# Vector and scalar potentials, Poincaré's theorem and Korn's inequality 

Chérif Amrouche, Philippe G. Ciarlet, Patrick Ciarlet

## To cite this version:

Chérif Amrouche, Philippe G. Ciarlet, Patrick Ciarlet. Vector and scalar potentials, Poincaré's theorem and Korn's inequality. 2008. hal-00297289

HAL Id: hal-00297289
https://hal.science/hal-00297289
Preprint submitted on 15 Jul 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Vector and scalar potentials, Poincaré's theorem and Korn's inequality 

Chérif Amrouche ${ }^{\mathrm{a}}$, Philippe G. Ciarlet ${ }^{\mathrm{b}}$, Patrick Ciarlet, Jr. ${ }^{\text {c }}$<br>${ }^{\text {a }}$ Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'Université, 64000 Pau, France<br>${ }^{\mathrm{b}}$ Department of Mathematics, City University of Hong Kong, 83, Tat Chee Avenue, Kowloon, Hong Kong<br>${ }^{\text {c }}$ Laboratoire POEMS, UMR 2706 CNRS/ENSTA/INRIA, École Nationale Supérieure de Techniques Avancées, 32, Boulevard Victor, 75739 Paris Cedex 15, France<br>Received ${ }^{* * * * * ; ~ a c c e p t e d ~ a f t e r ~ r e v i s i o n ~}+++++$<br>Presented by Philippe G. Ciarlet


#### Abstract

In this Note, we present several results concerning vector potentials and scalar potentials in a bounded, not necessarily simply-connected, three-dimensional domain. In particular, we consider singular potentials corresponding to data in negative order Sobolev spaces. We also give some applications to Poincaré's theorem and to Korn's inequality.

\section*{Résumé}

Dans cette note, nous présentons plusieurs résultats concernant les potentiels vecteurs et les potentiels scalaires dans des domaines bornés tridimensionnels, éventuellement multiplement connexes. En particulier, on considère des potentiels singuliers correspondant à des données dans des espaces de Sobolev d'exposant négatif. On donne également des applications au théorème de Poincaré et à l'inégalité de Korn.


## 1. Weak versions of a classical theorem of Poincaré

In this work, we assume that $\Omega$ is a bounded open connected of $\mathbb{R}^{3}$ with a Lipschitz-continuous boundary. The notation $X^{\prime}<,>_{X}$ denotes a duality pairing between a topological space $X$ and its dual $X^{\prime}$. The letter $C$ denotes a constant that is not necessarily the same at its various occurrences.

Email addresses: cherif.amrouche@univ-pau.fr (Chérif Amrouche), mapgc@cityu.edu.hk (Philippe G. Ciarlet), patrick.ciarlet@ensta.fr (Patrick Ciarlet, Jr.).

Theorem 1 Let $\boldsymbol{f} \in H^{-m}(\Omega)^{3}$ for some integer $m \geq 0$. Then, the following assertions are equivalent:
(i) ${ }_{H^{-m}(\Omega)}<\boldsymbol{f}, \boldsymbol{\varphi}>_{H_{0}^{m}(\Omega)}=0$ for any $\boldsymbol{\varphi} \in V_{m}=\left\{\boldsymbol{\varphi} \in H_{0}^{m}(\Omega)^{3}\right.$; $\left.\operatorname{div} \boldsymbol{\varphi}=0\right\}$,
(ii) $H_{H^{-m}(\Omega)}<\boldsymbol{f}, \varphi>_{H_{0}^{m}(\Omega)}=0$ for any $\varphi \in \mathcal{V}=\left\{\varphi \in \mathscr{D}(\Omega)^{3} ; \operatorname{div} \varphi=0\right\}$,
(iii) There exists a distribution $\chi \in H^{-m+1}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f}=\operatorname{grad} \chi$ in $\Omega$.

If $\Omega$ is in addition simply-connected, then the three previous statements are equivalent to:
(iv) curl $\boldsymbol{f}=\mathbf{0}$ in $\Omega$.

Proof. For the equivalence between (i), (ii) and (iii), we refer to [4]. The implication (iii) $\Longrightarrow$ (iv) clearly holds. It thus remains to prove that (iv) $\Longrightarrow$ (iii). To begin with, let $\boldsymbol{f} \in H^{-m}(\Omega)^{3}$ be such that curl $\boldsymbol{f}=\mathbf{0}$ in $\Omega$. We then use the same argument as in [6]. We know that there exist a unique $\boldsymbol{u} \in H_{0}^{m}(\Omega)^{3}$ and a unique $p \in H^{-m+1}(\Omega) / \mathbb{R}$ (see [3]) such that

$$
\begin{equation*}
\Delta^{m} \boldsymbol{u}+\operatorname{grad} p=\boldsymbol{f} \quad \text { and } \quad \operatorname{div} \boldsymbol{u}=0 \quad \text { in } \Omega \tag{1}
\end{equation*}
$$

Hence $\Delta^{m}$ curl $\boldsymbol{u}=\mathbf{0}$ in $\Omega$, so that the hypoellipticity of the polyharmonic operator $\Delta^{m}$ implies that curl $\boldsymbol{u} \in \mathcal{C}^{\infty}(\Omega)^{3}$. Since div $\boldsymbol{u}=0$, we deduce that $\Delta \boldsymbol{u}=\mathbf{c u r l}$ curl $\boldsymbol{u} \in \mathcal{C}^{\infty}(\Omega)^{3}$. This also implies that $\Delta^{m} \boldsymbol{u}$ belongs to $\mathcal{C}^{\infty}(\Omega)^{3}$ and is an irrotational vector field. By the classical Poincaré lemma, there exists $q \in \mathcal{C}^{\infty}(\Omega)^{3}$ such that $\Delta^{m} \boldsymbol{u}=\operatorname{grad} q$. Thus, we see that $\boldsymbol{f}=\operatorname{grad}(p+q)$ and thanks to [4] Proposition 2.10, the function $p+q$ belongs to the space $H^{-m+1}(\Omega)$.

We can give another proof of this implication (iv) $\Longrightarrow$ (iii) by using the following theorem:
Theorem 2 Assume that the sets $\Omega$ and $\mathbb{R}^{3} \backslash \Omega$ are simply-connected. Let $\boldsymbol{u} \in H_{0}^{m}(\Omega)^{3}$, $m \geq 0$, be a function that satisfies $\operatorname{div} \boldsymbol{u}=0$ in $\Omega$. Then there exists a vector potential $\boldsymbol{\psi}$ in $H_{0}^{m+1}(\Omega)^{3}$ such that

$$
\begin{equation*}
\boldsymbol{u}=\operatorname{curl} \boldsymbol{\psi}, \quad \operatorname{div} \Delta^{m+1} \boldsymbol{\psi}=0 \quad \text { in } \Omega \tag{2}
\end{equation*}
$$

and the following estimate holds:

$$
\begin{equation*}
\|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^{3}} \leq C\|\boldsymbol{u}\|_{H^{m}(\Omega)^{3}} . \tag{3}
\end{equation*}
$$

Proof. Let $\boldsymbol{u} \in H_{0}^{m}(\Omega)^{3}$ be such that $\operatorname{div} \boldsymbol{u}=0$ in $\Omega$ and let $\widetilde{\boldsymbol{u}}$ denote the extension of $\boldsymbol{u}$ by $\mathbf{0}$ in $\mathbb{R}^{3} \backslash \Omega$. Thus $\widetilde{\boldsymbol{u}} \in H_{0}^{m}\left(\mathbb{R}^{3}\right)^{3}$, div $\widetilde{\boldsymbol{u}}=0$ in $\mathbb{R}^{3}$, and there exist an open ball $B$ containing $\bar{\Omega}$ and a vector field $\boldsymbol{w} \in H_{0}^{m+1}(B)^{3}$ such that $\widetilde{\boldsymbol{u}}=\operatorname{curl} \boldsymbol{w}$ in $B$, and

$$
\|\boldsymbol{w}\|_{H^{m+1}(B)^{3}} \leq C\|\boldsymbol{u}\|_{H^{m}(B)^{3}}
$$

The open set $\Omega^{\prime}:=B \backslash \bar{\Omega}$ is bounded, has a Lipschitz-continuous boundary and is simply-connected. Furthermore, the vector field $\boldsymbol{w}^{\prime}:=\left.\boldsymbol{w}\right|_{\Omega^{\prime}}$ belongs to $H^{m+1}\left(\Omega^{\prime}\right)^{3}$ and satisfies curl $\boldsymbol{w}^{\prime}=\mathbf{0}$ in $\Omega^{\prime}$. Hence there exists a function $\chi^{\prime} \in H^{1}\left(\Omega^{\prime}\right)$ such that $\boldsymbol{w}^{\prime}=\operatorname{grad} \chi^{\prime}$ in $\Omega^{\prime}$. Hence in fact $\chi^{\prime} \in H^{m+2}\left(\Omega^{\prime}\right)$ and the estimate

$$
\left\|\chi^{\prime}\right\|_{H^{m+2}\left(\Omega^{\prime}\right)} \leq C\left\|\boldsymbol{w}^{\prime}\right\|_{H^{m+1}\left(\Omega^{\prime}\right)^{3}}
$$

holds. Since the function $\chi^{\prime} \in H^{m+2}\left(\Omega^{\prime}\right)$ can be extended to a function $\widetilde{\chi}$ in $H^{m+2}\left(\mathbb{R}^{3}\right)$, with

$$
\|\widetilde{\chi}\|_{H^{m+2}\left(\mathbb{R}^{3}\right)} \leq C\left\|\chi^{\prime}\right\|_{H^{m+2}\left(\Omega^{\prime}\right)} \leq C\left\|\boldsymbol{w}^{\prime}\right\|_{H^{m+1}\left(\Omega^{\prime}\right)^{3}}
$$

the vector field $\widetilde{\boldsymbol{\varphi}}:=\boldsymbol{w}-\operatorname{grad} \widetilde{\chi}$ belongs to the space $H^{m+1}(B)^{3}$ and satisfies $\left.\widetilde{\boldsymbol{\varphi}}\right|_{\Omega^{\prime}}=\mathbf{0}$. Then the restriction $\varphi:=\left.\widetilde{\boldsymbol{\varphi}}\right|_{\Omega}$ is in the space $H_{0}^{m+1}(\Omega)^{3}$, satisfies the estimate (3), and curl $\widetilde{\boldsymbol{\varphi}}=\mathbf{c u r l} \boldsymbol{w}=\widetilde{\boldsymbol{u}}$ in $B$. Thus $\boldsymbol{u}=\operatorname{curl} \boldsymbol{\varphi}$ in $\Omega$, with $\varphi \in H_{0}^{m+1}(\Omega)^{3}$. Let now $p \in H_{0}^{m+2}(\Omega)$ denote the unique solution of $\Delta^{m+2} p=\operatorname{div} \Delta^{m+1} \boldsymbol{\varphi}$, so that the estimate

$$
\|p\|_{H^{m+2}(\Omega)} \leq C\|\boldsymbol{\varphi}\|_{H^{m+1}(\Omega)^{3}}
$$

holds. Then the function $\boldsymbol{\psi}=\boldsymbol{\varphi}-\operatorname{grad} p$ satisfies (2)-(3).

We can now give another proof of the above implication (iv) $\Longrightarrow$ (iii): Consider again the solution $\boldsymbol{u} \in H_{0}^{m}(\Omega)^{3}$ of (1) and let $\boldsymbol{v} \in H_{0}^{m+1}(\Omega)^{3}$ denote the vector potential of $\boldsymbol{u}$ as given by Theorem 2. We then have $\Delta^{m}$ curl $\boldsymbol{u}=\mathbf{0}$. If $m=2 k$, with $k \geq 1$, then

$$
\begin{aligned}
H^{-m-1}(\Omega)^{3}<\Delta^{m} \text { curl } \boldsymbol{u}, \boldsymbol{v}>_{H_{0}^{m+1}(\Omega)^{3}} & =H^{-1}(\Omega)^{3}<\Delta^{k} \operatorname{curl} \boldsymbol{u}, \Delta^{k} \boldsymbol{v}>_{H_{0}^{1}(\Omega)^{3}} \\
& =\int_{\Omega} \Delta^{k} \boldsymbol{u} \cdot \Delta^{k} \operatorname{curl} \boldsymbol{v} d \boldsymbol{x}=\left\|\Delta^{k} \boldsymbol{u}\right\|_{L^{2}(\Omega)^{3}}^{2}
\end{aligned}
$$

This implies that $\Delta^{k} \boldsymbol{u}=\mathbf{0}$ in $\Omega$ and thus $\boldsymbol{u}=\mathbf{0}$ since $\boldsymbol{u} \in H_{0}^{m}(\Omega)^{3}$. The case $m=2 k+1$ follows by a similar argument.

## 2. Scalar Potentials

Let $\Gamma_{i}, 0 \leq i \leq I$, denote the connected components of the boundary $\Gamma$ of the domain $\Omega, \Gamma_{0}$ being the boundary of the only unbounded connected component of $\mathbb{R}^{3} \backslash \bar{\Omega}$. We do not assume that $\Omega$ is simplyconnected, but we suppose that there exist $J$ connected, oriented and open surfaces $\Sigma_{j}, 1 \leq j \leq J$, called "cuts", contained in $\Omega$, such that each surface $\Sigma_{j}$ is an open subset of a smooth manifold, the boundary of $\Sigma_{j}$ is contained in $\Gamma$ for $1 \leq j \leq J$, the intersection $\bar{\Sigma}_{i} \cap \bar{\Sigma}_{j}$ is empty for $i \neq j$, and finally the open set $\Omega^{\circ}=\Omega \backslash \bigcup_{j=1}^{J} \Sigma_{j}$ is simply-connected and pseudo-Lipschitz (see [1]). Finally, let $[\cdot]_{j}$ denote the jump of a function over $\Sigma_{j}$, for $1 \leq j \leq J$.

We then define the spaces

$$
H(\operatorname{curl}, \Omega)=\left\{\boldsymbol{v} \in L^{2}(\Omega)^{3} ; \operatorname{curl} \boldsymbol{v} \in L^{2}(\Omega)^{3}\right\}, \quad H(\operatorname{div}, \Omega)=\left\{\boldsymbol{v} \in L^{2}(\Omega)^{3} ; \operatorname{div} \boldsymbol{v} \in L^{2}(\Omega)\right\}
$$

which are provided with the graph norm, and their subspaces
$H_{0}(\operatorname{curl}, \Omega)=\{\boldsymbol{v} \in H(\operatorname{curl}, \Omega) ; \boldsymbol{v} \times \boldsymbol{n}=\mathbf{0} \quad$ on $\Gamma\}, \quad H_{0}(\operatorname{div}, \Omega)=\{\boldsymbol{v} \in H(\operatorname{div}, \Omega) ; \boldsymbol{v} \cdot \boldsymbol{n}=0 \quad$ on $\Gamma\}$.
For any function $q$ in $H^{1}\left(\Omega^{\circ}\right)$, grad $q$ is the gradient of $q$ in the sense of distributions in $\mathscr{D}^{\prime}\left(\Omega^{\circ}\right)$. It belongs to $L^{2}\left(\Omega^{\circ}\right)^{3}$ and therefore can be extended to $L^{2}(\Omega)^{3}$. In order to distinguish this extension from the gradient of $q$ in $\mathscr{D}^{\prime}(\Omega)$, we denote it by $\operatorname{grad} q$. We finally observe that the space

$$
K_{T}(\Omega):=\left\{\boldsymbol{w} \in H(\operatorname{curl}, \Omega) \cap H_{0}(\operatorname{div}, \Omega) ; \operatorname{curl} \boldsymbol{w}=\mathbf{0} \text { and } \operatorname{div} \boldsymbol{w}=0 \text { in } \Omega\right\}
$$

is of dimension equal to $J$. As shown in [1] Prop. 3.14, it is spanned by the functions $\operatorname{grad} q_{j}^{T}, 1 \leq j \leq J$, where each $q_{j}^{T} \in H^{1}\left(\Omega^{\circ}\right)$ is unique up to an additive constant and satisfies $\Delta q_{j}^{T}=0$ in $\Omega^{\circ}, \partial_{n} q_{j}^{T}=0$ on $\Gamma,\left[q_{j}^{T}\right]_{k}=\delta_{j k},\left[\partial_{n} q_{j}^{T}\right]_{k}=0$ and $H_{H^{-1 / 2}\left(\Sigma_{k}\right)}<\partial_{n} q_{j}^{T}, 1>_{H^{1 / 2}\left(\Sigma_{k}\right)}=\delta_{j k}$ for $1 \leq k \leq J$.
Theorem 3 For any function $\boldsymbol{f} \in L^{2}(\Omega)^{3}$ that satisfies

$$
\begin{equation*}
\operatorname{curl} \boldsymbol{f}=\mathbf{0} \text { in } \Omega \quad \text { and } \quad \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} d \boldsymbol{x}=0 \text { for all } \boldsymbol{v} \in K_{T}(\Omega) \tag{4}
\end{equation*}
$$

there exists a scalar potential $\chi$ in $H^{1}(\Omega)$ such that $\boldsymbol{f}=\operatorname{grad} \chi$ and the following estimate holds:

$$
\begin{equation*}
\|\chi\|_{H^{1}(\Omega)} \leq C\|f\|_{L^{2}(\Omega)^{3}} \tag{5}
\end{equation*}
$$

Proof. It suffices to show that, given any $\boldsymbol{v} \in H_{0}(\operatorname{div}, \Omega)$ such that $\operatorname{div} \boldsymbol{v}=0$ in $\Omega$, there holds $(\boldsymbol{f}, \boldsymbol{v})_{L^{2}(\Omega)^{3}}=0$. For such $\boldsymbol{v} \in H_{0}(\operatorname{div}, \Omega)$, let $\boldsymbol{z}=\sum_{j=1 H^{-1 / 2}\left(\Sigma_{j}\right)}^{J}<\boldsymbol{v} \cdot \boldsymbol{n}, 1>_{H^{1 / 2}\left(\Sigma_{j}\right)} \widetilde{\operatorname{grad}} q_{j}^{T}$ and $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{z}$. According to [1], Theorem 3.17, there exists a vector potential $\boldsymbol{\psi} \in L^{2}(\Omega)^{3}$ satisfying $\boldsymbol{w}=\operatorname{curl} \boldsymbol{\psi}, \operatorname{div} \boldsymbol{\psi}=0$ in $\Omega$ and $\boldsymbol{\psi} \times \boldsymbol{n}=\mathbf{0}$ on $\Gamma$. Hence

$$
\int_{\Omega} f \cdot v d x=\int_{\Omega} f \cdot \operatorname{curl} \psi d x=0
$$

The result is then a consequence of Theorem 1: there exists a function $\chi \in H^{1}(\Omega)$ satisfying $f=\operatorname{grad} \chi$ and the estimate (5) holds.

Remark 4 Any function $\boldsymbol{f} \in L^{2}(\Omega)^{3}$ that satisfies curl $\boldsymbol{f}=\mathbf{0}$ in $\Omega$ can be decomposed as:

$$
\boldsymbol{f}=\operatorname{grad} \chi+\widetilde{\operatorname{grad} p}, \quad \text { with } \quad \chi \in H^{1}(\Omega) \quad \text { and } \quad \widetilde{\operatorname{grad} p} \in K_{T}(\Omega)
$$

Such a result is alluded to in [7] (page 959) ; however it is not proven there.
The second condition in (4) is trivially satisfied when $\Omega$ is simply-connected since $K_{T}(\Omega)=\{\mathbf{0}\}$ in this case.
Theorem 5 For any distribution $\boldsymbol{f}$ in the dual space of $H_{0}(\operatorname{div}, \Omega)$ that satisfies

$$
\begin{equation*}
\text { curl } \boldsymbol{f}=\mathbf{0} \quad \text { in } \Omega \quad \text { and } \quad H_{0}(\operatorname{div}, \Omega)^{\prime}<\boldsymbol{f}, \boldsymbol{v}>_{H_{0}(\operatorname{div}, \Omega)}=0 \quad \text { for all } \quad \boldsymbol{v} \in K_{T}(\Omega) \tag{6}
\end{equation*}
$$

there exists a scalar potential $\chi$ in $L^{2}(\Omega)$ such that $\boldsymbol{f}=\operatorname{grad} \chi$ and the following estimate holds:

$$
\begin{equation*}
\|\chi\|_{L^{2}(\Omega)} \leq C\|\boldsymbol{f}\|_{H_{0}(\operatorname{div}, \Omega)^{\prime}} \tag{7}
\end{equation*}
$$

Proof. Let $\boldsymbol{f}$ be in the dual space of $H_{0}(\operatorname{div}, \Omega)$ with $\operatorname{curl} \boldsymbol{f}=\mathbf{0}$ in $\Omega$. We know that there exists $\boldsymbol{\psi} \in L^{2}(\Omega)^{3}$ and $\chi_{0} \in L^{2}(\Omega)$ such that $\boldsymbol{f}=\boldsymbol{\psi}+\boldsymbol{\operatorname { g r a d }} \chi_{0}$, with the estimate (see Proposition 1 of [5])

$$
\|\boldsymbol{\psi}\|_{L^{2}(\Omega)^{3}}+\left\|\chi_{0}\right\|_{L^{2}(\Omega)} \leq C\|\boldsymbol{f}\|_{H_{0}(\operatorname{div}, \Omega)^{\prime}}
$$

Observe that, thanks to the density of $\mathscr{D}(\Omega)^{3}$ in $H_{0}(\operatorname{div}, \Omega)$, we have $H_{0}(\operatorname{div}, \Omega)^{\prime}<\operatorname{grad} \chi_{0}, \boldsymbol{v}>_{H_{0}(\operatorname{div}, \Omega)}=$ 0 , for all $\boldsymbol{v} \in K_{T}(\Omega)$. Therefore, the function $\boldsymbol{\psi} \in L^{2}(\Omega)^{3}$ satisfies the conditions (4). By Theorem 3, there exists a function $p \in H^{1}(\Omega)$ such that $\psi=\operatorname{grad} p$, with the estimate

$$
\|p\|_{H^{1}(\Omega)} \leq C\|\boldsymbol{\psi}\|_{L^{2}(\Omega)^{3}} \leq C\|\boldsymbol{f}\|_{H_{0}(\operatorname{div}, \Omega)^{\prime}}
$$

Hence, the function $\chi=p+\chi_{0}$ satisfies the announced properties.
More generally, for any integer $m \geq 1$, let us introduce the space

$$
H_{0}^{m}(\operatorname{div}, \Omega)=\left\{\boldsymbol{v} \in H_{0}(\operatorname{div}, \Omega) ; \operatorname{div} \boldsymbol{v} \in H_{0}^{m}(\Omega)\right\}
$$

We can prove that $\mathscr{D}(\Omega)^{3}$ is dense in $H_{0}^{m}(\operatorname{div}, \Omega)$. Moreover, we can characterize its dual space, denoted by $H^{-m}(\operatorname{div}, \Omega)$ :

$$
H^{-m}(\operatorname{div}, \Omega)=\left\{\boldsymbol{\psi}+\operatorname{grad} \chi ; \boldsymbol{\psi} \in H_{0}(\operatorname{div}, \Omega)^{\prime}, \chi \in H^{-m}(\Omega)\right\}
$$

As a consequence of Theorem 5 , it is easy to prove that, for any distribution $f \in H^{-m}(\operatorname{div}, \Omega)$ that satisfies (6), there exists a scalar potential $\chi$ in $H^{-m}(\Omega)$ such that $f=\operatorname{grad} \chi$. We thus obtain an extension of part (iv) in Theorem 1 in the case where $\Omega$ is multiply-connected.

## 3. "Weak" vector potentials

First, we note that the continuous embeddings $H_{0}(\operatorname{curl}, \Omega)^{\prime} \hookrightarrow H^{-1}(\Omega)^{3}$ and $H_{0}(\operatorname{div}, \Omega)^{\prime} \hookrightarrow H^{-1}(\Omega)^{3}$ hold. Besides, for any $\boldsymbol{f} \in H^{-1}(\Omega)^{3}$, we know that there exist a unique $\boldsymbol{u} \in H_{0}^{1}(\Omega)^{3}$ such that div $\boldsymbol{u}=0$ in $\Omega$, and $\chi \in L^{2}(\Omega)$ such that $\boldsymbol{f}=\Delta \boldsymbol{u}+\operatorname{grad} \chi$ and the estimate

$$
\|\boldsymbol{u}\|_{H^{1}(\Omega)^{3}}+\|\chi\|_{L^{2}(\Omega) / \mathbb{R}} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)^{3}}
$$

holds. Letting $\boldsymbol{\xi}=\operatorname{curl} \boldsymbol{u}$, we obtain $\boldsymbol{f}=\operatorname{curl} \boldsymbol{\xi}+\operatorname{grad} \chi$. Since $\boldsymbol{\xi} \in L^{2}(\Omega)^{3}$ and $\chi \in L^{2}(\Omega)$, it follows that curl $\boldsymbol{\xi} \in H_{0}(\operatorname{curl}, \Omega)^{\prime}$ and $\operatorname{grad} \chi \in H_{0}(\operatorname{div}, \Omega)^{\prime}$. Therefore

$$
H^{-1}(\Omega)^{3}=H_{0}(\operatorname{curl}, \Omega)^{\prime}+H_{0}(\operatorname{div}, \Omega)^{\prime}
$$

Here, we consider the other kernel

$$
K_{N}(\Omega)=\left\{\boldsymbol{w} \in H_{0}(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega) ; \operatorname{curl} \boldsymbol{w}=\mathbf{0} \text { and } \operatorname{div} \boldsymbol{w}=0 \text { in } \Omega\right\}
$$

which is of dimension equal to $I$. It is spanned (see Proposition 3.18 of [1] for a proof) by the functions $\operatorname{grad} q_{i}^{N}, 1 \leq i \leq N$, where each $q_{i}^{N} \in H^{1}(\Omega)$ is the unique solution to the problem $\Delta q_{i}^{N}=0$ in $\Omega, q_{i}^{N}=0$ on $\Gamma_{0}, H_{H^{-1 / 2}\left(\Gamma_{0}\right)}<\partial_{n} q_{i}^{N}, 1>_{H^{1 / 2}\left(\Gamma_{0}\right)}=-1$, and $q_{i}^{N}=\mathrm{const}$ on $\Gamma_{k}, H_{H^{-1 / 2}\left(\Gamma_{k}\right)}<\partial_{n} q_{i}^{N}, 1>_{H^{1 / 2}\left(\Gamma_{k}\right)}=\delta_{i k}$, for $1 \leq k \leq I$.
Theorem 6 For any distribution $\boldsymbol{f}$ in the dual space $H_{0}(\mathbf{c u r l}, \Omega)^{\prime}$ that satisfies

$$
\begin{equation*}
\operatorname{div} \boldsymbol{f}=0 \quad \text { in } \Omega \quad \text { and } \quad H_{0}(\operatorname{curl}, \Omega)^{\prime}<\boldsymbol{f}, \boldsymbol{v}>_{H_{0}(\operatorname{curl}, \Omega)}=0 \quad \text { for all } \quad \boldsymbol{v} \in K_{N}(\Omega) \tag{8}
\end{equation*}
$$

there exists a vector potential $\boldsymbol{\xi}$ in $L^{2}(\Omega)^{3}$ such that

$$
\begin{equation*}
\boldsymbol{f}=\operatorname{curl} \boldsymbol{\xi}, \quad \text { with } \quad \operatorname{div} \boldsymbol{\xi}=0 \quad \text { in } \Omega \quad \text { and } \quad \boldsymbol{\xi} \cdot \boldsymbol{n}=0 \quad \text { on } \Gamma, \tag{9}
\end{equation*}
$$

and such that the following estimate holds:

$$
\begin{equation*}
\|\boldsymbol{\xi}\|_{L^{2}(\Omega)^{3}} \leq C\|\boldsymbol{f}\|_{H_{0}(\mathbf{c u r}, \Omega)^{\prime}} \tag{10}
\end{equation*}
$$

Proof. Let $\boldsymbol{f}$ be in the dual space $H_{0}(\operatorname{curl}, \Omega)^{\prime}$. According to Corollary 5 of [5], there exist $\boldsymbol{\psi} \in L^{2}(\Omega)^{3}$ and $\boldsymbol{\xi}_{0} \in L^{2}(\Omega)^{3}$ with $\operatorname{div} \boldsymbol{\xi}_{0}=0$ in $\Omega$ and $\boldsymbol{\xi}_{0} \cdot \boldsymbol{n}=0$ on $\Gamma$, such that $\boldsymbol{f}=\boldsymbol{\psi}+\mathbf{c u r l} \boldsymbol{\xi}_{0}$ and such that the estimate

$$
\|\boldsymbol{\psi}\|_{L^{2}(\Omega)^{3}}+\left\|\boldsymbol{\xi}_{0}\right\|_{L^{2}(\Omega)^{3}} \leq C\|\boldsymbol{f}\|_{H_{0}(\mathbf{c u r l}, \Omega)^{\prime}}
$$

holds. Thanks to the density of $\mathscr{D}(\Omega)^{3}$ in $H_{0}(\operatorname{curl}, \Omega)$, we deduce that for all $\boldsymbol{v} \in K_{N}(\Omega)$, we have $H_{0}(\operatorname{curl}, \Omega)^{\prime}<\operatorname{curl} \boldsymbol{\xi}_{0}, \boldsymbol{v}>_{H_{0}(\operatorname{curl}, \Omega)}=0$. Since div $\boldsymbol{f}=0$, it follows that div $\boldsymbol{\psi}=0$. Then, thanks to the orthogonality condition, $H_{0}(\operatorname{curl}, \Omega)^{\prime}<\boldsymbol{f}, \operatorname{grad} q_{i}^{N}>_{H_{0}(\operatorname{curl}, \Omega)}=0$ for all $i=1, \ldots, I$, the condition $H^{-1 / 2}\left(\Gamma_{i}\right)<\boldsymbol{\psi} \cdot \boldsymbol{n}, 1>_{H^{1 / 2}\left(\Gamma_{i}\right)}=0$ is satisfied for all $i=0, \ldots, I$. There thus exists a vector potential $\boldsymbol{\varphi} \in L^{2}(\Omega)^{3}$ (see Theorem 3.12 of [1]) such that $\boldsymbol{\psi}=\operatorname{curl} \boldsymbol{\varphi}$, with $\operatorname{div} \boldsymbol{\varphi}=0$ in $\Omega$ and $\boldsymbol{\varphi} \cdot \boldsymbol{n}=0$ on $\Gamma$, and such that the estimate

$$
\|\boldsymbol{\varphi}\|_{L^{2}(\Omega)^{3}} \leq C\|\boldsymbol{\psi}\|_{L^{2}(\Omega)^{3}} .
$$

holds. Hence, the vector function $\boldsymbol{\xi}=\boldsymbol{\xi}_{0}+\boldsymbol{\varphi}$ possesses the announced properties.
Remark 7 Theorem 6 has been established in [5] when $\Gamma$ is connected, in which case $K_{N}=\{\mathbf{0}\}$. For any integer $m \geq 1$, let us now introduce the space

$$
H_{0}^{m}(\operatorname{curl}, \Omega):=\left\{\boldsymbol{v} \in H_{0}(\operatorname{curl}, \Omega) ; \operatorname{curl} \boldsymbol{v} \in H_{0}^{m}(\Omega)^{3}\right\}
$$

We can prove that $\mathscr{D}(\Omega)^{3}$ is dense in $H_{0}^{m}(\operatorname{curl}, \Omega)$. Moreover, we can characterize its dual space as

$$
H^{-m}(\operatorname{curl}, \Omega)=\left\{\boldsymbol{\psi}+\operatorname{curl} \boldsymbol{\xi} ; \boldsymbol{\psi} \in H_{0}(\operatorname{curl}, \Omega)^{\prime}, \boldsymbol{\xi} \in H^{-m}(\Omega)^{3}\right\} .
$$

Like in Section 3, given any distribution $\boldsymbol{f} \in H^{-m}(\operatorname{curl}, \Omega)$, with $m \geq 1$, that satisfies (8), there exists a vector potential $\boldsymbol{\xi} \in H^{-m}(\Omega)^{3}$ such that $\boldsymbol{f}=\operatorname{curl} \boldsymbol{\xi}$. Finally, using the decomposition (1) with $m$ replaced by $m+1$, it is easy to prove, as in Section 3 , that

$$
H^{-m-1}(\Omega)^{3}=H^{-m}(\operatorname{curl}, \Omega)+H^{-m}(\operatorname{div}, \Omega), \quad \text { for } \quad m \geq 1
$$

## 4. Generalized Korn's Inequality

Finally, we consider tensor fields.

Theorem 8 Assume that $\Omega$ is simply-connected. Given any integer $m \geq 0$, let $\boldsymbol{e}=\left(e_{i j}\right) \in H^{-m}(\Omega)^{3 \times 3}$ be a symmetric matrix field that satisfies the following compatibility conditions for all $i, j, k, l \in\{1,2,3\}$ :

$$
\begin{equation*}
\mathscr{R}_{i j k l}:=\frac{\partial^{2} e_{i k}}{\partial x_{l} \partial x_{j}}+\frac{\partial^{2} e_{j l}}{\partial x_{k} \partial x_{i}}-\frac{\partial^{2} e_{j k}}{\partial x_{l} \partial x_{i}}-\frac{\partial^{2} e_{i l}}{\partial x_{k} \partial x_{j}}=0 \quad \text { in } \quad H^{-m-2}(\Omega) \tag{11}
\end{equation*}
$$

Then there exists a vector field $\boldsymbol{v} \in H^{-m+1}(\Omega)^{3}$ such that $e_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)$ and $\boldsymbol{v}$ is unique up to vector fields in the space $R(\Omega)=\left\{\boldsymbol{a}+\boldsymbol{b} \wedge \mathbf{i d}_{\Omega} ; \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{3}\right\}$.
Proof. Let $\boldsymbol{e}=\left(e_{i j}\right) \in H_{s}^{-m}(\Omega)^{3 \times 3}$ (the subscript $s$ denotes a symmetric matrix field), and let $f_{i j k}:=$ $\frac{\partial e_{i k}}{\partial x_{j}}-\frac{\partial e_{j k}}{\partial x_{i}}$. Then $f_{i j k} \in H^{-m-1}(\Omega)$ and, thanks to the compatibility conditions (11), we have $\frac{\partial}{\partial x_{l}} f_{i j k}=$ $\frac{\partial}{\partial x_{k}} f_{i j l}$. Hence the implication (iii) $\Longrightarrow$ (iv) in Theorem 1 shows that there exist distributions $p_{i j} \in$ $H^{-m}(\Omega)$, unique up to additive constants, such that $\frac{\partial}{\partial x_{k}} p_{i j}=f_{i j k}$. Besides, since $\frac{\partial}{\partial x_{k}} p_{i j}=-\frac{\partial}{\partial x_{k}} p_{j i}$, we can choose the distributions $p_{i j}$ in such a way that $p_{i j}+p_{j i}=0$. Noting that the distributions $q_{i j}:=e_{i j}+p_{i j}$ belong to $H^{-m}(\Omega)$ and satisfy $\frac{\partial}{\partial x_{k}} q_{i j}=\frac{\partial}{\partial x_{j}} q_{i k}$, we again resort to Theorem 1 to assert the existence of distributions $v_{i} \in H^{-m+1}(\Omega)$, unique up to additive constants, such that $\frac{\partial v_{i}}{\partial x_{j}}=q_{i j}$.

Define, for any integer $m \geq 0$, the following spaces:

$$
E(\Omega):=\left\{\boldsymbol{e} \in H_{s}^{-m}(\Omega)^{3 \times 3}, \mathscr{R}_{i j k l}(\boldsymbol{e})=0\right\} \quad \text { and } \quad \dot{H}^{-m+1}(\Omega)^{3}:=H^{-m+1}(\Omega)^{3} / R(\Omega)
$$

By the previous theorem, for any $\boldsymbol{e}=\left(e_{i j}\right) \in E(\Omega)$, there exists a unique $\dot{\boldsymbol{v}}=\left(\dot{v}_{i}\right) \in \dot{H}^{-m+1}(\Omega)^{3}$ such that $e_{i j}=\frac{1}{2}\left(\frac{\partial \dot{v}_{i}}{\partial x_{j}}+\frac{\partial \dot{v}_{j}}{\partial x_{i}}\right)$. We may thus define a linear mapping $\mathcal{F}: E(\Omega) \rightarrow \dot{H}^{-m+1}(\Omega)^{3}$ by $\mathcal{F}(\boldsymbol{e})=\dot{\boldsymbol{v}}$. Using the same method as in [6], we can then prove the following result:
Theorem 9 The linear mapping $\mathcal{F}: E(\Omega) \rightarrow \dot{H}^{-m+1}(\Omega)^{3}$ is an isomorphism. Besides, there exists a constant $C \geq 0$ such that

$$
\inf _{r \in R(\Omega)}\|\boldsymbol{v}+\boldsymbol{r}\|_{H^{-m+1}(\Omega)^{3}} \leq C \sum_{i, j}\left\|\boldsymbol{e}_{i j}(\boldsymbol{v})\right\|_{H^{-m}(\Omega)} \quad \text { for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^{3},
$$

and

$$
\|\boldsymbol{v}\|_{H^{-m+1}(\Omega)^{3}} \leq C\left(\|\boldsymbol{v}\|_{H^{-m}(\Omega)^{3}}+\sum_{i, j}\left\|\boldsymbol{e}_{i j}(\boldsymbol{v})\right\|_{H^{-m}(\Omega)}\right) \quad \text { for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^{3}
$$

where $e_{i j}(\boldsymbol{v})=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)$.

## 5. Acknowledgement

This work was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administration Region [Project No. 9041076, City U 100105]

## References

[1] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three dimensional nonsmooth domains, Math. Methods Appl. Sci. 21 (1998), pp. 823-864.
[2] C. Amrouche, P.G. Ciarlet, L. Gratie, S. Kesavan, On the characterization of matrix fields as linearized strain tensor fields, J. Math. Pures Appl. 86, (2006), pp. 116-132.
[3] C. Amroulhe, V. Girault, Problèmes généralisés de Stokes, Portug. Math. 49 (1992), pp. 464-503.
[4] C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, Czech. Math. Jour. 44, (1994), pp. 109-140.
[5] C. Bernardi, V. Girault, Espaces duaux des domaines des opérateurs divergence et rotationnel avec trace nulle, Publications du Laboratoire Jacques-Louis Lions R 03017 (2003).
[6] P.G. Ciarlet, P. Ciarlet Jr, Another approach to linearized elasticity and a new proof of Korn's inequality, Math. Models Methods Appl. Sci. 15 (2005), pp. 259-271.
[7] P. Fernandes and G. Gilardi. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, Math. Models Methods Appl. Sci. 7 (1997), pp. 957-991.

