ . From  $\mu|_{F^*} = | \cdot |_F^{-s_0}$ , we would deduce that  $\mu \circ N_{K/F} = | \cdot |_K^{-s_0}$ , i.e.  $\mu^\sigma = | \cdot |_K^{-s_0} \mu^{-1}$ , and  $\lambda\mu^\sigma = | \cdot |_K^{-s_0}$  would imply  $\lambda = \mu$ , which is absurd. Hence  $L_{rad(ex)}(\pi, s) = L(\lambda\mu^\sigma, s)[L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)]$ , and finally we have  $L_{As}(\pi, s) = L_1(\pi, s)L_{rad(ex)}(\pi, s) = L(\lambda|_{F^*}, s)L(\mu|_{F^*}, s)L(\lambda\mu^\sigma, s)$ .

If  $\lambda$  is equal to  $\mu$ , then  $L_1(\pi, s) = L(\lambda|_{F^*}, s)^2$ , and  $L_{rad(ex)}(\pi(\lambda, \mu), s) = L(\lambda \circ N_{K/F}, s) \vee L(\lambda|_{F^*}, s)$ . As  $L(\lambda \circ N_{K/F}, s) = L(\lambda|_{F^*}, s)L(\eta_{K/F}\lambda|_{F^*}, s)$ , we have  $L_{rad(ex)}(\pi(\lambda, \mu), s) = L(\lambda \circ N_{K/F}, s)$ . Again we have  $L_{As}(\pi, s) = L(\lambda|_{F^*}, s)L(\mu|_{F^*}, s)L(\lambda\mu^\sigma, s)$ .

In both cases, we have

$$\boxed{L_{As}(\pi(\lambda, \mu), s) = L(\lambda|_{F^*}, s)L(\mu|_{F^*}, s)L(\lambda\mu^\sigma, s).}$$

Eventually, comparing with equalities of subsection 4.1, we proved the following:

**Theorem 4.2.** *Let  $\rho \mapsto \pi(\rho)$  be the Langlands correspondence from two dimensional representations of  $W'_K$  to smooth irreducible infinite dimensional representations of  $G_2(K)$ , then if  $\rho$  is not primitive,  $\pi(\rho)$  is ordinary and we have the following equality of L-functions:*

$$\boxed{L_{As}(\pi(\rho), s) = L(M_{W'_K}^{W'_F}(\rho), s)}$$

As said in the introduction, combining Theorem 1.6 of [A-R] and Theorem of paragraph 1.5 in [He], one gets that  $L(M_{W'_K}^{W'_F}(\rho), s) = L_{As}(\pi(\rho), s)$  for  $\pi(\rho)$  a discrete series representation, so that we have actually the following:

**Theorem 4.3.** *Let  $\rho \mapsto \pi(\rho)$  be the Langlands correspondence from two dimensional representations of  $W'_K$  to smooth irreducible infinite dimensional representations of  $G_2(K)$ , we have the following equality of  $L$ -functions:*

$$L_{As}(\pi(\rho), s) = L(M_{W'_K}^{W'_F}(\rho), s)$$

## 5 Appendix. Dihedral supercuspidal distinguished representations

The aim of this section is to give a description of dihedral supercuspidal distinguished representations of  $G_2(K)$  in terms of Langlands parameter, it is done in Theorem 5.4.

### 5.1 Preliminary results

Let  $E$  be a local field,  $E'$  be a quadratic extension of  $E$ ,  $\chi$  a character of  $E^*$ ,  $\pi$  be a smooth irreducible infinite dimensional representation of  $G_2(E)$ , and  $\psi$  a non trivial character of  $E$ .

We note  $L(\chi, s)$  and  $\epsilon(\chi, s, \psi)$  the functions of the complex variable  $s$  defined before proposition 3.5 in [J-L]. We note  $\gamma(\chi, s, \psi)$  the ratio  $\epsilon(\chi, s, \psi)L(\chi, s)/L(\chi^{-1}, 1-s)$ .

We note  $L(\pi, s)$  and  $\epsilon(\pi, s, \psi)$  the functions of the complex variable  $s$  defined in Theorem 2.18 of [J-L]. We note  $\gamma(\pi, s, \psi)$  the ratio  $\epsilon(\pi, s, \psi)L(\pi, s)/L(\pi^\vee, 1-s)$ .

We note  $\lambda(E'/E, \psi)$  the Langlands-Deligne factor defined before proposition 1.3 in [J-L], it is equal to  $\epsilon(\eta_{E'/E}, 1/2, \psi)$ . As  $\eta_{E'/E}$  is equal to  $\eta_{E'/E}^{-1}$ , the factor  $\lambda(E'/E, \psi)$  is also equal to  $\gamma(\eta_{E'/E}, 1/2, \psi)$ .

From theorem 4.7 of [J-L], if  $\omega$  is a character of  $E'^*$ , then  $L(\pi(\omega), s)$  is equal to  $L(\omega, s)$ , and  $\epsilon(\pi, s, \psi)$  is equal to  $\lambda(E'/E, \psi)\epsilon(\omega, s, \psi)$ , hence  $\gamma(\pi, s, \psi)$  is equal to  $\lambda(E'/E, \psi)\gamma(\omega, s, \psi)$ .

We will need four results. The first is due to Frhlich and Queyrut, see [D] theorem 3.2 for a quick proof using a Poisson formula:

**Proposition 5.1.** *Let  $E$  be a local field,  $E'$  be a quadratic extension of  $E$ ,  $\chi'$  a character of  $E'^*$  trivial on  $E^*$ , and  $\psi'$  a non trivial character of  $E'$  trivial on  $E$ , then  $\gamma(\chi', 1/2, \psi') = 1$ .*

The second is a criterion of Hakim:

**Theorem 5.1.** ([Ha], Theorem 4.1) *Let  $\pi$  be an irreducible supercuspidal representation of  $G_2(K)$  with central character trivial on  $F^*$ , and  $\psi$  a nontrivial character of  $K$  trivial on  $F$ . Then  $\pi$  is distinguished if and only if  $\gamma(\pi \otimes \chi, 1/2, \psi) = 1$  for every character  $\chi$  of  $K^*$  trivial on  $F^*$ .*

The third is due to Flicker:

**Theorem 5.2.** ([F1], proposition 12) *Let  $\pi$  be a smooth irreducible distinguished representation of  $G_n(K)$ , then  $\pi^\sigma$  is isomorphic to  $\pi^\vee$ .*

The fourth is due to Kable in the case of  $G_n(K)$ , see [A-T] for a local proof in the case of  $G_2(K)$ :

**Theorem 5.3.** ([A-T], Proposition 3.1) *There exists no supercuspidal representation of  $G_2(K)$  which is distinguished and  $\eta_{K/F}$ -distinguished at the same time.*

### 5.2 Distinction criterion for dihedral supercuspidal representations

As a dihedral representation's parameter is a multiplicative character of a quadratic extension  $L$  of  $K$ , we first look at the propositionerties of the tower  $F \subset K \subset L$ . Three cases arise:

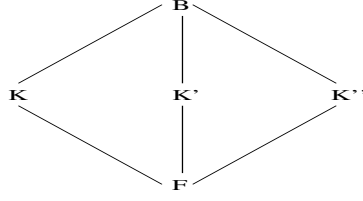


Figure 1:

1.  $L/F$  is biquadratic (hence Galois), it contains  $K$  and two other quadratic extensions  $F$ ,  $K'$  and  $K''$ .  
Its Galois group is isomorphic with  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , its non trivial elements are  $\sigma_{L/K}$ ,  $\sigma_{L/K'}$  and  $\sigma_{L/K''}$ . The conjugation  $\sigma_{L/K}$  extend  $\sigma_{K'/F}$  and  $\sigma_{K''/F}$ .
2.  $L/F$  is cyclic with Galois group isomorphic with  $\mathbb{Z}/4\mathbb{Z}$ , in this case we fix an element  $\tilde{\sigma}$  in  $G(L/F)$  extending  $\sigma$ , it is of order 4.
3.  $L/F$  non Galois. Then its Galois Closure  $M$  is quadratic over  $L$  and the Galois group of  $M$  over  $F$  is dihedral with order 8. To see this, we consider a morphism  $\tilde{\theta}$  from  $L$  to  $\bar{F}$  which extends  $\theta$ . Then if  $L' = \tilde{\theta}(L)$ ,  $L$  and  $L'$  are distinct, quadratic over  $K$  and generate  $M$  biquadratic over  $K$ .  $M$  is the Galois closure of  $L$  because any morphism from  $L$  into  $\bar{F}$ , either extends  $\theta$ , or the identity map of  $K$ , so that its image is either  $L$  or  $L'$ , so it is always included in  $M$ . Finally the Galois group  $M$  over  $F$  cannot be abelian (for  $L$  is not Galois over  $F$ ), it is of order 8, and it's not the quaternion group which only has one element of order 2, whereas here  $\sigma_{M/L}$  and  $\sigma_{M/L'}$  are of order 2. Hence it is the dihedral group of order 8 and we have the following lattice, where  $M/K'$  is cyclic of degree 4,  $M/K$  and  $B/F$  are biquadratic.

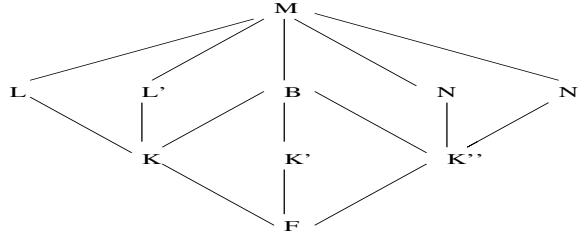


Figure 2:

We now prove the following proposition:

**Proposition 5.2.** *If a supercuspidal dihedral representation  $\pi$  of  $G_2(K)$  verifies  $\pi^\vee = \pi^\sigma$ , there exists a biquadratic extension  $B$  of  $F$ , containing  $K$ , such that if we call  $K'$  and  $K''$  the two other extensions between  $F$  and  $B$ , there is a character  $\omega$  of  $B$  trivial either on  $N_{B/K'}(B^*)$  or on  $N_{B/K''}(B^*)$ , such that  $\pi = \pi(\omega)$ .*

*Proof.* Let  $L$  be a quadratic extension of  $K$  and  $\omega$  a regular multiplicative of  $L$  such that  $\pi = \pi(\omega)$ , we note  $\sigma$  the conjugation of  $L$  over  $K$ , three cases show up:

1.  $L/F$  is biquadratic. The conjugations  $\sigma_{L/K'}$  and  $\sigma_{L/K''}$  both extend  $\sigma$ , hence from Theorem 1 of [G-L], we have  $\pi(\omega)^\sigma = \pi(\omega^{\sigma_{L/K'}})$ . The condition  $\pi^\vee = \pi^\sigma$  which one can also read  $\pi(\omega^{-1}) = \pi(\omega^{\sigma_{L/K'}})$ , is then equivalent from Appendix B, (2)b)1) of [G-L], to  $\omega^{\sigma_{L/K'}} = \omega^{-1}$  or  $\omega^{\sigma_{L/K''}} = \omega^{-1}$ . This is equivalent to  $\omega$  trivial on  $N_{L/K'}(L^*)$  or on  $N_{L/K''}(L^*)$ .

2.  $L/F$  is cyclic, the regularity of  $\omega$  makes the condition  $\pi(\omega^{-1}) = \pi(\omega)^\sigma$  impossible. Indeed one would have from Theorem 1 of [G-L]  $\pi(\omega^{\tilde{\sigma}}) = \pi(\omega^{-1})$ , which from Appendix B, (2)b)1) of [G-L] would imply  $\omega^{\tilde{\sigma}} = \omega$  or  $\omega^{\tilde{\sigma}^{-1}} = \omega$ . As  $\tilde{\sigma}^2 = \tilde{\sigma}^{-2} = \sigma$ , this would in turn imply  $\omega^\sigma = \omega$ , and  $\omega$  would be trivial on the kernel of  $N_{L/K}$  according to Hilbert's theorem 90.  $\pi^{vee}$  can therefore not be isomorphic to  $\pi^\sigma$ .
3.  $L/K$  is not Galois (which implies  $q \equiv 3[4]$  in the case  $p$  odd). Let  $\pi_{B/K}$  be the representation of  $G_2(B)$  which is the base change lift of  $\pi$  to  $B$ . As  $\pi_{B/K} = \pi(\omega \circ N_{M/L})$ , if  $\omega \circ N_{M/L} = \mu \circ N_{M/B}$  for a character  $\mu$  of  $B^*$ , then  $\pi(\omega) = \pi(\mu)$  (cf. [G-L], (3) of Appendix B) and we are brought back to case 1. Otherwise  $\omega \circ N_{M/L}$  is regular with respect to  $N_{M/B}$ . If  $\pi^\sigma = \pi^\vee$ , we would have  $\pi_{B/K}^{\sigma_{B/K'}} = \pi_{B/K}^\vee$  from Theorem 1 of [G-L]. That would contradict case 2 because  $M/K'$  is cyclic.

□

We described in the previous proposition representations  $\pi$  of  $G_2(K)$  verifying  $\pi^\vee = \pi^\sigma$ , now we characterize those who are  $G_2(F)$ -distinguished among them (from Theorem 5.2, a distinguished representation always satisfies the previous condition).

**Theorem 5.4.** *A dihedral supercuspidal representation  $\pi$  of  $G_2(K)$  is  $G_2(F)$ -distinguished if and only if there exists a quadratic extension  $B$  of  $K$  biquadratic over  $F$  such that if we call  $K'$  and  $K''$  the two other extensions between  $B$  and  $F$ , there is character  $\omega$  of  $B^*$  that does not factorize through  $N_{B/K}$  and trivial either on  $K'^*$  or on  $K''^*$ , such that  $\pi = \pi(\omega)$ .*

*Proof.* From Theorem 5.2 and Proposition 5.2, we can suppose that  $\pi = \pi(\omega)$ , for  $\omega$  a regular multiplicative character of a quadratic extension  $B$  of  $K$  biquadratic over  $F$ , with  $\omega$  trivial on  $N_{L/K'}(K'^*)$  or on  $N_{B/K''}(K''^*)$ . We will need the following:

**Lemma 5.1.** *Let  $B$  be a quadratic extension of  $K$  biquadratic over  $F$ , then  $F^*$  is a subset of  $N_{B/K}(B^*)$*

*Proof of Lemma 5.1.* The group  $N_{B/K}(B^*)$  contains the two groups  $N_{B/K}(K'^*)$  and  $N_{B/K}(K''^*)$ , which, as  $\sigma_{B/K}$  extends  $\sigma_{K'/F}$  and  $\sigma_{K''/F}$ , are respectively equal to  $N_{K'/F}(K'^*)$  and  $N_{K''/F}(K''^*)$ . But these two groups are distinct of index 2 in  $F^*$  from local classfield theory, thus they generate  $F^*$ , which is therefore contained in  $N_{B/K}(B^*)$ . □

We now choose  $\psi$  a non trivial character of  $K/F$  and note  $\psi_B$  the character  $\psi \circ \text{Tr}_{B/K}$ , it is trivial on  $K'$  and  $K''$ .

Suppose  $\omega$  trivial on  $K'$  or  $K''$ , then the restriction of the central character  $\eta_{B/K}\omega$  of  $\pi(\omega)$  is trivial on  $F^*$  according to Lemma 5.1.

As we have  $\gamma(\pi(\omega), 1/2, \psi) = \lambda(B/K, \psi)\gamma(\omega, 1/2, \psi_B) = \gamma(\eta_{B/K}, 1/2, \psi)\gamma(\omega, 1/2, \psi_B)$ , we deduce from Lemma 5.1 and Proposition 5.1 that  $\gamma(\pi(\omega), 1/2, \psi)$  is equal to one, hence from Theorem 5.1, the representation  $\pi(\omega)$  is distinguished.

Now suppose  $\omega|_{K'} = \eta_{B/K'}$  or  $\omega|_{K''} = \eta_{B/K''}$ , let  $\chi$  be a character of  $K^*$  extending  $\eta_{K/F}$ , then  $\pi(\omega) \otimes \chi = \pi(\omega\chi \circ N_{B/K})$ . As  $N_{B/K}|_{K'} = N_{K'/F}$  and  $N_{B/K}|_{K''} = N_{K''/F}$ , we have  $\chi \circ N_{B/K}|_{K'} = \eta_{B/K'}$  and  $\chi \circ N_{B/K}|_{K''} = \eta_{B/K''}$ , hence from what we've just seen,  $\pi(\omega) \otimes \chi$  is distinguished, i.e.  $\pi(\omega)$  is  $\eta_{K/F}$ -distinguished.

From Theorem 5.3,  $\pi$  cannot be distinguished and  $\eta_{K/F}$ -distinguished at the same time, and the theorem follows. □

We end with the following lemma:

**Lemma 5.2.** *Let  $B$  be a quadratic extension of  $K$  which is biquadratic over  $F$ . Call  $K'$  and  $K''$  the two other extensions between  $F$  and  $B$ , then the kernel of  $N_{B/K}$  is a subgroup of the group  $N_{B/K'}(B^*)N_{B/K''}(B^*)$ .*

*Proof.* If  $u$  belongs to  $\text{Ker}(N_{B/K})$ , it can be written  $x/\sigma_{B/K}(x)$  for some  $x$  in  $B^*$  according to Hilbert's Theorem 90. Hence we have  $u = (x\sigma_{B/K'}(x))/(\sigma_{B/K}(x)\sigma_{B/K'}(x)) = N_{B/K'}(x)/N_{B/K''}(\sigma_{B/K}(x))$ , and  $u$  belongs to  $N_{B/K'}(B^*)N_{B/K''}(B^*)$ .  $\square$

**Corollary 5.1.** *The (either/or) in Proposition 5.2 and Theorem 5.4 is exclusive*

*Proof.* In fact, in the situation of Lemma 5.2, a character  $\omega$  that is trivial on  $N_{B/K'}(B^*)$  and  $N_{B/K''}(B^*)$  factorizes through  $N_{B/K}$ , and  $\pi(\omega)$  is not supercuspidal.  $\square$

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