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# Ice formation in the Arctic during summer: false-bottoms

Phan Thanh Nam<sup>a,b</sup>, Pham Ngoc Dinh Alain<sup>a\*</sup>  
Pham Hoang Quan<sup>b</sup> and Dang Duc Trong<sup>b</sup>

<sup>a</sup>*Mathematics Department, MAPMO UMR 6628, BP 67-59, 45067 Orleans cedex, France*

<sup>b</sup>*Department of Mathematics, HoChiMinh City National University, Viet Nam*

## Abstract

The only source of ice formation in the Arctic during summer is a layer of ice called false-bottoms between an under-ice melt pond and the underlying ocean. Of interest is to give a mathematical model in order to determine the simultaneous growth and ablation of false-bottoms, which is governed by both of heat fluxes and salt fluxes. In one dimension, this problem may be considered mathematically as a two-phase Stefan problem with two free boundaries. Our main result is to prove the existence and uniqueness of the solution from the initial condition.

*Mathematics Subject Classification* 2000: 35R35, 35Q35.

*Keywords:* False-bottoms, Free boundary problem, Green's function, Contraction Mapping Principle.

## 1 Introduction

There are some different reservoirs of fresh water in the Arctic during summer (see, e.g., *Eicken et al.* [2]). First, melt water collects in surface melt pond (melting under the sun) which is the most important reservoir. Second, this melt water can percolate into the ice matrix to form an *under-ice melt pond* (see [5] for more detail). At the interface between this fresh water and the underlying salt water, double-diffusive convection of heat and salt leads to the formation of a layer of ice called *false-bottoms* (see Figure 1 below). Very early, *Nansen* [8] in 1897 noted that this is the only source of forming new ice in the Arctic during the summer. This phenomenon has been considered for a long time by many authors (see, e.g. [1, 2, 5, 6, 7, 9, 10]). However, it has been considered in geophysical view-point based on practical experiments rather than rigorously mathematical formulations.

One of the most interesting ones is the simultaneous growth and ablation of false-bottoms, which is governed by both of heat fluxes and salt fluxes. The ablation of the sea-ice interface is caused by dissolution rather than by melting. Note that salt water has the double properties: it does not freeze even for temperature less than  $0^{\circ}\text{C}$ , and it dissolves ice when it is in contact with ice. *McPhee et al.* [7] emphasized that properly describing heat and salt flux at the ice-ocean interface is essential for understanding and modeling the false-bottoms, and in particular without the double diffusion at this interface false bottoms would be so short-lived. The growth of the upper interface between a false bottom and a under-ice melt pond is governed by the purely thermodynamic condition at the interface.

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\*Author correspondence: alain.pham@univ-orleans.fr

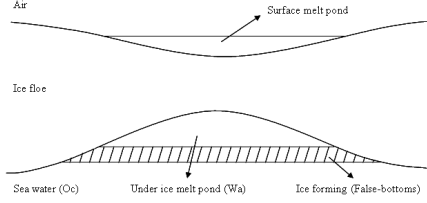


Figure 1: Ice formation in the Arctic during summer.

Recently, in 2003, *Notz et al.* [9] gave a model simulating successfully the simultaneous growth and ablation of false-bottoms. They formulated mathematically the problem by a system of partial differential equations and solved them numerically by using a numerical routine in Mathematica. Although their numerical result fits quite well to early experimental data from *Martin and Kauffman* [6], a rigorous proof of the existence and uniqueness of the solution for the system of equations is still unavailable. Our aim in this paper is to give a such a mathematical proof. More precisely, we shall represent the problem explicitly by a system of partial differential equations associated with free boundary conditions similar to [9], and then show that the system has a unique solution from given initial conditions.

Now we consider a one-dimensional model describing the simultaneous growth and ablation of the ice of false-bottoms. Here we have three environments: the ocean (Oc), the ice of false-bottoms (Fb) and the fresh water (Wa). Denote by  $T(x, t)$ ,  $S(x, t)$  the temperature and the salinity, and denote by  $h_0(t)$ ,  $h_u(t)$  the free boundaries at the interfaces ice-ocean (Fb-Oc) and ice-water (Fb-Wa), respectively (see Figure 2 below).

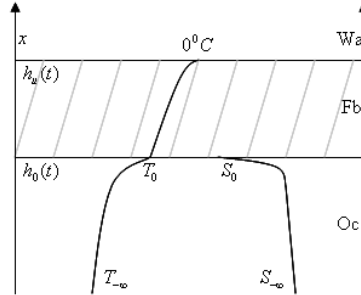


Figure 2: One-dimensional model.

At the interface Fb-Oc, we apply the first principle of thermodynamics, (i.e. variation of energy = variation of heat flux through the interface)

$$\Delta U = \Delta \Phi.$$

The net amount of heat transferred through the interface Fb-Oc in a section  $s$  is equal to

$$\Delta U = -dh_0 \rho_I L_f s,$$

where  $\rho_I$  is the density of the ice and  $L_f$  is the latent heat of fusion. On the other hand, the difference of the heat fluxes through a section  $s$  in the ice and the ocean during a time  $dt$  is

$$\Delta \Phi = (-\lambda_I T_x(h_0(t)+, t) + \lambda_O T_x(h_0(t)-, t)) s dt,$$

where  $\lambda_I$ ,  $\lambda_O$  are thermal conductivities of the ice and the ocean. Here the notations  $h_0(t)+$  and  $h_0(t)-$  stand for the right limit and the left limit at  $x = h_0(t)$ . Thus the law of

conservation of energy mentioned above, i.e.  $\Delta U = \Delta \Phi$ , leads to the Stefan condition for the heat balance at the interface

$$h'_0(t) = \tilde{\lambda}_I T_x(h_0(t)+, t) - \tilde{\lambda}_O T_x(h_0(t)-, t), \quad (1.1)$$

where

$$\tilde{\lambda}_I = \frac{\lambda_I}{\rho_I L_f} > 0, \quad \tilde{\lambda}_O = \frac{\lambda_O}{\rho_I L_f} > 0.$$

For simplicity, we can neglect the salt of the ice of false-bottoms. The water near the interface Fb-Oc is a mixture of melt water, which melts from the ice of false-bottoms, and sea water. This water freshens at the rate  $S_0(t)h'_0(t)$ , while salt diffuses into this water at the rate  $-DS_x(h_0(t)-, t)$ , where  $S_0(t) = S(h_0(t)-, t)$  is the salinity of the ocean at the interface and  $D > 0$  is the molecular diffusivity of salt in sea water. The balance of salt at this interface leads to the conservation condition

$$S_0(t)h'_0(t) = -DS_x(h_0(t)-, t). \quad (1.2)$$

Moreover, the interface temperature  $T_0 = T(h_0(t), t)$  and salinity  $S_0$  are connected via freezing-point relationship

$$T_0 + n_0 T'_0 = -m_0 S_0, \quad (1.3)$$

where  $m_0, n_0$  are non-zero constants. The establishment of equations (1.1) and (1.2) bases on *Martin and Kauffman* [6] and *Notz et al.* [9]. In (1.3), if  $n_0 = 0$  and  $m_0 = 0.054^\circ\text{C psu}^{-1}$  then we obtain the approximation  $T_0 \approx -m_0 S_0$ , which was used in [6, 9]. However, as we shall show later, it is reasonable to assume that the salinity is dependent on the gradient of the temperature at the freezing point.

At the interface Fb-Wa, we use the simplified scheme presented by *Grenfell and Maykut* [4], in which the temperature of the water in the under-ice melt pond is kept at  $0^\circ\text{C}$ . In particular, it leads to the boundary condition at the interface

$$T(h_u(t)-, t) = 0. \quad (1.4)$$

Furthermore, due to the neglect of heat fluxes from the fresh water above the false-bottoms, a thermodynamic condition similar to (1.1) reduces to the Stefan condition at the upper surface

$$h'_u(t) = \tilde{\lambda}_I T_x(h_u(t)-, t). \quad (1.5)$$

Finally, we have the following diffusion equations for salt and heat in the ocean and in the ice

$$T_t = D_I T_{xx}, \quad h_u(t) > x > h_0(t), \quad (1.6)$$

$$T_t = D_O T_{xx}, \quad h_0(t) > x > -\infty, \quad (1.7)$$

$$S_t = DS_{xx}, \quad h_0(t) > x > -\infty, \quad (1.8)$$

where  $D_I > 0$  and  $D_O > 0$  are the thermal diffusivity. Here all mentioned constants are given.

Assume that the initial conditions  $h_0(0) = h_0^0$ ,  $h_u(0) = h_u^0$ ,  $T(x, 0) = T^0(x)$  in  $h_u(0) > x > -\infty$  and  $S(x, 0) = S^0(x)$  in  $h_0(0) > x > -\infty$  are given. The problem is of finding

from the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$  a solution  $(h_0, h_u, T, S)$  of the system (1.1)-(1.8), where  $T = T(x, t)$  in  $t > 0$ ,  $h_u(t) > x > -\infty$  and  $S = S(x, t)$  in  $t > 0$ ,  $h_0(t) > x > -\infty$ . This is a one-dimensional two-phase Stefan problem with two free boundaries.

The main result of the present paper is to prove the existence and uniqueness a local solution for the problem. Rigorously, we suppose that the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$  satisfy the following three assumptions

- (H1)  $h_0^0 < h_u^0$ ;
- (H2)  $(T^0)_x$  is continuous and bounded in  $x \in (-\infty, h_0^0]$  and  $x \in [h_0^0, h_u^0]$ ;  $T^0$  is continuous at  $x = h_0^0$  and  $T^0(h_u^0-) = 0$ ;
- (H3)  $S^0$  is continuous and bounded in  $x \in (-\infty, h_0^0]$ .

**Remark 1.1.** *The condition (H1) means the ice layer of the false-bottom has already existed. It is of course a much more interesting problem to consider the behavior of the false-bottom at the starting time, which corresponds to condition  $h_0^0 = h_u^0$ , but it is extremely difficult. We mention that the difficulty does not only come from mathematical computation but also be recognized by physical experiments. For example, Notz et al. [9] simulated the experiment of Martin and Kauffman [6], in which they put fresh water at  $0^\circ\text{C}$  on top of salt water at its freezing point in order to simulate the evolution of a false-bottoms, but this model does not take salt transport through the false-bottom into account. Therefore they then started with a 5-cm layer of ice.*

**Remark 1.2.** *It is implicit from (1.1) and (1.5) that  $T_x(\cdot, t)$  is continuous at  $x = h_0(t)-$ ,  $h_0(t)+$  and  $h_u(t)-$ . Therefore, it is reasonable to assume that  $(T^0)_x$  is continuous also at  $x = h_0^0-$ ,  $h_0^0+$  and  $h_u^0-$  in (H2). Similarly, it is natural to assume that  $T^0$  is continuous at  $x = h_0^0$  and  $T^0(h_u^0-) = 0$  due to (1.3) and (1.4).*

Let  $\sigma > 0$ . We say that  $(h_0, h_u, T, S)$  is a solution of the system (1.1)-(1.8) in  $(0, \sigma)$  corresponding to the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$  if the following four conditions holds

- (C1)  $h_0(t)$  and  $h_u(t)$  are continuously differentiable in  $t \in [0, \sigma)$ ;  $h_0(0) = h_0^0 < h_u^0 = h_u(0)$  and  $h_0(t) < h_u(t)$  in  $t \in (0, \sigma)$ ;
- (C2)  $T_t, T_{xx}$  is continuous in  $t \in (0, \sigma)$ ,  $x \in (-\infty, h_0(t)) \cup (h_0(t), h_u(t))$ ;  $T_x(\cdot, t)$  is continuous at  $x = h_0(t)-$ ,  $x = h_0(t)+$  and  $x = h_u(t)-$ ;  $T(\cdot, t)$  is continuous at  $x = h_0(t)$ ;  $T(x, \cdot)$  is continuous at  $t = 0+$  and  $T(x, 0+) = T^0(x)$  in  $x \in (-\infty, h_u^0)$ ;
- (C3)  $S_t, S_{xx}$  is continuous in  $t \in (0, \sigma)$ ,  $x \in (-\infty, h_0(t))$ ;  $S_x(\cdot, t)$  is continuous at  $x = h_0(t)-$ ;  $S(x, \cdot)$  is continuous at  $t = 0$  and  $S(x, 0+) = S^0(x)$  in  $x \in (-\infty, h_0^0)$ ;
- (C4) Equations (1.1)-(1.8) hold in  $t \in (0, \sigma)$ .

In general, the conditions (C1)-(C4) assure a classical solution for the system of differential equations (1.1)-(1.8). Our main result is as follows.

**Theorem 1.1.** *Assume that  $(h_0^0, h_u^0, T^0, S^0)$  satisfy (H1)-(H3). Then there is a unique solution  $(h_0, h_u, T, S)$  of the system (1.1)-(1.8) in  $(0, \sigma)$  corresponding to the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$  for some  $\sigma > 0$ . Moreover, this solution can be extended uniquely whenever the condition  $h_u(\sigma) > h_0(\sigma)$  still holds.*

Our proof follows the approach of A.Friedman [3] (Chapter 8), which dealt with a classical one-phase Stefan problem with one free boundary. More precisely, we first reduce the problem to solving a system of nonlinear Volterra integral equations of the second kind and then solve this system by contraction principle. The remainder of the paper is divided into three sections. Section 2 is devoted to some preliminary results on Green functions and Volterra integral equations. In Section 3 we shall reformulate the problem to a system of

nonlinear integral equations. In Section 4 we apply the contraction principle to prove the existence and uniqueness of a local solution for this system.

## 2 Preliminaries

Let  $a(t), b(t)$  be continuously differentiable functions and  $b(t) > a(t)$  for all  $t \geq 0$ . Let  $\kappa > 0$  be a constant and let  $u(x, t)$  be a solution of the diffusion equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad b(t) > x > a(t). \quad (2.1)$$

We introduce the Green's function of equation (2.1),

$$G(x, t; \xi, \tau) = \frac{H(t - \tau)}{2\sqrt{\pi\kappa(t - \tau)}} \exp\left(-\frac{(x - \xi)^2}{4\kappa(t - \tau)}\right),$$

where  $H$  is the Heaviside function,

$$H(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

The following lemma is useful to reformulate the differential equation (2.1) to a Volterra integral equations of the second kind.

**Lemma 2.1.** *If  $u$  is a solution of (2.1) then for  $t > 0$  and  $a(t) < x < b(t)$  we have*

$$\begin{aligned} u(x, t) &= \int_0^t G(x, t; b(\tau), \tau) [\kappa u_\xi(b(\tau)-, \tau) + u(b(\tau), \tau)b'(\tau)] d\tau \\ &\quad - \int_0^t G(x, t; a(\tau), \tau) [\kappa u_\xi(a(\tau)+, \tau) + u(a(\tau), \tau)a'(\tau)] d\tau \\ &\quad - \kappa \int_0^t G_\xi(x, t; b(\tau), \tau) u(b(\tau), \tau) d\tau + \kappa \int_0^t G_\xi(x, t; a(\tau), \tau) u(a(\tau), \tau) d\tau \\ &\quad + \int_{a(0)}^{b(0)} G(x, t; \xi, 0) u(\xi, 0) d\xi. \end{aligned}$$

*Proof.* Note that

$$G_\tau + \kappa G_{\xi\xi} = 0 \text{ for all } \tau < t, \quad \text{and } G(x, t; \xi, t-) = \delta(x - \xi), \quad (2.2)$$

where  $\delta = H'$  is Dirac delta function.

Integrating the Green's identity, here  $u = u(\xi, \tau)$ ,

$$\kappa \frac{\partial}{\partial \xi} \left( G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right) - \frac{\partial}{\partial \tau} (uG) = 0$$

over the domain  $a(\tau) < \xi < b(\tau), 0 < \tau < t$ , we will obtain the desired result because

$$\begin{aligned}
& \int_0^t \int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \xi} \left( G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right) d\xi d\tau = \int_0^t \left[ G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right]_{\xi=a(\tau)}^{\xi=b(\tau)} d\tau \\
& = \int_0^t G(x, t; b(\tau), \tau) u_{\xi}(b(\tau)-, \tau) d\tau - \int_0^t G(x, t; a(\tau), \tau) u_{\xi}(a(\tau)+, \tau) d\tau \\
& \quad - \int_0^t G_{\xi}(x, t; b(\tau), \tau) u(b(\tau), \tau) d\tau + \int_0^t G_{\xi}(x, t; a(\tau), \tau) u(a(\tau), \tau) d\tau,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau} (uG) d\xi d\tau \\
& = \int_0^t \left\{ \frac{\partial}{\partial \tau} \left( \int_{a(\tau)}^{b(\tau)} uG d\xi \right) - [uG]_{\xi=b(\tau)} b'(\tau) + [uG]_{\xi=a(\tau)} a'(\tau) \right\} d\tau \\
& = \left[ \int_{a(\tau)}^{b(\tau)} uG d\xi \right]_{\tau=0}^{\tau=t-} - \int_0^t [uG]_{\xi=b(\tau)} b'(\tau) d\tau + \int_0^t [uG]_{\xi=a(\tau)} a'(\tau) d\tau \\
& = u(x, t) - \int_{a(0)}^{b(0)} G(x, t; \xi, 0) u(\xi, 0) d\xi - \int_0^t G(x, t; b(\tau), \tau) u(b(\tau), \tau) b'(\tau) d\tau \\
& \quad + \int_0^t G(x, t; a(\tau), \tau) u(a(\tau), \tau) a'(\tau) d\tau.
\end{aligned}$$

Here we have made use the following equality, with  $v = uG$ ,

$$\int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi = \frac{\partial}{\partial \tau} \left( \int_{a(\tau)}^{b(\tau)} v(\xi, \tau) d\xi \right) - v(b(\tau), \tau) b'(\tau) + v(a(\tau), \tau) a'(\tau).$$

In fact, in the case  $a(t) \equiv 0$  we have

$$\begin{aligned}
& \int_0^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi = \int_0^\infty H(b(\tau) - \xi) \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi \\
&= \int_0^\infty \frac{\partial}{\partial \tau} [H(b(\tau) - \xi)v(\xi, \tau)] d\xi - \int_0^\infty \frac{\partial}{\partial \tau} [H(b(\tau) - \xi)] v(\xi, \tau) d\xi \\
&= \frac{\partial}{\partial \tau} \left( \int_0^\infty H(b(\tau) - \xi)v(\xi, \tau) d\xi \right) - \int_0^\infty \delta(b(\tau) - \xi) b'(\tau) v(\xi, \tau) d\xi \\
&= \frac{\partial}{\partial \tau} \left( \int_0^{b(\tau)} v(\xi, \tau) d\xi \right) - v(b(\tau), \tau) b'(\tau).
\end{aligned}$$

If  $a(t)$  is not constant, we can write

$$\int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi = \int_0^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi - \int_0^{a(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi,$$

and apply the above result. This completes the proof.  $\square$

**Remark 2.1.** *The result in Lemma 2.1 still holds for  $b(t) \equiv +\infty$  or  $a(t) \equiv -\infty$ . For example, if  $a(t) \equiv -\infty$  then the formula in Lemma 2.1 reduces to*

$$\begin{aligned}
u(x, t) &= \int_0^t G(x, t; b(\tau), \tau) [\kappa u_\xi(b(\tau), \tau) + u(b(\tau), \tau) b'(\tau)] d\tau \\
&\quad - \kappa \int_0^t G_\xi(x, t; b(\tau), \tau) u(b(\tau), \tau) d\tau + \int_{-\infty}^{b(0)} G(x, t; \xi, 0) u(\xi, 0) d\xi.
\end{aligned}$$

We shall need also an useful lemma giving the jump relation at the boundary (see Friedman [3], page 217, Lemma 1).

**Lemma 2.2.** *Let  $p(t)$  be continuous and let  $s(t) > 0$  satisfy the Lipschitz condition,  $0 \leq t \leq \sigma$ . Then, for  $0 < t \leq \sigma$ ,*

$$\lim_{x \rightarrow s(t)^-} \kappa \int_0^t p(\tau) G_x(x, t; s(\tau), \tau) d\tau = \frac{1}{2} p(t) + \kappa \int_0^t p(\tau) G_x(s(t), t; s(\tau), \tau) d\tau.$$

**Remark 2.2.** *In applications of the above lemma, sometimes we need to note that  $G_\xi = -G_x$ . Moreover, for the right limit we have*

$$\lim_{x \rightarrow s(t)^+} \kappa \int_0^t p(\tau) G_x(x, t; s(\tau), \tau) d\tau = -\frac{1}{2} p(t) + \kappa \int_0^t p(\tau) G_x(s(t), t; s(\tau), \tau) d\tau.$$



Finally, we state a simple version of the uniqueness for a system of linear Volterra integral equations of the second kind.

**Lemma 2.3.** *Let  $n \in \mathbb{N}$ ,  $\sigma > 0$ ,  $p > 1$ ,  $q > 1$ ,  $1/p + 1/q = 1$ . Assume that  $\tau \mapsto W_j(t, \tau)$  is measurable in  $(0, t)$  for all  $t \in (0, \sigma]$  and*

$$\int_0^t |W_j(t, \tau)|^p d\tau \leq \text{const.}, \quad \forall t \in (0, \sigma], j = \overline{1, n}.$$

Then the system

$$\Psi_i(t) = \sum_{j=1}^n \left( \int_0^t W_j(t, \tau) \Psi_j(\tau) d\tau \right), \quad \forall t \in (0, \sigma], j = \overline{1, n},$$

has a unique solution  $\{\Psi_j\}_{j=1}^n = 0$  in  $L^q(0, \sigma)$ .

*Proof.* Using Holder inequality one has

$$\begin{aligned} |\Psi_i(t)| &\leq \sum_{j=1}^n \left| \int_0^t W_j(t, \tau) \Psi_j(\tau) d\tau \right| \\ &\leq \sum_{j=1}^n \left( \int_0^t |W_j(t, \tau)|^p d\tau \right)^{1/p} \left( \int_0^t |\Psi_j(\tau)|^q d\tau \right)^{1/q} \\ &\leq \text{const.} \sum_{j=1}^n \left( \int_0^t |\Psi_j(\tau)|^q d\tau \right)^{1/q}. \end{aligned}$$

Therefore,

$$\sum_{j=1}^n |\Psi_j(\tau)|^q \leq \text{const.} \sum_{j=1}^n \left( \int_0^t |\Psi_j(\tau)|^q d\tau \right) = \text{const.} \int_0^t \left( \sum_{j=1}^n |\Psi_j(\tau)|^q \right) d\tau,$$

and it follows from Gronwall's Lemma that  $\sum_{j=1}^n |\Psi_j(\tau)|^q = 0$ . Thus  $\{\Psi_j\}_{j=1}^n = 0$ .  $\square$

**Remark 2.3.** *If, for example,*

$$|W_j(t, \tau)| \leq \frac{\text{const.}}{\sqrt{t - \tau}}, \quad j = \overline{1, n}.$$

then we can choose any  $p \in (1, 2)$  in order to apply Lemma 2.3.

### 3 Reduction to integral equations

Denote by  $G_1, G_2, G_3$  the Green function  $G$  in Lemma 2.1 corresponding to  $\kappa = D, D_O, D_I$ , respectively. We shall reformulate the problem (1.1)-(1.8) to a system of integral equations of time-dependent functions  $(v_0, v_1, v_2, v_3)$  where

$$v_0(t) = T_0'(t), \quad v_1(t) = T_x(h_0(t)-, t), \quad v_2(t) = T_x(h_0(t)+, t), \quad v_3(t) = T_x(h_u(t)-, t). \quad (3.1)$$

Assume that  $(h_0, h_u, T, S)$  is a solution of the system (1.1)-(1.8) in  $(0, \sigma)$  corresponding to the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$  for some  $\sigma > 0$ . Applying Lemma 2.1 to equation (1.8) with  $h_0(t) > x > -\infty$  and using condition (1.2), one has

$$S(x, t) = -D \int_0^t G_{1\xi}(x, t; h_0(\tau), \tau) S_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_1(x, t; \xi, 0) S^0(\xi) d\xi. \quad (3.2)$$

In particular, we see from (3.2) that  $S(x, t)$  is determined completely by  $h_0$  and  $S_0(t)$ . Taking  $x \rightarrow h_0(t)-$  in (3.2) and using the jump relation in Lemma 2.2, we get

$$S_0(t) = -2D \int_0^t G_{1\xi}(h_0(t), t; h_0(\tau), \tau) S_0(\tau) d\tau + 2 \int_{-\infty}^{h_0^0} G_1(h_0(t), t; \xi, 0) S^0(\xi) d\xi. \quad (3.3)$$

Note that for  $v_0(t) = T_0'(t)$  the condition (1.3) can be rewritten as

$$T^0(h_0^0) + \int_0^t v_0(\tau) d\tau + n_0 v_0(t) = -m_0 S_0(t). \quad (3.4)$$

Here we have used  $T_0(0) = T(h_0(0), 0) = T_0(h_0^0)$ . We deduce from (3.3) and (3.4) the first integral equation

$$\begin{aligned} v_0(t) = & -\frac{1}{n_0} T^0(h_0^0) - \frac{1}{n_0} \int_0^t v_0(\tau) d\tau - \frac{2m_0}{n_0} \int_{-\infty}^{h_0^0} G_1(h_0(t), t; \xi, 0) S^0(\xi) d\xi \\ & - \frac{2D}{n_0} \int_0^t G_{1\xi}(h_0(t), t; h_0(\tau), \tau) \left[ T^0(h_0^0) + \int_0^\tau v_0(s) ds + n_0 v_0(\tau) \right] d\tau. \end{aligned} \quad (3.5)$$

We next apply Lemma 2.1 to equation (1.7) for  $h_0(t) > x > -\infty$  to get

$$\begin{aligned} T(x, t) = & \int_0^t G_2(x, t; h_0(\tau), \tau) [D_O v_1(\tau) + T_0(\tau) h_0'(\tau)] d\tau \\ & - D_O \int_0^t G_{2\xi}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0) T^0(\xi) d\xi, \end{aligned} \quad (3.6)$$

where  $v_1(t) = T_x(h_0(t)-, t)$ . We now differentiate both sides of (3.6) with respect to  $x$ , then

take  $x \rightarrow h_0(t)-$ . To go into the details, because  $D_O G_{2\xi x} = -D_O G_{2\xi\xi} = G_{2\tau}$ , we have

$$\begin{aligned}
& -D_O \int_0^t G_{2\xi x}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau = - \int_0^t G_{2\tau}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau \\
& = - \int_0^t \left[ \frac{\partial}{\partial \tau} (G_2(x, t; h_0(\tau), \tau)) - G_{2\xi}(x, t; h_0(\tau), \tau) h_0'(\tau) \right] T_0(\tau) d\tau \\
& = G_2(x, t; h_0^0, 0) T_0(0) + \int_0^t G_2(x, t; h_0(\tau), \tau) T_0'(\tau) d\tau \\
& \quad - \int_0^t G_{2x}(x, t; h_0(\tau), \tau) T_0(\tau) h_0'(\tau) d\tau.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \int_{-\infty}^{h_0^0} G_{2x}(x, t; \xi, 0) T^0(\xi) d\xi = - \int_{-\infty}^{h_0^0} G_{2\xi}(x, t; \xi, 0) T^0(\xi) d\xi \\
& = -G_2(x, t; h_0^0, 0) T^0(h_0^0) + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi.
\end{aligned}$$

Thus it follows from (3.6) that

$$\begin{aligned}
T_x(x, t) &= D_O \int_0^t G_{2x}(x, t; h_0(\tau), \tau) v_1(\tau) d\tau \\
&\quad + \int_0^t G_2(x, t; h_0(\tau), \tau) v_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi \quad (3.7)
\end{aligned}$$

for all  $h_0(t) > x > -\infty$ . Here we have used the compatible condition  $T_0(0) = T(h_0(0), 0) = T^0(h_0^0)$  and replaced  $T_0'(\tau)$  by  $v_0(\tau)$ . Taking  $x \rightarrow h_0(t)-$  in (3.7) and using Lemma 2.1 for the first term of the right hand side, we have the second integral equation

$$\begin{aligned}
v_1(t) &= 2D_O \int_0^t G_{2x}(h_0(t), t; h_0(\tau), \tau) v_1(\tau) d\tau \\
&\quad + 2 \int_0^t G_2(h_0(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{-\infty}^{h_0^0} G_2(h_0(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \quad (3.8)
\end{aligned}$$

Now we consider the heat distribution in false-bottoms. Applying Lemma 2.1 to equation

(1.6) for  $h_u(t) > x > h_0(t)$  one has

$$\begin{aligned}
T(x, t) &= D_I \int_0^t G_3(x, t; h_u(\tau), \tau) v_3(\tau) d\tau \\
&\quad - \int_0^t G_3(x, t; h_0(\tau), \tau) [D_I v_2(\tau) + T_0(\tau) h'_0(\tau)] d\tau \\
&\quad + D_I \int_0^t G_{3\xi}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau + \int_{h_0^0}^{h_u^0} G_3(x, t; \xi, 0) T^0(\xi) d\xi, \quad (3.9)
\end{aligned}$$

where  $v_2(t) = T_x(h_0(t)+, t)$  and  $v_3(t) = T_x(h_u(t)-, t) = \frac{1}{\lambda_I} h'_u(t)$ . Let us differentiate both sides of (3.9) with respect to  $x$ . We have

$$\begin{aligned}
&D_I \int_0^t G_{3\xi x}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau = \int_0^t G_{3\tau}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau \\
&= \int_0^t \left[ \frac{\partial}{\partial \tau} (G_3(x, t; h_0(\tau), \tau)) - G_{3\xi}(x, t; h_0(\tau), \tau) h'_0(\tau) \right] T_0(\tau) d\tau \\
&= -G_3(x, t; h_0^0, 0) T_0(0) - \int_0^t G_3(x, t; h_0(\tau), \tau) T'_0(\tau) d\tau \\
&\quad + \int_0^t G_{3x}(x, t; h_0(\tau), \tau) T_0(\tau) h'_0(\tau) d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_{h_0^0}^{h_u^0} G_{3x}(x, t; \xi, 0) T^0(\xi) d\xi = - \int_{h_0^0}^{h_u^0} G_{3\xi}(x, t; \xi, 0) T^0(\xi) d\xi \\
&= -G_3(x, t; h_u^0, 0) T^0(h_u^0-) + G_3(x, t; h_0^0, 0) T^0(h_0^0-) + \int_{h_0^0}^{h_u^0} G_3(x, t; \xi, 0) T'_\xi(\xi) d\xi.
\end{aligned}$$

Using the compatible conditions  $T^0(h_u^0-) = 0$  and  $T_0(0) = T^0(h_0^0)$  and replacing  $T'_0(\tau)$  by  $v_0(\tau)$ , we find from (3.9) that

$$\begin{aligned}
T_x(x, t) &= D_I \int_0^t G_{3x}(x, t; h_u(\tau), \tau) v_3(\tau) d\tau - D_I \int_0^t G_{3x}(x, t; h_0(\tau), \tau) v_2(\tau) d\tau \\
&\quad - \int_0^t G_3(x, t; h_0(\tau), \tau) v_0(\tau) d\tau + \int_{h_0^0}^{h_u^0} G_3(x, t; \xi, 0) T'_\xi(\xi) d\xi \quad (3.10)
\end{aligned}$$

for all  $h_u(t) > x > h_0(t)$ . Taking  $x \rightarrow h_0(t)+$  and  $x \rightarrow h_u(t)-$  in (3.10) and using Lemma 2.2 again we obtain the last two integral equations

$$\begin{aligned} v_2(t) &= 2D_I \int_0^t G_{3x}(h_0(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_0(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_0(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_0(t), t; \xi, 0) T_\xi^0(\xi) d\xi, \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} v_3(t) &= 2D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_u(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \end{aligned} \quad (3.12)$$

Note that due to the interface condition (1.1) and (1.5), one has

$$h_0(t) = h_0^0 + \tilde{\lambda}_I \int_0^t v_2(\tau) d\tau - \tilde{\lambda}_O \int_0^t v_1(\tau) d\tau. \quad (3.13)$$

and

$$h_u(t) = h_u^0 + \tilde{\lambda}_I \int_0^t v_3(\tau) d\tau. \quad (3.14)$$

Therefore the equations (3.5), (3.8), (3.11) and (3.12) form a system of nonlinear integral equations of the form  $v = Pv$ , where  $v = (v_0, v_1, v_2, v_3)$  and  $Pv = (P_0v, P_1v, P_2v, P_3v)$ . Note that the time-dependent function  $v = (v_0, v_1, v_2, v_3)$  is first only continuous in  $(0, \sigma)$  by its definition in (3.1). However, because it is a solution of the system of integral equations (3.5), (3.8), (3.11) and (3.12), it is indeed continuous in  $[0, \sigma]$ . We thus have proved the direct part of the following statement.

**Proposition 3.1.** *The problem of finding a solution  $(h_0, h_u, T, S)$  of (1.1)-(1.8) in  $(0, \sigma)$  is equivalent to the problem of finding a continuous solution  $v = (v_0, v_1, v_2, v_3)$  in  $[0, \sigma]$  for the system (3.5), (3.8), (3.11) and (3.12) such that  $h_u(t) > h_0(t)$  in  $(0, \sigma)$ , where  $h_0$  and  $h_u$  are defined by (3.13) and (3.14).*

We of course just need to prove the converse part of the statement.

*Proof.* Suppose that  $v = (v_0, v_1, v_2, v_3)$  is a continuous solution in  $[0, \sigma]$  for the system (3.5), (3.8), (3.11) and (3.12) such that  $h_u(t) > h_0(t)$  in  $(0, \sigma)$ ,  $h_0$  and  $h_u$  are defined by (3.13) and (3.14). We shall now recover the solution  $(h_0, h_u, T, S)$  of (1.1)-(1.8).

Since  $h_0$  and  $h_u$  have already been defined by (3.13) and (3.14), it remains to determine  $T$  and  $S$ . We define  $T_0(t)$  by

$$T_0(t) = T^0(h_0^0) + \int_0^t v_0(\tau) d\tau. \quad (3.15)$$

and determine naturally  $T$  by (3.6) and (3.9). Similarly, we define  $S_0$  by (3.4) and determine  $S$  by (3.2). We need to prove that  $(h_0, h_u, T, S)$  satisfies (C1)-(C4).

**Step 1.** We first prove (C1)-(C3) except the behavior of  $T$  and  $S$  at the interfaces  $x = h_0(t)$  and  $x = h_u(t)-$ . In fact, the condition (C1) follows the definition of  $h_0$  in (3.13), the definition of  $h_u$  in (3.14), and the condition  $h_u(t) > h_0(t)$  in definition of the solution  $v = (v_0, v_1, v_2, v_3)$ .

Due to definition of  $S$  in the integral forms (3.2),  $S_t$  and  $S_{xx}$  are continuous in  $t \in (0, \sigma)$ ,  $x \in (-\infty, h_0(t))$ . Moreover, getting  $t \rightarrow 0+$  in (3.2) and using  $\lim_{t \rightarrow 0+} G_1(x, t; \xi, 0) = \delta(x - \xi)$ , we obtain the initial condition  $S(x, 0+) = S^0(x)$  for all  $x \in (-\infty, h_0^0)$ .

Similarly, due to definition of  $T$  in (3.6) and (3.9),  $T_t$  and  $T_{xx}$  are continuous in  $t \in (0, \sigma)$ ,  $x \in (-\infty, h_0(t)) \cup (h_0(t), h_u(t))$ , and  $T(x, 0+) = T^0(x)$  for all  $x \in (-\infty, h_0^0) \cup (h_0^0, h_u^0)$ .

**Step 2.** We next check three diffusion equations (1.6)-(1.8). To prove (1.8) from the definition of  $S$  in (3.2), we simply verify that that each term in the right-hand side of (3.2) is a homogeneous solution of the operator  $(\partial/\partial t - D\partial^2/\partial x^2)$  in  $t \in (0, \sigma)$ ,  $x \in (-\infty, h_0(t))$ . In fact, for the second term one it follows from the properties of Green function that

$$\begin{aligned} & \left( \frac{\partial}{\partial t} - D \frac{\partial^2}{\partial x^2} \right) \left( \int_{-\infty}^{h_0^0} G_1(x, t; \xi, 0) S^0(\xi) d\xi \right) \\ &= \int_{-\infty}^{h_0^0} \left( \frac{\partial}{\partial t} - D \frac{\partial^2}{\partial x^2} \right) (G_1(x, t; \xi, 0)) S^0(\xi) d\xi = 0. \end{aligned}$$

For the first term, we have to be more careful because in general differentiating with respect to  $t$  a function of the form  $t \mapsto \int_0^t K(t, \tau) d\tau$  may cause a jump,

$$\frac{d}{dt} \left( \int_0^t K(t, \tau) d\tau \right) = \lim_{\tau \rightarrow t-} K(t, \tau) + \int_0^t K_t(t, \tau) d\tau.$$

However, in this case the jump  $\lim_{\tau \rightarrow t-} K(t, \tau)$  vanishes since

$$\lim_{\tau \rightarrow t-} G_1(x, t; \xi, \tau) = \lim_{\tau \rightarrow t-} G_{1\xi}(x, t; \xi, \tau) = 0, \quad \forall x \neq \xi.$$

Therefore, the first term can be treat similarly to the second term. Thus (1.8) holds. The proofs for (1.6) and (1.7) are the same.

**Step 3.** We now prove that  $S(h_0(t)-, t) = S_0(t)$ , which in particular accomplish (C3), and verify the Stefan condition (1.2).

In fact, (3.3) holds due to (3.4) and (3.5). On the other hand, taking  $x \rightarrow h_0(t)-$  in (3.2) and using Lemma 2.2 one has

$$\begin{aligned} S(h_0(t)-, t) &= \frac{1}{2}S_0(t) - D \int_0^t G_{1\xi}(h_0(t), t; h_0(\tau), \tau) S_0(\tau) d\tau \\ &\quad + \int_{-\infty}^{h_0^0} G_1(h_0(t), t; \xi, 0) S^0(\xi) d\xi. \end{aligned}$$

It follows from the latter equation and (3.3) that  $S(h_0(t)-, t) = S_0(t)$ .

Applying Lemma 2.1 to equation (1.8) for  $h_0(t) > x > -\infty$  and using  $S(h_0(t)-, t) = S_0(t)$ , we have

$$\begin{aligned} S(x, t) &= \int_0^t G_1(x, t; h_0(\tau), \tau) [DS_x(h_0(\tau)-, \tau) + S_0(\tau)h_0'(\tau)] d\tau \\ &\quad - D \int_0^t G_{1\xi}(x, t; h_0(\tau), \tau) S_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_1(x, t; \xi, 0) S^0(\xi) d\xi. \end{aligned}$$

Comparing the latter equation to the original definition of  $S$  in (3.2), we obtain

$$\int_0^t G_1(x, t; h_0(\tau), \tau) \Psi_1(\tau) d\tau = 0, \quad x \in (-\infty, h_0(t)), \quad (3.16)$$

where  $\Psi_1(t) = DS_x(h_0(t)-, t) + S_0(t)h_0'(t)$ . We shall deduce from (3.16) that  $\Psi_1 \equiv 0$ , which is equivalent to (1.2). Indeed, differentiating (3.16) with respect to  $x$ , then taking  $x \rightarrow h_0(t)-$  and using Lemma 2.2 we get

$$\Psi_1(t) = -2 \int_0^t G_{1x}(h_0(t), t; h_0(\tau), \tau) \Psi_1(\tau) d\tau.$$

Note that this is a linear Volterra integral equation of the second kind and

$$|G_{1x}(h_0(t), t; h_0(\tau), \tau)| \leq \frac{\text{const.}}{\sqrt{t-\tau}}.$$

Therefore, it follows from Lemma 2.3 that  $\Psi_1(t) = 0$ .

**Step 4.** We prove that  $T_x(h_0(t)-, t) = v_1(t)$ ,  $T_x(h_0(t)+, t) = v_2(t)$  and  $T_x(h_u(t)-, t) = v_3(t)$ , which in particular imply the Stefan conditions (1.1) and (1.5). Note that we have already had from (3.15) that  $v_0(t) = T_0'(t)$  and  $T_0(0) = T^0(h_0^0)$ .

To prove  $T_x(h_u(t)-, t) = v_3(t)$  from the definition of  $T$  in (3.9), we use the same process of getting (3.12) from (3.9). In fact, differentiating both sides of (3.9) with respect to  $x$  and

then taking  $x \rightarrow h_u(t)-$  we have

$$\begin{aligned}
T_x(h_u(t)-, t) &= \frac{1}{2}v_3(t) + D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau)v_3(\tau)d\tau \\
&\quad - D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau)v_2(\tau)d\tau \\
&\quad - \int_0^t G_3(h_u(t), t; h_0(\tau), \tau)T'_0(\tau)d\tau + \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0)T_\xi^0(\xi)d\xi.
\end{aligned}$$

Comparing the latter equation to (3.12) and using  $v_0(\tau) = T'_0(\tau)$ , we obtain  $T_x(h_u(t)-, t) = v_3(t)$ . By the same way we find that  $T_x(h_0(t)+, t) = v_2(t)$  and  $T_x(h_0(t)-, t) = v_1(t)$ .

**Step 5.** Finally, we verify that  $T(h_0(t), t) = T_0(t)$  and  $T(h_u(t)-, t) = 0$  to accomplish (C2) and (C4).

Applying Lemma 2.1 to equation (1.7) for  $h_0(t) > x > -\infty$  one has

$$\begin{aligned}
T(x, t) &= \int_0^t G_2(x, t; h_0(\tau), \tau) [D_O T_x(h_0(\tau)-, \tau) + T(h_0(\tau)-, \tau)h'_0(\tau)] d\tau \\
&\quad - D_O \int_0^t G_{2\xi}(x, t; h_0(\tau), \tau)T(h_0(\tau)-, \tau)d\tau + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0)T^0(\xi)d\xi.
\end{aligned}$$

Comparing the latter equation to the original definition of  $T$  in (3.6) and using  $T_x(h_0(t)-, t) = v_1(t)$ , we obtain

$$\int_0^t [G_2(x, t; h_0(\tau), \tau)h'_0(\tau) - D_O G_{2\xi}(x, t; h_0(\tau), \tau)] \Psi_2(\tau)d\tau = 0, \quad (3.17)$$

where  $\Psi_2(t) = T(h_0(t)-, t) - T_0(t)$ . We now prove  $\Psi_2 \equiv 0$  by using the same technique of dealing  $\Psi_1 \equiv 0$ . Taking  $x \rightarrow h_u(t)-$  in (3.17) and using Lemma 2.2, we find that

$$\Psi_2(t) = 2 \int_0^t [G_2(x, t; h_0(\tau), \tau)h'_0(\tau) - D_O G_{2\xi}(x, t; h_0(\tau), \tau)] \Psi_2(\tau)d\tau.$$

Note that this is a linear Volterra integral equation of the second kind and

$$|G_2(x, t; h_0(\tau), \tau)h'_0(\tau) - D_O G_{2\xi}(x, t; h_0(\tau), \tau)| \leq \frac{\text{const.}}{\sqrt{t-\tau}}.$$

Therefore, it follows from Lemma 2.3 that  $\Psi_2 \equiv 0$ . Thus  $T(h_0(t)-, t) = T_0(t)$ .

We shall use the same argument, in fact a little more complicated one, to deduce that  $T(h_0(t)+, t) = T_0(t)$  and  $T(h_u(t), t) = 0$ . Applying Lemma 2.1 to equation (1.6) for  $h_u(t) >$



$x > h_0(t)$  and comparing to the original definition of  $T$  in (3.9), one has

$$\begin{aligned} & \int_0^t [G_3(x, t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(x, t; h_0(\tau), \tau)] \Psi_3(\tau) d\tau \\ &= \int_0^t [G_3(x, t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(x, t; h_u(\tau), \tau)] \Psi_4(\tau) d\tau, \end{aligned} \quad (3.18)$$

where  $\Psi_3(\tau) = T(h_0(t)+, t) - T_0(t)$  and  $\Psi_4(t) = T(h_u(t)-, t)$ . Taking  $x \rightarrow h_0(t)+$  and  $x \rightarrow h_u(t)-$  in (3.18), we have

$$\begin{aligned} \Psi_3(t) &= 2 \int_0^t [G_3(h_0(t), t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(h_0(t), t; h_0(\tau), \tau)] \Psi_3(\tau) d\tau \\ &- 2 \int_0^t [G_3(h_0(t), t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(h_0(t), t; h_u(\tau), \tau)] \Psi_4(\tau) d\tau, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} -\Psi_4(t) &= -2 \int_0^t [G_3(h_u(t), t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(h_u(t), t; h_0(\tau), \tau)] \Psi_3(\tau) d\tau \\ &+ 2 \int_0^t [G_3(h_u(t), t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(h_u(t), t; h_u(\tau), \tau)] \Psi_4(\tau) d\tau \end{aligned} \quad (3.20)$$

Note that equations (3.19) and (3.20) form a system of linear Volterra integral equations of the second kind and

$$\begin{aligned} |G_3(h_u(t), t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(h_u(t), t; h_0(\tau), \tau)| &\leq \frac{\text{const.}}{\sqrt{t-\tau}}, \\ |G_3(h_0(t), t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(h_0(t), t; h_u(\tau), \tau)| &\leq \frac{\text{const.}}{\sqrt{t-\tau}}. \end{aligned}$$

We therefore deduce from Lemma 2.3 that  $\Psi_3 \equiv \Psi_4 \equiv 0$ . Thus  $T(h_u(t)-, t) = 0$ , i.e. (1.4), and  $T(h_0(t), t) = T_0(t)$ , which accomplishes (C2) and also implies (1.3) in viewing (3.4). The proof is completed.  $\square$

**Remark 3.1.** We learned the technique of proving  $\Psi_1 \equiv 0$  by (3.16) from Friedman [3]. However, there is a minor mistake in [3]. The formula (1.28) in page 221, i.e.

$$\int_0^t G_\xi(x, t; s(\tau), \tau)u(s(\tau), \tau) d\tau = 0,$$

should be

$$\int_0^t [G_\xi(x, t; s(\tau), \tau) - G(x, t; s(\tau), \tau)s'(\tau)] u(s(\tau), \tau) d\tau = 0.$$

Fortunately, the conclusion  $u(s(t), t) = 0$  still holds by the same argument.

## 4 Solving integral equations

Now we accomplish the proof of Theorem 1.1. We want to prove the system (3.5), (3.8), (3.11) and (3.12) has a unique local solution  $v = (v_0, v_1, v_2, v_3)$ , and this solution can be extended uniquely by prolongation whenever the condition  $h_u(t) > h_0(t)$  still holds. Note that this system is of the form  $v = Pv := (P_0v, P_1v, P_2v, P_3v)$ , where  $P_0v, P_1v, P_2v$  and  $P_3v$  are the right-hand sides of (3.5), (3.8), (3.11) and (3.12) respectively.

Denote by  $C(\sigma, M)$  the space of functions  $v = (v_0, v_1, v_2, v_3)$  continuous in  $[0, \sigma]$  and

$$\|v\|_{[0, \sigma]} := \max_{t \in [0, \sigma], j=0,3} |v_j(t)| \leq M.$$

When  $v = 0$  then  $P(0)$  depends only on the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$  and is continuous on  $[0, \infty)$ . Fix  $M > 0$  large enough, says  $M \geq 2 \|P(0)\|_{[0,1]}$ .

**Proposition 4.1.** *There exists  $\sigma > 0$  depending on  $(h_0^0, h_u^0, T^0, S^0)$  such that  $P$  maps  $C(t_0, M)$  into itself and is a contraction.*

Note that  $C(\sigma, M)$  is a complete metric space. Therefore it follows from Proposition 4.1 and the Contraction Mapping Principle that  $P$  has a unique fixed point  $v$  in  $C(\sigma, M)$ . It gives also the unique solution  $(h_0, h_u, T, S)$  of (1.1)-(1.8) in  $(0, \sigma)$  due to Proposition 3.1.

*Proof.* First at all, let us consider  $\sigma > 0$  small enough, says

$$\sigma \leq \frac{h_u^0 - h_0^0}{2(2\tilde{\lambda}_I + \tilde{\lambda}_O)M}.$$

Then for  $v \in C(\sigma, M)$ , it follows from (3.13)-(3.14) that

$$\begin{aligned} h_u(t) - h_0(\tau) &\geq h_u^0 - h_0^0 - |h_u(t) - h_u(0)| - |h_0(\tau) - h_0(0)| \\ &\geq h_u^0 - h_0^0 - \tilde{\lambda}_I M t - (\tilde{\lambda}_O + \tilde{\lambda}_I) M \tau \geq \frac{h_u^0 - h_0^0}{2} > 0 \end{aligned} \quad (4.1)$$

for all  $0 \leq t \leq \sigma$ . In particular  $h_u(t) > h_0(t)$  for all  $t \in [0, \sigma]$  and hence  $P$  is well-defined on  $C(\sigma, M)$ . We now estimate  $\|Pv - P\tilde{v}\|_{[0, \sigma]}$  for  $v, \tilde{v} \in C(\sigma, M)$ . In what follows, denote by  $\tilde{h}_u$  and  $\tilde{h}_0$  the functions given by (3.13)-(3.14) where  $v$  is replaced by  $\tilde{v}$ . For simplicity, we shall always denote by  $C_0 > 0$  an universal constant depending only on the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$ . We shall go to the details for  $P_3$ , and  $P_0, P_1, P_2$  can be treated by the same way. Recall that

$$\begin{aligned} (P_3v)(t) &= 2D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_u(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \end{aligned}$$

We need some preliminary estimates related to the Green's function  $G_3$ .

**Lemma 4.1.** For all  $v, \tilde{v} \in C(\sigma, M)$ ,  $0 \leq \tau < t \leq \sigma$  and  $\xi \in \mathbb{R}$ , we have

$$\left| G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \right| \leq \frac{C_0}{\sqrt{t-\tau}} \|v - \tilde{v}\|_{[0, \sigma]}, \quad (4.2)$$

$$\left| G_3(h_u(t), t; h_0(\tau), \tau) - G_3(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \right| \leq \frac{C_0}{\sqrt{t-\tau}} \|v - \tilde{v}\|_{[0, \sigma]}, \quad (4.3)$$

$$\left| G_{3x}(h_u(t), t; h_0(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \right| \leq \frac{C_0}{\sqrt{t-\tau}} \|v - \tilde{v}\|_{[0, \sigma]}, \quad (4.4)$$

$$\begin{aligned} & \left| G_{3x}(h_u(t), t; \xi, \tau) - G_{3x}(\tilde{h}_u(t), t; \xi, \tau) \right| \leq \\ & \leq C_0 \left[ \exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right) + \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right) \right] \|v - \tilde{v}\|_{[0, \sigma]}. \end{aligned} \quad (4.5)$$

*Proof.* Considering

$$G_{3x}(h_u(t), t; h_u(\tau), \tau) = -\frac{h_u(t) - h_u(\tau)}{4\sqrt{\pi D_I^3(t-\tau)^3}} \exp\left(-\frac{(h_u(t) - h_u(\tau))^2}{4D_I(t-\tau)}\right)$$

as a function of one-variable  $(h_u(t) - h_u(\tau))$  and using Lagrange formula one has

$$\begin{aligned} & G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \\ & = -\frac{[(h_u(t) - h_u(\tau)) - (\tilde{h}_u(t) - \tilde{h}_u(\tau))]}{4\sqrt{\pi D_I^3(t-\tau)^3}} \exp\left(-\frac{\theta^2}{4D_I(t-\tau)}\right) \left(1 - \frac{2\theta^2}{4D_I(t-\tau)}\right) \end{aligned}$$

for some  $\theta$  between  $(h_u(t) - h_u(\tau))$  and  $(\tilde{h}_u(t) - \tilde{h}_u(\tau))$ . Thus (4.2) follows

$$|(h_u(t) - h_u(\tau)) - (\tilde{h}_u(t) - \tilde{h}_u(\tau))| \leq \tilde{\lambda}_I(t-\tau) \|v - \tilde{v}\|_{[0, \sigma]},$$

and the elementary inequality

$$e^{-z}|1-2z| \leq \frac{|1-2z|}{1+z} \leq 2, \quad \text{where } z = \frac{\theta^2}{4D_I(t-\tau)} \geq 0.$$

To prove (4.3) we shall use the identity

$$\begin{aligned} & G_3(h_u(t), t; h_0(\tau), \tau) - G_3(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \\ & = -\frac{[(h_u(t) - h_0(\tau)) - (\tilde{h}_u(t) - \tilde{h}_0(\tau))]\theta_1}{4\sqrt{\pi D_I^3(t-\tau)^3}} \exp\left(-\frac{\theta_1^2}{4D_I(t-\tau)}\right) \end{aligned}$$

for some  $\theta_1$  between  $(h_u(t) - h_0(\tau))$  and  $(\tilde{h}_u(t) - \tilde{h}_0(\tau))$ . Note from (4.1) that  $|\theta_1| \geq \frac{h_u^0 - h_0^0}{2}$ . Therefore (4.3) follows

$$|(h_u(t) - h_0(\tau)) - (\tilde{h}_u(t) - \tilde{h}_0(\tau))| \leq (2\tilde{\lambda}_I + \tilde{\lambda}_O)\sigma \|v - \tilde{v}\|_{[0, \sigma]}$$

and the elementary inequality

$$e^{-z}z \leq 1, \quad \text{where } z = \frac{\theta_1^2}{4D_I(t-\tau)} \geq 0.$$

We can also prove (4.4) by the same argument. In fact, we write

$$\begin{aligned} & G_{3x}(h_u(t), t; h_0(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \\ &= -\frac{[(h_u(t) - h_0(\tau)) - (\tilde{h}_u(t) - \tilde{h}_0(\tau))]}{4\sqrt{\pi D_I^3(t - \tau)^3}} \exp\left(-\frac{\theta_2^2}{4D_I(t - \tau)}\right) \left(1 - \frac{2\theta_2^2}{4D_I(t - \tau)}\right) \end{aligned}$$

for some  $\theta_2$  between  $(h_u(t) - h_0(\tau))$  and  $(\tilde{h}_u(t) - \tilde{h}_0(\tau))$  and use

$$e^{-z}z|1 - 2z| \leq \frac{z|1 - 2z|}{1 + z + \frac{1}{2}z^2} \leq 4, \quad \text{where } z = \frac{\theta_2^2}{4D_I(t - \tau)} \geq 0.$$

Finally, for the last inequality (4.5) we write

$$G_3(h_u(t), t; \xi, 0) - G_3(\tilde{h}_u(t), t; \xi, 0) = -\frac{[h_u(t) - \tilde{h}_u(t)]\theta_3}{4\sqrt{\pi D_I^3 t^3}} \exp\left(-\frac{\theta_3^2}{4D_I t}\right)$$

for some  $\theta_3$  between  $(h_u(t) - \xi)$  and  $(\tilde{h}_u(t) - \xi)$ . Note that

$$\exp\left(-\frac{\theta_3^2}{8D_I t}\right) \leq \max\left\{\exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right), \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right)\right\}.$$

Thus (4.5) follows

$$e^{-z}\sqrt{z} \leq \frac{\sqrt{z}}{1 + z} \leq \frac{1}{2}, \quad \text{where } z = \frac{\theta_3^2}{8D_I t} \geq 0,$$

and the estimate  $|h_u(t) - \tilde{h}_u(t)| \leq \tilde{\lambda}_I t \|v - \tilde{v}\|_{[0, \sigma]}$ .  $\square$

We now already to estimate  $\|P_3 v - P_3 \tilde{v}\|_{[0, \sigma]}$  for  $v, \tilde{v} \in C(\sigma, M)$ . Recall that  $P_3$  is the sum of four terms

$$\begin{aligned} (P_3 v)(t) &= 2D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_u(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \end{aligned}$$

For the first term of  $P_3$ , using (4.2) we have

$$\begin{aligned} & |G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \tilde{v}_3(\tau)| \\ &\leq |G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau)| \cdot |v_3(\tau)| \\ &\quad + |G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau)| \cdot |v_3(\tau) - \tilde{v}_3(\tau)| \\ &\leq \frac{C_0 M}{\sqrt{t - \tau}} \|v - \tilde{v}\|_{[0, \sigma]} + \frac{C_0}{\sqrt{t - \tau}} \|v - \tilde{v}\|_{[0, \sigma]} = \frac{C_0(M + 1)}{\sqrt{t - \tau}} \|v - \tilde{v}\|_{[0, \sigma]} \end{aligned}$$

for all  $0 \leq \tau \leq t \leq \sigma$ . Therefore

$$\begin{aligned} & \int_0^t \left| G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \tilde{v}_3(\tau) \right| d\tau \\ &\leq C_0(M + 1)\sqrt{\sigma} \|v - \tilde{v}\|_{[0, \sigma]}, \quad t \in [0, \sigma]. \end{aligned}$$

We have also the same estimate for the second term and the third term of  $P_3$  due to (4.3) and (4.4). For the last term of  $P_3$ , it follows from (4.5) and the boundness of  $T_x^0$  that

$$\begin{aligned} & \int_{h_0^0}^{h_u^0} \left| G_3(h_u(t), t; \xi, 0) - G_3(\tilde{h}_u(t), t; \xi, 0) \right| T_\xi^0(\xi) d\xi \\ & \leq C_0 \|v - \tilde{v}\|_{[0, \sigma]} \int_{h_0^0}^{h_u^0} \left[ \exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right) + \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right) \right] d\xi. \end{aligned}$$

Note that by changing of variables

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[ \exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right) + \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right) \right] d\xi \\ & = 2 \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{8D_I t}\right) d\xi = 2\sqrt{\pi 8D_I t}. \end{aligned}$$

Thus

$$\int_{h_0(0)}^{h_u(0)} \left| G_3(h_u(t), t; \xi, 0) - G_3(\tilde{h}_u(t), t; \xi, 0) \right| T_\xi^0(\xi) d\xi \leq 2C_0 \sqrt{\pi 8D_I \sigma} \|v - \tilde{v}\|_{[0, \sigma]}.$$

In summary, we have

$$\|P_3 v - P_3 \tilde{v}\|_{[0, \sigma]} \leq C_0 \sqrt{\sigma} \|v - \tilde{v}\|_{[0, \sigma]},$$

where  $C_0$  stands for a constant depending only on the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$ . We have also the same estimates for  $P_0, P_1, P_2$ . Thus for  $\sigma > 0$  small enough, says  $\sigma \leq \frac{1}{4C_0^2}$ , one has

$$\|Pv - P\tilde{v}\|_{[0, \sigma]} \leq \frac{1}{2} \|v - \tilde{v}\|_{[0, \sigma]}, \quad \forall v, \tilde{v} \in C(\sigma, M). \quad (4.6)$$

Note that  $M \geq 2\|P(0)\|_{[0, 1]} \geq 2\|P(0)\|_{[0, \sigma]}$ . Thus it follows from (4.6) that  $P$  maps from  $C(\sigma, M)$  into itself and is a contraction.  $\square$

To finish the proof of Theorem 1.1, we prove that the solution can be extended uniquely by prolongation whenever  $h_u(\sigma) > h_0(\sigma)$ . Proposition 3.1 allows us consider the problem of finding  $v = (v_0, v_1, v_2, v_3)$  instead of the problem of finding  $(h_0, h_u, T, S)$ . Assume that  $v = (v_0, v_1, v_2, v_3)$  is a solution of the system (3.5), (3.8), (3.11) and (3.12) in  $[0, \sigma]$  such that  $h_u(t) > h_0(t)$  for all  $t \in [0, \sigma]$ . From such solution  $v$ , we can extend naturally  $(h_0, h_u, T, S)$  in  $[0, \sigma]$  by using the scheme of the proof of Proposition 3.1. We now want to check that  $(h_0(\sigma), h_u(\sigma), T(\cdot, \sigma), S(\cdot, \sigma))$  satisfy (H1)-(H3).

In fact, the condition (H1) is automatically satisfied. For (H2), the continuity of  $T_x(\cdot, \sigma)$  in  $(-\infty, h_0(\sigma)) \cup (h_0(\sigma), h_u(\sigma))$  is guaranteed by (3.7) and (3.10), which still hold for  $t = \sigma$ . The behavior of  $T$  at the interface is also insured by the relationships  $T_x(h_0(\sigma)-, \sigma) = v_1(\sigma)$ ,

$T_x(h_0(\sigma)+, \sigma) = v_2(\sigma)$ ,  $T_x(h_u(\sigma)-, \sigma) = v_3(\sigma)$ ,  $T(h_0(\sigma), \sigma) = T_0(\sigma)$  and  $T(h_u(\sigma)-, \sigma) = 0$ . Moreover, we deduce from (3.7) that

$$\limsup_{x \rightarrow -\infty} |T_x(x, \sigma)| = \limsup_{x \rightarrow -\infty} \left| \int_{-\infty}^{h_0(0)} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi \right| \leq \sup_{h_0(0) > \xi > -\infty} |T_\xi^0(\xi)|$$

and the boundness of  $T_x$  follows. Here we have used a property of the Green's function

$$\int_{-\infty}^{\infty} |G_2(x, t; \xi, 0)| d\xi = 1.$$

Thus (H2) is indeed true. Similarly, (H3) holds.

By considering  $(h_0(\sigma), h_u(\sigma), T(\cdot, \sigma), S(\cdot, \sigma))$  as the new initial condition, we use Proposition 4.1 to extend uniquely the solution  $v = (v_0, v_1, v_2, v_3)$  of the system (3.5), (3.8), (3.11) and (3.12) in  $[0, \sigma^*]$  such that  $h_u(t) > h_0(t)$  for all  $t \in (0, \sigma^*)$  for some  $\sigma^* > \sigma$ . Finally, it follows from Proposition 3.1 that the extended solution  $v$  gives the extended solution  $(h_0, h_u, T, S)$  of the system (1.1)-(1.8) in  $(0, \sigma^*)$  corresponding to the initial conditions  $(h_0^0, h_u^0, T^0, S^0)$ . The proof is completed.

**Remark 4.1.** We may also replace the infinite domain  $h_u(0) > x > -\infty$  of the initial conditions by a finite interval  $h_u^0 > x > L$  where  $L$  is a deep point in the ocean satisfying  $L < h_0(t)$  for  $0 \leq t \leq \sigma$ . However, in this case it is necessary to require the history information  $(T(L, \cdot), S(L, \cdot))$ ,  $0 < t < \sigma$ , at the point  $L$ . We can find the same result by using the Green's function for the half-plane  $x > L$ ,

$$K_j(x, t; \xi, \tau) = G_j(x, t; \xi, \tau) - G_j(2L - x, t; \xi, \tau), \quad j = 1, 2, 3.$$

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