



HAL
open science

Distinguished representations and exceptional poles of the Asai-L-function

Nadir Matringe

► **To cite this version:**

Nadir Matringe. Distinguished representations and exceptional poles of the Asai-L-function. 2008.
hal-00299528v3

HAL Id: hal-00299528

<https://hal.science/hal-00299528v3>

Preprint submitted on 19 Jul 2008 (v3), last revised 4 Nov 2008 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distinguished representations and exceptional poles of the Asai-L-function

Nadir MATRINGE

July 19, 2008

Abstract

Let K/F be a quadratic extension of p -adic fields. We show that a generic irreducible representation of $GL(n, K)$ is distinguished if and only if its Rankin-Selberg Asai L-function has an exceptional pole at zero. We use this result to compute Asai L-functions of ordinary irreducible representations of $GL(2, K)$. In the appendix, we describe supercuspidal dihedral representations of $GL(2, K)$ in terms of Langlands parameter.

1 Introduction

For K/F a quadratic extension of local fields, let σ be the conjugation relative to this extension, and $\eta_{K/F}$ be the character of F^* with kernel norms of K^* . The conjugation σ extends naturally to an automorphism of $GL(n, K)$, and we still note this automorphism σ . If π is a representation of $GL(n, K)$, we note π^σ the representation $g \mapsto \pi(\sigma(g))$.

If π is a smooth irreducible representation of $GL(n, K)$, and χ a character of F^* , the dimension of the space of linear forms on its space, which transform by χ under $GL(n, F)$ (with respect to the action $[(L, g) \mapsto L \circ \pi(g)]$), is known to be at most one (Proposition 11, [F1]). One says that π is χ -distinguished if this dimension is one, and says that π is distinguished if it is 1-distinguished. Jacquet conjectured two results about distinguished representations of $GL(n, K)$. Let π be a smooth irreducible representation of $GL(n, K)$ and π^\vee its contragredient. The first conjecture states that it is equivalent for π with central character trivial on F^* to be isomorphic to $\pi^{\vee\sigma}$ and for π to be distinguished or $\eta_{K/F}$ -distinguished. In [K], Kable proved it for discrete series representations, using Asai L -functions.

The second conjecture, which is proved in [K], states that if π is a discrete series representation, then it cannot be distinguished and $\eta_{K/F}$ -distinguished at the same time.

One of the key points in Kable's proof is that if a discrete series representation of $GL(n, K)$ is such that its Asai L -function has a pole at zero, then it is distinguished, Theorem 1.4 of [A-K-T] shows that it is actually an equivalence. This theorem actually shows that Asai L -functions of tempered distinguished representations admit a pole at zero.

In this article, using a result of Youngbin Ok which states that for a distinguished representation, linear forms invariant under the affine subgroup of $GL(n, F)$ are actually $GL(n, F)$ -invariant (which generalises Corollary 1.2 of [A-K-T]), we prove in Theorem 3.1 that a generic representation is distinguished if and only if its Asai L -function admits an exceptional pole at zero. A pole at zero is always exceptional for Asai L -functions of discrete series representations (see explanation before Proposition 3.4). As a first application, we give in Proposition 3.6 a formula for Asai L -functions of supercuspidal representations of $GL(n, K)$.

There are actually three different ways to define Asai L -functions: one via the local Langlands correspondence and in terms of Langlands parameters denoted by $L_W(\pi, s)$, the one we use via the theory of Rankin-Selberg integrals denoted by $L_{As}(\pi, s)$, and the Langlands-Shahidi method applied to a suitable unitary group, denoted by $L_{As,2}(\pi, s)$ (see [A-R]). It is expected that the above three L -functions are equal.

For a discrete series representation π , it is shown in [He] that $L_W(\pi, s) = L_{As,2}(\pi, s)$, and in

[A-R] that $L_{As}(\pi, s) = L_{As,2}(\pi, s)$, both proofs using global methods.

As a second application of our principal result, we show (by local methods) in Theorem 4.2 of section 4 that for an ordinary representation (i.e. corresponding through Langlands correspondance to an imprimitive 2 dimensional representation of the Weil-Deligne group) π of $GL(2, K)$, we have $L_W(\pi, s) = L_{As}(\pi, s)$. We recall that for odd residual characteristic, every smooth irreducible infinite dimensional representation of $GL(2, K)$ is ordinary.

In the appendix (section 5), we describe in Theorem 5.4 distinguished dihedral supercuspidal representations, this description is used in section 4 for the computation of L_{As} for such representations.

2 Preliminaries

Let E_1 be a field, and E_2 a finite galois extension of E_1 , we note $Gal(E_2/E_1)$ the Galois group of E_2 over E_1 , and we note Tr_{E_2/E_1} (respectively N_{E_2/E_1}) the trace (respectively the norm) application from E_2 to E_1 . If E_2 is quadratic over E_1 , we note σ_{E_2/E_1} the non trivial element of $Gal(E_2/E_1)$.

In the rest of this paper, the letter F will always designate a non archimedean local field of characteristic zero in a fixed algebraic closure \bar{F} , and the letter K a quadratic extension of F in \bar{F} . We note q_F and q_K the cardinality of their residual fields, R_K and R_F their integer rings, P_K and P_F the maximal ideals of R_K and R_F , and U_K and U_F their unit groups. We also note v_K and v_F the respective normalized valuations, and $|\cdot|_K$ and $|\cdot|_F$ the respective absolute values. We fix an element δ of $K - F$ such that $\delta^2 \in F$, hence $K = F(\delta)$.

Let ψ be a non trivial character of K trivial on F , it is of the form $x \mapsto \psi' \circ Tr_{K/F}(\delta x)$ for some non trivial character ψ' of F .

Whenever G is an algebraic group defined over F , we note $G(K)$ its K -points and $G(F)$ its F -points. The group $GL(n)$ is noted G_n , its standard maximal unipotent subgroup is noted N_n . If π is a representation of a group, we also note π its isomorphism class. Let μ be a character of F^* , we say that a representation π of $G_n(K)$ is μ -distinguished if it admits on its space V_π a linear form L , which verifies the following: for v in V and h in $G_n(K)$, then $L(\pi(h)v) = \mu(det(h))L(v)$. If $\mu = 1$, we say that π is distinguished.

We note $K_n(F)$ the maximal compact subgroup $G_n(R_F)$ of $G_n(F)$, and for $r \geq 1$, we note $K_{n,r}(F)$, the congruence subgroup $I_n + M_n(P_F^r)$.

The character ψ defines a character of $N_n(K)$ that we still note ψ , given by $\psi(n) = \psi(\sum_{i=1}^{n-1} n_{i,i+1})$.

We now recall standard results from [F2].

Let π be a generic smooth irreducible representation of $G_n(K)$, we note π^\vee its smooth contra-redient, and c_π its central character.

We note $D(F^n)$ the space of smooth functions with compact support on F^n , and $D_0(F^n)$ the subspace of $D(F^n)$ of functions vanishing at zero. We note ρ the natural action of $G_n(F)$ on $D(F^n)$, given by $\rho(g)\phi(x_1, \dots, x_n) = \phi((x_1, \dots, x_n)g)$, and we note η the line vector $(0, \dots, 0, 1)$ of length n .

If W belongs to the Whittaker model $W(\pi, \psi)$ of π , and ϕ belongs to $D(F^n)$, the following integral converges for s of real part large enough:

$$\int_{N_n(F) \backslash G_n(F)} W(g)\phi(\eta g) |det(g)|_F^s dg.$$

This integral as a function of s has a meromorphic extension to \mathbb{C} which we note $\Psi(W, \phi, s)$. For s of real part large enough, the function $\Psi(W, \phi, s)$ is a rational function in q_F^{-s} , which actually has a Laurent series development.

The \mathbb{C} -vector space generated by these functions is in fact a fractional ideal $I(\pi)$ of $\mathbb{C}[q_F^{-s}, q_F^s]$. This ideal $I(\pi)$ is principal, and has a unique generator of the form $1/P(q_F^{-s})$, where P is a polynomial with $P(0) = 1$.

Definition 2.1. We note $L_{As}(\pi, s)$ the generator of $I(\pi)$ defined just above, and call it the Asai L -function of π .

Remark 2.1. If P belongs to $\mathbb{C}[X]$ and has constant term equal to one, then the function of the complex variable $L_P : s \mapsto 1/P(q_F^{-s})$ is called an Euler factor. It is a meromorphic function on \mathbb{C} and admits $(2i\pi/Ln(q_F))\mathbb{Z}$ as a period subgroup. Hence if s_0 is a pole of L_P , the elements $s_0 + (2i\pi/Ln(q_F))\mathbb{Z}$ are also poles of L_P , with same multiplicities, we identify s_0 and $s_0 + (2i\pi/Ln(q_F))\mathbb{Z}$ when we talk about poles. A pole s_0 then corresponds to a root α_0 of P by the formula $q^{-s_0} = \alpha_0$, its multiplicity in L_P equal to the multiplicity of α_0 in P .

Let w_n be the matrix of $G_n(\mathbb{Z})$ with ones on the antidiagonal, and zeroes elsewhere. For W in $W(\pi, \psi)$, we note \tilde{W} the function $g \mapsto W(w_n {}^t g^{-1})$ which belongs to $W(\pi^\vee, \psi^{-1})$, and we note $\widehat{\phi}$ the Fourier transform (with respect to ψ' an its associate autodual Haar measure) of ϕ in $D(F^n)$.

Theorem 2.1. (Functional equation) (Th. of [F2])

There exists an epsilon factor $\epsilon_{As}(\pi, s, \psi)$ which is, up to scalar, a (maybe negative) power of q^s , such that the following functional equation is satisfied for any W in $W(\pi, \psi)$ and any ϕ in $D(F^n)$:

$$\Psi(\tilde{W}, \widehat{\phi}, 1-s)/L_{As}(\pi^\vee, 1-s) = c_\pi(-1)^{n-1} \epsilon_{As}(\pi, s, \psi) \Psi(W, \phi, s)/L_{As}(\pi, s).$$

3 Poles of the Asai L -function and distinguishedness

Now suppose $L_{As}(\pi, s)$ has a pole at s_0 , its order d is the highest order pole of the family of functions of $I(\pi)$.

Then we have the following Laurent expansion at s_0 :

$$\Psi(W, \phi, s) = B_{s_0}(W, \phi)/(q^s - q^{s_0})^d + \text{smaller order terms.} \quad (1)$$

The residue $B_{s_0}(W, \phi)$ defines a non zero bilinear form on $W(\pi, \psi) \times D(F^n)$, satisfying the quasi-invariance:

$$B_{s_0}(\pi(g)W, \rho(g)\phi) = |\det(g)|_F^{-s_0} B_{s_0}(W, \phi).$$

Following [C-P] for the split case $K = F \times F$, we state the following definition:

Definition 3.1. A pole of the Asai L -function $L_{As}(\pi, s)$ at s_0 is called exceptional if the associated bilinear form B_{s_0} vanishes on $W(\pi, \psi) \times D_0(F^n)$.

As an immediate consequence, if s_0 is an exceptional pole of $L_{As}(\pi, s)$, then B_{s_0} is of the form $B_{s_0}(W, \phi) = \lambda_{s_0}(W)\phi(0)$, where λ_{s_0} is a non zero $|\det(\cdot)|_F^{-s_0}$ invariant linear form on $W(\pi, \psi)$. Hence we have:

Proposition 3.1. Let π be a generic irreducible representation of $G_n(K)$, and suppose its Asai L -function has an exceptional pole at zero, then π is distinguished.

We note $P_n(F)$ the affine subgroup of $G_n(F)$, given by matrices with last row equal to η . For more convenience, we introduce a second L -function: for W in $W(\pi, \psi)$, by standard arguments, the following integral is convergent for $Re(s)$ large, and defines a rational function in q^{-s} , which has a Laurent series development:

$$\int_{N_n(F) \backslash P_n(F)} W(p) |\det(p)|_F^s dp.$$

We note $\Psi_1(W, s)$ the corresponding Laurent series. By standard arguments again, the vector space generated by these functions is a fractional ideal $I_1(\pi)$ of $\mathbb{C}[q_F^{-s}, q_F^s]$, which has a unique

generator of the form $1/Q(q^{-s})$, where Q is a polynomial with $Q(0) = 1$. We note $L_1(\pi, s)$ this generator.

Lemma 3.1. (*[J-P-S] p. 393*)

Let W be in $W(\pi, \psi)$, one can choose ϕ with support small enough around $(0, \dots, 0, 1)$ such that $\Psi(W, \phi, s) = \Psi_1(W, s - 1)$.

Proof. As we gave a reference, we only sketch the proof. We first recall the following integration formula (cf. proof of the proposition in paragraph 4 of [F]), for $Re(s) \gg 0$:

$$\Psi(W, \phi, s) = \int_{K_n(F)} \int_{N_n(F) \backslash P_n(F)} W(pk) |det(p)|_F^{s-1} dp \int_{F^*} \phi(\eta ak) c_\pi(a) |a|_F^{ns} d^* adk. \quad (2)$$

Choosing r large enough for W to be right invariant under $K_{n,r}(F)$, we take ϕ a positive multiple of the characteristic function of $\eta K_{n,r}(F)$, and conclude from equation (2). \square

Hence we have the inclusion $I_1(\pi) \subset I(\pi)$, which implies that $L_1(\pi, s) = L_{As}(\pi, s)R(q^s, q^{-s})$ for some R in $\mathbb{C}[q_F^{-s}, q_F^s]$. But because L_1 and L_{As} are both Euler factors, R is actually just a polynomial in q^{-s} , with constant term equal to one. Noting $L_{rad(ex)}(\pi, s)$ its inverse (which is an Euler factor), we have $L_{As}(\pi, s) = L_1(\pi, s)L_{rad(ex)}(\pi, s)$, we will say that L_1 divides L_{As} . The explanation for the notation $L_{rad(ex)}$ is given in Remark 3.1.

We now give a characterisation of exceptional poles:

Proposition 3.2. *A pole of $L_{As}(\pi, s)$ is exceptional if and only if it is a pole of the function $L_{rad(ex)}(\pi, s)$ defined just above.*

Proof. From equation (2), it becomes clear that the vector space generated by the integrals $\Psi(W, \phi, s)$ with W in $W(\pi, \psi)$ and ϕ in $D_0(F^n)$, is contained in $I_1(\pi)$, but because of Lemma 3.1, those two vector spaces are equal. Hence $L_1(\pi, s)$ is a generator of the ideal generated as a vector space by the functions $\Psi(W, \phi, s)$ with W in $W(\pi, \psi)$ and ϕ in $D_0(F^n)$.

From equation (1), if s_0 is an exceptional pole, a function $\Psi(W, \phi, s)$, with ϕ in $D_0(F^n)$, cannot have a pole of highest order at s_0 , hence we have one implication.

Now if the order of the pole s_0 for $L_{As}(\pi, s)$ is strictly greater than the one of $L_1(\pi, s)$, then the first residual term corresponding to a pole of highest order of the Laurent development of any function $\Psi(W, \phi, s)$ with $\phi(0) = 0$ must be zero, and zero is exceptional. \square

Lemma 3.1 also implies:

Proposition 3.3. *The functional $\Lambda_{\pi, s} : W \mapsto \Psi_1(W, s - 1)/L_{As}(\pi, s)$ defines a (maybe null) linear form on $W(\pi, \psi)$ which transforms by $|det(\cdot)|_F^{1-s}$ under the affine subgroup $P_n(F)$. For fixed W in $W(\pi, \psi)$, then $s \mapsto \Lambda_{\pi, s}(W)$ is a polynomial of q^{-s} .*

Now we are able to prove the converse of Proposition 3.1:

Theorem 3.1. *A generic irreducible representation π of $G_n(K)$ is distinguished if and only if $L_{As}(s, \pi)$ admits an exceptional pole at zero.*

Proof. We only need to prove that if π is distinguished, then $L_{As}(s, \pi)$ admits an exceptional pole at zero, so we suppose π distinguished.

From equation (2), for $Re(s) \ll 0$, and π distinguished (so that c_π has trivial restriction to F^*), one has:

$$\Psi(\tilde{W}, \hat{\phi}, 1 - s) = \int_{K_n(F)} \int_{N_n(F) \backslash P_n(F)} \tilde{W}(pk) |det(p)|_F^{-s} dp \int_{F^*} \hat{\phi}(\eta ak) |a|_F^{n(1-s)} d^* adk. \quad (3)$$

This implies that:

$$\Psi(\tilde{W}, \widehat{\phi}, 1-s)/L_{As}(\pi^\vee, 1-s) = \int_{K_n(F)} \Lambda_{\pi^\vee, 1-s}(\pi^\vee(k)\tilde{W}) \int_{F^*} \widehat{\phi}(\eta ak) |a|_F^{n(1-s)} d^* adk. \quad (4)$$

The second member of the equality is actually a finite sum: $\sum_i \lambda_i \Lambda_{\pi^\vee, 1-s}(\pi^\vee(k_i)\tilde{W}) \int_{F^*} \widehat{\phi}(\eta ak_i) |a|_F^{n(1-s)} d^* a$, where the λ_i 's are positive constants and the k_i 's are elements of $K_n(F)$ independant of s .

Note that there exists a positive constant ϵ , such that for $Re(s) < \epsilon$, the integral $\int_{F^*} \widehat{\phi}(\eta ak_i) |a|_F^{n(1-s)} d^* a$ is absolutely convergent, and defines a holomorphic function. So we have an equality (equality 4) of analytic functions (actually of polynomials in q^{-s}), hence it is true for all s such that $Re(s) < \epsilon$. For $s = 0$, we get:

$$\Psi(\tilde{W}, \widehat{\phi}, 1)/L_{As}(\pi^\vee, 1) = \int_{K_n(F)} \Lambda_{\pi^\vee, 1}(\pi^\vee(k)\tilde{W}) \int_{F^*} \widehat{\phi}(\eta ak) |a|_F^n d^* adk.$$

But as π is distinguished, so is π^\vee , and as $\Lambda_{\pi^\vee, 1}$ is a $P_n(F)$ -invariant linear form on $W(\pi^\vee, \psi^{-1})$, it follows from theorem 3.1.2 of [Ok] that it is actually $G_n(F)$ -invariant.

Finally

$$\Psi(\tilde{W}, \widehat{\phi}, 1)/L_{As}(\pi^\vee, 1) = \Lambda_{\pi^\vee, 1}(\tilde{W}) \int_{K_n(F)} \int_{F^*} \widehat{\phi}(\eta ak) |a|_F^n d^* adk$$

which is equal to:

$$\Lambda_{\pi^\vee, 1}(\tilde{W}) \int_{P_n(F) \backslash G_n(F)} \widehat{\phi}(\eta g) d_\mu g$$

where d_μ is up to scalar the unique $|\det(\cdot)|^{-1}$ invariant measure on $P_n(F) \backslash G_n(F)$. But as

$$\int_{P_n(F) \backslash G_n(F)} \widehat{\phi}(\eta g) d_\mu g = \int_{F^n} \widehat{\phi}(x) dx = \phi(0),$$

we deduce from the functional equation that $\Psi(W, \phi, 0)/L_{As}(\pi, 0) = 0$ whenever $\phi(0) = 0$.

As one can choose W , and ϕ vanishing at zero, such that $\Psi(W, \phi, s)$ is the constant function equal to 1 (see the proof of Theorem 1.4 in [A-K-T]), hence $L_{As}(\pi, s)$ has a pole at zero, which must be exceptional. \square

For a discrete series representation π , it follows from Lemma 2 of [K], that the integrals of the form

$$\int_{N_n(F) \backslash P_n(F)} W(p) |\det(p)|_F^{s-1} dp.$$

converge absolutely for $Re(s) > -\epsilon$ for some positive ϵ , hence as functions of s , they cannot have a pole at zero.

This implies that $L_1(\pi, s)$ has no pole at zero, hence Theorem 3.1 in this case gives:

Proposition 3.4. (*[K], Theorem 4*)

A discrete series representation π of $G_n(K)$ is distinguished if and only if $L_{As}(s, \pi)$ admits a pole at zero.

Let s_0 be in \mathbb{C} . We notice that if π is a generic irreducible representation of $G_n(K)$, it is $|\cdot|_F^{-s_0}$ -distinguished if and only if $\pi \otimes |\cdot|_K^{s_0/2}$ is distinguished, but as $L_{As}(s, \pi \otimes |\cdot|_K^{s_0/2})$ is equal to $L_{As}(s + s_0, \pi)$, Theorem 3.1 becomes:

Theorem 3.2. *A generic irreducible representation π of $G_n(K)$ is $|\cdot|_F^{-s_0}$ -distinguished if and only if $L_{As}(s, \pi)$ admits an exceptional pole at s_0 .*

Remark 3.1. Let P and Q be two polynomials in $\mathbb{C}[X]$ with constant term 1, we say that the Euler factor $L_P(s) = 1/P(q_F^{-s})$ divides $L_Q(s) = 1/Q(q_F^{-s})$ if and only if P divides Q . We note $L_P \vee L_Q$ the Euler factor $1/(P \vee Q)(q_F^{-s})$, where the l.c.m $P \vee Q$ is chosen such that $(P \vee Q)(0) = 1$. We define the g.c.d $L_P \wedge L_Q$ the same way.

It follows from equation (2) that if $c_{\pi|_{F^*}}$ is ramified, then $L_{As}(\pi, s) = L_1(\pi, s)$. It also follows from the same equation that if $c_{\pi|_{F^*}} = | \cdot |_F^{-s_1}$ for some s_1 in \mathbb{C} , then $L_{rad(ex)}(\pi, s)$ divides $1/(1 - q_F^{s_1 - ns})$. Anyway, $L_{rad(ex)}(\pi, s)$ has simple poles.

Now we can explain the notation $L_{rad(ex)}$. We refer to [C-P] where the case $K = F \times F$ is treated. In fact, in the latter, $L_{ex}(\pi, s)$ is the function $1/P_{ex}(\pi, q_F^{-s})$, with $P_{ex}(\pi, q_F^{-s}) = \prod_{s_i} (1 - q_F^{s_i - s})^{d_i}$, where the s_i 's are the exceptional poles of $L_{As}(\pi, s)$ and the d_i 's their order in $L_{As}(\pi, s)$. Hence $L_{rad(ex)}(\pi, s) = 1/P_{rad(ex)}(\pi, q_F^{-s})$, where $P_{rad(ex)}(\pi, X)$ is the unique generator with constant term equal to one, of the radical of the ideal generated by $P_{ex}(\pi, X)$ in $\mathbb{C}[X]$.

We proved:

Proposition 3.5. *Let π be an irreducible generic representation of $G_n(K)$, the Euler factor $L_{rad(ex)}(\pi, s)$ has simple poles, it is therefore equal to $\prod 1/(1 - q_F^{s_0 - s})$ where the product is taken over the $q_F^{s_0}$'s such that π is $| \cdot |_F^{-s_0}$ -distinguished.*

Suppose now that π is supercuspidal, then the restriction to $P_n(K)$ of any W in $W(\pi, \psi)$ has compact support modulo $N_n(K)$, hence $\Psi_1(W, s - 1)$ is a polynomial in q^{-s} , and $L_1(\pi, s)$ is equal to 1. Hence Proposition 3.5 becomes:

Proposition 3.6. *Let π be an irreducible supercuspidal representation of $G_n(K)$, then $L_{As}(\pi, s) = \prod 1/(1 - q^{s_0 - s})$ where the product is taken over the q^{s_0} 's such that π is $| \cdot |_F^{-s_0}$ -distinguished.*

4 Asai L -functions of $GL(2)$

4.1 Asai L -functions for imprimitive Weil-Deligne representations of dimension 2

The aim of this paragraph is to compute $L_W(\rho, s)$ (see the introduction) when ρ is an imprimitive two dimensional representation of the Weil-Deligne group of K .

We note W_K (resp. W_F) the Weil group of K (resp. F), I_K (resp. I_F) the inertia subgroup of W_K (resp. W_F), W'_K (resp. W'_F) the group $W_K \times SL(2, \mathbb{C})$ (resp. $W_F \times SL(2, \mathbb{C})$) and I'_K (resp. I'_F) the group $I_K \times SL(2, \mathbb{C})$ (resp. $I_F \times SL(2, \mathbb{C})$). We note ϕ_F a Frobenius element of W_F , and we also note ϕ'_F the element (ϕ_F, I_2) of W'_F .

We note $sp(n)$ the unique (up to isomorphism) complex irreducible representation of $SL(2, \mathbb{C})$ of dimension n .

If ρ is a finite dimensional representation of W'_K , we note $M_{W'_K}^{W'_F}(\rho)$ the representation of W'_F induced multiplicatively from ρ . We recall its definition:

If V is the space of ρ , then the space of $M_{W'_K}^{W'_F}(\rho)$ is $V \otimes V$. Noting τ an element of $W_F - W_K$, and σ the element (τ, I) of W'_F , we have:

$$M_{W'_K}^{W'_F}(\rho)(h)(v_1 \otimes v_2) = \rho(h)v_1 \otimes \rho^\sigma(h)v_2$$

for h in W'_K , v_1 and v_2 in V .

$$M_{W'_K}^{W'_F}(\rho)(\sigma)(v_1 \otimes v_2) = \rho(\sigma^2)v_2 \otimes v_1$$

for v_1 and v_2 in V .

We refer to paragraph 7 of [P] for definition and basic propositionerties of multiplicative induction in the general case.

Definition 4.1. The function $L_W(\rho, s)$ is by definition the usual L -function of the representation $M_{W'_K}^{W'_F}(\rho)$, i.e. $L_W(\rho, s) = L(M_{W'_K}^{W'_F}(\rho), s)$.

i) If ρ is of the form $\text{Ind}_{W'_B}^{W'_K}(\omega)$ for some multiplicative character ω of a biquadratic extension B of F , we note K' and K'' the two other extensions between F and B . If we call σ_1 an element of W'_K which is not in $W'_{K'} \cup W'_{K''}$ and σ_3 an element of $W'_{K''}$ which is not in $W'_{K'} \cup W'_K$, then $\sigma_2 = \sigma_3\sigma_1$ is an element of $W'_{K'}$, which is not in $W'_K \cup W'_{K''}$. The elements $(1, \sigma_1, \sigma_2, \sigma_3)$ are representatives of W'_F/W'_B , and 1 and σ_3 are representatives of W'_F/W'_K .

If one identifies ω with a character (still called ω) of B^* , then ω^{σ_1} identifies with $\omega \circ \sigma_{B/K}$, ω^{σ_2} with $\omega \circ \sigma_{B/K'}$ and ω^{σ_3} with $\omega \circ \sigma_{B/K''}$. One then verifies that if a belongs to W_B , one has:

- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(a)] = \omega\omega^{\sigma_2} + \omega\omega^{\sigma_3} + \omega^{\sigma_1}\omega^{\sigma_2} + \omega^{\sigma_1}\omega^{\sigma_3}$.
- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(\sigma_1 a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_1 a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_1 a)] = 0$.
- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(\sigma_2 a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_2 a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_2 a)] = \omega(\sigma_2 a \sigma_2 a) + \omega^{\sigma_1}(\sigma_2 a \sigma_2 a)$.
- $\text{Tr}[M_{W'_K}^{W'_F}(\rho)(\sigma_3 a)] = \text{Tr}[\text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_3 a)] + \text{Tr}[\text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_3 a)] = \omega(\sigma_3 a \sigma_3 a) + \omega^{\sigma_1}(\sigma_3 a \sigma_3 a)$.

Hence we have the isomorphism

$$M_{W'_K}^{W'_F}(\rho) \simeq \text{Ind}_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega)) \oplus \text{Ind}_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega)).$$

From this we deduce that

$$L(M_{W'_K}^{W'_F}(\rho), s) = L(\omega|_{K'^*}, s)L(\omega|_{K''^*}, s).$$

ii) Let L be a quadratic extension of F , such that $\rho = \text{Ind}_{W'_L}^{W'_K}(\chi)$, with χ regular, is not isomorphic to a representation of the form $\text{Ind}_{W'_B}^{W'_K}(\omega)$ as in i), then

$$L(M_{W'_K}^{W'_F}(\rho), s) = 1.$$

Indeed, we show that $M_{W'_K}^{W'_F}(\rho)^{I'_F} = \{0\}$. If it wasn't the case, the representation $(M_{W'_K}^{W'_F}(\rho), V)$ would admit a I'_F -fixed vector, and so would its contragredient V^* . Now in the subspace of I'_F -fixed vectors of V^* , choosing an eigenvector of $M_{W'_K}^{W'_F}(\rho)(\phi_F)$, we would deduce the existence of a linear form L on $(M_{W'_K}^{W'_F}(\rho), V)$ which transforms under W'_F by an unramified character μ of W'_F . If we identify μ with a character μ' of F^* , the restriction of μ to W'_K corresponds to $\mu' \circ N_{K/F}$ of K^* , so we can write it as $\theta\theta^\sigma$, where θ is a character of W'_K corresponding to an extension of μ' to K^* . As the restriction of $M_{W'_K}^{W'_F}$ to W'_K is isomorphic to $\rho \otimes \rho^\sigma$, we deduce that $\theta^{-1}\rho \otimes (\theta^{-1}\rho)^\sigma$ is W'_K distinguished, that is $\theta\rho^{vee} \simeq (\theta^{-1}\rho)^\sigma$. But from the proof of Theorem 5.2, this would imply that $\theta^{-1}\rho$ hence ρ , could be induced from a character of a biquadratic extension of F , which we supposed is not the case.

iii) Suppose $\rho = sp(2)$ acts on the space \mathbb{C}^2 with canonical basis (e_1, e_2) by the natural action $\rho[h, M](v) = M(v)$ for h in W_K , M in $SL(2, \mathbb{C})$ and v in \mathbb{C}^2 . Then the space of $M_{W'_K}^{W'_F}(\rho)$ is $V \otimes V$ and $SL(2, \mathbb{C})$ acts on it as $sp(2) \otimes sp(2)$. Decomposing $V \otimes V$ as the direct sum $\text{Alt}(V) \oplus$

$Sym(V)$, we see that $SL(2, \mathbb{C})$ acts as 1 on $Alt(V)$, and $M_{W'_K}^{W'_F}(\rho) \left[1, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \right] (e_1 \otimes e_1) = x^2 e_1 \otimes e_1$. Hence the representation of $SL(2, \mathbb{C})$ on $Sym(V)$ must be $sp(3)$. The Weil group W_F acts as $\eta_{K/F}$ on $Alt(V)$ and trivially on $Sym(V)$, finally $M_{W'_K}^{W'_F}(\rho)$ is isomorphic to $sp(3) \oplus \eta_{K/F}$. Tensoring with a character χ , we have $M_{W'_K}^{W'_F}(\chi sp(2)) = \chi|_{F^*} M_{W'_K}^{W'_F}(sp(2)) = \chi|_{F^*} \eta_{K/F} \oplus \chi|_{F^*} sp(3)$. Hence one has the following equality:

$$L(M_{W'_K}^{W'_F}(\chi sp(2)), s) = L(\chi|_{F^*} \eta_{K/F}, s) L(\chi|_{F^*}, s+1).$$

iv) If $\rho = \lambda \oplus \mu$, with λ and μ two characters of W'_K , then from [P], Lemma 7.1, we have $M_{W'_K}^{W'_F}(\rho) = \lambda|_{F^*} \oplus \mu|_{F^*} \oplus Ind_{W'_K}^{W'_F}(\lambda\mu^\sigma)$. Hence we have

$$L(M_{W'_K}^{W'_F}(\rho)) = L(\lambda|_{F^*}, s) L(\mu|_{F^*}, s) L(\lambda\mu^\sigma, s).$$

4.2 Asai L -functions for ordinary representations of $GL(2)$

In this subsection, we compute Asai L -functions for ordinary (i.e. non exceptional) representations of $G_2(K)$, and prove (Theorem 4.2) that they are equal to the corresponding functions L_W of imprimitive representations of W'_K .

In order to compute L_{As} , we first compute L_1 , but this latter computation is easy because Kirillov models of infinite dimensional irreducible representations of $G_2(K)$ are well-known (see [B], Th. 4.7.2 and 4.7.3).

Let π be an irreducible infinite dimensional (hence generic) representation of $G_2(K)$, we have the following situations for the computation of $L_1(\pi, s)$.

i) and ii) If π is supercuspidal, its Kirillov model consists of functions with compact support on K^* , hence

$$L_1(\pi, s) = 1.$$

iii) If $\pi = \sigma(\chi)$ ($\sigma(\chi|_K, \chi|_K^{-1/2})$ in [B]) is a special series representation of $G_2(K)$, twist of the Steinberg representation by the character χ of K^* , the Kirillov model of π consists of functions of $D(K)$ multiplied by $\chi|_K$. Hence their restrictions to F are functions of $D(F)$ multiplied by $\chi|_F^2$, and the ideal $I_1(\pi)$ is generated by functions of s of the form

$$\int_{F^*} \phi(t) \chi(t) |t|_F^{s-1} |t|_F^2 d^*t = \int_{F^*} \phi(t) \chi(t) |t|_F^{s+1} d^*t,$$

for ϕ in $D(F)$, hence we have

$$L_1(\pi, s) = L(\chi|_{F^*}, s+1).$$

iv) If $\pi = \pi(\lambda, \mu)$ is the principal series representation (λ and μ being two characters of K^* , with $\lambda\mu^{-1}$ different from $| \cdot |$ and $| \cdot |^{-1}$) corresponding to the representation $\lambda \oplus \mu$ of W'_K .

If $\lambda \neq \mu$, the Kirillov model of π is given by functions of the form $| \cdot |_K^{1/2} \chi \phi_1 + | \cdot |_K^{1/2} \mu \phi_2$, for ϕ_1 and ϕ_2 in $D(K)$, and

$$L_1(\pi, s) = L(\lambda|_{F^*}, s) \vee L(\mu|_{F^*}, s).$$

If $\lambda = \mu$, the Kirillov model of π is given by functions of the form $| \cdot |_K^{1/2} \lambda \phi_1 + | \cdot |_K^{1/2} \lambda v_K(t) \phi_2$, for ϕ_1 and ϕ_2 in $D(K)$, and

$$L_1(\pi, s) = L(\lambda|_{F^*}, s)^2.$$

In order to compute $L_{rad(ex)}$ for ordinary representations, we need to know when they are distinguished by a character $|\cdot|_F^{-s_0}$ for some $s_0 \in \mathbb{C}$, we will then use Theorem 3.2. The answer is given by the following, which is a mix of Theorem 5.4 and proposition B.17 of [F-H]:

Theorem 4.1. a) *A dihedral supercuspidal representation π of $G_2(K)$ is $|\cdot|_F^{-s_0}$ -distinguished if and only if there exists a quadratic extension B of K , biquadratic over F (hence there are two other extensions between F and B that we call K' and K''), and a character of B^* regular with respect to $N_{B/K}$ which restricts either to K' as $|\cdot|_{K'}^{-s_0}$ or to K'' as $|\cdot|_{K''}^{-s_0}$, such that π is equal to $\pi(\omega)$.*

b) *Let μ be a character of K^* , then the sepcial series representation $\sigma(\mu)$ is $|\cdot|_F^{-s_0}$ -distinguished if and only if μ restricts to F^* as $\eta_{K/F}|\cdot|_F^{-s_0}$.*

c) *Let λ and μ be two characters of K^* , with $\lambda\mu^{-1}$ and $\lambda^{-1}\mu$ different from $|\cdot|_K$, then the principal series representation $\pi(\lambda, \mu)$ is $|\cdot|_F^{-s_0}$ -distinguished if and only if either λ and μ restrict as $|\cdot|_F^{-s_0}$ to F^* or $\lambda\mu^\sigma$ is equal to $|\cdot|_{K'}^{-s_0}$.*

Proof. Let π be a representation, it is $|\cdot|_F^{-s_0}$ -distinguished if and only if $\pi \otimes |\cdot|_K^{s_0/2}$ is distinguished because $|\cdot|_K^{-s_0/2}$ extends $|\cdot|_F^{-s_0}$, it then suffices to apply Theorem 5.4 and proposition B.17 of [F-H]. We give the full proof for case a). Suppose π is dihedral supercuspidal and $\pi \otimes |\cdot|_K^{s_0/2}$ is distinguished. From Theorem 5.4, the representation $\pi \otimes |\cdot|_K^{s_0/2}$ must be of the form $\pi(\omega)$, for ω a character of quadratic extension B of K , biquadratic over F , such that if we call K' and K'' two other extensions between F and B , ω doesn't factorize through $N_{B/K}$ and restricts either trivially on K'^* , or trivially on K''^* . But π is equal to $\pi(\omega) \otimes |\cdot|_K^{-s_0/2} = \pi(\omega|_B^{-s_0/2})$ because $|\cdot|_B = |\cdot|_K \circ N_{B/K}$. As $|\cdot|_B^{-s_0/2}$ restricts to K' (resp. K'') as $|\cdot|_{K'}^{-s_0}$ (resp. $|\cdot|_{K''}^{-s_0}$), case a) follows. \square

We are now able to compute $L_{rad(ex)}$, hence L_{As} for ordinary representations.

i) Suppose that $\pi = \pi(Ind_{W'_B}^{W'_K}(\omega)) = \pi(\omega)$ is supercuspidal, with Langlands parameter $Ind_{W'_B}^{W'_K}(\omega)$, where ω is a multiplicative character of a biquadratic extension B over F that doesn't factorize through $N_{B/K}$.

We note K' and K'' the two other extensions between B and F . Here $L_1(\pi, s)$ is equal to one.

We have the following series of equivalences:

$$\begin{aligned} s_0 \text{ is a pole of } L_{As}(\pi(\omega), s) &\iff \pi(\omega) \text{ is } |\cdot|_F^{-s_0} \text{ -distinguished} \\ &\iff \omega|_{K'^*} = |\cdot|_{K'}^{-s_0} \text{ or } \omega|_{K''^*} = |\cdot|_{K''}^{-s_0} \\ &\iff s_0 \text{ is a pole of } L(\omega|_{K'^*}, s) \text{ or of } L(\omega|_{K''^*}, s) \\ &\iff s_0 \text{ is a pole of } L(\omega|_{K'^*}, s) \vee L(\omega|_{K''^*}, s) \end{aligned}$$

As both functions $L_{As}(\pi(\omega), s)$ and $L(\omega|_{K'^*}, s) \vee L(\omega|_{K''^*}, s)$ have simple poles and are Euler factors, they are equal. Now suppose that $L(\omega|_{K'^*}, s)$ and $L(\omega|_{K''^*}, s)$ have a common pole s_0 , this would imply that $\omega|_{K'^*} = |\cdot|_{K'}^{-s_0}$ and $\omega|_{K''^*} = |\cdot|_{K''}^{-s_0}$, which would mean that $\omega|_B^{s_0/2}$ is trivial on $K'^*K''^*$. According to Lemma 5.2, this would contradict the fact that ω does not factorize through $N_{B/K}$, hence $L(\omega|_{K'^*}, s) \vee L(\omega|_{K''^*}, s) = L(\omega|_{K'^*}, s)L(\omega|_{K''^*}, s)$. Finally we proved:

$$\boxed{L_{As}(\pi(\omega), s) = L(\omega|_{K'^*}, s)L(\omega|_{K''^*}, s).}$$

ii) Suppose that π is a supercuspidal representation, corresponding to an imprimitive representation of W'_K that cannot be induced from a character of the Weil-Deligne group of a biquadratic extension of F . Then necessarily π cannot be $|\cdot|_F^{-s_0}$ -distinguished, for any complex number s_0 of \mathbb{C} .

If it was the case, from Theorem 4.1, it would correspond to a Weil representation $\pi(\omega)$ for some multiplicative character of a biquadratic extension of F , which cannot be. Hence $L_{rad(ex)}(\pi, s)$ has no pole and is equal to one because it is an Euler factor, so we proved that:

$$\boxed{L_{As}(\pi, s) = 1.}$$

iii) If π is equal to $\sigma(\chi)$, then $L_1(\pi, s) = L(\chi|_{F^*}, s + 1)$. We want to compute $L_{rad(ex)}(\pi, s)$, we have the following series of equivalences:

$$\begin{aligned} s_0 \text{ is an exceptional pole of } L_{As}(\sigma(\chi), s) &\iff \sigma(\chi) \text{ is } | \cdot |_F^{-s_0} \text{ - distinguished} \\ &\iff \chi|_{F^*} = \eta_{K/F} | \cdot |_F^{-s_0} \\ &\iff s_0 \text{ is a pole of } L(\chi|_{F^*} \eta_{K/F}, s) \end{aligned}$$

As both functions $L_{rad(ex)}(\pi, s)$ and $L(\chi|_{F^*} \eta_{K/F}, s)$ have simple poles and are Euler factors, they are equal, we thus have:

$$\boxed{L_{As}(\sigma(\chi)) = L(\chi|_{F^*}, s + 1)L(\chi|_{F^*} \eta_{K/F}, s).}$$

iv) If $\pi = \pi(\lambda, \mu)$, we first compute $L_{rad(ex)}(\pi, s)$. We have the following series of equivalences:

$$\begin{aligned} s_0 \text{ is an exceptional pole of } L_{As}(\pi(\lambda, \mu), s) &\iff \pi(\lambda, \mu) \text{ is } | \cdot |_F^{-s_0} \text{ - distinguished} \\ &\iff \lambda\mu^\sigma = | \cdot |_K^{-s_0} \text{ or, } \lambda|_{F^*} = | \cdot |_F^{-s_0} \text{ and } \mu|_{F^*} = | \cdot |_F^{-s_0} \\ &\iff s_0 \text{ is a pole of } L(\lambda\mu^\sigma, s) \text{ or of } L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s) \\ &\iff s_0 \text{ is a pole of } L(\lambda\mu^\sigma, s) \vee [L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)] \end{aligned}$$

As both functions $L_{rad(ex)}(\pi(\lambda, \mu), s)$ and $L(\lambda\mu^\sigma, s) \vee [L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)]$ have simple poles and are Euler factors, they are equal.

If $\lambda \neq \mu$, then $L_1(\pi, s) = L(\lambda|_{F^*}, s) \vee L(\mu|_{F^*}, s)$. But $L(\lambda\mu^\sigma, s)$ and $L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)$ have no common pole. If there was a common pole s_0 , one would have $\lambda\mu^\sigma = | \cdot |_K^{-s_0}$, $\lambda|_{F^*} = | \cdot |_F^{-s_0}$ and $\mu|_{F^*} = | \cdot |_F^{-s_0}$. From $\mu|_{F^*} = | \cdot |_F^{-s_0}$, we would deduce that $\mu \circ N_{K/F} = | \cdot |_K^{-s_0}$, i.e. $\mu^\sigma = | \cdot |_K^{-s_0} \mu^{-1}$, and $\lambda\mu^\sigma = | \cdot |_K^{-s_0}$ would imply $\lambda = \mu$, which is absurd. Hence $L_{rad(ex)}(\pi, s) = L(\lambda\mu^\sigma, s)[L(\lambda|_{F^*}, s) \wedge L(\mu|_{F^*}, s)]$, and finally we have $L_{As}(\pi, s) = L_1(\pi, s)L_{rad(ex)}(\pi, s) = L(\lambda|_{F^*}, s)L(\mu|_{F^*}, s)L(\lambda\mu^\sigma, s)$.

If λ is equal to μ , then $L_1(\pi, s) = L(\lambda|_{F^*}, s)^2$, and $L_{rad(ex)}(\pi(\lambda, \mu), s) = L(\lambda \circ N_{K/F}, s) \vee L(\lambda|_{F^*}, s)$. As $L(\lambda \circ N_{K/F}, s) = L(\lambda|_{F^*}, s)L(\eta_{K/F}\lambda|_{F^*}, s)$, we have $L_{rad(ex)}(\pi(\lambda, \mu), s) = L(\lambda \circ N_{K/F}, s)$. Again we have $L_{As}(\pi, s) = L(\lambda|_{F^*}, s)L(\mu|_{F^*}, s)L(\lambda\mu^\sigma, s)$.

In both cases, we have

$$\boxed{L_{As}(\pi(\lambda, \mu), s) = L(\lambda|_{F^*}, s)L(\mu|_{F^*}, s)L(\lambda\mu^\sigma, s).}$$

Eventually, comparing with equalities of subsection 4.1, we proved the following:

Theorem 4.2. *Let $\rho \mapsto \pi(\rho)$ be the Langlands correspondence from two dimensional representations of W'_K to smooth irreducible infinite dimensional representations of $G_2(K)$, then if ρ is not primitive, $\pi(\rho)$ is ordinary and we have the following equality of L-functions:*

$$\boxed{L_{As}(\pi(\rho), s) = L(M_{W'_K}^{W'_F}(\rho), s)}$$

As said in the introduction, combining Theorem 1.6 of [A-R] and Theorem of paragraph 1.5 in [He], one gets that $L(M_{W'_K}^{W'_F}(\rho), s) = L_{As}(\pi(\rho), s)$ for $\pi(\rho)$ a discrete series representation, so that we have actually the following:

Theorem 4.3. *Let $\rho \mapsto \pi(\rho)$ be the Langlands correspondence from two dimensional representations of W'_K to smooth irreducible infinite dimensional representations of $G_2(K)$, we have the following equality of L -functions:*

$$L_{As}(\pi(\rho), s) = L(M_{W'_K}^{W'_F}(\rho), s)$$

5 Appendix. Dihedral supercuspidal distinguished representations

The aim of this section is to give a description of dihedral supercuspidal distinguished representations of $G_2(K)$ in terms of Langlads parameter, it is done in Theorem 5.4.

5.1 Preliminary results

Let E be a local field, E' be a quadratic extension of E , χ a character of E^* , π be a smooth irreducible infinite dimensional representation of $G_2(E)$, and ψ a non trivial character of E .

We note $L(\chi, s)$ and $\epsilon(\chi, s, \psi)$ the functions of the complex variable s defined before proposition 3.5 in [J-L]. We note $\gamma(\chi, s, \psi)$ the ratio $\epsilon(\chi, s, \psi)L(\chi, s)/L(\chi^{-1}, 1 - s)$.

We note $L(\pi, s)$ and $\epsilon(\pi, s, \psi)$ the functions of the complex variable s defined in Theorem 2.18 of [J-L]. We note $\gamma(\pi, s, \psi)$ the ratio $\epsilon(\pi, s, \psi)L(\pi, s)/L(\pi^\vee, 1 - s)$.

We note $\lambda(E'/E, \psi)$ the Langlands-Deligne factor defined before proposition 1.3 in [J-L], it is equal to $\epsilon(\eta_{E'/E}, 1/2, \psi)$. As $\eta_{E'/E}$ is equal to $\eta_{E'/E}^{-1}$, the factor $\lambda(E'/E, \psi)$ is also equal to $\gamma(\eta_{E'/E}, 1/2, \psi)$.

From theorem 4.7 of [J-L], if ω is a character of E'^* , then $L(\pi(\omega), s)$ is equal to $L(\omega, s)$, and $\epsilon(\pi, s, \psi)$ is equal to $\lambda(E'/E, \psi)\epsilon(\omega, s, \psi)$, hence $\gamma(\pi, s, \psi)$ is equal to $\lambda(E'/E, \psi)\gamma(\omega, s, \psi)$.

We will need four results. The first is due to Frhlich and Queyrut, see [D] theorem 3.2 for a quick proof using a Poisson formula:

Proposition 5.1. *Let E be a local field, E' be a quadratic extension of E , χ' a character of E'^* trivial on E^* , and ψ' a non trivial character of E' trivial on E , then $\gamma(\chi', 1/2, \psi') = 1$.*

The second is a criterion of Hakim:

Theorem 5.1. *([Ha], Theorem 4.1) Let π be an irreducible supercuspidal representation of $G_2(K)$ with central character trivial on F^* , and ψ a nontrivial character of K trivial on F . Then π is distinguished if and only if $\gamma(\pi \otimes \chi, 1/2, \psi) = 1$ for every character χ of K^* trivial on F^* .*

The third is due to Flicker:

Theorem 5.2. *([F1], proposition 12) Let π be a smooth irreducible distinguished representation of $G_n(K)$, then π^σ is isomorphic to π^\vee .*

The fourth is due to Kable in the case of $G_n(K)$, see [A-T] for a local proof in the case of $G_2(K)$:

Theorem 5.3. *([A-T], Proposition 3.1) There exists no supercuspidal representation of $G_2(K)$ which is distinguished and $\eta_{K/F}$ -distinguished at the same time.*

5.2 Distinction criterion for dihedral supercuspidal representations

As a dihedral representation's parameter is a multiplicative character of a quadratic extension L of K , we first look at the propositionerties of the tower $F \subset K \subset L$. Three cases arise:

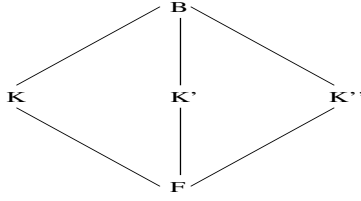


Figure 1:

1. L/F is biquadratic (hence Galois), it contains K and two other quadratic extensions F , K' and K'' .

Its Galois group is isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, its non trivial elements are $\sigma_{L/K}$, $\sigma_{L/K'}$ and $\sigma_{L/K''}$. The conjugation $\sigma_{L/K}$ extend $\sigma_{K'/F}$ and $\sigma_{K''/F}$.

2. L/F is cyclic with Galois group isomorphic with $\mathbb{Z}/4\mathbb{Z}$, in this case we fix an element $\tilde{\sigma}$ in $G(L/F)$ extending σ , it is of order 4.

3. L/F non Galois. Then its Galois Closure M is quadratic over L and the Galois group of M over F is dihedral with order 8. To see this, we consider a morphism $\tilde{\theta}$ from L to \bar{F} which extends θ . Then if $L' = \tilde{\theta}(L)$, L and L' are distinct, quadratic over K and generate M biquadratic over K . M is the Galois closure of L because any morphism from L into \bar{F} , either extends θ , or the identity map of K , so that its image is either L or L' , so it is always included in M . Finally the Galois group M over F cannot be abelian (for L is not Galois over F), it is of order 8, and it's not the quaternion group which only has one element of order 2, whereas here $\sigma_{M/L}$ and $\sigma_{M/L'}$ are of order 2. Hence it is the dihedral group of order 8 and we have the following lattice, where M/K' is cyclic of degree 4, M/K and B/F are biquadratic.

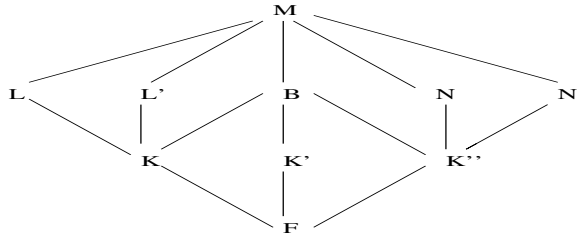


Figure 2:

We now prove the following proposition:

Proposition 5.2. *If a supercuspidal dihedral representation π of $G_2(K)$ verifies $\pi^\vee = \pi^\sigma$, there exists a biquadratic extension B of F , containing K , such that if we call K' and K'' the two other extensions between F and B , there is a character ω of B trivial either on $N_{B/K'}(B^*)$ or on $N_{B/K''}(B^*)$, such that $\pi = \pi(\omega)$.*

Proof. Let L be a quadratic extension of K and ω a regular multiplicative of L such that $\pi = \pi(\omega)$, we note σ the conjugation of L over K , three cases show up:

1. L/F is biquadratic. The conjugations $\sigma_{L/K'}$ and $\sigma_{L/K''}$ both extend σ , hence from Theorem 1 of [G-L], we have $\pi(\omega)^\sigma = \pi(\omega^{\sigma_{L/K'}})$. The condition $\pi^\vee = \pi^\sigma$ which one can also read $\pi(\omega^{-1}) = \pi(\omega^{\sigma_{L/K'}})$, is then equivalent from Appendix B, (2)b)1) of [G-L], to $\omega^{\sigma_{L/K'}} = \omega^{-1}$ or $\omega^{\sigma_{L/K''}} = \omega^{-1}$. This is equivalent to ω trivial on $N_{L/K'}(L^*)$ or on $N_{L/K''}(L^*)$.

2. L/F is cyclic, the regularity of ω makes the condition $\pi(\omega^{-1}) = \pi(\omega)^\sigma$ impossible. Indeed one would have from Theorem 1 of [G-L] $\pi(\omega^{\tilde{\sigma}}) = \pi(\omega^{-1})$, which from Appendix B, (2)b)1) of [G-L] would imply $\omega^{\tilde{\sigma}} = \omega$ or $\omega^{\tilde{\sigma}^{-1}} = \omega$. As $\tilde{\sigma}^2 = \tilde{\sigma}^{-2} = \sigma$, this would in turn imply $\omega^\sigma = \omega$, and ω would be trivial on the kernel of $N_{L/K}$ according to Hilbert's theorem 90. π^{vee} can therefore not be isomorphic to π^σ .
3. L/K is not Galois (which implies $q \equiv 3[4]$ in the case p odd). Let $\pi_{B/K}$ be the representation of $G_2(B)$ which is the base change lift of π to B . As $\pi_{B/K} = \pi(\omega \circ N_{M/L})$, if $\omega \circ N_{M/L} = \mu \circ N_{M/B}$ for a character μ of B^* , then $\pi(\omega) = \pi(\mu)$ (cf. [G-L], (3) of Appendix B) and we are brought back to case 1. Otherwise $\omega \circ N_{M/L}$ is regular with respect to $N_{M/B}$. If $\pi^\sigma = \pi^\vee$, we would have $\pi_{B/K}^{\sigma_{B/K'}} = \pi_{B/K}^\vee$ from Theorem 1 of [G-L]. That would contradict case 2 because M/K' is cyclic. □

We described in the previous proposition representations π of $G_2(K)$ verifying $\pi^\vee = \pi^\sigma$, now we characterize those who are $G_2(F)$ -distinguished among them (from Theorem 5.2, a distinguished representation always satisfies the previous condition).

Theorem 5.4. *A dihedral supercuspidal representation π of $G_2(K)$ is $G_2(F)$ -distinguished if and only if there exists a quadratic extension B of K biquadratic over F such that if we call K' and K'' the two other extensions between B and F , there is character ω of B^* that does not factorize through $N_{B/K}$ and trivial either on K'^* or on K''^* , such that $\pi = \pi(\omega)$.*

Proof. From Theorem 5.2 and Proposition 5.2, we can suppose that $\pi = \pi(\omega)$, for ω a regular multiplicative character of a quadratic extension B of K biquadratic over F , with ω trivial on $N_{L/K'}(K'^*)$ or on $N_{B/K''}(K''^*)$. We will need the following:

Lemma 5.1. *Let B be a quadratic extension of K biquadratic over F , then F^* is a subset of $N_{B/K}(B^*)$*

Proof of Lemma 5.1. The group $N_{B/K}(B^*)$ contains the two groups $N_{B/K}(K'^*)$ and $N_{B/K}(K''^*)$, which, as $\sigma_{B/K}$ extends $\sigma_{K'/F}$ and $\sigma_{K''/F}$, are respectively equal to $N_{K'/F}(K'^*)$ and $N_{K''/F}(K''^*)$. But these two groups are distinct of index 2 in F^* from local classfield theory, thus they generate F^* , which is therefore contained in $N_{B/K}(B^*)$. □

We now choose ψ a non trivial character of K/F and note ψ_B the character $\psi \circ \text{Tr}_{B/K}$, it is trivial on K' and K'' .

Suppose ω trivial on K' or K'' , then the restriction of the central character $\eta_{B/K}\omega$ of $\pi(\omega)$ is trivial on F^* according to Lemma 5.1.

As we have $\gamma(\pi(\omega), 1/2, \psi) = \lambda(B/K, \psi)\gamma(\omega, 1/2, \psi_B) = \gamma(\eta_{B/K}, 1/2, \psi)\gamma(\omega, 1/2, \psi_B)$, we deduce from Lemma 5.1 and Proposition 5.1 that $\gamma(\pi(\omega), 1/2, \psi)$ is equal to one, hence from Theorem 5.1, the representation $\pi(\omega)$ is distinguished.

Now suppose $\omega|_{K'} = \eta_{B/K'}$ or $\omega|_{K''} = \eta_{B/K''}$, let χ be a character of K^* extending $\eta_{K/F}$, then $\pi(\omega) \otimes \chi = \pi(\omega\chi \circ N_{B/K})$. As $N_{B/K|_{K'}} = N_{K'/F}$ and $N_{B/K|_{K''}} = N_{K''/F}$, we have $\chi \circ N_{B/K|_{K'}} = \eta_{B/K'}$ and $\chi \circ N_{B/K|_{K''}} = \eta_{B/K''}$, hence from what we've just seen, $\pi(\omega) \otimes \chi$ is distinguished, i.e. $\pi(\omega)$ is $\eta_{K/F}$ -distinguished.

From Theorem 5.3, π cannot be distinguished and $\eta_{K/F}$ -distinguished at the same time, and the theorem follows. □

We end with the following lemma:

Lemma 5.2. *Let B be a quadratic extension of K which is biquadratic over F . Call K' and K'' the two other extensions between F and B , then the kernel of $N_{B/K}$ is a subgroup of the group $N_{B/K'}(B^*)N_{B/K''}(B^*)$.*

Proof. If u belongs to $\text{Ker}(N_{B/K})$, it can be written $x/\sigma_{B/K}(x)$ for some x in B^* according to Hilbert's Theorem 90. Hence we have $u = (x\sigma_{B/K'}(x))/(\sigma_{B/K}(x)\sigma_{B/K'}(x)) = N_{B/K'}(x)/N_{B/K''}(\sigma_{B/K}(x))$, and u belongs to $N_{B/K'}(B^*)N_{B/K''}(B^*)$. \square

Corollary 5.1. *The (either/or) in Proposition 5.2 and Theorem 5.4 is exclusive*

Proof. In fact, in the situation of Lemma 5.2, a character ω that is trivial on $N_{B/K'}(B^*)$ and $N_{B/K''}(B^*)$ factorizes through $N_{B/K}$, and $\pi(\omega)$ is not supercuspidal. \square

Acknowledgements

I would like to thank Corinne Blondel and Paul Gérardin for many helpful comments. I also thank Jeffrey Hakim who allowed me to have access to Youngbin Ok's PHD thesis.

References

- [A-T] U.K. Anandavardhanan and R.Tandon , *On Distinguishness*, Pacific J. Math., **206** (2002), 269-286.
- [A-K-T] U.K. Anandavardhanan, A.C. Kable and R.Tandon *Distinguished representations and poles of twisted tensor L-functions*, Proc. Amer. Math. Soc., **132** (2004), No. 10, 2875-2883.
- [A-R] U.K. Anandavardhanan and C.S. Rajan, *Distinguished representations, base change, and reducibility for unitary groups*, Int. Math. Res. Not., **14** (2005), No. 14, 841-854.
- [B] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.
- [C-P] J.W. Cogdell, I.I. Piatetski-Shapiro, *Derivatives and L-functions for $GL(n)$* , to appear in The Heritage of B. Moisezon, IMCP.
- [D] P. Deligne, *Les constantes locales de l'equation fonctionnelle de la fonction L d'Artin d'une representation orthogonale*, Invent. Math., **35**, (1976), pp. 299-316.
- [F] Y. Flicker, *Twisted tensors and Euler products*, Bull. Soc. Math. France, **116** no.3 (1988), p. 295-313.
- [F1] Y. Flicker, *On distinguished representations*, J. Reine Angew. Math., **418** (1991), 139-172.
- [F2] Y. Flicker, Appendix of *On zeroes of the twisted tensor L-function*, Math. Ann., **297**, (1993), p.199-219.
- [F-H] Y. Flicker and J. Hakim, *Quaternionic distinguished representations*, Amer. J. Math., **116**, (1994), 683-736.
- [G-L] P. Grardin and J-P. Labesse, *The solution of a base change problem for $GL(2)$* , in Automorphic Forms, Representations and L-functions (Corvallis), AMS Proc. Symp. Pure. Math., **33** (1979), 115-133.
- [Ha] J. Hakim, *Distinguished p-adic Representations*, Duke Math. J., **62** (1991), 1-22.
- [He] G. Henmiart, *Correspondance de Langlands et fonctions L des carrs extrieur et symtrique*, preprint, 2003, Institut des Hautes Etudes Scientifiques.
- [J-L] H. Jacquet and R. Langlands, *Automorphic forms on $GL(2)$* , Lect. Notes Math., **114**, Springer, 1970.
- [J-P-S] H.Jacquet, I.I.Piatetskii-Shapiro and J.A.Shalika, *Rankin-Selberg Convolutions*, Amer. J. Math., **105** (1983), 367-464.

- [K] A.C. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math., **126**, (2004), 789-820.
- [P] D. Prasad, *Invariant linear forms for representations of $GL(2)$ over a local field*, Amer. J. Math., **114**, 1317-1363 (1992).
- [Ok] Y. Ok, *Distinction and Gamma factors at $1/2$: Supercuspidal Case*, Thesis, Columbia University (1997)