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APPROXIMATE VOLUME AND INTEGRATION FOR BASIC SEMI-ALGEBRAIC SETS*

D. HENRION[†], J. B. LASSERRE[‡], AND C. SAVORGNAN[§]

Abstract. Given a basic compact semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$, we introduce a methodology that generates a sequence converging to the volume of \mathbf{K} . This sequence is obtained from optimal values of a hierarchy of either semidefinite or linear programs. Not only the volume but also every finite vector of moments of the probability measure that is uniformly distributed on \mathbf{K} can be approximated as closely as desired, and so permits to approximate the integral on \mathbf{K} of any given polynomial; extension to integration against some weight functions is also provided. Finally, some numerical issues associated with the algorithms involved are briefly discussed.

Key words. Computational geometry; volume; integration; \mathbf{K} -moment problem; semidefinite programming

AMS subject classifications. 14P10, 11E25, 12D15, 90C25

1. Introduction. Computing the volume and/or integrating on a subset $\mathbf{K} \subset \mathbb{R}^n$ is a challenging problem with many important applications. One possibility is to use basic Monte Carlo techniques that generate points uniformly in a box containing \mathbf{K} and then count the proportion of points falling into \mathbf{K} . To the best of our knowledge, all other approximate (deterministic or randomized) or exact techniques deal with polytopes or convex bodies only. Similarly, powerful cubature formulas exist for numerical integration against a weight function on simple sets (like e.g. simplex, box), but not for arbitrary semi-algebraic sets.

The purpose of this paper is to introduce a deterministic technique that potentially applies to any basic compact semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$. It is deterministic (no randomization) and differs from previous ones in the literature essentially dedicated to *convex* bodies (and more particularly, convex polytopes). Indeed, one treats the original problem as an infinite dimensional optimization (and even linear programming (LP)) problem whose unknown is the Lebesgue measure on \mathbf{K} . Next, by restricting to finitely many of its moments, and using a certain characterization on the \mathbf{K} -moment problem, one ends up in solving a hierarchy of semidefinite programming (SDP) problems whose size is parametrized by the number of moments considered; the dual LP has a simple interpretation and from this viewpoint, convexity of \mathbf{K} does not help much. For a certain choice of the criterion to optimize, one obtains a monotone non increasing sequence of upper bounds on the volume of \mathbf{K} . Convergence to the exact value invokes results on the \mathbf{K} -moment problem by Putinar [36]. Importantly, there is *no* convexity and not even connectedness assumption on \mathbf{K} , as this plays no role in the \mathbf{K} -moment problem. Alternatively, using a different characterization of the

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\mathbf{K} -moment problem due to Krivine [23], one may solve a hierarchy of LP (instead of SDP) problems whose size is also parametrized by the number of moments. Our contribution is a new addition to the already very long list of applications of the moment approach (some of them described in e.g. Landau [24] and Lasserre [28]) and semidefinite programming [45]. In principle, the method also permits to approximate any finite number of moments of the uniform distribution on \mathbf{K} , and so provides a means to approximate the integral of a polynomial on \mathbf{K} . Extension to integration against a weight function is also proposed.

Background. Computing or even approximating the volume of a convex body is hard theoretically and in practice as well. Even if $\mathbf{K} \subset \mathbb{R}^n$ is a convex polytope, exact computation of its volume or integration over \mathbf{K} is a computational challenge. Computational complexity of these problems is discussed in e.g. Bollobás [7] and Dyer and Frieze [11]. Any deterministic algorithm with polynomial time complexity that would compute an upper bound $\overline{\text{vol}}(\mathbf{K})$ and a lower bound $\underline{\text{vol}}(\mathbf{K})$ on $\text{vol}(\mathbf{K})$ cannot yield an upper bound on the ratio $\overline{\text{vol}}(\mathbf{K})/\underline{\text{vol}}(\mathbf{K})$ better than polynomial in the dimension n . Methods for exact volume computation use either triangulations or simplicial decompositions depending on whether the polytope has a half-space description or a vertex description. See e.g. Cohen and Hickey, [10], Lasserre [25], Lawrence [32] and see Büeler et al. [8] for a comparison. Another set of methods which use generating functions are described in e.g. Barvinok [3] and Lasserre and Zeron [30]. Concerning integration on simple sets (e.g. simplex, box) via cubature formulas, the interested reader is referred to Gautschi [14, 15] and Trefethren [43].

A *convex body* $\mathbf{K} \subset \mathbb{R}^n$ is a compact convex subset with nonempty interior. A strong separation *oracle* answers either $x \in \mathbf{K}$ or $x \notin \mathbf{K}$, and in the latter case produces a hyperplane separating x from \mathbf{K} . A negative result states that for every polynomial-time algorithm for computing the volume of a convex body $\mathbf{K} \subset \mathbb{R}^n$ given by a well-guaranteed separation oracle, there is a constant $c > 0$ such that $\overline{\text{vol}}(\mathbf{K})/\underline{\text{vol}}(\mathbf{K}) \leq (cn/\log n)^n$ cannot be guaranteed for $n \geq 2$. However, Lovász [33] proved that there is a polynomial-time algorithm that produces $\overline{\text{vol}}(\mathbf{K})$ and $\underline{\text{vol}}(\mathbf{K})$ satisfying $\overline{\text{vol}}(\mathbf{K})/\underline{\text{vol}}(\mathbf{K}) \leq n^n (n+1)^{n/2}$, whereas Elekes [13] proved that for $0 < \epsilon < 2$ there is no polynomial-time algorithm that produces $\overline{\text{vol}}(\mathbf{K})$ and $\underline{\text{vol}}(\mathbf{K})$ satisfying $\overline{\text{vol}}(\mathbf{K})/\underline{\text{vol}}(\mathbf{K}) \leq (2 - \epsilon)^n$.

If one accepts randomized algorithms that fail with small probability, then the situation is more favorable. Indeed, the celebrated Dyer, Frieze and Kanan probabilistic approximation algorithm [12] computes the volume to fixed arbitrary relative precision ϵ , in time polynomial in ϵ^{-1} . The latter algorithm uses approximation schemes based on rapidly mixing Markov chains and isoperimetric inequalities. See also hit-and-run algorithms for sampling points according to a given distribution, described in e.g. Belisle [5], Belisle et al. [6], and Smith [41]

Contribution. This paper is concerned with computing (or rather approximating) the volume of a compact basic semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$ defined by

$$\mathbf{K} := \{ x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \dots, m \} \quad (1.1)$$

for some polynomials $(g_j)_{j=1}^m \subset \mathbb{R}[x]$. Hence \mathbf{K} is possibly non-convex and non-connected. Therefore, in view of the above discussion, this is quite a challenging problem.

(a) We present a numerical scheme that depends on a parameter p , a polynomial that is nonnegative on \mathbf{K} (e.g. $p \equiv 1$). For each parameter p , it provides converging approximations of moments of the measure uniformly supported on \mathbf{K} (with mass equal to $\text{vol}(\mathbf{K})$). For the choice $p \equiv 1$ one obtains a *monotone* non-increasing sequence of upper bounds that converges to $\text{vol}(\mathbf{K})$.

(b) The way we see the problem dates back to the 19th century pioneer work in the one-dimensional case by Chebyshev [9], Markov [34] and Stieltjes [42], where given n moments $s_k = \int_a^b t^k f(t) dt$, $k = 0, \dots, n-1$, and $a < c < d < b$, one wishes to approximate the integral $\int_c^d f(t) dt$ and analyzes asymptotics as $n \rightarrow \infty$; characterizing feasible sequence (s_k) is referred to as the Markov moment problem (and L -moment problem if in addition one requires $0 \leq f \leq L$ for some scalar L). For an historical account on this problem as well as other developments, the interested reader is referred to e.g. Krein [21], Krein and Nudelman [22], Karlin and Studden [20] and Putinar [37].

Our method combines a simple idea, easy to describe, with relatively recent powerful results on the \mathbf{K} -moment problem described in e.g. [38, 39, 40]. It only requires knowledge of a set \mathbf{B} (containing \mathbf{K}) simple enough so that the moments of the Lebesgue measure on \mathbf{B} can be obtained easily. For instance $\mathbf{B} := \{x \in \mathbb{R}^n : \|x\|_p \leq a\}$ with $p = 2$ (the scaled n -dimensional ball) or $p = \infty$ (the scaled n -dimensional box) and $a \in \mathbb{R}$ a given constant. Then computing $\text{vol}(\mathbf{K})$ is equivalent to computing the mass of the Borel measure μ which is the restriction to \mathbf{K} of the Lebesgue measure on \mathbf{B} . This in turn is translated into an infinite dimensional LP problem P with parameter p (some polynomial nonnegative on \mathbf{K}) and with the Borel measure μ as unknown. Then, from certain results on the \mathbf{K} -moment problem and its dual theory of the representation of polynomials positive on \mathbf{K} , problem P can be approximated by an appropriate hierarchy of semidefinite programs (SDP) whose size depends on the number d of moments of μ considered. One obtains approximations of the moments of μ which converge to the exact value as $d \rightarrow \infty$. For the choice $p \equiv 1$ of the parameter p , one even obtains a non-increasing sequence of upper bounds converging to $\text{vol}(\mathbf{K})$. Asymptotic convergence is ensured by invoking results of Putinar [36] on the \mathbf{K} -moment problem. Alternatively, one may replace the SDP hierarchy with an LP hierarchy and now invoke results of Krivine [23] for convergence.

Interestingly, the dual of each SDP relaxation defines a strengthening of P^* , the LP dual of P , and highlights why the problem of computing the volume is difficult. Indeed, one has to approximate from above the function f ($= p$ on \mathbf{K} and 0 on $\mathbf{B} \setminus \mathbf{K}$) by a polynomial h of bounded degree, so as to minimize the integral $\int_{\mathbf{B}} (h - f) dx$. From this viewpoint, convexity of \mathbf{K} plays no particular role and so, does not help much.

(c) Let $d \in \mathbb{N}$ be fixed, arbitrary. One obtains an approximation of the moments of degree up to d of the measure μ on \mathbf{K} , as closely as desired. Therefore, this technique also provides a sequence of approximations that converges to $\int_{\mathbf{K}} q dx$ for *any* polynomial q of degree at most d (in contrast, Monte Carlo simulation is for a given q). Finally, we also propose a similar approximation scheme for integrating a polynomial on \mathbf{K} against a nonnegative weight function $w(x)$. The only required data are moments of the measure $d\nu = w dx$ on a simple set \mathbf{B} (e.g. box or simplex) containing \mathbf{K} , which can be obtained by usual cubature formulas for integration.

On the practical side, at each step d of the hierarchy, the computational workload is that of solving an SDP problem of increasing size. In principle, this can be done

in time polynomial in the input size of the SDP problem, at given relative accuracy. However, in view of the present status and limitations of current SDP solvers, so far the method is restricted to problems of small dimension n if one wishes to obtain good approximations. The alternative LP hierarchy might be preferable for larger size problems, even if proved to be less efficient when used in other contexts where the moment approach applies, see e.g. [27, 31].

Preliminary results on simple problems for which $\text{vol}(\mathbf{K})$ is known show that indeed convexity plays no particular role. In addition, as for interpolation problems, the choice of the basis of polynomials is crucial from the viewpoint of numerical precision. This is illustrated on a trival example on the real line where, as expected, the basis of Chebyshev polynomials is far better than the usual monomial basis. In fact, it is conjectured that trigonometric polynomials would be probably the best choice. Finally, the choice of the parameter p is also very important and unfortunately, the choice of $p \equiv 1$ which guarantees a monotone convergence to $\text{vol}(\mathbf{K})$ is not the best choice at all. Best results are obtained when p is negative outside \mathbf{K} .

So far, for convex polytopes, this method is certainly not competitive with exact specific methods as those described in e.g. [8]. It rather should be viewed as a relatively simple deterministic methodology that applies to a very general context for which even getting good bounds on $\text{vol}(\mathbf{K})$ is very difficult, and for which the only alternative presently available seems to be brute force Monte Carlo.

2. Notation, definitions and preliminary results. Let $\mathbb{R}[x]$ be the ring of real polynomials in the variables $x = (x_1, \dots, x_n)$, and let $\Sigma^2[x] \subset \mathbb{R}[x]$ be the subset of sums of squares (SOS) polynomials. Denote $\mathbb{R}[x]_d \subset \mathbb{R}[x]$ be the set of polynomials of degree at most d , which forms a vector space of dimension $s(d) = \binom{n+d}{d}$. If $f \in \mathbb{R}[x]_d$, write $f(x) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha$ in the usual canonical basis (x^α) , and denote by $\mathbf{f} = (f_\alpha) \in \mathbb{R}^{s(d)}$ its vector of coefficients. Similarly, denote by $\Sigma^2[x]_d \subset \Sigma^2[x]$ the subset of SOS polynomials of degree at most $2d$.

Moment matrix. Let $\mathbf{y} = (y_\alpha)$ be a sequence indexed in the canonical basis (x^α) of $\mathbb{R}[x]$, let $L_{\mathbf{y}} : \mathbb{R}[x] \rightarrow \mathbb{R}$ be the linear functional

$$f \quad (= \sum_{\alpha} f_{\alpha} x^{\alpha}) \quad \mapsto \quad L_{\mathbf{y}}(f) = \sum_{\alpha} f_{\alpha} y_{\alpha},$$

and let $M_d(\mathbf{y})$ be the symmetric matrix with rows and columns indexed in the canonical basis (x^α) , and defined by:

$$M_d(\mathbf{y})(\alpha, \beta) := L_{\mathbf{y}}(x^{\alpha+\beta}) = y_{\alpha+\beta},$$

for every $\alpha, \beta \in \mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| (= \sum_i \alpha_i) \leq d\}$.

A sequence $\mathbf{y} = (y_\alpha)$ is said to have a *representing* finite Borel measure μ if $y_\alpha = \int x^\alpha d\mu$ for every $\alpha \in \mathbb{N}^n$. A necessary (but not sufficient) condition is that $M_d(\mathbf{y}) \succeq 0$ for every $d \in \mathbb{N}$. However, if in addition, $|y_\alpha| \leq M$ for some M and for every $\alpha \in \mathbb{N}^n$, then \mathbf{y} has a representing measure on $[-1, 1]^n$.

Localizing matrix. Similarly, with $\mathbf{y} = (y_\alpha)$ and $g \in \mathbb{R}[x]$ written as

$$x \mapsto g(x) = \sum_{\gamma \in \mathbb{N}^n} g_{\gamma} x^{\gamma},$$

let $M_d(g\mathbf{y})$ be the symmetric matrix with rows and columns indexed in the canonical

basis (x^α) , and defined by:

$$M_d(g\mathbf{y})(\alpha, \beta) := L_{\mathbf{y}}(g(x)x^{\alpha+\beta}) = \sum_{\gamma} g_{\gamma} y_{\alpha+\beta+\gamma},$$

for every $\alpha, \beta \in \mathbb{N}_d^n$. A necessary (but not sufficient) condition for \mathbf{y} to have a representing measure with support contained in the level set $\{x : g(x) \geq 0\}$ is that $M_d(g\mathbf{y}) \succeq 0$ for every $d \in \mathbb{N}$.

2.1. Moment conditions and representation theorems. The following results from the \mathbf{K} -moment problem and its dual theory of polynomials positive on \mathbf{K} provide the rationale behind the hierarchy of SDP relaxations introduced in [26], and potential applications in many different contexts. See e.g. [28] and the many references therein.

SOS-based representations. Let $Q(g) \subset \mathbb{R}[x]$ be the quadratic module generated by polynomials $(g_j)_{j=1}^m \subset \mathbb{R}[x]$, that is,

$$Q(g) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j : (\sigma_j)_{j=1}^m \subset \Sigma^2[x] \right\}. \quad (2.1)$$

ASSUMPTION 2.1. *The set $\mathbf{K} \subset \mathbb{R}^n$ in (1.1) is compact and the quadratic polynomial $x \mapsto a^2 - \|x\|^2$ belongs to $Q(g)$ for some given constant $a \in \mathbb{R}$.*

THEOREM 2.2 (Putinar's Positivstellensatz [36]). *Let Assumption 2.1 hold.*

(a) *If $f \in \mathbb{R}[x]$ is strictly positive on \mathbf{K} , then $f \in Q(g)$. That is:*

$$f = \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \quad (2.2)$$

for some SOS polynomials $(\sigma_j)_{j=1}^m \subset \Sigma^2[x]$.

(b) *If $\mathbf{y} = (y_\alpha)$ is such that for every $d \in \mathbb{N}$,*

$$M_d(\mathbf{y}) \succeq 0; \quad M_d(g_j \mathbf{y}) \succeq 0, \quad j = 1, \dots, m, \quad (2.3)$$

then \mathbf{y} has a representing finite Borel measure μ supported on \mathbf{K} .

Given $f \in \mathbb{R}[x]$, or $\mathbf{y} = (y_\alpha) \subset \mathbb{R}$, checking whether (2.2) holds for SOS $(\sigma_j) \subset \Sigma^2[x]$ with a priori bounded degree, or checking whether (2.3) holds with d fixed, reduces to solving an SDP.

Another type of representation. Let $\mathbf{K} \subseteq \mathbf{B}$ be as in (1.1) and assume for simplicity that the g_j s have been scaled to satisfy $0 \leq g_j \leq 1$ on \mathbf{K} , for every $j = 1, \dots, m$. In addition, assume that the family of polynomials $(1, g_1, \dots, g_m)$ generates the algebra $\mathbb{R}[x]$. For every $\alpha \in \mathbb{N}^m$, let g^α and $(1-g)^\beta$ denote the polynomials

$$x \mapsto g(x)^\alpha := g_1(x)^{\alpha_1} \cdots g_m(x)^{\alpha_m},$$

and

$$x \mapsto (1-g(x))^\beta := (1-g_1(x))^{\beta_1} \cdots (1-g_m(x))^{\beta_m}.$$

the following result is due to Krivine [23] but is explicit in e.g. Vasilescu [46].

THEOREM 2.3.

(a) *If $f \in \mathbb{R}[x]$ is strictly positive on \mathbf{K} , then*

$$f = \sum_{\alpha, \beta \in \mathbb{N}^m} c_{\alpha\beta} g^\alpha (1-g)^\beta \quad (2.4)$$

for finitely many nonnegative scalars $(c_{\alpha\beta}) \subset \mathbb{R}_+$.

(b) *If $\mathbf{y} = (y_\alpha)$ is such that*

$$L_{\mathbf{y}}(g^\alpha (1-g)^\beta) \geq 0, \quad (2.5)$$

for every $\alpha, \beta \in \mathbb{N}^m$, then \mathbf{y} has a representing finite Borel measure μ supported on \mathbf{K} .

Theorem 2.3 extends the well-known Hausdorff moment conditions on the hypercube $[0, 1]^n$, as well as Handelman representation [17] for convex polytopes $\mathbf{K} \subset \mathbb{R}^n$. Observe that checking whether (2.4), resp. (2.5), holds with α, β bounded a priori, reduces to solving an LP in the variables $(c_{\alpha\beta})$, resp. (y_α) .

2.2. A preliminary result. Given any two measures μ_1, μ_2 on a Borel σ -algebra \mathcal{B} , the notation $\mu_1 \leq \mu_2$ means $\mu_1(\mathbf{C}) \leq \mu_2(\mathbf{C})$ for every $\mathbf{C} \in \mathcal{B}$.

LEMMA 2.4. *Let Assumption 2.1 hold and let $\mathbf{y}_1 = (y_{1\alpha})$ and $\mathbf{y}_2 = (y_{2\alpha})$ be two moment sequences with respective representing measures μ_1 and μ_2 on \mathbf{K} . If*

$$M_d(\mathbf{y}_2 - \mathbf{y}_1) \succeq 0; \quad M_d(g_j(\mathbf{y}_2 - \mathbf{y}_1)) \succeq 0, \quad j = 1, \dots, m,$$

for every $d \in \mathbb{N}$, then $\mu_1 \leq \mu_2$.

Proof. As $M_d(\mathbf{y}_2 - \mathbf{y}_1) \succeq 0$ and $M_d(g_j(\mathbf{y}_2 - \mathbf{y}_1)) \succeq 0$ for $j = 1, \dots, m$ and $d \in \mathbb{N}$, by Theorem 2.2, the sequence $\mathbf{y}_0 := \mathbf{y}_2 - \mathbf{y}_1$ has a representing Borel measure μ_0 on \mathbf{K} . From $y_{0\alpha} + y_{1\alpha} = y_{2\alpha}$ for every $\alpha \in \mathbb{N}^n$, we conclude that

$$\int x^\alpha d\mu_0 + \int x^\alpha d\mu_1 = \int x^\alpha d\mu_2, \quad \forall \alpha \in \mathbb{N}^n,$$

and as \mathbf{K} is compact, by the Stone-Weierstrass theorem,

$$\int f d\mu_0 + \int f d\mu_1 = \int f d\mu_2$$

for every continuous function f on \mathbf{K} , which in turn implies $\mu_0 + \mu_1 = \mu_2$, i.e., the desired result $\mu_1 \leq \mu_2$. \square

3. Main result. We first introduce an infinite-dimensional LP problem P whose unique optimal solution is the restriction μ of the normalized Lebesgue measure on \mathbf{B} (hence with $\mu(\mathbf{K}) = \text{vol}(\mathbf{K})/2^n$) and whose dual has a clear interpretation. We then define a hierarchy of SDP problems (alternatively, a hierarchy of LP problems) to approximate any finite sequence of moments of μ , as closely as desired.

3.1. An infinite-dimensional linear program P . After possibly some normalization of the defining polynomials, assume with no loss of generality that $\mathbf{K} \subset \mathbf{B} \subseteq [-1, 1]^n$ with \mathbf{B} a set over which integration w.r.t. the Lebesgue measure is easy. For instance, \mathbf{B} is the box $[-1, 1]^n$ or \mathbf{B} is the euclidean unit ball.

Let \mathcal{B} be the Borel σ -algebra of Borel subsets of \mathbf{B} , and let μ_2 be the Lebesgue measure on \mathbf{B} , normalized so that $2^n \mu_2(\mathbf{B}) = \text{vol}(\mathbf{B})$. Therefore, if $\text{vol}(\mathbf{C})$ denotes the n -dimensional volume of $\mathbf{C} \in \mathcal{B}$, then $\mu_2(\mathbf{C}) = \text{vol}(\mathbf{C})/2^n$ for every $\mathbf{C} \in \mathcal{B}$.

Also, the notation $\mu_1 \ll \mu_2$ means that μ_1 is absolutely continuous w.r.t. μ_2 , and $L_1(\mu_2)$ is the set of all functions integrable w.r.t. μ_2 . By the Radon-Nikodym theorem, there exists a nonnegative measurable function $f \in L_1(\mu_2)$ such that $\mu_1(\mathbf{C}) = \int_{\mathbf{C}} f d\mu_2$ for every $\mathbf{C} \in \mathcal{B}$, and f is called the Radon-Nikodym derivative of μ_1 w.r.t. μ_2 . In particular, $\mu_1 \leq \mu_2$ obviously implies $\mu_1 \ll \mu_2$. For $\mathbf{K} \in \mathcal{B}$, let $M(\mathbf{K})$ be the set of finite Borel measures on \mathbf{K} .

THEOREM 3.1. *Let $\mathbf{K} \in \mathcal{B}$ with $\mathbf{K} \subseteq \mathbf{B}$ and let $p \in \mathbb{R}[x]$ be positive almost everywhere on \mathbf{K} . Consider the following infinite-dimensional LP problem:*

$$P : \sup_{\mu_1} \left\{ \int p d\mu_1 : \mu_1 \leq \mu_2; \mu_1 \in M(\mathbf{K}) \right\} \quad (3.1)$$

with optimal value denoted $\sup P$ (and $\max P$ if the supremum is achieved).

Then the restriction μ_1^* of μ_2 to \mathbf{K} is the unique optimal solution of P and $\max P = \int p d\mu_1^* = \int_{\mathbf{K}} p d\mu_2$. In particular, if $p \equiv 1$ then $\max P = \text{vol}(\mathbf{K})/2^n$.

Proof. Let μ_1^* be the restriction of μ_2 to \mathbf{K} (i.e. $\mu_1^*(\mathbf{C}) = \mu_2(\mathbf{C} \cap \mathbf{K})$, $\forall \mathbf{C} \in \mathcal{B}$). Observe that μ_1^* is a feasible solution of P . Next, let μ_1 be any feasible solution of P . As $\mu_1 \leq \mu_2$ then

$$\mu_1(\mathbf{C} \cap \mathbf{K}) \leq \mu_2(\mathbf{C} \cap \mathbf{K}) = \mu_1^*(\mathbf{C} \cap \mathbf{K}), \quad \forall \mathbf{C} \in \mathcal{B},$$

and so, $\mu_1 \leq \mu_1^*$ because μ_1 and μ_1^* are supported on \mathbf{K} . Therefore, as $p \geq 0$ on \mathbf{K} , $\int p d\mu_1 \leq \int p d\mu_1^*$ which proves that μ_1^* is an optimal solution of P .

Next suppose that $\mu_1 \neq \mu_1^*$ is another optimal solution of P . As $\mu_1 \leq \mu_1^*$ then $\mu_1 \ll \mu_1^*$ and so, by the Radon-Nikodym theorem, there exists a nonnegative measurable function $f \in L_1(\mu_1^*)$ such that

$$\mu_1(\mathbf{C}) = \int_{\mathbf{C}} d\mu_1 = \int_{\mathbf{C}} f(x) d\mu_1^*(x), \quad \forall \mathbf{C} \in \mathcal{B} \cap \mathbf{K}.$$

Next, as $\mu_1 \leq \mu_1^*$, $\mu_1^* - \mu_1 =: \mu_0$ is a finite Borel measure on \mathbf{K} which satisfies

$$0 \leq \mu_0(\mathbf{C}) = \int_{\mathbf{C}} (1 - f(x)) d\mu_1^*(x), \quad \forall \mathbf{C} \in \mathcal{B} \cap \mathbf{K},$$

and so $1 \geq f(x)$ for almost all $x \in \mathbf{K}$. But then, since $\int p d\mu_1 = \int p d\mu_1^*$,

$$0 = \int p d\mu_0 = \int_{\mathbf{K}} p(x)(1 - f(x)) d\mu_1^*(x),$$

which (recalling $p > 0$ almost everywhere on \mathbf{K}) implies that $f(x) = 1$ for almost-all $x \in \mathbf{K}$. And so $\mu_1 = \mu_1^*$. \square

3.2. The dual of P . Let \mathcal{F} be the Banach space of continuous functions on \mathbf{B} (equipped with the sup norm) and \mathcal{F}_+ its positive cone, i.e., the elements $f \in \mathcal{F}$ which are nonnegative on \mathbf{B} . The dual of P reads:

$$P^* : \inf_{f \in \mathcal{F}_+} \left\{ \int f d\mu_2 : f \geq p \text{ on } \mathbf{K} \right\} \quad (3.2)$$

with optimal value denoted $\inf P^*$ ($\min P^*$ is the infimum is achieved).

Hence, a minimizing sequence of P^* aims at approximating from above the function f ($= p$ on \mathbf{K} and 0 on $\mathbf{B} \setminus \mathbf{K}$) by a sequence (f_ℓ) of continuous functions so as to minimize $\int f_\ell d\mu_2$.

Let $x \mapsto d(x, \mathbf{K})$ be the euclidean distance to the set \mathbf{K} and with $\epsilon_\ell > 0$, let $\mathbf{K}_\ell := \{x \in \mathbf{B} : d(x, \mathbf{K}) < \epsilon_\ell\}$ be an open bounded outer approximation of \mathbf{K} , so that $\mathbf{B} \setminus \mathbf{K}_\ell$ is closed (hence compact) with $\epsilon_\ell \rightarrow 0$ as $\ell \rightarrow \infty$. By Urysohn's Lemma [1, A4.2, p. 379], there exists a sequence $(f_\ell) \subset \mathcal{F}_+$ such that $0 \leq f_\ell \leq 1$ on \mathbf{B} , $f_\ell = 0$ on $\mathbf{B} \setminus \mathbf{K}_\ell$, and $f_\ell = 1$ on \mathbf{K} . Therefore,

$$\int f_\ell d\mu_2 = \text{vol}(\mathbf{K})/2^n + \int_{\mathbf{K}_\ell \setminus \mathbf{K}} f_\ell d\mu_2,$$

and so $\int f_\ell d\mu_2 \rightarrow \text{vol}(\mathbf{K})/2^n$ as $\ell \rightarrow \infty$. Hence, for the choice of the parameter $p \equiv 1$, $\text{vol}(\mathbf{K})/2^n$ is the optimal value of both P and P^* .

3.3. A hierarchy of semidefinite relaxations for computing the volume of \mathbf{K} . Let $\mathbf{y}_2 = (y_{2\alpha})$ be the sequence of all moments of μ_2 . For example, if $\mathbf{B} = [-1, 1]^n$, then

$$y_{2\alpha} = 2^{-n} \prod_{j=1}^n \left(\frac{2((1 + \alpha_j) \bmod 2)}{1 + \alpha_j} \right), \quad \forall \alpha \in \mathbb{N}^n.$$

Let \mathbf{K} be a compact semi-algebraic set as in (1.1) and let $r_j = \lceil (\deg g_j)/2 \rceil$, $j = 1, \dots, m$. Let $p \in \mathbb{R}[x]$ be a given polynomial positive almost everywhere on \mathbf{K} , and let $r_0 := \lceil (\deg p)/2 \rceil$. For $d \geq \max_j r_j$, consider the following semidefinite program:

$$Q_d : \begin{cases} \sup_{\mathbf{y}_1} & L_{\mathbf{y}_1}(p) \\ \text{s.t.} & M_d(\mathbf{y}_1) \succeq 0 \\ & M_d(\mathbf{y}_2 - \mathbf{y}_1) \succeq 0 \\ & M_{d-r_j}(g_j \mathbf{y}_1) \succeq 0, \quad j = 1, \dots, m \end{cases} \quad (3.3)$$

with optimal value denoted $\sup Q_d$ (and $\max Q_d$ if the supremum is achieved).

Observe that $\sup Q_d \geq \max P$ for every d . Indeed, the sequence \mathbf{y}_1^* of moments of the Borel measure μ_1^* (restriction of μ_2 to \mathbf{K} and unique optimal solution of P) is a feasible solution of Q_d for every d .

THEOREM 3.2. *Let Assumption 2.1 hold and consider the hierarchy of semidefinite programs (Q_d) in (3.3). Then:*

(a) Q_d has an optimal solution (i.e. $\sup Q_d = \max Q_d$) and

$$\max Q_d \downarrow \int_{\mathbf{K}} p d\mu_2, \quad \text{as } d \rightarrow \infty.$$

(b) Let $\mathbf{y}_1^d = (y_{1\alpha}^d)$ be an optimal solution of Q_d , then

$$\lim_{d \rightarrow \infty} y_{1\alpha}^d = \int_{\mathbf{K}} x^\alpha d\mu_2, \quad \forall \alpha \in \mathbb{N}^n. \quad (3.4)$$

Proof. (a) and (b). Recall that $\mathbf{B} \subseteq [-1, 1]^n$. By definition of μ_2 , observe that $|y_{2\alpha}| \leq 1$ for every $\alpha \in \mathbb{N}_{2d}^n$, and from $M_d(\mathbf{y}_2 - \mathbf{y}_1) \succeq 0$, the diagonal elements $y_{22\alpha} - y_{12\alpha}$ are nonnegative. Hence $y_{12\alpha} \leq y_{22\alpha}$ for every $\alpha \in \mathbb{N}_d^n$ and therefore,

$$\max[y_{10}, \max_{i=1, \dots, n} L_{\mathbf{y}_1}(x_i^{2d})] \leq 1.$$

By [29, Lemma 1], this in turn implies that $|y_{1\alpha}| \leq 1$ for every $\alpha \in \mathbb{N}_{2d}^n$, and so the feasible set of Q_d is closed, bounded, hence compact, which in turn implies that Q_d is solvable (i.e., has an optimal solution).

Let \mathbf{y}_1^d be an optimal solution of Q_d and by completing with zeros, make \mathbf{y}_1^d an element of the unit ball \mathbf{B}_∞ of l_∞ (the Banach space of bounded sequences, equipped with the sup-norm). As l_∞ is the topological dual of l_1 , by the Banach-Alaoglu Theorem, \mathbf{B}_∞ is weak \star compact, and even weak \star sequentially compact; see e.g. Ash [1]. Therefore, there exists $\mathbf{y}_1^* \in \mathbf{B}_\infty$ and a subsequence $\{d_k\} \subset \mathbb{N}$ such that $\mathbf{y}_1^{d_k} \rightarrow \mathbf{y}_1^*$ as $k \rightarrow \infty$, for the weak \star topology $\sigma(l_\infty, l_1)$. In particular,

$$\lim_{k \rightarrow \infty} y_{1\alpha}^{d_k} = y_{1\alpha}^*, \quad \forall \alpha \in \mathbb{N}^n. \quad (3.5)$$

Next let $d \in \mathbb{N}$ be fixed, arbitrary. From the pointwise convergence (3.5) we also obtain $M_d(\mathbf{y}_1^*) \succeq 0$ and $M_d(\mathbf{y}_2 - \mathbf{y}_1^*) \succeq 0$. Similarly, $M_{d-r_j}(g_j \mathbf{y}_1^*) \succeq 0$ for every $j = 1, \dots, m$. As d was arbitrary, by Theorem 2.2, \mathbf{y}_1^* has a representing measure μ_1 supported on $\mathbf{K} \subset \mathbf{B}$. In particular, from (3.5), as $k \rightarrow \infty$,

$$\max P \leq \max Q_{d_k} = L_{\mathbf{y}_1^{d_k}}(p) \downarrow L_{\mathbf{y}_1^*}(p) = \int p d\mu_1.$$

Next, as both μ_1 and μ_2 are supported on $[-1, 1]^n$, and $M_d(\mathbf{y}_2 - \mathbf{y}_1^*) \succeq 0$ for every d , one has $|y_{2\alpha} - y_{1\alpha}^*| \leq 1$ for every $\alpha \in \mathbb{N}^n$. Hence $\mathbf{y}_2 - \mathbf{y}_1^*$ has a representing measure on $[-1, 1]^n$. As in the proof of Lemma 2.4¹, we conclude that $\mu_1 \leq \mu_2$. Therefore μ_1 is admissible for problem P, with value $L_{\mathbf{y}_1^*}(p) = \int p d\mu_1 \geq \max P$. Therefore, μ_1 must be an optimal solution of P (hence unique) and by Theorem 3.1, $L_{\mathbf{y}_1^*} = \int p d\mu_1 = \int_{\mathbf{K}} p d\mu_2$. As the converging subsequence $\{d_k\}$ was arbitrary, it follows that in fact the whole sequence \mathbf{y}_1^d converges to \mathbf{y}_1^* for the weak \star topology $\sigma(l_\infty, l_1)$. And so (3.4) holds. This proves (a) and (b). \square

Writing $M_d(\mathbf{y}_1) = \sum_{\alpha} A_{\alpha} y_{1\alpha}$, and $M_{d-r_j}(g_j \mathbf{y}_1) = \sum_{\alpha} B_{\alpha}^j y_{1\alpha}$ for appropriate real symmetric matrices $(A_{\alpha}, B_{\alpha}^j)$, the dual of Q_d reads:

$$Q_d^* : \begin{cases} \inf_{X, Y, Z_j} & \langle M_d(\mathbf{y}_2), Y \rangle \\ \text{s.t.} & \langle A_{\alpha}, Y - X \rangle - \sum_{j=1}^m \langle B_{\alpha}^j, Z_j \rangle = p_{\alpha} \\ & X, Y, Z_j \succeq 0, \end{cases}$$

where $\langle X, Y \rangle = \text{trace}(XY)$ is the standard inner product of real symmetric matrices, and $X \succeq 0$ stands for X is positive semidefinite. This can be reformulated as:

$$Q_d^* : \begin{cases} \inf_{h, \sigma_0, \dots, \sigma_m} & \int h d\mu_2 \\ \text{s.t.} & h - p = \sigma_0 + \sum_{j=1}^m \sigma_j g_j \\ & h \in \Sigma^2[x]_d, \sigma_0 \in \Sigma^2[x]_d, \sigma_j \in \Sigma^2[x]_{d-r_j}. \end{cases} \quad (3.6)$$

The constraint of this semidefinite program states that the polynomial $h - p$ is written in Putinar's form (2.2) and so $h - p \geq 0$ on \mathbf{K} . In addition, $h \geq 0$ because it is a sum of squares.

¹If $\mathbf{K} \subset [-1, 1]^n$ then in Lemma 2.4, the condition $M_d(\mathbf{y}_2 - \mathbf{y}_1) \succeq 0, \forall d \in \mathbb{N}$, is sufficient.

This interpretation of Q_d^* also shows why computing $\text{vol}(\mathbf{K})$ is difficult. Indeed, when $p \equiv 1$, to get a good upper bound on $\text{vol}(\mathbf{K})$, one needs to obtain a good *polynomial* approximation $h \in \mathbb{R}[x]$ of the indicator function $I_{\mathbf{K}}(x)$ on \mathbf{B} . In general, high degree of h will be necessary to attenuate side effects on the boundary of \mathbf{B} and \mathbf{K} , a well-known issue in interpolation with polynomials.

PROPOSITION 3.3. *If \mathbf{K} and $\mathbf{B} \setminus \mathbf{K}$ have a nonempty interior, there is no duality gap, that is, both optimal values of Q_d and Q_d^* are equal. In addition, Q_d^* has an optimal solution $(h^*, (\sigma_j^*))$.*

Proof. Let μ_1 be the uniform distribution on \mathbf{K} , i.e., the restriction of μ_2 to \mathbf{K} , and let $\mathbf{y}_1 = (y_{1\alpha})$ be its sequence of moments up to degree $2d$. As \mathbf{K} has nonempty interior, then clearly $M_d(\mathbf{y}_1) \succ 0$ and $M_{d-r_j}(g_j \mathbf{y}_1) \succ 0$ for every $j = 1, \dots, m$. If $\mathbf{B} \setminus \mathbf{K}$ also has nonempty interior then $M_d(\mathbf{y}_2 - \mathbf{y}_1) \succ 0$ because with $f \in \mathbb{R}[x]_d$ with coefficient vector \mathbf{f} ,

$$\langle \mathbf{f}, M_d(\mathbf{y}_2 - \mathbf{y}_1) \mathbf{f} \rangle = \int_{\mathbf{B} \setminus \mathbf{K}} f(x)^2 d\mu_2, \quad \forall f \in \mathbb{R}[x]_d.$$

Therefore Slater's condition holds for Q_d and the result follows from a standard result of duality in semidefinite programming; see e.g. [45]. \square

REMARK 3.4. *Let $f \in \mathbb{R}[x]$ and suppose that one wants to approximate the integral $J^* := \int_{\mathbf{K}} f d\mu_2$. Then for d sufficiently large, an optimal solution of Q_d allows to approximate J^* . Indeed,*

$$J^* = \int_{\mathbf{K}} f d\mu_2 = \int f d\mu_1 = L_{\mathbf{y}_1^*}(f) = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} y_{1\alpha}^*,$$

where \mathbf{y}_1^* is the moment sequence of μ_1 , the unique optimal solution of \mathbf{P} (the restriction of μ_2 to \mathbf{K}). And so, from (3.4), $L_{\mathbf{y}_1^d}(f) \approx J^*$ when d is sufficiently large.

3.4. A hierarchy of linear programs. Let $\mathbf{K} \subset \mathbf{B} \subseteq [-1, 1]^n$ be as in (1.1) and assume for simplicity that the g_j s have been scaled to satisfy $0 \leq g_j \leq 1$ on \mathbf{K} for every $j = 1, \dots, m$. In addition, assume that the family of polynomials $(1, g_1, \dots, g_m)$ generates the algebra $\mathbb{R}[x]$. For $d \in \mathbb{N}$, consider the following linear program:

$$L_d : \begin{cases} \sup_{\mathbf{y}_1} & y_{10} \\ \text{s.t.} & L_{\mathbf{y}_2 - \mathbf{y}_1} \left(\prod_{i=1}^n (1 + x_i)^{\alpha_i} (1 - x_i)^{\beta_i} \right) \geq 0, \quad \alpha, \beta \in \mathbb{N}_d^n \\ & L_{\mathbf{y}_1}(g^{\alpha} (1 - g)^{\beta}) \geq 0, \quad \alpha, \beta \in \mathbb{N}_d^n \end{cases} \quad (3.7)$$

with optimal value denoted $\sup L_d$ (and $\max L_d$ if $\sup L_d$ is finite). Notice that $\sup L_d \geq \text{vol}(\mathbf{K})/2^n$ for all d . Indeed, the sequence \mathbf{y}_1^* of moments of the Borel measure μ_1^* (restriction of μ_2 to \mathbf{K} and unique optimal solution of \mathbf{P}) is a feasible solution of L_d for every d .

THEOREM 3.5. *For the hierarchy of linear programs (L_d) in (3.7), the following holds:*

- (a) L_d has an optimal solution (i.e. $\sup L_d = \max L_d$) and $\max L_d \downarrow \text{vol}(\mathbf{K})/2^n$ as $d \rightarrow \infty$.
- (b) Let \mathbf{y}_1^d be an optimal solution of L_d . Then (3.4) holds.

Proof. We first prove that L_d has finite value. L_d always has a feasible solution \mathbf{y}_1 , namely the moment vector associated with the Borel measure μ_1 , the restriction of μ_2 to \mathbf{K} , and so $\sup L_d \geq \text{vol}(\mathbf{K})/2^n$. Next, from the constraint $L_{\mathbf{y}_2 - \mathbf{y}_1}(\bullet) \geq 0$ with $\alpha = \beta = 0$, we obtain $y_{10} \leq y_{20} \leq 1$. Hence $\sup L_d \leq 1$ and therefore, the linear program L_d has an optimal solution \mathbf{y}_1^d . Fix $\gamma \in \mathbb{N}^n$ and $\epsilon > 0$, arbitrary. As $|x^\gamma| \leq 1 < 1 + \epsilon$ on \mathbf{B} (hence on \mathbf{K}), by Theorem 2.3(a),

$$1 + \epsilon \pm x^\gamma = \sum_{\alpha, \beta \in \mathbb{N}^m} c_{\alpha\beta}^\gamma g^\alpha (1 - g)^\beta,$$

for some $(c_{\alpha\beta}^\gamma) \subset \mathbb{R}_+$ with $|\alpha|, |\beta| \leq s_\gamma$. Hence, as soon as $d \geq s_\gamma$, applying $L_{\mathbf{y}_1^d}$ yields

$$(1 + \epsilon) y_{10}^d \pm y_{1\gamma}^d = \sum_{\alpha, \beta \in \mathbb{N}^m} c_{\alpha\beta}^\gamma L_{\mathbf{y}_1^d} (g^\alpha (1 - g)^\beta) \geq 0,$$

and so

$$\forall \gamma \in \mathbb{N}^n : \quad |y_{1\gamma}^d| \leq (1 + \epsilon) y_{10}^d \leq 1 + \epsilon, \quad \forall d \geq s_\gamma. \quad (3.8)$$

Complete \mathbf{y}_1^d with zeros to make it an element of \mathbb{R}^∞ . Because of (3.8), using a standard diagonal element, there exists a subsequence (d_k) and an element $\mathbf{y}_1^* \in (1 + \epsilon) \mathbf{B}_\infty$ (where \mathbf{B}_∞ is the unit ball of l_∞) such that (3.5) holds. Now with $\alpha, \beta \in \mathbb{N}^m$ fixed, arbitrary, (3.5) yields $L_{\mathbf{y}_1^*} (g^\alpha (1 - g)^\beta) \geq 0$. Hence by Theorem 2.3(b), \mathbf{y}_1^* has a representing measure μ_1 supported on \mathbf{K} . Next, let $\mathbf{y}_0 := \mathbf{y}_2 - \mathbf{y}_1^*$. Again, (3.5) yields:

$$L_{\mathbf{y}_0} \left(\prod_{i=1}^n (1 + x_i)^{\alpha_i} (1 - x_i)^{\beta_i} \right) \geq 0, \quad \forall \alpha, \beta \in \mathbb{N}^n,$$

and so by Theorem 2.3(b), \mathbf{y}_0 is the moment vector of some Borel measure μ_0 supported on $[-1, 1]^n$. As measures on compact sets are identified with their moments, and $y_{0\alpha} + y_{1\alpha}^* = y_{2\alpha}$ for every $\alpha \in \mathbb{N}^n$, it follows that $\mu_0 + \mu_1 = \mu_2$, and so $\mu_1 \leq \mu_2$. Therefore, μ_1 is an admissible solution to P with parameter $p \equiv 1$, and with value $\mu_1(\mathbf{K}) = y_{10}^* \geq \text{vol}(\mathbf{K})/2^n$. Hence, μ_1 is the unique optimal solution to P with value $\mu_1(\mathbf{K}) = \text{vol}(\mathbf{K})/2^n$.

Finally, by using (3.5) and following the same argument as in the proof of Theorem 3.2, one obtains the desired result (3.4). \square

Remark 3.4 also applies to the LP relaxations (3.7).

3.5. Integration against a weight function. With $\mathbf{K} \subset \mathbf{B}$ as in (1.1) suppose now that one wishes to approximate the integral

$$J^* := \int_{\mathbf{K}} f(x) w(x) dx, \quad (3.9)$$

for some given nonnegative *weight* function $w : \mathbb{R}^n \rightarrow \mathbb{R}$, and where $f \in \mathbb{R}[x]_d$ is some nonnegative polynomial. One makes the following assumption:

ASSUMPTION 3.6. *One knows the moments $\mathbf{y}_2 = (y_{2\alpha})$ of the Borel measure $d\mu_2 = w dx$ on \mathbf{B} , that is:*

$$y_{2\alpha} = \int_{\mathbf{B}} x^\alpha d\mu_2 \left(= \int_{\mathbf{B}} x^\alpha w(x) dx \right), \quad \alpha \in \mathbb{N}^n. \quad (3.10)$$

Indeed, for many weight functions w , and given $d \in \mathbb{N}$, one may compute the moments $\mathbf{y}_2 = (y_{2\alpha})$ of μ_2 via *cubature* formula, exact up to degree d . In practice, one only knows finitely many moments of μ_2 , say up to degree d , fixed.

Consider the hierarchy of semidefinite programs

$$Q_d : \begin{cases} \sup_{\mathbf{y}_1} & L_{\mathbf{y}_1}(f) \\ \text{s.t.} & M_d(\mathbf{y}_1) \quad \succeq 0 \\ & M_d(\mathbf{y}_2 - \mathbf{y}_1) \quad \succeq 0 \\ & M_{d-r_j}(g_j \mathbf{y}_1) \quad \succeq 0, \quad j = 1, \dots, m \end{cases} \quad (3.11)$$

with \mathbf{y}_2 as in Assumption 3.6.

THEOREM 3.7. *Let Assumption 2.1 and 3.6 hold and consider the hierarchy of semidefinite programs (Q_d) in (3.11) with \mathbf{y}_2 as in (3.10). Then Q_d is solvable and $\max Q_d \downarrow J^*$ as $d \rightarrow \infty$.*

The proof is almost a verbatim copy of that of Theorem 3.2.

4. Numerical experiments and discussion. In this section we report some numerical experiments carried out with Matlab and the package GloptiPoly 3 for manipulating and solving generalized problems of moments [18]. The SDP problems were solved with SeDuMi 1.1R3 [35]. Univariate Chebyshev polynomials were manipulated with the *chebfun* package [4].

The single-interval example below permit to visualize the numerical behavior of the algorithm. The folium example illustrates that, as expected, the non-convexity of \mathbf{K} does not seem to penalize the moment approach. Finally, our experience reveals that the choice of alternative polynomial bases affects the quality of the approximations.

4.1. Single interval. Consider the elementary one-dimensional set $\mathbf{K} = [0, \frac{1}{2}] = \{x \in \mathbb{R} : g_1(x) = x(\frac{1}{2} - x) \geq 0\}$ included in the unit interval $\mathbf{B} = [-1, 1]$. We want to approximate $\text{vol}(\mathbf{K}) = \frac{1}{2}$. Moments of the Lebesgue measure μ_2 on \mathbf{B} are given by $\mathbf{y}_2 = (2, 0, 2/3, 0, 2/5, 0, 2/7, \dots)$.

Here is a simple Matlab script using GloptiPoly 3 instructions to input and solve the SDP relaxation Q_d of the LP moment problem P with $p \equiv 1$:

```
>> d = 10; % degree
>> mpol x0 x1
>> m0 = meas(x0); m1 = meas(x1);
>> g1 = x1*(1/2-x1);
>> dm = (1+(0:d))'; y2 = ((+1).^dm-(-1).^dm)./dm;
>> y0 = mom(mmon(x0,d)); y1 = mom(mmon(x1,d));
>> P = msdp(max(mass(m1)), g1>=0, y0==y2-y1); % input moment problem
>> msol(P); % solve SDP relaxation
>> y1 = double(mvec(m1)); % retrieve moment vector
```

The volume estimate is then the first entry in vector \mathbf{y}_1 . Note in particular the use of the moment constraint $\mathbf{y}_0 == \mathbf{y}_2 - \mathbf{y}_1$ which ensures that moments \mathbf{y}_0 of μ_0 will be substituted by linear combinations of moments \mathbf{y}_1 of μ_1 (decision variables) and moments \mathbf{y}_2 of μ_2 (given).

Figure 4.1 displays three approximation sequences of $\text{vol}(\mathbf{K})$ obtained by solving SDP relaxations (3.11) of increasing degrees $d = 2, \dots, 50$ of the infinite-dimensional LP moment problem P with three different parameters p :

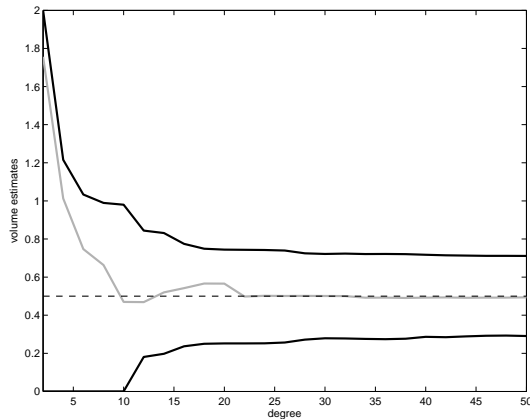


FIG. 4.1. Three sequences of approximations of $\text{vol}[0, \frac{1}{2}]$ obtained by solving SDP relaxations of increasing degree.

- the upper curve (in black) is a monotone non increasing sequence of upper bounds obtained by maximizing $\int d\mu_1$, the mass of μ_1 , using the objective function $\max(\text{mass}(\mathbf{m1}))$ in the above script;
- the medium curve (in gray) is a sequence of approximations obtained by maximizing $\int p d\mu_1$ with $p := g_1$, using the objective function $\max(\mathbf{g1})$ in the above script;
- the lower curve (in black) is a monotone non decreasing sequence of lower bounds on $\text{vol}(\mathbf{K})$ obtained by computing upper bounds on the volume of $\mathbf{B} \setminus \mathbf{K}$, using the objective function $\max(\text{mass}(\mathbf{m1}))$ and the support constraint $\mathbf{g1} \leq 0$ in the above script. The volume estimate is then $2 - \mathbf{y1}(1)$.

We observe a much faster convergence when maximizing $\int g_1 d\mu_1$ instead of $\int d\mu_1$; the upper and lower curves apparently exhibit slow convergence.

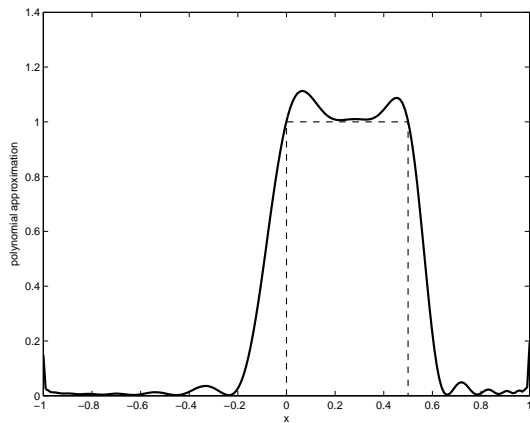


FIG. 4.2. Positive polynomial approximation of degree 50 (solid) of the indicator function $I_{[0, \frac{1}{2}]}$ (dashed) on $[-1, 1]$.

To analyze these phenomena, we use solutions of the dual SDP problems, provided automatically by the primal-dual interior-point method implemented in the

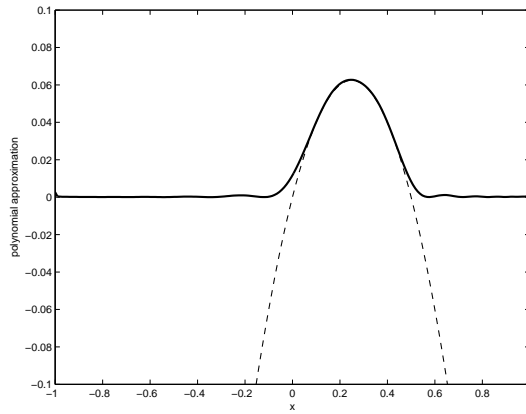


FIG. 4.3. Positive polynomial approximation of degree 50 (solid) of the positive piecewise-polynomial function $\max(0, g_1)$ on $[-1, 1]$. Polynomial g_1 is represented in dashed line.

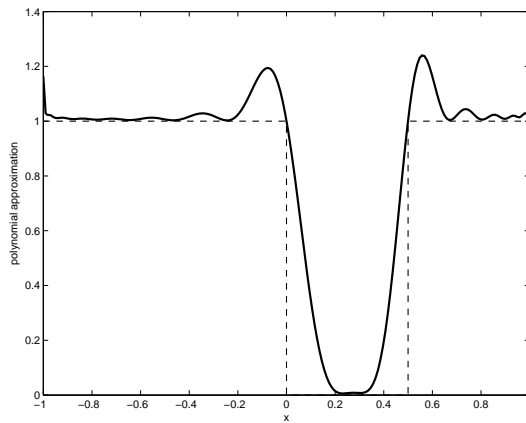


FIG. 4.4. Positive polynomial approximation of degree 50 (solid) of the complementary indicator function $1 - I_{[0, \frac{1}{2}]}$ (dashed) on $[-1, 1]$.

SDP solver SeDuMi. On Figure 4.2 we represent the degree-50 positive polynomial approximation h of the indicator function $I_{\mathbf{K}}$ on \mathbf{B} , which minimizes $\int_{\mathbf{B}} h dx$ while satisfying $h - 1 \geq 0$ on \mathbf{K} and $h \geq 0$ on $\mathbf{B} \setminus \mathbf{K}$ (yielding the volume estimate of the upper curve in Figure 4.1). On Figure 4.3, we represent the degree-50 polynomial approximation h of the piecewise-polynomial function $\max(0, g_1)$, which minimizes $\int_{\mathbf{B}} h dx$ while satisfying $h - g_1 \geq 0$ on \mathbf{K} and $h \geq 0$ on $\mathbf{B} \setminus \mathbf{K}$ (yielding the volume estimate of the medium curve in Figure 4.1). On Figure 4.4 we represent the degree-50 polynomial approximation h of the complementary indicator function $1 - I_{\mathbf{K}}$, which minimizes $\int_{\mathbf{B}} h dx$ while satisfying $h - 1 \geq 0$ on $\mathbf{B} \setminus \mathbf{K}$ and $h \geq 0$ on \mathbf{K} (yielding the volume estimate of the lower curve in Figure 4.1). We observe the characteristic oscillation phenomena near the boundary, typical of polynomial approximation problems [44]. The continuous function $\max(0, g_1)$ is easier to approximate than discontinuous indicator functions, and this partly explains the better convergence of the medium approximation on Figure 4.1.

On Figures 4.2 and 4.4, one observes relatively large oscillations near the boundary

points $x \in \{-1, 0, \frac{1}{2}, 1\}$ which significantly corrupt the quality of the volume approximation. To some extent, these oscillations can be reduced by using a Chebyshev polynomial basis instead of the standard power basis.

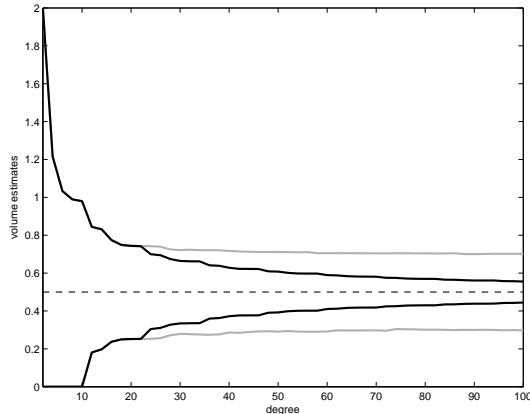


FIG. 4.5. Upper and lower bounds on $\text{vol}[0, \frac{1}{2}]$ obtained by solving SDP relaxations in the Chebyshev basis (black) and power basis (gray).

Figure 4.5 displays upper and lower bounds on the volume, computed up to degree 100, with the power basis (in gray) and with the Chebyshev basis (in black). Note that in order to input and solve SDP problems in the Chebyshev basis, we used our own implementation and the `chebfun` package since GloptiPoly 3 supports only the power basis. In Figure 4.5 we see that above degree 20 the quality of the bounds obtained with the power basis deteriorates, which suggests that the SDP solver encounters some numerical problems rather than convergence becoming slower (which is confirmed when changing to Chebyshev basis; see below). It seems that the SDP solver is not able to improve the bounds, most likely due to the symmetric Hankel structure of the moment matrices in the power basis: indeed, it is known that the conditioning (ratio of extreme singular values) of positive definite Hankel matrices is an exponential function of the matrix size [19]. When the smallest singular values reach machine precision, the SDP solver is not able to optimize the objective function any further.

In Figures 4.6 and 4.7 one observes that the degree-100 polynomial approximation $h(x)$ of the indicator function and its complement are tighter in the Chebyshev basis (black) than in the power basis (gray). Firstly, we observe that the degree-100 approximations in the power basis do not significantly differ from the degree-50 approximations in the same basis, represented in Figures 4.2 and 4.4. This is consistent with the very flat behavior of the right half of the upper and lower curves (in gray) in Figure 4.5. Secondly, some coefficients of $h(x)$ in the power basis have large magnitude $h(x) = 1.0019 + 3.6161x - 29.948x^2 + \dots + 88123x^{49} + 54985x^{50} + \dots - 1018.4x^{99} + 26669x^{100}$ with the Euclidean norm of the coefficient vector greater than 10^6 . In contrast, the polynomial $h(x)$ obtained in the Chebyshev basis $h(x) = 0.1862t_0(x) + 0.093432t_1(x) - 0.30222t_2(x) + \dots + 0.0055367t_{49}(x) - 0.020488t_{50}(x) + \dots - 0.0012267t_{99}(x) + 0.0011190t_{100}(x)$ has a coefficient vector of Euclidean norm around 0.57627, where $t_k(x)$ denotes the k -th Chebyshev polynomial. Thirdly, oscillations around points $x = 0$ and $x = 1/2$ did not disappear with the Chebyshev basis, but the peaks are much thinner than with the power basis. Finally,

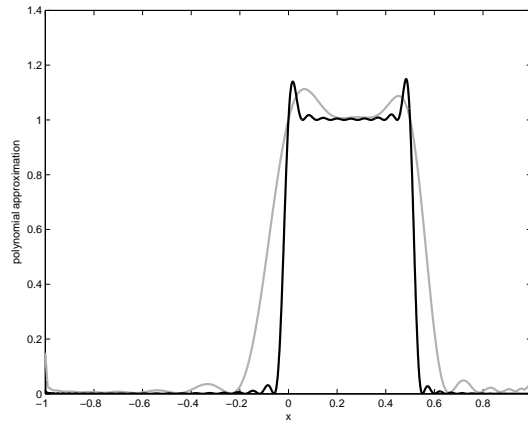


FIG. 4.6. Positive polynomial approximation of degree 100 of the indicator function $I_{[0, \frac{1}{2}]}$ in the Chebyshev basis (black) and power basis (gray).

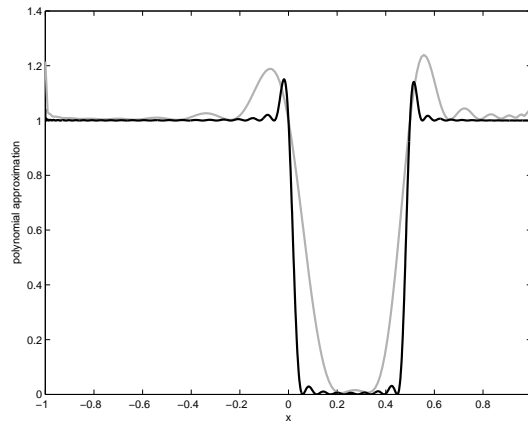


FIG. 4.7. Positive polynomial approximation of degree 100 of the indicator function $1 - I_{[0, \frac{1}{2}]}$ in the Chebyshev basis (black) and power basis (gray).

the oscillations near the interval ends $x = -1$ and $x = 1$ are almost suppressed, a well-known property of Chebyshev polynomials which have a denser root distribution near the interval ends.

From these simple observations, we conjecture that a polynomial basis with a dense root distribution near the boundary of the semi-algebraic sets \mathbf{K} and \mathbf{B} should ensure a better convergence of the hierarchy of volume estimates.

Finally, Figure 4.8 displays the CPU time required to solve the SDP problems (with SeDuMi, in the power basis in gray and in the Chebyshev basis in black) as a function of the degree, showing a polynomial dependence slightly slower than cubic in the power basis (due to the sparsity of moment matrices) and slightly faster than cubic in the Chebyshev basis. For example, solving the SDP problem of degree 100 takes about 2.5 seconds of CPU time on our standard desktop computer.

4.2. Bean. Consider $\mathbf{K} = \{x \in \mathbb{R}^2 : g_1(x) = x_1(x_1^2 + x_2^2) - (x_1^4 + x_1^2 x_2^2 + x_2^4) \geq 0\}$ displayed in Figure 4.9, which is a surface delimited by an algebraic curve $g_1(x) = 0$

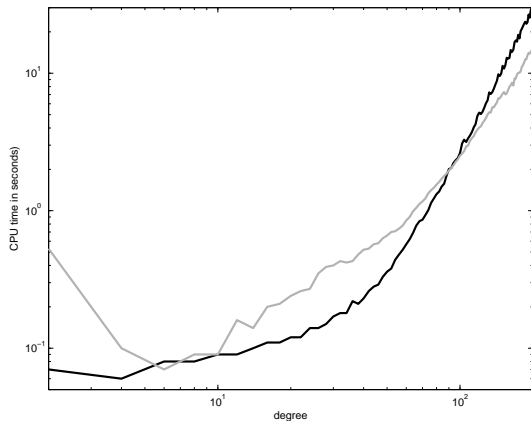


FIG. 4.8. CPU time required to solve the SDP relaxations (Chebyshev basis in black, power basis in gray) as a function of the degree.

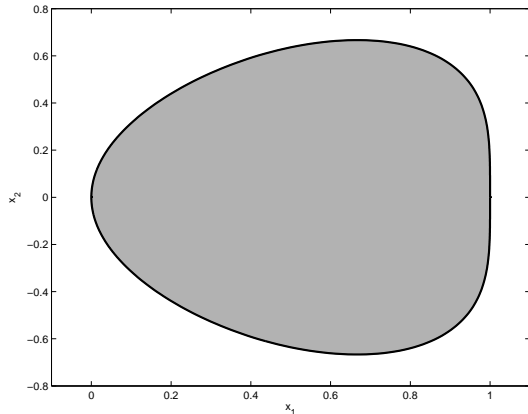


FIG. 4.9. Bean surface.

of genus zero, hence rationally parametrizable. From the parametrization $x_1(t) = (1 + t^2)/(1 + t^2 + t^4)$, $x_2(t) = tx_1(t)$, $t \in \mathbb{R}$, obtained with the `algcures` package of `Maple`, we can calculate

$$\begin{aligned} \text{vol}(\mathbf{K}) &= \int_{\mathbf{K}} dx_1 dx_2 = \int_{\mathbb{R}} x_1(t) dx_2(t) = \int_{\mathbb{R}} \frac{(1-t)(1+t)(1+t^2)(1+3t^2+t^4)}{(1+t+t^2)^3(1-t+t^2)^3} dt \\ &= \frac{7\sqrt{3}\pi}{36} \approx 1.0581 \end{aligned}$$

with the help of the `int` integration routine of `Maple`. Similarly, we can calculate symbolically the first moments of the Lebesgue measure μ_1 on \mathbf{K} , namely $y_{100} = \text{vol}(\mathbf{K})$, $y_{110} = \frac{23}{42} \text{vol}(\mathbf{K})$, $y_{101} = 0$, $y_{120} = \frac{23}{63} \text{vol}(\mathbf{K})$, $y_{111} = 0$, $y_{102} = \frac{113}{1008} \text{vol}(\mathbf{K})$ etc. Observe that $\mathbf{K} \subseteq \mathbf{B} = [-1, 1]^2$.

On Figure 4.10 we represent a degree-20 positive polynomial approximation h of the indicator function $I_{\mathbf{K}}$ on \mathbf{B} obtained by solving an SDP problem with 231 unknown moments. We observe the typical oscillations near the boundary regions, but we can recognize the shape of Figure 4.9.

In Table 4.1 we give relative errors in percentage observed when solving successive SDP relaxations (in the power basis) of the LP moment problems of maximizing

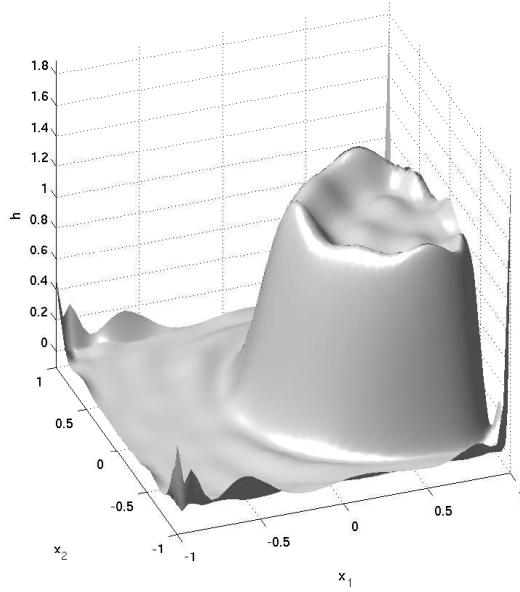


FIG. 4.10. Positive polynomial approximation of degree 20 of the indicator function of the bean surface.

degree	2	4	6	8	10	12	14	
error	78%	63%	13%	0.83%	9.1%	0.80%	3.31%	
degree	16	18	20	22	24	26	28	30
error	3.8%	3.3%	2.6%	5.6%	4.1%	4.1%	3.9%	3.7%

TABLE 4.1

Relative error when approximating the volume of the bean surface, as a function of the degree of the SDP relaxation.

$\int g_1 d\mu_1$. Note that the error sequence is not monotonically decreasing since we do not maximize $\int d\mu_1$ and a good approximation can be obtained with few moments. Above degree 16, the approximation stagnates around 4%. Most likely this is due to the use of the power basis, as already observed in the previous univariate examples. For example, at degree 20, one obtains the 6 first moment approximation

$$y_{200}^{20} = 1.10, y_{210}^{20} = 0.589, y_{201}^{20} = 0.00, y_{220}^{20} = 0.390, y_{211}^{20} = 0.00, y_{202}^{20} = 0.122$$

to be compared with the exact numerical values

$$y_{200} = 1.06, y_{210} = 0.579, y_{201} = 0.00, y_{220} = 0.386, y_{211} = 0.00, y_{202} = 0.119.$$

Increasing the degree does not provide a better approximation. It is expected that a change of basis (e.g. multivariate Chebyshev or trigonometric) can be useful in this context.

4.3. Folium. Consider $\mathbf{K} = \{x \in \mathbb{R}^2 : g_1(x) = -(x_1^2 + x_2^2)^3 + 4x_1^2x_2^2 \geq 0\}$ displayed in Figure 4.11, which is a surface delimited by an algebraic curve of polar

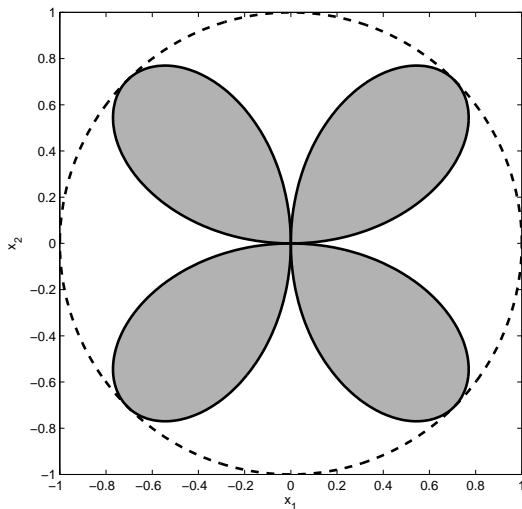


FIG. 4.11. Folium surface.

equation $\rho = \sin(2\theta)$. The surface is contained in the unit disk \mathbf{B} , on which the Lebesgue measure has moments

$$y_{2\alpha} = \frac{(1 + (-1)^{\alpha_1})(1 + (-1)^{\alpha_2})\Gamma(\frac{1}{2}(1 + \alpha_1))\Gamma(\frac{1}{2}(1 + \alpha_2))}{\Gamma(\frac{1}{2}(4 + \alpha_1 + \alpha_2))}, \quad \forall \alpha \in \mathbb{N}^2,$$

where Γ denotes the gamma function. The area is $\text{vol}(\mathbf{K}) = \frac{1}{2} \int_0^{2\pi} \sin^2(2\theta)d\theta = \frac{1}{2}\pi$ and so, $\text{vol}(\mathbf{K} \setminus \mathbf{B}) = \pi - \text{vol}(\mathbf{K}) = \frac{1}{2}\pi$.

In Table 4.2 we give relative errors in percentage observed when solving successive SDP relaxations (in the power basis) of the LP moment problems of maximizing $\int g_1 d\mu_1$. We observe that nonconvexity of \mathbf{K} does not play any special role. The quality of estimates does not really improve for degrees greater than 20. Here too, an alternative polynomial basis with dense root distribution near the boundaries of \mathbf{K} and \mathbf{B} would certainly help.

degree	4	6	8	10	12	14	16
error	87%	19%	14%	9.4%	4.3%	4.5%	5.9%
degree	18	20	22	24	26	28	30
error	1.2%	5.3%	5.9%	7.2%	8.7%	9.0%	8.8%

TABLE 4.2

Relative error when approximating the volume of the folium surface, as a function of the degree of the SDP relaxation.

Figure 4.12 displays a degree-20 positive polynomial approximation h of the indicator function $I_{\mathbf{K}}$ on \mathbf{B} obtained by solving an SDP problem with 231 unknown moments. For visualization purposes, $\max(5/4, h)$ rather than h is displayed. Again typical oscillations occur near the boundary regions, but we can recognize the shape of Figure 4.11.

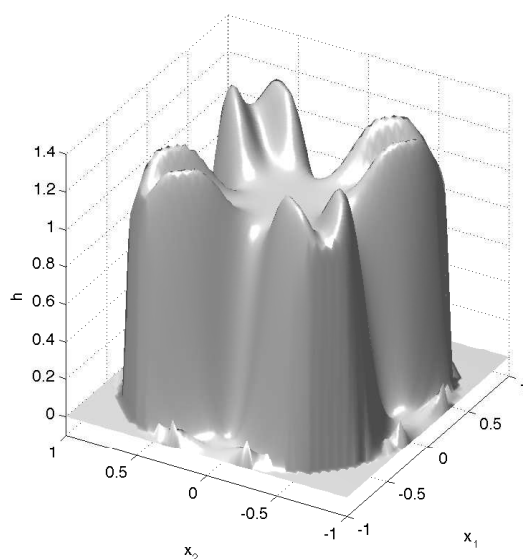


FIG. 4.12. Positive polynomial approximation of degree 20 of the indicator function of the folium surface.

5. Concluding Remarks. The methodology presented in this paper is general enough and applies to compact basic semi-algebraic sets which are neither necessarily convex nor connected. Its efficiency is related to the degree needed to obtain a good polynomial approximation of the indicator function of \mathbf{K} (on a simple set that contains \mathbf{K}) and from this viewpoint, convexity of \mathbf{K} does not help much. On the other hand, the method is limited by the size of problems that SDP solvers presently available can handle. Moreover, the impact of the choice of the polynomial basis (e.g., Chebyshev or trigonometric) on the quality of the solution of the SDP relaxations deserves further investigation for a better understanding. Therefore, in view of the present status of SDP solvers and since in general high accuracy will require high degree, the method can provide good approximations for problems of small dimension (typically $n = 2$ or $n = 3$). However, if one is satisfied with cruder bounds then one may consider problems in higher dimensions.

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