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Abstract

In this Note, we present several results concerning vector potentials and scalar potentials in a bounded, not necessarily simply-connected, three-dimensional domain. In particular, we consider singular potentials corresponding to data in negative order Sobolev spaces. We also give some applications to Poincaré's theorem and to Korn's inequality.

Résumé

Dans cette note, nous présentons plusieurs résultats concernant les potentiels vecteurs et les potentiels scalaires dans des domaines bornés tridimensionnels, éventuellement multiplement connexes. En particulier, on considère des potentiels singuliers correspondant à des données dans des espaces de Sobolev d'exposant négatif. On donne également des applications au théorème de Poincaré et à l'inégalité de Korn.

1. Weak versions of a classical theorem of Poincaré

In this work, we assume that Ω is a bounded open connected of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. The notation ${}_{X'}<,>_X$ denotes a duality pairing between a topological space X and its dual X'. The letter C denotes a constant that is not necessarily the same at its various occurrences.

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Theorem 1 *Let $\boldsymbol{f} \in H^{-m}(\Omega)^3$ for some integer $m \geq 0$. Then, the following assertions are equivalent:*
(i) ${}_{H^{-m}(\Omega)} < \boldsymbol{f},\ \boldsymbol{\varphi} >_{H_0^m(\Omega)} = 0$ *for any* $\boldsymbol{\varphi} \in V_m = \{\boldsymbol{\varphi} \in H_0^m(\Omega)^3;\ \mathrm{div}\ \boldsymbol{\varphi} = 0\}$,
(ii) ${}_{H^{-m}(\Omega)} < \boldsymbol{f},\ \boldsymbol{\varphi} >_{H_0^m(\Omega)} = 0$ *for any* $\boldsymbol{\varphi} \in \mathcal{V} = \{\boldsymbol{\varphi} \in \mathscr{D}(\Omega)^3;\ \mathrm{div}\ \boldsymbol{\varphi} = 0\}$,
(iii) *There exists a distribution $\chi \in H^{-m+1}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\ \chi$ in Ω.*

If Ω is in addition simply-connected, then the three previous statements are equivalent to:
(iv) $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ *in* Ω.

Proof. For the equivalence between (i), (ii) and (iii), we refer to [4]. The implication (iii) $\Longrightarrow$ (iv) clearly holds. It thus remains to prove that (iv) $\Longrightarrow$ (iii). To begin with, let $\boldsymbol{f} \in H^{-m}(\Omega)^3$ be such that $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ in Ω. We then use the same argument as in [6]. We know that there exist a unique $\boldsymbol{u} \in H_0^m(\Omega)^3$ and a unique $p \in H^{-m+1}(\Omega)/\mathbb{R}$ (see [3]) such that

$$\Delta^m \boldsymbol{u} + \mathbf{grad}\ p = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{1}$$

Hence $\Delta^m \mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$ in Ω, so that the hypoellipticity of the polyharmonic operator Δ^m implies that $\mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)^3$. Since $\mathrm{div}\ \boldsymbol{u} = 0$, we deduce that $\Delta \boldsymbol{u} = \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)^3$. This also implies that $\Delta^m \boldsymbol{u}$ belongs to $\mathcal{C}^\infty(\Omega)^3$ and is an irrotational vector field. By the classical Poincaré lemma, there exists $q \in \mathcal{C}^\infty(\Omega)^3$ such that $\Delta^m \boldsymbol{u} = \mathbf{grad}\ q$. Thus, we see that $\boldsymbol{f} = \mathbf{grad}\ (p+q)$ and thanks to [4] Proposition 2.10, the function $p+q$ belongs to the space $H^{-m+1}(\Omega)$. $\square$

We can give another proof of this implication (iv) $\Longrightarrow$ (iii) by using the following theorem:
Theorem 2 *Assume that the sets Ω and $\mathbb{R}^3 \setminus \Omega$ are simply-connected. Let $\boldsymbol{u} \in H_0^m(\Omega)^3$, $m \geq 0$, be a function that satisfies* $\mathrm{div}\ \boldsymbol{u} = 0$ *in Ω. Then there exists a vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$ such that*

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \qquad \mathrm{div}\ \Delta^{m+1} \boldsymbol{\psi} = 0 \quad \textit{in } \Omega, \tag{2}$$

and the following estimate holds:

$$\|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C \|\boldsymbol{u}\|_{H^m(\Omega)^3}. \tag{3}$$

Proof. Let $\boldsymbol{u} \in H_0^m(\Omega)^3$ be such that $\mathrm{div}\ \boldsymbol{u} = 0$ in Ω and let $\widetilde{\boldsymbol{u}}$ denote the extension of $\boldsymbol{u}$ by $\mathbf{0}$ in $\mathbb{R}^3 \setminus \Omega$. Thus $\widetilde{\boldsymbol{u}} \in H_0^m(\mathbb{R}^3)^3$, $\mathrm{div}\ \widetilde{\boldsymbol{u}} = 0$ in $\mathbb{R}^3$, and there exist an open ball B containing $\overline{\Omega}$ and a vector field $\boldsymbol{w} \in H_0^{m+1}(B)^3$ such that $\widetilde{\boldsymbol{u}} = \mathbf{curl}\ \boldsymbol{w}$ in B, and

$$\|\boldsymbol{w}\|_{H^{m+1}(B)^3} \leq C \|\boldsymbol{u}\|_{H^m(B)^3}.$$

The open set $\Omega' := B \setminus \overline{\Omega}$ is bounded, has a Lipschitz-continuous boundary and is simply-connected. Furthermore, the vector field $\boldsymbol{w}' := \boldsymbol{w}|_{\Omega'}$ belongs to $H^{m+1}(\Omega')^3$ and satisfies $\mathbf{curl}\ \boldsymbol{w}' = \mathbf{0}$ in Ω'. Hence there exists a function $\chi' \in H^1(\Omega')$ such that $\boldsymbol{w}' = \mathbf{grad}\ \chi'$ in Ω'. Hence in fact $\chi' \in H^{m+2}(\Omega')$ and the estimate

$$\|\chi'\|_{H^{m+2}(\Omega')} \leq C \|\boldsymbol{w}'\|_{H^{m+1}(\Omega')^3}$$

holds. Since the function $\chi' \in H^{m+2}(\Omega')$ can be extended to a function $\widetilde{\chi}$ in $H^{m+2}(\mathbb{R}^3)$, with

$$\|\widetilde{\chi}\|_{H^{m+2}(\mathbb{R}^3)} \leq C \|\chi'\|_{H^{m+2}(\Omega')} \leq C \|\boldsymbol{w}'\|_{H^{m+1}(\Omega')^3},$$

the vector field $\widetilde{\boldsymbol{\varphi}} := \boldsymbol{w} - \mathbf{grad}\ \widetilde{\chi}$ belongs to the space $H^{m+1}(B)^3$ and satisfies $\widetilde{\boldsymbol{\varphi}}|_{\Omega'} = \mathbf{0}$. Then the restriction $\boldsymbol{\varphi} := \widetilde{\boldsymbol{\varphi}}|_\Omega$ is in the space $H_0^{m+1}(\Omega)^3$, satisfies the estimate (3), and $\mathbf{curl}\ \widetilde{\boldsymbol{\varphi}} = \mathbf{curl}\ \boldsymbol{w} = \widetilde{\boldsymbol{u}}$ in B. Thus $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\varphi}$ in Ω, with $\boldsymbol{\varphi} \in H_0^{m+1}(\Omega)^3$. Let now $p \in H_0^{m+2}(\Omega)$ denote the unique solution of $\Delta^{m+2} p = \mathrm{div}\ \Delta^{m+1} \boldsymbol{\varphi}$, so that the estimate

$$\|p\|_{H^{m+2}(\Omega)} \leq C \|\boldsymbol{\varphi}\|_{H^{m+1}(\Omega)^3}$$

holds. Then the function $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad}\ p$ satisfies (2)-(3). $\square$

We can now give another proof of the above implication (iv) $\Longrightarrow$ (iii): Consider again the solution $\boldsymbol{u} \in H_0^m(\Omega)^3$ of (1) and let $\boldsymbol{v} \in H_0^{m+1}(\Omega)^3$ denote the vector potential of $\boldsymbol{u}$ as given by Theorem 2. We then have $\Delta^m \mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$. If $m = 2k$, with $k \geq 1$, then

$$\begin{aligned} {}_{H^{-m-1}(\Omega)^3} < \Delta^m \mathbf{curl}\ \boldsymbol{u},\ \boldsymbol{v} >_{H_0^{m+1}(\Omega)^3} &= {}_{H^{-1}(\Omega)^3} < \Delta^k \mathbf{curl}\ \boldsymbol{u},\ \Delta^k \boldsymbol{v} >_{H_0^1(\Omega)^3} \\ &= \int_\Omega \Delta^k \boldsymbol{u} \cdot \Delta^k \mathbf{curl}\ \boldsymbol{v}\ d\boldsymbol{x}\ = \|\Delta^k \boldsymbol{u}\|^2_{L^2(\Omega)^3}. \end{aligned}$$

This implies that $\Delta^k \boldsymbol{u} = \mathbf{0}$ in Ω and thus $\boldsymbol{u} = \mathbf{0}$ since $\boldsymbol{u} \in H_0^m(\Omega)^3$. The case $m = 2k+1$ follows by a similar argument. $\square$

2. Scalar Potentials

Let Γ_i, $0 \leq i \leq I$, denote the connected components of the boundary Γ of the domain Ω, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. We do not assume that Ω is simply-connected, but we suppose that there exist J connected, oriented and open surfaces Σ_j, $1 \leq j \leq J$, called "cuts", contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ for $1 \leq j \leq J$, the intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^J \Sigma_j$ is simply-connected and pseudo-Lipschitz (see [1]). Finally, let $[\cdot]_j$ denote the jump of a function over Σ_j, for $1 \leq j \leq J$.

We then define the spaces

$$H(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{v} \in L^2(\Omega)^3\},\quad H(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathrm{div}\ \boldsymbol{v} \in L^2(\Omega)\},$$

which are provided with the graph norm, and their subspaces

$$H_0(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in H(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma\},\quad H_0(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\}.$$

For any function q in $H^1(\Omega^\circ)$, $\mathbf{grad}\ q$ is the gradient of q in the sense of distributions in $\mathscr{D}'(\Omega^\circ)$. It belongs to $L^2(\Omega^\circ)^3$ and therefore can be extended to $L^2(\Omega)^3$. In order to distinguish this extension from the gradient of q in $\mathscr{D}'(\Omega)$, we denote it by $\widetilde{\mathbf{grad}}\ q$. We finally observe that the space

$$K_T(\Omega) := \{\boldsymbol{w} \in H(\mathbf{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and } \mathrm{div}\ \boldsymbol{w} = 0 \text{ in } \Omega\}$$

is of dimension equal to J. As shown in [1] Prop. 3.14, it is spanned by the functions $\widetilde{\mathbf{grad}}\ q_j^T$, $1 \leq j \leq J$, where each $q_j^T \in H^1(\Omega^\circ)$ is unique up to an additive constant and satisfies $\Delta q_j^T = 0$ in Ω°, $\partial_n q_j^T = 0$ on Γ, $[q_j^T]_k = \delta_{jk}$, $[\partial_n q_j^T]_k = 0$ and ${}_{H^{-1/2}(\Sigma_k)} < \partial_n q_j^T\ , 1 >_{H^{1/2}(\Sigma_k)} = \delta_{jk}$ for $1 \leq k \leq J$.

Theorem 3 *For any function $\boldsymbol{f} \in L^2(\Omega)^3$ that satisfies*

$$\mathbf{curl}\ \boldsymbol{f} = \mathbf{0} \textit{ in } \Omega \qquad \textit{and} \qquad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\ d\boldsymbol{x} = 0 \textit{ for all } \boldsymbol{v} \in K_T(\Omega), \tag{4}$$

there exists a scalar potential χ in $H^1(\Omega)$ such that $\boldsymbol{f} = \mathbf{grad}\ \chi$ and the following estimate holds:

$$\|\chi\|_{H^1(\Omega)} \leq C \|\boldsymbol{f}\|_{L^2(\Omega)^3}. \tag{5}$$

Proof. It suffices to show that, given any $\boldsymbol{v} \in H_0(\mathrm{div}, \Omega)$ such that $\mathrm{div}\ \boldsymbol{v} = 0$ in Ω, there holds $(\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)^3} = 0$. For such $\boldsymbol{v} \in H_0(\mathrm{div}, \Omega)$, let $\boldsymbol{z} = \sum_{j=1}^J {}_{H^{-1/2}(\Sigma_j)} < \boldsymbol{v} \cdot \boldsymbol{n},\ 1 >_{H^{1/2}(\Sigma_j)}\ \widetilde{\mathbf{grad}}\ q_j^T$ and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$. According to [1], Theorem 3.17, there exists a vector potential $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfying $\boldsymbol{w} = \mathbf{curl}\ \boldsymbol{\psi}$, $\mathrm{div}\ \boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Hence

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\ d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\ \boldsymbol{\psi}\ d\boldsymbol{x} = 0.$$

The result is then a consequence of Theorem 1: there exists a function $\chi \in H^1(\Omega)$ satisfying $\boldsymbol{f} = \mathbf{grad}\ \chi$ and the estimate (5) holds. $\square$

Remark 4 *Any function $\boldsymbol{f} \in L^2(\Omega)^3$ that satisfies $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ in Ω can be decomposed as:*

$$\boldsymbol{f} = \mathbf{grad}\ \chi + \widetilde{\mathbf{grad}}\ p, \quad \textit{with} \quad \chi \in H^1(\Omega) \quad \textit{and} \quad \widetilde{\mathbf{grad}}\ p \in K_T(\Omega).$$

Such a result is alluded to in [7] (page 959) ; however it is not proven there.
The second condition in (4) is trivially satisfied when Ω is simply-connected since $K_T(\Omega) = \{\mathbf{0}\}$ in this case.

Theorem 5 *For any distribution $\boldsymbol{f}$ in the dual space of $H_0(div, \Omega)$ that satisfies*

$$\mathbf{curl}\ \boldsymbol{f} = \mathbf{0} \quad \textit{in } \Omega \quad \textit{and} \quad {}_{H_0(\mathrm{div},\Omega)'} < \boldsymbol{f}\ ,\ \boldsymbol{v} >_{H_0(\mathrm{div},\Omega)} = 0 \quad \textit{for all} \quad \boldsymbol{v} \in K_T(\Omega), \tag{6}$$

there exists a scalar potential χ in $L^2(\Omega)$ such that $\boldsymbol{f} = \mathbf{grad}\ \chi$ and the following estimate holds:

$$\|\chi\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{7}$$

Proof. Let $\boldsymbol{f}$ be in the dual space of $H_0(\mathrm{div}, \Omega)$ with $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ in Ω. We know that there exists $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0$, with the estimate (see Proposition 1 of [5])

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

Observe that, thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathrm{div}, \Omega)$, we have ${}_{H_0(\mathrm{div},\Omega)'} < \mathbf{grad}\ \chi_0,\ \boldsymbol{v} >_{H_0(\mathrm{div},\Omega)} = 0$, for all $\boldsymbol{v} \in K_T(\Omega)$. Therefore, the function $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfies the conditions (4). By Theorem 3, there exists a function $p \in H^1(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{grad}\ p$, with the estimate

$$\|p\|_{H^1(\Omega)} \leq C\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

Hence, the function $\chi = p + \chi_0$ satisfies the announced properties. $\square$

More generally, for any integer $m \geq 1$, let us introduce the space

$$H_0^m(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in H_0(\mathrm{div}, \Omega);\ \mathrm{div}\boldsymbol{v} \in H_0^m(\Omega)\}.$$

We can prove that $\mathscr{D}(\Omega)^3$ is dense in $H_0^m(\mathrm{div}, \Omega)$. Moreover, we can characterize its dual space, denoted by $H^{-m}(\mathrm{div}, \Omega)$:

$$H^{-m}(\mathrm{div}, \Omega) = \{\boldsymbol{\psi} + \mathbf{grad}\ \chi;\ \boldsymbol{\psi} \in H_0(\mathrm{div}, \Omega)',\ \chi \in H^{-m}(\Omega)\}.$$

As a consequence of Theorem 5, it is easy to prove that, for any distribution $\boldsymbol{f} \in H^{-m}(\mathrm{div}, \Omega)$ that satisfies (6), there exists a scalar potential χ in $H^{-m}(\Omega)$ such that $\boldsymbol{f} = \mathbf{grad}\ \chi$. We thus obtain an extension of part (iv) in Theorem 1 in the case where Ω is multiply-connected.

3. "Weak" vector potentials

First, we note that the continuous embeddings $H_0(\mathbf{curl}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$ and $H_0(\mathrm{div}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$ hold. Besides, for any $\boldsymbol{f} \in H^{-1}(\Omega)^3$, we know that there exist a unique $\boldsymbol{u} \in H_0^1(\Omega)^3$ such that div $\boldsymbol{u} = 0$ in Ω, and $\chi \in L^2(\Omega)$ such that $\boldsymbol{f} = \Delta\boldsymbol{u} + \mathbf{grad}\ \chi$ and the estimate

$$\|\boldsymbol{u}\|_{H^1(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)^3}$$

holds. Letting $\boldsymbol{\xi} = \mathbf{curl}\ \boldsymbol{u}$, we obtain $\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi} + \mathbf{grad}\ \chi$. Since $\boldsymbol{\xi} \in L^2(\Omega)^3$ and $\chi \in L^2(\Omega)$, it follows that $\mathbf{curl}\ \boldsymbol{\xi} \in H_0(\mathbf{curl}, \Omega)'$ and $\mathbf{grad}\ \chi \in H_0(\mathrm{div}, \Omega)'$. Therefore

$$H^{-1}(\Omega)^3 = H_0(\mathbf{curl}, \Omega)' + H_0(\mathrm{div}, \Omega)'.$$

Here, we consider the other kernel

$$K_N(\Omega) = \{\boldsymbol{w} \in H_0(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and div } \boldsymbol{w} = 0 \text{ in } \Omega\}$$

which is of dimension equal to I. It is spanned (see Proposition 3.18 of [1] for a proof) by the functions $\mathbf{grad}\ q_i^N$, $1 \le i \le N$, where each $q_i^N \in H^1(\Omega)$ is the unique solution to the problem $\Delta q_i^N = 0$ in Ω, $q_i^N = 0$ on Γ_0, ${}_{H^{-1/2}(\Gamma_0)} < \partial_n q_i^N\ ,\ 1 >_{H^{1/2}(\Gamma_0)} = -1$, and $q_i^N = const$ on Γ_k, ${}_{H^{-1/2}(\Gamma_k)} < \partial_n q_i^N\ ,\ 1 >_{H^{1/2}(\Gamma_k)} = \delta_{ik}$, for $1 \le k \le I$.

Theorem 6 *For any distribution $\boldsymbol{f}$ in the dual space $H_0(\mathbf{curl}, \Omega)'$ that satisfies*

$$\mathrm{div}\ \boldsymbol{f} = 0 \quad \textit{in } \Omega \quad \textit{and} \quad {}_{H_0(\mathbf{curl},\Omega)'} < \boldsymbol{f}\ ,\ \boldsymbol{v} >_{H_0(\mathbf{curl},\Omega)} = 0 \quad \textit{for all} \quad \boldsymbol{v} \in K_N(\Omega), \tag{8}$$

there exists a vector potential $\boldsymbol{\xi}$ in $L^2(\Omega)^3$ such that

$$\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi}, \quad \textit{with} \quad \mathrm{div}\ \boldsymbol{\xi} = 0 \quad \textit{in } \Omega \quad \textit{and} \quad \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \quad \textit{on } \Gamma, \tag{9}$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{L^2(\Omega)^3} \le C \|\boldsymbol{f}\|_{H_0(\mathbf{curl},\Omega)'}. \tag{10}$$

Proof. Let $\boldsymbol{f}$ be in the dual space $H_0(\mathbf{curl}, \Omega)'$. According to Corollary 5 of [5], there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\boldsymbol{\xi}_0 \in L^2(\Omega)^3$ with div $\boldsymbol{\xi}_0 = 0$ in Ω and $\boldsymbol{\xi}_0 \cdot \boldsymbol{n} = 0$ on Γ, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\ \boldsymbol{\xi}_0$ and such that the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\boldsymbol{\xi}_0\|_{L^2(\Omega)^3} \le C \|\boldsymbol{f}\|_{H_0(\mathbf{curl},\Omega)'}$$

holds. Thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathbf{curl}, \Omega)$, we deduce that for all $\boldsymbol{v} \in K_N(\Omega)$, we have ${}_{H_0(\mathbf{curl},\Omega)'} < \mathbf{curl}\ \boldsymbol{\xi}_0\ ,\ \boldsymbol{v} >_{H_0(\mathbf{curl},\Omega)} = 0$. Since div $\boldsymbol{f} = 0$, it follows that div $\boldsymbol{\psi} = 0$. Then, thanks to the orthogonality condition, ${}_{H_0(\mathbf{curl},\Omega)'} < \boldsymbol{f}\ ,\ \mathbf{grad}\ q_i^N >_{H_0(\mathbf{curl},\Omega)} = 0$ for all $i = 1, \dots, I$, the condition ${}_{H^{-1/2}(\Gamma_i)} < \boldsymbol{\psi} \cdot \boldsymbol{n}\ ,\ 1 >_{H^{1/2}(\Gamma_i)} = 0$ is satisfied for all $i = 0, \dots, I$. There thus exists a vector potential $\boldsymbol{\varphi} \in L^2(\Omega)^3$ (see Theorem 3.12 of [1]) such that $\boldsymbol{\psi} = \mathbf{curl}\ \boldsymbol{\varphi}$, with div $\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ, and such that the estimate

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \le C \|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

holds. Hence, the vector function $\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \boldsymbol{\varphi}$ possesses the announced properties. □

Remark 7 Theorem 6 has been established in [5] when Γ is connected, in which case $K_N = \{\mathbf{0}\}$.

For any integer $m \ge 1$, let us now introduce the space

$$H_0^m(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in H_0(\mathbf{curl}, \Omega);\ \mathbf{curl}\ \boldsymbol{v} \in H_0^m(\Omega)^3\}.$$

We can prove that $\mathscr{D}(\Omega)^3$ is dense in $H_0^m(\mathbf{curl}, \Omega)$. Moreover, we can characterize its dual space as

$$H^{-m}(\mathbf{curl}, \Omega) = \{\boldsymbol{\psi} + \mathbf{curl}\ \boldsymbol{\xi};\ \boldsymbol{\psi} \in H_0(\mathbf{curl}, \Omega)',\ \boldsymbol{\xi} \in H^{-m}(\Omega)^3\}.$$

Like in Section 3, given any distribution $\boldsymbol{f} \in H^{-m}(\mathbf{curl}, \Omega)$, with $m \ge 1$, that satisfies (8), there exists a vector potential $\boldsymbol{\xi} \in H^{-m}(\Omega)^3$ such that $\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi}$. Finally, using the decomposition (1) with m replaced by $m+1$, it is easy to prove, as in Section 3 , that

$$H^{-m-1}(\Omega)^3 = H^{-m}(\mathbf{curl}, \Omega) + H^{-m}(\mathbf{div}, \Omega), \quad \text{for } m \ge 1.$$

4. Generalized Korn's Inequality

Finally, we consider *tensor fields.*

Theorem 8 *Assume that Ω is simply-connected. Given any integer $m \geq 0$, let $\boldsymbol{e} = (e_{ij}) \in H^{-m}(\Omega)^{3\times 3}$ be a symmetric matrix field that satisfies the following compatibility conditions for all $i, j, k, l \in \{1, 2, 3\}$:*

$$\mathscr{R}_{ijkl} := \frac{\partial^2 e_{ik}}{\partial x_l \partial x_j} + \frac{\partial^2 e_{jl}}{\partial x_k \partial x_i} - \frac{\partial^2 e_{jk}}{\partial x_l \partial x_i} - \frac{\partial^2 e_{il}}{\partial x_k \partial x_j} = 0 \quad in \quad H^{-m-2}(\Omega). \tag{11}$$

Then there exists a vector field $\boldsymbol{v} \in H^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ and $\boldsymbol{v}$ is unique up to vector fields in the space $R(\Omega) = \{\boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega;\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$.

Proof. Let $\boldsymbol{e} = (e_{ij}) \in H_s^{-m}(\Omega)^{3\times 3}$ (the subscript s denotes a symmetric matrix field), and let $f_{ijk} := \frac{\partial e_{ik}}{\partial x_j} - \frac{\partial e_{jk}}{\partial x_i}$. Then $f_{ijk} \in H^{-m-1}(\Omega)$ and, thanks to the compatibility conditions (11), we have $\frac{\partial}{\partial x_l} f_{ijk} = \frac{\partial}{\partial x_k} f_{ijl}$. Hence the implication (iii) $\Longrightarrow$ (iv) in Theorem 1 shows that there exist distributions $p_{ij} \in H^{-m}(\Omega)$, unique up to additive constants, such that $\frac{\partial}{\partial x_k} p_{ij} = f_{ijk}$. Besides, since $\frac{\partial}{\partial x_k} p_{ij} = -\frac{\partial}{\partial x_k} p_{ji}$, we can choose the distributions p_{ij} in such a way that $p_{ij} + p_{ji} = 0$. Noting that the distributions $q_{ij} := e_{ij} + p_{ij}$ belong to $H^{-m}(\Omega)$ and satisfy $\frac{\partial}{\partial x_k} q_{ij} = \frac{\partial}{\partial x_j} q_{ik}$, we again resort to Theorem 1 to assert the existence of distributions $v_i \in H^{-m+1}(\Omega)$, unique up to additive constants, such that $\frac{\partial v_i}{\partial x_j} = q_{ij}$. $\square$

Define, for any integer $m \geq 0$, the following spaces:

$$E(\Omega) := \{\boldsymbol{e} \in H_s^{-m}(\Omega)^{3\times 3},\ \mathscr{R}_{ijkl}(\boldsymbol{e}) = 0\} \quad \text{and} \quad \dot{H}^{-m+1}(\Omega)^3 := H^{-m+1}(\Omega)^3 / R(\Omega)$$

By the previous theorem, for any $\boldsymbol{e} = (e_{ij}) \in E(\Omega)$, there exists a unique $\dot{\boldsymbol{v}} = (\dot{v}_i) \in \dot{H}^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\frac{\partial \dot{v}_i}{\partial x_j} + \frac{\partial \dot{v}_j}{\partial x_i})$. We may thus define a linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ by $\boldsymbol{\mathcal{F}(e) = \dot{v}}$. Using the same method as in [6], we can then prove the following result:

Theorem 9 *The linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ is an isomorphism. Besides, there exists a constant $C \geq 0$ such that*

$$\inf_{\boldsymbol{r}\in R(\Omega)} \|\boldsymbol{v} + \boldsymbol{r}\|_{H^{-m+1}(\Omega)^3} \leq C \sum_{i,j} \|\boldsymbol{e}_{ij}(\boldsymbol{v})\|_{H^{-m}(\Omega)} \quad \text{for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^3,$$

and

$$\|\boldsymbol{v}\|_{H^{-m+1}(\Omega)^3} \leq C(\|\boldsymbol{v}\|_{H^{-m}(\Omega)^3} + \sum_{i,j} \|\boldsymbol{e}_{ij}(\boldsymbol{v})\|_{H^{-m}(\Omega)}) \quad \text{for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^3$$

where $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$.

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