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Ice formation in the Arctic during summer: false-bottoms

Phan Thanh Nam^{a,b}, Pham Ngoc Dinh Alain^{a*}
Pham Hoang Quan^b and Dang Duc Trong^b

^a*Mathematics Department, MAPMO UMR 6628, BP 67-59, 45067 Orleans cedex, France*

^b*Department of Mathematics, HoChiMinh City National University, Viet Nam*

Abstract

The only source of ice formation in the Arctic during summer is a layer of ice called false-bottoms between an under-ice melt pond and the underlying ocean. Of interest is to give a mathematical model in order to determine the simultaneous growth and ablation of false-bottoms, which is governed by both of heat fluxes and salt fluxes. In one dimension, this problem may be considered mathematically as a two-phase Stefan problem with two free boundaries. Our main result is to prove the existence and uniqueness of the solution from the initial condition.

Mathematics Subject Classification 2000: 35R35, 35Q35.

Keywords: False-bottoms, Free boundary problem, Green's function, Contraction Mapping Principle.

1 Introduction

There are some different reservoirs of fresh water in the Arctic during summer (see, e.g., *Eicken et al.* [2]). First, melt water collects in surface melt pond (melting under the sun) which is the most important reservoir. Second, this melt water can percolate into the ice matrix to form an *under-ice melt pond* (see [5] for more detail). At the interface between this fresh water and the underlying salt water, double-diffusive convection of heat and salt leads to the formation of a layer of ice called *false-bottoms* (see Figure 1 below). Very early, *Nansen* [8] in 1897 noted that this is the only source of forming new ice in the Arctic during the summer. This phenomenon has been considered for a long time by many authors (see, e.g. [1, 2, 5, 6, 7, 9, 10]). However, it has been considered in geophysical view-point based on practical experiments rather than rigorously mathematical formulations.

One of the most interesting ones is the simultaneous growth and ablation of false-bottoms, which is governed by both of heat fluxes and salt fluxes. The ablation of the sea-ice interface is caused by dissolution rather than by melting. Note that salt water has the double properties: it does not freeze even for temperature less than 0°C , and it dissolves ice when it is in contact with ice. *McPhee et al.* [7] emphasized that properly describing heat and salt flux at the ice-ocean interface is essential for understanding and modeling the false-bottoms, and in particular without the double diffusion at this interface false bottoms would be so short-lived. The growth of the upper interface between a false bottom and a under-ice melt pond is governed by the purely thermodynamic condition at the interface.

*Author correspondence: alain.pham@univ-orleans.fr

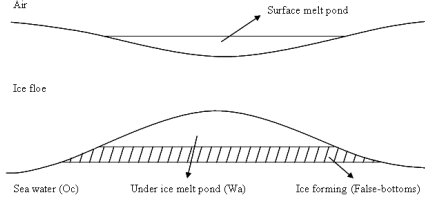


Figure 1: Ice formation in the Arctic during summer.

Recently, in 2003, *Notz et al.* [9] gave a model simulating successfully the simultaneous growth and ablation of false-bottoms. They formulated mathematically the problem by a system of partial differential equations and solved them numerically by using a numerical routine in Mathematica. Although their numerical result fits quite well to early experimental data from *Martin and Kauffman* [6], a rigorous proof of the existence and uniqueness of the solution for the system of equations is still unavailable. Our aim in this paper is to give a such a mathematical proof. More precisely, we shall represent the problem explicitly by a system of partial differential equations associated with free boundary conditions similar to [9], and then show that the system has a unique solution from given initial conditions.

Now we consider a one-dimensional model describing the simultaneous growth and ablation of the ice of false-bottoms. Here we have three environments: the ocean (Oc), the ice of false-bottoms (Fb) and the fresh water (Wa). Denote by $T(x, t)$, $S(x, t)$ the temperature and the salinity, and denote by $h_0(t)$, $h_u(t)$ the free boundaries at the interfaces ice-ocean (Fb-Oc) and ice-water (Fb-Wa), respectively (see Figure 2 below).

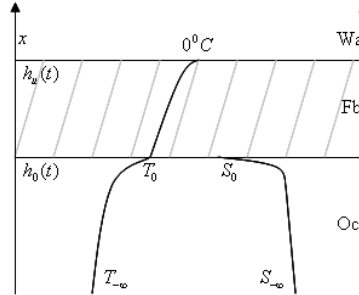


Figure 2: One-dimensional model.

At the interface Fb-Oc, we apply the first principle of thermodynamics, (i.e. variation of energy = variation of heat flux through the interface)

$$\Delta U = \Delta \Phi.$$

The net amount of heat transferred through the interface Fb-Oc in a section s is equal to

$$\Delta U = -dh_0 \rho_I L_f s,$$

where ρ_I is the density of the ice and L_f is the latent heat of fusion. On the other hand, the difference of the heat fluxes through a section s in the ice and the ocean during a time dt is

$$\Delta \Phi = (-\lambda_I T_x(h_0(t)+, t) + \lambda_O T_x(h_0(t)-, t)) s dt,$$

where λ_I , λ_O are thermal conductivities of the ice and the ocean. Here the notations $h_0(t)+$ and $h_0(t)-$ stand for the right limit and the left limit at $x = h_0(t)$. Thus the law of

conservation of energy mentioned above, i.e. $\Delta U = \Delta \Phi$, leads to the Stefan condition for the heat balance at the interface

$$h'_0(t) = \tilde{\lambda}_I T_x(h_0(t)+, t) - \tilde{\lambda}_O T_x(h_0(t)-, t), \quad (1.1)$$

where

$$\tilde{\lambda}_I = \frac{\lambda_I}{\rho_I L_f} > 0, \quad \tilde{\lambda}_O = \frac{\lambda_O}{\rho_I L_f} > 0.$$

For simplicity, we can neglect the salt of the ice of false-bottoms. The water near the interface Fb-Oc is a mixture of melt water, which melts from the ice of false-bottoms, and sea water. This water freshens at the rate $S_0(t)h'_0(t)$, while salt diffuses into this water at the rate $-DS_x(h_0(t)-, t)$, where $S_0(t) = S(h_0(t)-, t)$ is the salinity of the ocean at the interface and $D > 0$ is the molecular diffusivity of salt in sea water. The balance of salt at this interface leads to the conservation condition

$$S_0(t)h'_0(t) = -DS_x(h_0(t)-, t). \quad (1.2)$$

Moreover, the interface temperature $T_0 = T(h_0(t), t)$ and salinity S_0 are connected via freezing-point relationship

$$T_0 + n_0 T'_0 = -m_0 S_0, \quad (1.3)$$

where m_0, n_0 are non-zero constants. The establishment of equations (1.1) and (1.2) bases on *Martin and Kauffman* [6] and *Notz et al.* [9]. In (1.3), if $n_0 = 0$ and $m_0 = 0.054^\circ\text{C psu}^{-1}$ then we obtain the approximation $T_0 \approx -m_0 S_0$, which was used in [6, 9]. However, as we shall show later, it is reasonable to assume that the salinity is dependent on the gradient of the temperature at the freezing point.

At the interface Fb-Wa, we use the simplified scheme presented by *Grenfell and Maykut* [4], in which the temperature of the water in the under-ice melt pond is kept at 0°C . In particular, it leads to the boundary condition at the interface

$$T(h_u(t)-, t) = 0. \quad (1.4)$$

Furthermore, due to the neglect of heat fluxes from the fresh water above the false-bottoms, a thermodynamic condition similar to (1.1) reduces to the Stefan condition at the upper surface

$$h'_u(t) = \tilde{\lambda}_I T_x(h_u(t)-, t). \quad (1.5)$$

Finally, we have the following diffusion equations for salt and heat in the ocean and in the ice

$$T_t = D_I T_{xx}, \quad h_u(t) > x > h_0(t), \quad (1.6)$$

$$T_t = D_O T_{xx}, \quad h_0(t) > x > -\infty, \quad (1.7)$$

$$S_t = DS_{xx}, \quad h_0(t) > x > -\infty, \quad (1.8)$$

where $D_I > 0$ and $D_O > 0$ are the thermal diffusivity. Here all mentioned constants are given.

Assume that the initial conditions $h_0(0) = h_0^0$, $h_u(0) = h_u^0$, $T(x, 0) = T^0(x)$ in $h_u(0) > x > -\infty$ and $S(x, 0) = S^0(x)$ in $h_0(0) > x > -\infty$ are given. The problem is of finding

from the initial conditions (h_0^0, h_u^0, T^0, S^0) a solution (h_0, h_u, T, S) of the system (1.1)-(1.8), where $T = T(x, t)$ in $t > 0$, $h_u(t) > x > -\infty$ and $S = S(x, t)$ in $t > 0$, $h_0(t) > x > -\infty$. This is a one-dimensional two-phase Stefan problem with two free boundaries.

The main result of the present paper is to prove the existence and uniqueness a local solution for the problem. Rigorously, we suppose that the initial conditions (h_0^0, h_u^0, T^0, S^0) satisfy the following three assumptions

- (H1) $h_0^0 < h_u^0$;
- (H2) $(T^0)_x$ is continuous and bounded in $x \in (-\infty, h_0^0]$ and $x \in [h_0^0, h_u^0]$; T^0 is continuous at $x = h_0^0$ and $T^0(h_u^0-) = 0$;
- (H3) S^0 is continuous and bounded in $x \in (-\infty, h_0^0]$.

Remark 1.1. *The condition (H1) means the ice layer of the false-bottom has already existed. It is of course a much more interesting problem to consider the behavior of the false-bottom at the starting time, which corresponds to condition $h_0^0 = h_u^0$, but it is extremely difficult. We mention that the difficulty does not only come from mathematical computation but also be recognized by physical experiments. For example, Notz et al. [9] simulated the experiment of Martin and Kauffman [6], in which they put fresh water at 0°C on top of salt water at its freezing point in order to simulate the evolution of a false-bottoms, but this model does not take salt transport through the false-bottom into account. Therefore they then started with a 5-cm layer of ice.*

Remark 1.2. *It is implicit from (1.1) and (1.5) that $T_x(\cdot, t)$ is continuous at $x = h_0(t)-$, $h_0(t)+$ and $h_u(t)-$. Therefore, it is reasonable to assume that $(T^0)_x$ is continuous also at $x = h_0^0-$, h_0^0+ and h_u^0- in (H2). Similarly, it is natural to assume that T^0 is continuous at $x = h_0^0$ and $T^0(h_u^0-) = 0$ due to (1.3) and (1.4).*

Let $\sigma > 0$. We say that (h_0, h_u, T, S) is a solution of the system (1.1)-(1.8) in $(0, \sigma)$ corresponding to the initial conditions (h_0^0, h_u^0, T^0, S^0) if the following four conditions holds

- (C1) $h_0(t)$ and $h_u(t)$ are continuously differentiable in $t \in [0, \sigma)$; $h_0(0) = h_0^0 < h_u^0 = h_u(0)$ and $h_0(t) < h_u(t)$ in $t \in (0, \sigma)$;
- (C2) T_t, T_{xx} is continuous in $t \in (0, \sigma)$, $x \in (-\infty, h_0(t)) \cup (h_0(t), h_u(t))$; $T_x(\cdot, t)$ is continuous at $x = h_0(t)-$, $x = h_0(t)+$ and $x = h_u(t)-$; $T(\cdot, t)$ is continuous at $x = h_0(t)$; $T(x, \cdot)$ is continuous at $t = 0+$ and $T(x, 0+) = T^0(x)$ in $x \in (-\infty, h_u^0)$;
- (C3) S_t, S_{xx} is continuous in $t \in (0, \sigma)$, $x \in (-\infty, h_0(t))$; $S_x(\cdot, t)$ is continuous at $x = h_0(t)-$; $S(x, \cdot)$ is continuous at $t = 0$ and $S(x, 0+) = S^0(x)$ in $x \in (-\infty, h_0^0)$;
- (C4) Equations (1.1)-(1.8) hold in $t \in (0, \sigma)$.

In general, the conditions (C1)-(C4) assure a classical solution for the system of differential equations (1.1)-(1.8). Our main result is as follows.

Theorem 1.1. *Assume that (h_0^0, h_u^0, T^0, S^0) satisfy (H1)-(H3). Then there is a unique solution (h_0, h_u, T, S) of the system (1.1)-(1.8) in $(0, \sigma)$ corresponding to the initial conditions (h_0^0, h_u^0, T^0, S^0) for some $\sigma > 0$. Moreover, this solution can be extended uniquely whenever the condition $h_u(\sigma) > h_0(\sigma)$ still holds.*

Our proof follows the approach of A.Friedman [3] (Chapter 8), which dealt with a classical one-phase Stefan problem with one free boundary. More precisely, we first reduce the problem to solving a system of nonlinear Volterra integral equations of the second kind and then solve this system by contraction principle. The remainder of the paper is divided into three sections. Section 2 is devoted to some preliminary results on Green functions and Volterra integral equations. In Section 3 we shall reformulate the problem to a system of

nonlinear integral equations. In Section 4 we apply the contraction principle to prove the existence and uniqueness of a local solution for this system.

2 Preliminaries

Let $a(t), b(t)$ be continuously differentiable functions and $b(t) > a(t)$ for all $t \geq 0$. Let $\kappa > 0$ be a constant and let $u(x, t)$ be a solution of the diffusion equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad b(t) > x > a(t). \quad (2.1)$$

We introduce the Green's function of equation (2.1),

$$G(x, t; \xi, \tau) = \frac{H(t - \tau)}{2\sqrt{\pi\kappa(t - \tau)}} \exp\left(-\frac{(x - \xi)^2}{4\kappa(t - \tau)}\right),$$

where H is the Heaviside function,

$$H(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

The following lemma is useful to reformulate the differential equation (2.1) to a Volterra integral equations of the second kind.

Lemma 2.1. *If u is a solution of (2.1) then for $t > 0$ and $a(t) < x < b(t)$ we have*

$$\begin{aligned} u(x, t) &= \int_0^t G(x, t; b(\tau), \tau) [\kappa u_\xi(b(\tau)-, \tau) + u(b(\tau), \tau)b'(\tau)] d\tau \\ &\quad - \int_0^t G(x, t; a(\tau), \tau) [\kappa u_\xi(a(\tau)+, \tau) + u(a(\tau), \tau)a'(\tau)] d\tau \\ &\quad - \kappa \int_0^t G_\xi(x, t; b(\tau), \tau) u(b(\tau), \tau) d\tau + \kappa \int_0^t G_\xi(x, t; a(\tau), \tau) u(a(\tau), \tau) d\tau \\ &\quad + \int_{a(0)}^{b(0)} G(x, t; \xi, 0) u(\xi, 0) d\xi. \end{aligned}$$

Proof. Note that

$$G_\tau + \kappa G_{\xi\xi} = 0 \text{ for all } \tau < t, \quad \text{and } G(x, t; \xi, t-) = \delta(x - \xi), \quad (2.2)$$

where $\delta = H'$ is Dirac delta function.

Integrating the Green's identity, here $u = u(\xi, \tau)$,

$$\kappa \frac{\partial}{\partial \xi} \left(G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right) - \frac{\partial}{\partial \tau} (uG) = 0$$

over the domain $a(\tau) < \xi < b(\tau), 0 < \tau < t$, we will obtain the desired result because

$$\begin{aligned}
& \int_0^t \int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \xi} \left(G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right) d\xi d\tau = \int_0^t \left[G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right]_{\xi=a(\tau)}^{\xi=b(\tau)} d\tau \\
& = \int_0^t G(x, t; b(\tau), \tau) u_{\xi}(b(\tau)-, \tau) d\tau - \int_0^t G(x, t; a(\tau), \tau) u_{\xi}(a(\tau)+, \tau) d\tau \\
& \quad - \int_0^t G_{\xi}(x, t; b(\tau), \tau) u(b(\tau), \tau) d\tau + \int_0^t G_{\xi}(x, t; a(\tau), \tau) u(a(\tau), \tau) d\tau,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau} (uG) d\xi d\tau \\
& = \int_0^t \left\{ \frac{\partial}{\partial \tau} \left(\int_{a(\tau)}^{b(\tau)} uG d\xi \right) - [uG]_{\xi=b(\tau)} b'(\tau) + [uG]_{\xi=a(\tau)} a'(\tau) \right\} d\tau \\
& = \left[\int_{a(\tau)}^{b(\tau)} uG d\xi \right]_{\tau=0}^{\tau=t-} - \int_0^t [uG]_{\xi=b(\tau)} b'(\tau) d\tau + \int_0^t [uG]_{\xi=a(\tau)} a'(\tau) d\tau \\
& = u(x, t) - \int_{a(0)}^{b(0)} G(x, t; \xi, 0) u(\xi, 0) d\xi - \int_0^t G(x, t; b(\tau), \tau) u(b(\tau), \tau) b'(\tau) d\tau \\
& \quad + \int_0^t G(x, t; a(\tau), \tau) u(a(\tau), \tau) a'(\tau) d\tau.
\end{aligned}$$

Here we have made use the following equality, with $v = uG$,

$$\int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi = \frac{\partial}{\partial \tau} \left(\int_{a(\tau)}^{b(\tau)} v(\xi, \tau) d\xi \right) - v(b(\tau), \tau) b'(\tau) + v(a(\tau), \tau) a'(\tau).$$

In fact, in the case $a(t) \equiv 0$ we have

$$\begin{aligned}
& \int_0^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi = \int_0^\infty H(b(\tau) - \xi) \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi \\
&= \int_0^\infty \frac{\partial}{\partial \tau} [H(b(\tau) - \xi)v(\xi, \tau)] d\xi - \int_0^\infty \frac{\partial}{\partial \tau} [H(b(\tau) - \xi)] v(\xi, \tau) d\xi \\
&= \frac{\partial}{\partial \tau} \left(\int_0^\infty H(b(\tau) - \xi)v(\xi, \tau) d\xi \right) - \int_0^\infty \delta(b(\tau) - \xi) b'(\tau) v(\xi, \tau) d\xi \\
&= \frac{\partial}{\partial \tau} \left(\int_0^{b(\tau)} v(\xi, \tau) d\xi \right) - v(b(\tau), \tau) b'(\tau).
\end{aligned}$$

If $a(t)$ is not constant, we can write

$$\int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi = \int_0^{b(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi - \int_0^{a(\tau)} \frac{\partial}{\partial \tau} (v(\xi, \tau)) d\xi,$$

and apply the above result. This completes the proof. \square

Remark 2.1. *The result in Lemma 2.1 still holds for $b(t) \equiv +\infty$ or $a(t) \equiv -\infty$. For example, if $a(t) \equiv -\infty$ then the formula in Lemma 2.1 reduces to*

$$\begin{aligned}
u(x, t) &= \int_0^t G(x, t; b(\tau), \tau) [\kappa u_\xi(b(\tau), \tau) + u(b(\tau), \tau) b'(\tau)] d\tau \\
&\quad - \kappa \int_0^t G_\xi(x, t; b(\tau), \tau) u(b(\tau), \tau) d\tau + \int_{-\infty}^{b(0)} G(x, t; \xi, 0) u(\xi, 0) d\xi.
\end{aligned}$$

We shall need also an useful lemma giving the jump relation at the boundary (see Friedman [3], page 217, Lemma 1).

Lemma 2.2. *Let $p(t)$ be continuous and let $s(t) > 0$ satisfy the Lipschitz condition, $0 \leq t \leq \sigma$. Then, for $0 < t \leq \sigma$,*

$$\lim_{x \rightarrow s(t)^-} \kappa \int_0^t p(\tau) G_x(x, t; s(\tau), \tau) d\tau = \frac{1}{2} p(t) + \kappa \int_0^t p(\tau) G_x(s(t), t; s(\tau), \tau) d\tau.$$

Remark 2.2. *In applications of the above lemma, sometimes we need to note that $G_\xi = -G_x$. Moreover, for the right limit we have*

$$\lim_{x \rightarrow s(t)^+} \kappa \int_0^t p(\tau) G_x(x, t; s(\tau), \tau) d\tau = -\frac{1}{2} p(t) + \kappa \int_0^t p(\tau) G_x(s(t), t; s(\tau), \tau) d\tau.$$

Finally, we state a simple version of the uniqueness for a system of linear Volterra integral equations of the second kind.

Lemma 2.3. *Let $n \in \mathbb{N}$, $\sigma > 0$, $p > 1$, $q > 1$, $1/p + 1/q = 1$. Assume that $\tau \mapsto W_j(t, \tau)$ is measurable in $(0, t)$ for all $t \in (0, \sigma]$ and*

$$\int_0^t |W_j(t, \tau)|^p d\tau \leq \text{const.}, \quad \forall t \in (0, \sigma], j = \overline{1, n}.$$

Then the system

$$\Psi_i(t) = \sum_{j=1}^n \left(\int_0^t W_j(t, \tau) \Psi_j(\tau) d\tau \right), \quad \forall t \in (0, \sigma], j = \overline{1, n},$$

has a unique solution $\{\Psi_j\}_{j=1}^n = 0$ in $L^q(0, \sigma)$.

Proof. Using Holder inequality one has

$$\begin{aligned} |\Psi_i(t)| &\leq \sum_{j=1}^n \left| \int_0^t W_j(t, \tau) \Psi_j(\tau) d\tau \right| \\ &\leq \sum_{j=1}^n \left(\int_0^t |W_j(t, \tau)|^p d\tau \right)^{1/p} \left(\int_0^t |\Psi_j(\tau)|^q d\tau \right)^{1/q} \\ &\leq \text{const.} \sum_{j=1}^n \left(\int_0^t |\Psi_j(\tau)|^q d\tau \right)^{1/q}. \end{aligned}$$

Therefore,

$$\sum_{j=1}^n |\Psi_j(\tau)|^q \leq \text{const.} \sum_{j=1}^n \left(\int_0^t |\Psi_j(\tau)|^q d\tau \right) = \text{const.} \int_0^t \left(\sum_{j=1}^n |\Psi_j(\tau)|^q \right) d\tau,$$

and it follows from Gronwall's Lemma that $\sum_{j=1}^n |\Psi_j(\tau)|^q = 0$. Thus $\{\Psi_j\}_{j=1}^n = 0$. \square

Remark 2.3. *If, for example,*

$$|W_j(t, \tau)| \leq \frac{\text{const.}}{\sqrt{t - \tau}}, \quad j = \overline{1, n}.$$

then we can choose any $p \in (1, 2)$ in order to apply Lemma 2.3.

3 Reduction to integral equations

Denote by G_1, G_2, G_3 the Green function G in Lemma 2.1 corresponding to $\kappa = D, D_O, D_I$, respectively. We shall reformulate the problem (1.1)-(1.8) to a system of integral equations of time-dependent functions (v_0, v_1, v_2, v_3) where

$$v_0(t) = T_0'(t), \quad v_1(t) = T_x(h_0(t)-, t), \quad v_2(t) = T_x(h_0(t)+, t), \quad v_3(t) = T_x(h_u(t)-, t). \quad (3.1)$$

Assume that (h_0, h_u, T, S) is a solution of the system (1.1)-(1.8) in $(0, \sigma)$ corresponding to the initial conditions (h_0^0, h_u^0, T^0, S^0) for some $\sigma > 0$. Applying Lemma 2.1 to equation (1.8) with $h_0(t) > x > -\infty$ and using condition (1.2), one has

$$S(x, t) = -D \int_0^t G_{1\xi}(x, t; h_0(\tau), \tau) S_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_1(x, t; \xi, 0) S^0(\xi) d\xi. \quad (3.2)$$

In particular, we see from (3.2) that $S(x, t)$ is determined completely by h_0 and $S_0(t)$. Taking $x \rightarrow h_0(t)-$ in (3.2) and using the jump relation in Lemma 2.2, we get

$$S_0(t) = -2D \int_0^t G_{1\xi}(h_0(t), t; h_0(\tau), \tau) S_0(\tau) d\tau + 2 \int_{-\infty}^{h_0^0} G_1(h_0(t), t; \xi, 0) S^0(\xi) d\xi. \quad (3.3)$$

Note that for $v_0(t) = T_0'(t)$ the condition (1.3) can be rewritten as

$$T^0(h_0^0) + \int_0^t v_0(\tau) d\tau + n_0 v_0(t) = -m_0 S_0(t). \quad (3.4)$$

Here we have used $T_0(0) = T(h_0(0), 0) = T_0(h_0^0)$. We deduce from (3.3) and (3.4) the first integral equation

$$\begin{aligned} v_0(t) = & -\frac{1}{n_0} T^0(h_0^0) - \frac{1}{n_0} \int_0^t v_0(\tau) d\tau - \frac{2m_0}{n_0} \int_{-\infty}^{h_0^0} G_1(h_0(t), t; \xi, 0) S^0(\xi) d\xi \\ & - \frac{2D}{n_0} \int_0^t G_{1\xi}(h_0(t), t; h_0(\tau), \tau) \left[T^0(h_0^0) + \int_0^\tau v_0(s) ds + n_0 v_0(\tau) \right] d\tau. \end{aligned} \quad (3.5)$$

We next apply Lemma 2.1 to equation (1.7) for $h_0(t) > x > -\infty$ to get

$$\begin{aligned} T(x, t) = & \int_0^t G_2(x, t; h_0(\tau), \tau) [D_O v_1(\tau) + T_0(\tau) h_0'(\tau)] d\tau \\ & - D_O \int_0^t G_{2\xi}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0) T^0(\xi) d\xi, \end{aligned} \quad (3.6)$$

where $v_1(t) = T_x(h_0(t)-, t)$. We now differentiate both sides of (3.6) with respect to x , then

take $x \rightarrow h_0(t)-$. To go into the details, because $D_O G_{2\xi x} = -D_O G_{2\xi\xi} = G_{2\tau}$, we have

$$\begin{aligned}
& -D_O \int_0^t G_{2\xi x}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau = - \int_0^t G_{2\tau}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau \\
& = - \int_0^t \left[\frac{\partial}{\partial \tau} (G_2(x, t; h_0(\tau), \tau)) - G_{2\xi}(x, t; h_0(\tau), \tau) h_0'(\tau) \right] T_0(\tau) d\tau \\
& = G_2(x, t; h_0^0, 0) T_0(0) + \int_0^t G_2(x, t; h_0(\tau), \tau) T_0'(\tau) d\tau \\
& \quad - \int_0^t G_{2x}(x, t; h_0(\tau), \tau) T_0(\tau) h_0'(\tau) d\tau.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \int_{-\infty}^{h_0^0} G_{2x}(x, t; \xi, 0) T^0(\xi) d\xi = - \int_{-\infty}^{h_0^0} G_{2\xi}(x, t; \xi, 0) T^0(\xi) d\xi \\
& = -G_2(x, t; h_0^0, 0) T^0(h_0^0) + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi.
\end{aligned}$$

Thus it follows from (3.6) that

$$\begin{aligned}
T_x(x, t) &= D_O \int_0^t G_{2x}(x, t; h_0(\tau), \tau) v_1(\tau) d\tau \\
&\quad + \int_0^t G_2(x, t; h_0(\tau), \tau) v_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi \quad (3.7)
\end{aligned}$$

for all $h_0(t) > x > -\infty$. Here we have used the compatible condition $T_0(0) = T(h_0(0), 0) = T^0(h_0^0)$ and replaced $T_0'(\tau)$ by $v_0(\tau)$. Taking $x \rightarrow h_0(t)-$ in (3.7) and using Lemma 2.1 for the first term of the right hand side, we have the second integral equation

$$\begin{aligned}
v_1(t) &= 2D_O \int_0^t G_{2x}(h_0(t), t; h_0(\tau), \tau) v_1(\tau) d\tau \\
&\quad + 2 \int_0^t G_2(h_0(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{-\infty}^{h_0^0} G_2(h_0(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \quad (3.8)
\end{aligned}$$

Now we consider the heat distribution in false-bottoms. Applying Lemma 2.1 to equation

(1.6) for $h_u(t) > x > h_0(t)$ one has

$$\begin{aligned}
T(x, t) &= D_I \int_0^t G_3(x, t; h_u(\tau), \tau) v_3(\tau) d\tau \\
&\quad - \int_0^t G_3(x, t; h_0(\tau), \tau) [D_I v_2(\tau) + T_0(\tau) h'_0(\tau)] d\tau \\
&\quad + D_I \int_0^t G_{3\xi}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau + \int_{h_0^0}^{h_u^0} G_3(x, t; \xi, 0) T^0(\xi) d\xi, \quad (3.9)
\end{aligned}$$

where $v_2(t) = T_x(h_0(t)+, t)$ and $v_3(t) = T_x(h_u(t)-, t) = \frac{1}{\lambda_I} h'_u(t)$. Let us differentiate both sides of (3.9) with respect to x . We have

$$\begin{aligned}
&D_I \int_0^t G_{3\xi x}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau = \int_0^t G_{3\tau}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau \\
&= \int_0^t \left[\frac{\partial}{\partial \tau} (G_3(x, t; h_0(\tau), \tau)) - G_{3\xi}(x, t; h_0(\tau), \tau) h'_0(\tau) \right] T_0(\tau) d\tau \\
&= -G_3(x, t; h_0^0, 0) T_0(0) - \int_0^t G_3(x, t; h_0(\tau), \tau) T'_0(\tau) d\tau \\
&\quad + \int_0^t G_{3x}(x, t; h_0(\tau), \tau) T_0(\tau) h'_0(\tau) d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_{h_0^0}^{h_u^0} G_{3x}(x, t; \xi, 0) T^0(\xi) d\xi = - \int_{h_0^0}^{h_u^0} G_{3\xi}(x, t; \xi, 0) T^0(\xi) d\xi \\
&= -G_3(x, t; h_u^0, 0) T^0(h_u^0-) + G_3(x, t; h_0^0, 0) T^0(h_0^0-) + \int_{h_0^0}^{h_u^0} G_3(x, t; \xi, 0) T'_\xi(\xi) d\xi.
\end{aligned}$$

Using the compatible conditions $T^0(h_u^0-) = 0$ and $T_0(0) = T^0(h_0^0)$ and replacing $T'_0(\tau)$ by $v_0(\tau)$, we find from (3.9) that

$$\begin{aligned}
T_x(x, t) &= D_I \int_0^t G_{3x}(x, t; h_u(\tau), \tau) v_3(\tau) d\tau - D_I \int_0^t G_{3x}(x, t; h_0(\tau), \tau) v_2(\tau) d\tau \\
&\quad - \int_0^t G_3(x, t; h_0(\tau), \tau) v_0(\tau) d\tau + \int_{h_0^0}^{h_u^0} G_3(x, t; \xi, 0) T'_\xi(\xi) d\xi \quad (3.10)
\end{aligned}$$

for all $h_u(t) > x > h_0(t)$. Taking $x \rightarrow h_0(t)+$ and $x \rightarrow h_u(t)-$ in (3.10) and using Lemma 2.2 again we obtain the last two integral equations

$$\begin{aligned} v_2(t) &= 2D_I \int_0^t G_{3x}(h_0(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_0(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_0(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_0(t), t; \xi, 0) T_\xi^0(\xi) d\xi, \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} v_3(t) &= 2D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_u(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \end{aligned} \quad (3.12)$$

Note that due to the interface condition (1.1) and (1.5), one has

$$h_0(t) = h_0^0 + \tilde{\lambda}_I \int_0^t v_2(\tau) d\tau - \tilde{\lambda}_O \int_0^t v_1(\tau) d\tau. \quad (3.13)$$

and

$$h_u(t) = h_u^0 + \tilde{\lambda}_I \int_0^t v_3(\tau) d\tau. \quad (3.14)$$

Therefore the equations (3.5), (3.8), (3.11) and (3.12) form a system of nonlinear integral equations of the form $v = Pv$, where $v = (v_0, v_1, v_2, v_3)$ and $Pv = (P_0v, P_1v, P_2v, P_3v)$. Note that the time-dependent function $v = (v_0, v_1, v_2, v_3)$ is first only continuous in $(0, \sigma)$ by its definition in (3.1). However, because it is a solution of the system of integral equations (3.5), (3.8), (3.11) and (3.12), it is indeed continuous in $[0, \sigma]$. We thus have proved the direct part of the following statement.

Proposition 3.1. *The problem of finding a solution (h_0, h_u, T, S) of (1.1)-(1.8) in $(0, \sigma)$ is equivalent to the problem of finding a continuous solution $v = (v_0, v_1, v_2, v_3)$ in $[0, \sigma]$ for the system (3.5), (3.8), (3.11) and (3.12) such that $h_u(t) > h_0(t)$ in $(0, \sigma)$, where h_0 and h_u are defined by (3.13) and (3.14).*

We of course just need to prove the converse part of the statement.

Proof. Suppose that $v = (v_0, v_1, v_2, v_3)$ is a continuous solution in $[0, \sigma]$ for the system (3.5), (3.8), (3.11) and (3.12) such that $h_u(t) > h_0(t)$ in $(0, \sigma)$, h_0 and h_u are defined by (3.13) and (3.14). We shall now recover the solution (h_0, h_u, T, S) of (1.1)-(1.8).

Since h_0 and h_u have already been defined by (3.13) and (3.14), it remains to determine T and S . We define $T_0(t)$ by

$$T_0(t) = T^0(h_0^0) + \int_0^t v_0(\tau) d\tau. \quad (3.15)$$

and determine naturally T by (3.6) and (3.9). Similarly, we define S_0 by (3.4) and determine S by (3.2). We need to prove that (h_0, h_u, T, S) satisfies (C1)-(C4).

Step 1. We first prove (C1)-(C3) except the behavior of T and S at the interfaces $x = h_0(t)$ and $x = h_u(t)-$. In fact, the condition (C1) follows the definition of h_0 in (3.13), the definition of h_u in (3.14), and the condition $h_u(t) > h_0(t)$ in definition of the solution $v = (v_0, v_1, v_2, v_3)$.

Due to definition of S in the integral forms (3.2), S_t and S_{xx} are continuous in $t \in (0, \sigma)$, $x \in (-\infty, h_0(t))$. Moreover, getting $t \rightarrow 0+$ in (3.2) and using $\lim_{t \rightarrow 0+} G_1(x, t; \xi, 0) = \delta(x - \xi)$, we obtain the initial condition $S(x, 0+) = S^0(x)$ for all $x \in (-\infty, h_0^0)$.

Similarly, due to definition of T in (3.6) and (3.9), T_t and T_{xx} are continuous in $t \in (0, \sigma)$, $x \in (-\infty, h_0(t)) \cup (h_0(t), h_u(t))$, and $T(x, 0+) = T^0(x)$ for all $x \in (-\infty, h_0^0) \cup (h_0^0, h_u^0)$.

Step 2. We next check three diffusion equations (1.6)-(1.8). To prove (1.8) from the definition of S in (3.2), we simply verify that that each term in the right-hand side of (3.2) is a homogeneous solution of the operator $(\partial/\partial t - D\partial^2/\partial x^2)$ in $t \in (0, \sigma)$, $x \in (-\infty, h_0(t))$. In fact, for the second term one it follows from the properties of Green function that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - D \frac{\partial^2}{\partial x^2} \right) \left(\int_{-\infty}^{h_0^0} G_1(x, t; \xi, 0) S^0(\xi) d\xi \right) \\ &= \int_{-\infty}^{h_0^0} \left(\frac{\partial}{\partial t} - D \frac{\partial^2}{\partial x^2} \right) (G_1(x, t; \xi, 0)) S^0(\xi) d\xi = 0. \end{aligned}$$

For the first term, we have to be more careful because in general differentiating with respect to t a function of the form $t \mapsto \int_0^t K(t, \tau) d\tau$ may cause a jump,

$$\frac{d}{dt} \left(\int_0^t K(t, \tau) d\tau \right) = \lim_{\tau \rightarrow t-} K(t, \tau) + \int_0^t K_t(t, \tau) d\tau.$$

However, in this case the jump $\lim_{\tau \rightarrow t-} K(t, \tau)$ vanishes since

$$\lim_{\tau \rightarrow t-} G_1(x, t; \xi, \tau) = \lim_{\tau \rightarrow t-} G_{1\xi}(x, t; \xi, \tau) = 0, \quad \forall x \neq \xi.$$

Therefore, the first term can be treat similarly to the second term. Thus (1.8) holds. The proofs for (1.6) and (1.7) are the same.

Step 3. We now prove that $S(h_0(t)-, t) = S_0(t)$, which in particular accomplish (C3), and verify the Stefan condition (1.2).

In fact, (3.3) holds due to (3.4) and (3.5). On the other hand, taking $x \rightarrow h_0(t)-$ in (3.2) and using Lemma 2.2 one has

$$\begin{aligned} S(h_0(t)-, t) &= \frac{1}{2}S_0(t) - D \int_0^t G_{1\xi}(h_0(t), t; h_0(\tau), \tau) S_0(\tau) d\tau \\ &\quad + \int_{-\infty}^{h_0^0} G_1(h_0(t), t; \xi, 0) S^0(\xi) d\xi. \end{aligned}$$

It follows from the latter equation and (3.3) that $S(h_0(t)-, t) = S_0(t)$.

Applying Lemma 2.1 to equation (1.8) for $h_0(t) > x > -\infty$ and using $S(h_0(t)-, t) = S_0(t)$, we have

$$\begin{aligned} S(x, t) &= \int_0^t G_1(x, t; h_0(\tau), \tau) [DS_x(h_0(\tau)-, \tau) + S_0(\tau)h_0'(\tau)] d\tau \\ &\quad - D \int_0^t G_{1\xi}(x, t; h_0(\tau), \tau) S_0(\tau) d\tau + \int_{-\infty}^{h_0^0} G_1(x, t; \xi, 0) S^0(\xi) d\xi. \end{aligned}$$

Comparing the latter equation to the original definition of S in (3.2), we obtain

$$\int_0^t G_1(x, t; h_0(\tau), \tau) \Psi_1(\tau) d\tau = 0, \quad x \in (-\infty, h_0(t)), \quad (3.16)$$

where $\Psi_1(t) = DS_x(h_0(t)-, t) + S_0(t)h_0'(t)$. We shall deduce from (3.16) that $\Psi_1 \equiv 0$, which is equivalent to (1.2). Indeed, differentiating (3.16) with respect to x , then taking $x \rightarrow h_0(t)-$ and using Lemma 2.2 we get

$$\Psi_1(t) = -2 \int_0^t G_{1x}(h_0(t), t; h_0(\tau), \tau) \Psi_1(\tau) d\tau.$$

Note that this is a linear Volterra integral equation of the second kind and

$$|G_{1x}(h_0(t), t; h_0(\tau), \tau)| \leq \frac{\text{const.}}{\sqrt{t-\tau}}.$$

Therefore, it follows from Lemma 2.3 that $\Psi_1(t) = 0$.

Step 4. We prove that $T_x(h_0(t)-, t) = v_1(t)$, $T_x(h_0(t)+, t) = v_2(t)$ and $T_x(h_u(t)-, t) = v_3(t)$, which in particular imply the Stefan conditions (1.1) and (1.5). Note that we have already had from (3.15) that $v_0(t) = T_0'(t)$ and $T_0(0) = T^0(h_0^0)$.

To prove $T_x(h_u(t)-, t) = v_3(t)$ from the definition of T in (3.9), we use the same process of getting (3.12) from (3.9). In fact, differentiating both sides of (3.9) with respect to x and

then taking $x \rightarrow h_u(t)-$ we have

$$\begin{aligned} T_x(h_u(t)-, t) &= \frac{1}{2}v_3(t) + D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau)v_3(\tau)d\tau \\ &\quad - D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau)v_2(\tau)d\tau \\ &\quad - \int_0^t G_3(h_u(t), t; h_0(\tau), \tau)T'_0(\tau)d\tau + \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0)T_\xi^0(\xi)d\xi. \end{aligned}$$

Comparing the latter equation to (3.12) and using $v_0(\tau) = T'_0(\tau)$, we obtain $T_x(h_u(t)-, t) = v_3(t)$. By the same way we find that $T_x(h_0(t)+, t) = v_2(t)$ and $T_x(h_0(t)-, t) = v_1(t)$.

Step 5. Finally, we verify that $T(h_0(t), t) = T_0(t)$ and $T(h_u(t)-, t) = 0$ to accomplish (C2) and (C4).

Applying Lemma 2.1 to equation (1.7) for $h_0(t) > x > -\infty$ one has

$$\begin{aligned} T(x, t) &= \int_0^t G_2(x, t; h_0(\tau), \tau) [D_O T_x(h_0(\tau)-, \tau) + T(h_0(\tau)-, \tau)h'_0(\tau)] d\tau \\ &\quad - D_O \int_0^t G_{2\xi}(x, t; h_0(\tau), \tau)T(h_0(\tau)-, \tau)d\tau + \int_{-\infty}^{h_0^0} G_2(x, t; \xi, 0)T^0(\xi)d\xi. \end{aligned}$$

Comparing the latter equation to the original definition of T in (3.6) and using $T_x(h_0(t)-, t) = v_1(t)$, we obtain

$$\int_0^t [G_2(x, t; h_0(\tau), \tau)h'_0(\tau) - D_O G_{2\xi}(x, t; h_0(\tau), \tau)] \Psi_2(\tau)d\tau = 0, \quad (3.17)$$

where $\Psi_2(t) = T(h_0(t)-, t) - T_0(t)$. We now prove $\Psi_2 \equiv 0$ by using the same technique of dealing $\Psi_1 \equiv 0$. Taking $x \rightarrow h_u(t)-$ in (3.17) and using Lemma 2.2, we find that

$$\Psi_2(t) = 2 \int_0^t [G_2(x, t; h_0(\tau), \tau)h'_0(\tau) - D_O G_{2\xi}(x, t; h_0(\tau), \tau)] \Psi_2(\tau)d\tau.$$

Note that this is a linear Volterra integral equation of the second kind and

$$|G_2(x, t; h_0(\tau), \tau)h'_0(\tau) - D_O G_{2\xi}(x, t; h_0(\tau), \tau)| \leq \frac{\text{const.}}{\sqrt{t-\tau}}.$$

Therefore, it follows from Lemma 2.3 that $\Psi_2 \equiv 0$. Thus $T(h_0(t)-, t) = T_0(t)$.

We shall use the same argument, in fact a little more complicated one, to deduce that $T(h_0(t)+, t) = T_0(t)$ and $T(h_u(t), t) = 0$. Applying Lemma 2.1 to equation (1.6) for $h_u(t) >$

$x > h_0(t)$ and comparing to the original definition of T in (3.9), one has

$$\begin{aligned} & \int_0^t [G_3(x, t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(x, t; h_0(\tau), \tau)] \Psi_3(\tau) d\tau \\ &= \int_0^t [G_3(x, t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(x, t; h_u(\tau), \tau)] \Psi_4(\tau) d\tau, \end{aligned} \quad (3.18)$$

where $\Psi_3(\tau) = T(h_0(t)+, t) - T_0(t)$ and $\Psi_4(t) = T(h_u(t)-, t)$. Taking $x \rightarrow h_0(t)+$ and $x \rightarrow h_u(t)-$ in (3.18), we have

$$\begin{aligned} \Psi_3(t) &= 2 \int_0^t [G_3(h_0(t), t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(h_0(t), t; h_0(\tau), \tau)] \Psi_3(\tau) d\tau \\ &- 2 \int_0^t [G_3(h_0(t), t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(h_0(t), t; h_u(\tau), \tau)] \Psi_4(\tau) d\tau, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} -\Psi_4(t) &= -2 \int_0^t [G_3(h_u(t), t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(h_u(t), t; h_0(\tau), \tau)] \Psi_3(\tau) d\tau \\ &+ 2 \int_0^t [G_3(h_u(t), t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(h_u(t), t; h_u(\tau), \tau)] \Psi_4(\tau) d\tau \end{aligned} \quad (3.20)$$

Note that equations (3.19) and (3.20) form a system of linear Volterra integral equations of the second kind and

$$\begin{aligned} |G_3(h_u(t), t; h_0(\tau), \tau)h'_0(\tau) - D_I G_{3\xi}(h_u(t), t; h_0(\tau), \tau)| &\leq \frac{\text{const.}}{\sqrt{t-\tau}}, \\ |G_3(h_0(t), t; h_u(\tau), \tau)h'_u(\tau) - D_I G_{3\xi}(h_0(t), t; h_u(\tau), \tau)| &\leq \frac{\text{const.}}{\sqrt{t-\tau}}. \end{aligned}$$

We therefore deduce from Lemma 2.3 that $\Psi_3 \equiv \Psi_4 \equiv 0$. Thus $T(h_u(t)-, t) = 0$, i.e. (1.4), and $T(h_0(t), t) = T_0(t)$, which accomplishes (C2) and also implies (1.3) in viewing (3.4). The proof is completed. \square

Remark 3.1. We learned the technique of proving $\Psi_1 \equiv 0$ by (3.16) from Friedman [3]. However, there is a minor mistake in [3]. The formula (1.28) in page 221, i.e.

$$\int_0^t G_\xi(x, t; s(\tau), \tau)u(s(\tau), \tau) d\tau = 0,$$

should be

$$\int_0^t [G_\xi(x, t; s(\tau), \tau) - G(x, t; s(\tau), \tau)s'(\tau)] u(s(\tau), \tau) d\tau = 0.$$

Fortunately, the conclusion $u(s(t), t) = 0$ still holds by the same argument.

4 Solving integral equations

Now we accomplish the proof of Theorem 1.1. We want to prove the system (3.5), (3.8), (3.11) and (3.12) has a unique local solution $v = (v_0, v_1, v_2, v_3)$, and this solution can be extended uniquely by prolongation whenever the condition $h_u(t) > h_0(t)$ still holds. Note that this system is of the form $v = Pv := (P_0v, P_1v, P_2v, P_3v)$, where P_0v, P_1v, P_2v and P_3v are the right-hand sides of (3.5), (3.8), (3.11) and (3.12) respectively.

Denote by $C(\sigma, M)$ the space of functions $v = (v_0, v_1, v_2, v_3)$ continuous in $[0, \sigma]$ and

$$\|v\|_{[0, \sigma]} := \max_{t \in [0, \sigma], j=0,3} |v_j(t)| \leq M.$$

When $v = 0$ then $P(0)$ depends only on the initial conditions (h_0^0, h_u^0, T^0, S^0) and is continuous on $[0, \infty)$. Fix $M > 0$ large enough, says $M \geq 2 \|P(0)\|_{[0,1]}$.

Proposition 4.1. *There exists $\sigma > 0$ depending on (h_0^0, h_u^0, T^0, S^0) such that P maps $C(t_0, M)$ into itself and is a contraction.*

Note that $C(\sigma, M)$ is a complete metric space. Therefore it follows from Proposition 4.1 and the Contraction Mapping Principle that P has a unique fixed point v in $C(\sigma, M)$. It gives also the unique solution (h_0, h_u, T, S) of (1.1)-(1.8) in $(0, \sigma)$ due to Proposition 3.1.

Proof. First at all, let us consider $\sigma > 0$ small enough, says

$$\sigma \leq \frac{h_u^0 - h_0^0}{2(2\tilde{\lambda}_I + \tilde{\lambda}_O)M}.$$

Then for $v \in C(\sigma, M)$, it follows from (3.13)-(3.14) that

$$\begin{aligned} h_u(t) - h_0(\tau) &\geq h_u^0 - h_0^0 - |h_u(t) - h_u(0)| - |h_0(\tau) - h_0(0)| \\ &\geq h_u^0 - h_0^0 - \tilde{\lambda}_I M t - (\tilde{\lambda}_O + \tilde{\lambda}_I) M \tau \geq \frac{h_u^0 - h_0^0}{2} > 0 \end{aligned} \quad (4.1)$$

for all $0 \leq t \leq \sigma$. In particular $h_u(t) > h_0(t)$ for all $t \in [0, \sigma]$ and hence P is well-defined on $C(\sigma, M)$. We now estimate $\|Pv - P\tilde{v}\|_{[0, \sigma]}$ for $v, \tilde{v} \in C(\sigma, M)$. In what follows, denote by \tilde{h}_u and \tilde{h}_0 the functions given by (3.13)-(3.14) where v is replaced by \tilde{v} . For simplicity, we shall always denote by $C_0 > 0$ an universal constant depending only on the initial conditions (h_0^0, h_u^0, T^0, S^0) . We shall go to the details for P_3 , and P_0, P_1, P_2 can be treated by the same way. Recall that

$$\begin{aligned} (P_3v)(t) &= 2D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_u(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \end{aligned}$$

We need some preliminary estimates related to the Green's function G_3 .

Lemma 4.1. For all $v, \tilde{v} \in C(\sigma, M)$, $0 \leq \tau < t \leq \sigma$ and $\xi \in \mathbb{R}$, we have

$$\left| G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \right| \leq \frac{C_0}{\sqrt{t-\tau}} \|v - \tilde{v}\|_{[0, \sigma]}, \quad (4.2)$$

$$\left| G_3(h_u(t), t; h_0(\tau), \tau) - G_3(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \right| \leq \frac{C_0}{\sqrt{t-\tau}} \|v - \tilde{v}\|_{[0, \sigma]}, \quad (4.3)$$

$$\left| G_{3x}(h_u(t), t; h_0(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \right| \leq \frac{C_0}{\sqrt{t-\tau}} \|v - \tilde{v}\|_{[0, \sigma]}, \quad (4.4)$$

$$\begin{aligned} & \left| G_{3x}(h_u(t), t; \xi, \tau) - G_{3x}(\tilde{h}_u(t), t; \xi, \tau) \right| \leq \\ & \leq C_0 \left[\exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right) + \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right) \right] \|v - \tilde{v}\|_{[0, \sigma]}. \end{aligned} \quad (4.5)$$

Proof. Considering

$$G_{3x}(h_u(t), t; h_u(\tau), \tau) = -\frac{h_u(t) - h_u(\tau)}{4\sqrt{\pi D_I^3(t-\tau)^3}} \exp\left(-\frac{(h_u(t) - h_u(\tau))^2}{4D_I(t-\tau)}\right)$$

as a function of one-variable $(h_u(t) - h_u(\tau))$ and using Lagrange formula one has

$$\begin{aligned} & G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \\ & = -\frac{[(h_u(t) - h_u(\tau)) - (\tilde{h}_u(t) - \tilde{h}_u(\tau))]}{4\sqrt{\pi D_I^3(t-\tau)^3}} \exp\left(-\frac{\theta^2}{4D_I(t-\tau)}\right) \left(1 - \frac{2\theta^2}{4D_I(t-\tau)}\right) \end{aligned}$$

for some θ between $(h_u(t) - h_u(\tau))$ and $(\tilde{h}_u(t) - \tilde{h}_u(\tau))$. Thus (4.2) follows

$$|(h_u(t) - h_u(\tau)) - (\tilde{h}_u(t) - \tilde{h}_u(\tau))| \leq \tilde{\lambda}_I(t-\tau) \|v - \tilde{v}\|_{[0, \sigma]},$$

and the elementary inequality

$$e^{-z}|1-2z| \leq \frac{|1-2z|}{1+z} \leq 2, \quad \text{where } z = \frac{\theta^2}{4D_I(t-\tau)} \geq 0.$$

To prove (4.3) we shall use the identity

$$\begin{aligned} & G_3(h_u(t), t; h_0(\tau), \tau) - G_3(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \\ & = -\frac{[(h_u(t) - h_0(\tau)) - (\tilde{h}_u(t) - \tilde{h}_0(\tau))]\theta_1}{4\sqrt{\pi D_I^3(t-\tau)^3}} \exp\left(-\frac{\theta_1^2}{4D_I(t-\tau)}\right) \end{aligned}$$

for some θ_1 between $(h_u(t) - h_0(\tau))$ and $(\tilde{h}_u(t) - \tilde{h}_0(\tau))$. Note from (4.1) that $|\theta_1| \geq \frac{h_u^0 - h_0^0}{2}$. Therefore (4.3) follows

$$|(h_u(t) - h_0(\tau)) - (\tilde{h}_u(t) - \tilde{h}_0(\tau))| \leq (2\tilde{\lambda}_I + \tilde{\lambda}_O)\sigma \|v - \tilde{v}\|_{[0, \sigma]}$$

and the elementary inequality

$$e^{-z}z \leq 1, \quad \text{where } z = \frac{\theta_1^2}{4D_I(t-\tau)} \geq 0.$$

We can also prove (4.4) by the same argument. In fact, we write

$$\begin{aligned} & G_{3x}(h_u(t), t; h_0(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_0(\tau), \tau) \\ &= -\frac{[(h_u(t) - h_0(\tau)) - (\tilde{h}_u(t) - \tilde{h}_0(\tau))]}{4\sqrt{\pi D_I^3(t - \tau)^3}} \exp\left(-\frac{\theta_2^2}{4D_I(t - \tau)}\right) \left(1 - \frac{2\theta_2^2}{4D_I(t - \tau)}\right) \end{aligned}$$

for some θ_2 between $(h_u(t) - h_0(\tau))$ and $(\tilde{h}_u(t) - \tilde{h}_0(\tau))$ and use

$$e^{-z}z|1 - 2z| \leq \frac{z|1 - 2z|}{1 + z + \frac{1}{2}z^2} \leq 4, \quad \text{where } z = \frac{\theta_2^2}{4D_I(t - \tau)} \geq 0.$$

Finally, for the last inequality (4.5) we write

$$G_3(h_u(t), t; \xi, 0) - G_3(\tilde{h}_u(t), t; \xi, 0) = -\frac{[h_u(t) - \tilde{h}_u(t)]\theta_3}{4\sqrt{\pi D_I^3 t^3}} \exp\left(-\frac{\theta_3^2}{4D_I t}\right)$$

for some θ_3 between $(h_u(t) - \xi)$ and $(\tilde{h}_u(t) - \xi)$. Note that

$$\exp\left(-\frac{\theta_3^2}{8D_I t}\right) \leq \max\left\{\exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right), \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right)\right\}.$$

Thus (4.5) follows

$$e^{-z}\sqrt{z} \leq \frac{\sqrt{z}}{1 + z} \leq \frac{1}{2}, \quad \text{where } z = \frac{\theta_3^2}{8D_I t} \geq 0,$$

and the estimate $|h_u(t) - \tilde{h}_u(t)| \leq \tilde{\lambda}_I t \|v - \tilde{v}\|_{[0, \sigma]}$. \square

We now already to estimate $\|P_3 v - P_3 \tilde{v}\|_{[0, \sigma]}$ for $v, \tilde{v} \in C(\sigma, M)$. Recall that P_3 is the sum of four terms

$$\begin{aligned} (P_3 v)(t) &= 2D_I \int_0^t G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) d\tau - 2D_I \int_0^t G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) d\tau \\ &\quad - 2 \int_0^t G_3(h_u(t), t; h_0(\tau), \tau) v_0(\tau) d\tau + 2 \int_{h_0^0}^{h_u^0} G_3(h_u(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \end{aligned}$$

For the first term of P_3 , using (4.2) we have

$$\begin{aligned} & |G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \tilde{v}_3(\tau)| \\ &\leq |G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau)| \cdot |v_3(\tau)| \\ &\quad + |G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau)| \cdot |v_3(\tau) - \tilde{v}_3(\tau)| \\ &\leq \frac{C_0 M}{\sqrt{t - \tau}} \|v - \tilde{v}\|_{[0, \sigma]} + \frac{C_0}{\sqrt{t - \tau}} \|v - \tilde{v}\|_{[0, \sigma]} = \frac{C_0(M + 1)}{\sqrt{t - \tau}} \|v - \tilde{v}\|_{[0, \sigma]} \end{aligned}$$

for all $0 \leq \tau \leq t \leq \sigma$. Therefore

$$\begin{aligned} & \int_0^t \left| G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) - G_{3x}(\tilde{h}_u(t), t; \tilde{h}_u(\tau), \tau) \tilde{v}_3(\tau) \right| d\tau \\ &\leq C_0(M + 1)\sqrt{\sigma} \|v - \tilde{v}\|_{[0, \sigma]}, \quad t \in [0, \sigma]. \end{aligned}$$

We have also the same estimate for the second term and the third term of P_3 due to (4.3) and (4.4). For the last term of P_3 , it follows from (4.5) and the boundness of T_x^0 that

$$\begin{aligned} & \int_{h_0^0}^{h_u^0} \left| G_3(h_u(t), t; \xi, 0) - G_3(\tilde{h}_u(t), t; \xi, 0) \right| T_\xi^0(\xi) d\xi \\ & \leq C_0 \|v - \tilde{v}\|_{[0, \sigma]} \int_{h_0^0}^{h_u^0} \left[\exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right) + \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right) \right] d\xi. \end{aligned}$$

Note that by changing of variables

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\exp\left(-\frac{(h_u(t) - \xi)^2}{8D_I t}\right) + \exp\left(-\frac{(\tilde{h}_u(t) - \xi)^2}{8D_I t}\right) \right] d\xi \\ & = 2 \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{8D_I t}\right) d\xi = 2\sqrt{\pi 8D_I t}. \end{aligned}$$

Thus

$$\int_{h_0(0)}^{h_u(0)} \left| G_3(h_u(t), t; \xi, 0) - G_3(\tilde{h}_u(t), t; \xi, 0) \right| T_\xi^0(\xi) d\xi \leq 2C_0 \sqrt{\pi 8D_I \sigma} \|v - \tilde{v}\|_{[0, \sigma]}.$$

In summary, we have

$$\|P_3 v - P_3 \tilde{v}\|_{[0, \sigma]} \leq C_0 \sqrt{\sigma} \|v - \tilde{v}\|_{[0, \sigma]},$$

where C_0 stands for a constant depending only on the initial conditions (h_0^0, h_u^0, T^0, S^0) . We have also the same estimates for P_0, P_1, P_2 . Thus for $\sigma > 0$ small enough, says $\sigma \leq \frac{1}{4C_0^2}$, one has

$$\|Pv - P\tilde{v}\|_{[0, \sigma]} \leq \frac{1}{2} \|v - \tilde{v}\|_{[0, \sigma]}, \quad \forall v, \tilde{v} \in C(\sigma, M). \quad (4.6)$$

Note that $M \geq 2\|P(0)\|_{[0, 1]} \geq 2\|P(0)\|_{[0, \sigma]}$. Thus it follows from (4.6) that P maps from $C(\sigma, M)$ into itself and is a contraction. \square

To finish the proof of Theorem 1.1, we prove that the solution can be extended uniquely by prolongation whenever $h_u(\sigma) > h_0(\sigma)$. Proposition 3.1 allows us consider the problem of finding $v = (v_0, v_1, v_2, v_3)$ instead of the problem of finding (h_0, h_u, T, S) . Assume that $v = (v_0, v_1, v_2, v_3)$ is a solution of the system (3.5), (3.8), (3.11) and (3.12) in $[0, \sigma]$ such that $h_u(t) > h_0(t)$ for all $t \in [0, \sigma]$. From such solution v , we can extend naturally (h_0, h_u, T, S) in $[0, \sigma]$ by using the scheme of the proof of Proposition 3.1. We now want to check that $(h_0(\sigma), h_u(\sigma), T(\cdot, \sigma), S(\cdot, \sigma))$ satisfy (H1)-(H3).

In fact, the condition (H1) is automatically satisfied. For (H2), the continuity of $T_x(\cdot, \sigma)$ in $(-\infty, h_0(\sigma)) \cup (h_0(\sigma), h_u(\sigma))$ is guaranteed by (3.7) and (3.10), which still hold for $t = \sigma$. The behavior of T at the interface is also insured by the relationships $T_x(h_0(\sigma)-, \sigma) = v_1(\sigma)$,

$T_x(h_0(\sigma)+, \sigma) = v_2(\sigma)$, $T_x(h_u(\sigma)-, \sigma) = v_3(\sigma)$, $T(h_0(\sigma), \sigma) = T_0(\sigma)$ and $T(h_u(\sigma)-, \sigma) = 0$. Moreover, we deduce from (3.7) that

$$\limsup_{x \rightarrow -\infty} |T_x(x, \sigma)| = \limsup_{x \rightarrow -\infty} \left| \int_{-\infty}^{h_0(0)} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi \right| \leq \sup_{h_0(0) > \xi > -\infty} |T_\xi^0(\xi)|$$

and the boundness of T_x follows. Here we have used a property of the Green's function

$$\int_{-\infty}^{\infty} |G_2(x, t; \xi, 0)| d\xi = 1.$$

Thus (H2) is indeed true. Similarly, (H3) holds.

By considering $(h_0(\sigma), h_u(\sigma), T(\cdot, \sigma), S(\cdot, \sigma))$ as the new initial condition, we use Proposition 4.1 to extend uniquely the solution $v = (v_0, v_1, v_2, v_3)$ of the system (3.5), (3.8), (3.11) and (3.12) in $[0, \sigma^*]$ such that $h_u(t) > h_0(t)$ for all $t \in (0, \sigma^*)$ for some $\sigma^* > \sigma$. Finally, it follows from Proposition 3.1 that the extended solution v gives the extended solution (h_0, h_u, T, S) of the system (1.1)-(1.8) in $(0, \sigma^*)$ corresponding to the initial conditions (h_0^0, h_u^0, T^0, S^0) . The proof is completed.

Remark 4.1. We may also replace the infinite domain $h_u(0) > x > -\infty$ of the initial conditions by a finite interval $h_u^0 > x > L$ where L is a deep point in the ocean satisfying $L < h_0(t)$ for $0 \leq t \leq \sigma$. However, in this case it is necessary to require the history information $(T(L, \cdot), S(L, \cdot))$, $0 < t < \sigma$, at the point L . We can find the same result by using the Green's function for the half-plane $x > L$,

$$K_j(x, t; \xi, \tau) = G_j(x, t; \xi, \tau) - G_j(2L - x, t; \xi, \tau), \quad j = 1, 2, 3.$$

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