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Abstract : in this paper we prove local existence and uniqueness of regular solutions for a quasistatic model arising in micromagnetism theory. Moreover we show global existence of regular solutions for small data in the 2D case for the Landau-Lifschitz equation. These results extend those already obtained by the authors in the whole space.

AMS Subjects : 35K20, 35Q60.

1 Introduction

The micromagnetism theory is the study of electromagnetic phenomena occuring in soft magnetic material (see [4]). A soft magnetic material is characterized by a spontaneous magnetisation represented by a magnetic moment denoted by $u(t,x)$. This vector field is defined in $[0,T]\times\Omega$, where Ω is the domain where the material is confined. It links the magnetic field H and the magnetic induction by the relation $B = H + \bar{u}$, where $\bar{u}$ is the extension of u by zero outside Ω. Furthermore the norm of u is constant and equal to 1 in $[0,T]\times\Omega$.

The behavior of u is governed by the following Landau-Lifschitz type equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - u\wedge(\Delta u + H(u)) + u\wedge\Big(u\wedge(\Delta u + H(u))\Big) = 0 & \text{in } [0,T]\times\Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } [0,T]\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where ν is the outward unitary normal on $\partial\Omega$.

In this equation, $H(u)$ represents the magnetic field generated by u. The operator $v\mapsto H(v)$ is defined for v in $L^2(\Omega)$ by :

$$\begin{cases} \operatorname{curl} H(v) = 0 \text{ in } I\!R^3 \text{ (from stationary Maxwell equation)}, \\ \operatorname{div}(H(v)+\bar{v}) = 0 \text{ in } I\!R^3 \text{ (since } \operatorname{div} B = 0 \text{ according to Faraday's law)}, \\ H(v)\in L^2(I\!R^3), \end{cases} \tag{1.2}$$

where $\bar{v}$ is the extension of v by zero outside Ω.

We assume that the initial data u_0 satisfies the hypothesis $(\mathcal{H})$ below

$$(\mathcal{H})\begin{cases} u_0 \in H^2(\Omega), \\ \dfrac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ |u_0| \equiv 1. \end{cases}$$

Our first result is one of local existence and uniqueness of a regular solution for (1.1).

Theorem 1.1 *Assuming that u_0 satisfies $(\mathcal{H})$, there exists a time $T^* > 0$ depending only on the size of the data and there exists a unique u such that for all $T < T^*$,*

$$\begin{cases} u \in \mathcal{C}^0([0,T]; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)), \\ \\ |u(x,t)| = 1 \text{ in } [0,T] \times \Omega, \\ \\ u \text{ satisfies (1.1).} \end{cases}$$

We next prove the following stability result of the solution.

Theorem 1.2 *Under assumption $(\mathcal{H})$, the regular solution given by theorem 1.1 depends continuously on u_0 for the topology of $\mathcal{C}^0([0,T]; H^2(\Omega))$.*

Finally, we consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = u \wedge \Delta u - u \wedge (u \wedge \Delta u) \text{ in } [0,T] \times \Omega, \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } [0,T] \times \partial\Omega, \\ \\ u(0,x) = u_0(x) \text{ in } \Omega, \\ \\ |u(t,x)| = 1 \text{ in } [0,T] \times \Omega. \end{cases} \tag{1.3}$$

Adapting the proof of Theorem 1.1 we obtain that

Theorem 1.3 *Under assumption $(\mathcal{H})$, there exists a time $T^* > 0$ and there exists a unique $u \in \mathcal{C}^0([0,T]; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$ for all $T < T^*$ such that u satisfies (1.3).*

In the 2 dimensional case, we can improve this result and we obtain a theorem of global existence for small data.

Theorem 1.4 *Assuming that $\Omega \subset I\!R^2$, there exists $\delta > 0$ such that if u_0 satisfies $(\mathcal{H})$ and if $\|\nabla u_0\|_{H^1(\Omega)} \leq \delta$ then the regular solution of (1.3) with initial data u_0 exists on $I\!R^+$.*

The paper is organized as follows. In section 2, we prove technical Lemmas. Section 3 is devoted to the proof of Theorem 1.1. We prove the stability theorem in section 4. In the last part, we establish Theorem 1.4. The proof of Theorem 1.3 is a simple adaptation of Theorem 1.1 and is left to the reader.

2 Preliminary results

2.1 Regularity results

Lemma 2.1 *Let Ω be a bounded regular open set. There exists a constant C such that for all $u \in H^2(\Omega)$ such that $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$,*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.1}$$

$$\|\nabla u\|_{H^1(\Omega)} \leq C\left(\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{2.2}$$

and for $u \in H^3(\Omega)$ *such that* $\dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$,

$$\|\nabla u\|_{H^2(\Omega)} \leq C\left(\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} + \|\nabla\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{2.3}$$

Proof. The first inequality results from the regularity of the operator $A = I - \Delta$ with domain

$$D(A) = \left\{u \in H^2(\Omega), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}$$

see for example [7].

Furthermore, in [8], we find the following result

Proposition 2.1 *Let* Ω *be a bounded regular open set of* $I\!R^d$, $d \leq 3$. *Then, there exists a constant* C *such that for all* $V \in H^m(\Omega)$ *such that* $V \cdot \nu = 0$ *on* $\partial\Omega$,

$$\|V\|_{H^m(\Omega)} \leq C\left(\|V\|_{L^2(\Omega)} + \|\operatorname{div} V\|_{H^{m-1}(\Omega)} + \|\operatorname{curl} V\|_{H^{m-1}(\Omega)}\right).$$

We set $V = \nabla u$ and since $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, we can apply Proposition 2.1 to conclude the proof of Lemma 2.1.

Using Lemma 2.1 and the classical interpolation inequality, we rewrite Sobolev and Gagliardo-Nirenberg inequalities on the following form:

Lemma 2.2 *Let* Ω *be a regular bounded domain of* $I\!R^3$. *There exists a constant* C *such that for all* $u \in H^2(\Omega)$ *such that* $\dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$,

$$\|u\|_{L^\infty(\Omega)} \leq C\left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{2.4}$$

$$\|\nabla u\|_{L^6(\Omega)} \leq C\left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{2.5}$$

$$\|\nabla u\|^2_{L^4(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}\left(\|u\|^2_{L^2} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{2.6}$$

and for all $u \in H^3(\Omega)$ *such that* $\dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$,

$$\|D^2 u\|_{L^3(\Omega)} \leq C\left(\left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} + \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{4}}\|\nabla\Delta u\|^{\frac{1}{2}}_{L^2(\Omega)}\right). \tag{2.7}$$

Proof. In the estimates (2.4)-(2.5) we use (2.1) and classical embedding theorem.
Estimate (2.6) is the well known Gagliardo-Nirenberg inequality, and (2.7) is the usual embeding of $H^{1/2}(\Omega)$ in $L^3(\Omega)$.

2.2 Study of the operator H

We consider the operator $u \mapsto H(u)$ defined by (1.2). It satisfies

$$\begin{cases} H(u) \in L^2(\mathbb{R}^3), & \\ \operatorname{curl} H(u) = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}\left(H(u) + \bar{u}\right) = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\bar{u}$ is the extension of u by zero outside $\overline{\Omega}$.
We observe that $u \mapsto -H(u)$ is the orthogonal projection of $\bar{u}$ on the vector fields of gradients in $L^2(\mathbb{R}^3)$. Classicaly, we have

$$\|H(u)\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad 1 < p < +\infty. \tag{2.8}$$

Following Ladyshenskaya [10] page 196 we can derive the following regularity result

Lemma 2.3 *Let $p \in]1, +\infty[$. Then, if u belongs to $W^{1,p}(\Omega)$ (resp. $W^{2,p}(\Omega)$), the restriction of $H(u)$ to Ω belongs to $W^{1,p}(\Omega)$ (resp. $W^{2,p}(\Omega)$) and there exists a constant C such that*

$$\|H(u)\|_{W^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{2.9}$$

and

$$\|H(u)\|_{W^{2,p}(\Omega)} \leq C\|u\|_{W^{2,p}(\Omega)}. \tag{2.10}$$

Proof : as $\operatorname{curl} H = 0$ in $\mathbb{R}^3$ we can assume that H is gradient vector field

$$H = -\nabla\psi.$$

So we have to solve

$$\begin{cases} -\Delta\psi = -\operatorname{div} u & \text{in } \Omega, \\ -\Delta\psi = 0 & \text{in } \Omega', \\ [\psi]_{|\partial\Omega} = 0, \quad \left[\dfrac{\partial\psi}{\partial\nu}\right]_{|\partial\Omega} = u \cdot \nu, & \end{cases} \tag{2.11}$$

where $\Omega' = {}^C\overline{\Omega}$ and $[\psi]\,|_{\partial\Omega}$ is the jump of ψ across $\partial\Omega$.

First step : $W^{1,p}$ regularity.

The main idea is to reduce the problem to an homogeneous problem in $\mathbb{R}^3$.
By classical properties of the trace operator, for u belonging to $W^{k,p}(\Omega)$, there exists a function ψ_1 in $W^{k+1,p}(\Omega_2 \setminus \overline{\Omega})$ such that

$$\psi_1|_{\partial\Omega} = 0\ , \ \psi_1|_{\partial\Omega_2'} = 0\ , \ \left.\frac{\partial\psi_1}{\partial\nu}\right|_{\partial\Omega_2'} = 0\ , \ \left.\frac{\partial\psi_1}{\partial\nu}\right|_{\partial\Omega} = u \cdot \nu.$$

Obviously, one has, for some constante c

$$\|\psi_1\|_{W^{k+1,p}(\Omega_2\setminus\overline{\Omega})} \leq c\,\|u\|_{W^{k,p}(\Omega)}\,.$$

So, it is equivalent to find a ψ solution of (2.11) or to find a φ solution of the following homogeneous problem

$$\begin{cases} -\Delta\varphi = -\operatorname{div} u & \text{in } \Omega, \\ -\Delta(\varphi+\psi_1) = 0 & \text{in } \Omega_2 \setminus \overline{\Omega}, \\ -\Delta\varphi = 0 & \text{in } \Omega_2' \\ [\varphi]|_{\partial\Omega_2} = 0 \quad \left[\dfrac{\partial\varphi}{\partial\nu}\right]\Big|_{\partial\Omega_2} = 0, \end{cases} \tag{2.12}$$

In the sequel we use the following notations : $f_1 = -\operatorname{div} u \, \mathbb{I}_\Omega$, $f_2 = \Delta\psi_1 \, \mathbb{I}_{\Omega_2\setminus\overline{\Omega}}$ and $f = f_1 + f_2$. With these notations, φ is solution of

$$-\Delta\varphi = f \quad \text{in } \mathbb{R}^3, \tag{2.13}$$

and classical regularity results for the Dirichlet problem imply that there exists $\varphi \in W^{2,p}(\mathbb{R}^3)$ as soon as $f \in L^p(\mathbb{R}^3)$. So $\nabla\psi = \nabla\varphi$ belongs to $W^{1,p}(\Omega)$.

Second step : $W^{2,p}$ regularity.

If we assume that $f_1 \in W^{1,p}(\Omega)$ and that $f_2 \in W^{1,p}(\Omega_2 \setminus \overline{\Omega})$, we can show that φ belongs to $W^{3,p}(\Omega) \bigcap W^{3,p}(\Omega_2 \setminus \overline{\Omega}) \bigcap W^{3,p}(\Omega_2')$.
Let us differentiate (2.13) in Ω, in $\Omega_2 \setminus \overline{\Omega}$ and in Ω_2' , we find

$$\begin{cases} -\Delta\dfrac{\partial\varphi}{\partial x} = \dfrac{\partial f_1}{\partial x} \text{ in } \Omega \\ -\Delta\dfrac{\partial\varphi}{\partial x} = \dfrac{\partial f_2}{\partial x} \text{ in } \Omega_2 \setminus \overline{\Omega}. \\ -\Delta\dfrac{\partial\varphi}{\partial x} = 0 \text{ in } \Omega_2'. \end{cases}$$

We observe that as $[\varphi]\,|_\Omega = 0$ and $\left[\dfrac{\partial\varphi}{\partial\nu}\right]\Big|_\Omega = 0$, then $\left[\dfrac{\partial\varphi}{\partial x}\right]\Big|_\Omega = 0$.

It remains to study the regularity of $\left[\dfrac{\partial}{\partial\nu}\left(\dfrac{\partial\varphi}{\partial x}\right)\right]\Big|_\Omega$.

We claim that $\left[\dfrac{\partial}{\partial\nu}\left(\dfrac{\partial\varphi}{\partial x}\right)\right]\Big|_{\Omega_2}$ has the same regularity as $(f_2 - f_1)|_{\partial\Omega_2} = [\Delta\varphi]|_{\partial\Omega_2}$. This is obvious in the half space, and for regular open set, we have by local chart the following properties.

- For any tangential vector field τ, $\left[\dfrac{\partial}{\partial\tau}\left(\dfrac{\partial\varphi}{\partial x}\right)\right]\Big|_{\partial\Omega_2} = 0$
- $\left[\dfrac{\partial}{\partial\nu}\left(\dfrac{\partial\varphi}{\partial x}\right)\right]\Big|_{\partial\Omega_2}$ is equal to some linear expression between $\left[\dfrac{\partial}{\partial\tau}\left(\dfrac{\partial\varphi}{\partial x}\right)\right]\Big|_{\partial\Omega_2} = 0$ and $[\Delta\varphi]|_{\partial\Omega_2}$.

So $\left[\dfrac{\partial}{\partial\nu}\left(\dfrac{\partial\varphi}{\partial x}\right)\right]\Big|\,|_{\partial\Omega_2}$ has the same regularity as $(f_2 - f_1)|_{\partial\Omega_2}$.

We have proved that $\dfrac{\partial\varphi}{\partial x}$ satisfies

$$
\begin{cases}
-\Delta \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial f_1}{\partial x} \text{ in } \Omega, \\[2ex]
-\Delta \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial f_2}{\partial x} \text{ in } \Omega_2 \setminus \overline{\Omega}, \\[2ex]
-\Delta \dfrac{\partial \varphi}{\partial x} = 0 \text{ in } \Omega_2' \\[2ex]
\left[\dfrac{\partial \varphi}{\partial x}\right]\Big|_{\partial\Omega_2} = 0\ , \ \left[\dfrac{\partial}{\partial \nu}\left(\dfrac{\partial \varphi}{\partial x}\right)\right]\Big|_{\partial\Omega_2} = g,
\end{cases}
$$

where g belongs to $W^{1-1/p,p}(\partial\Omega_2)$. Thus we conclude as in the previous step.

2.3 Comparison Lemma

We recall without proof a classical comparison Lemma.

Lemma 2.4 *Let $f : I\!R \times I\!R \longrightarrow I\!R$, $\mathcal{C}^1$ be non decreasing in its second variable. Assume moreover that $y : I \subset I\!R \longrightarrow I\!R$ is a continuous function satisfying :*

$$\forall\, t > 0, \ \ y(t) \leq y_0 + \int_0^t f(\tau, y(\tau))d\tau.$$

Let $z : J \longrightarrow I\!R$ be the solution of

$$
\begin{cases}
z'(t) = f(t, z(t)), \\
z(0) = y_0,
\end{cases}
$$

Then $\forall\, t > 0, \ \ y(t) \leq z(t)$.

3 Proof of Theorem 1.1

Taking formaly the inner product in $L^2(\Omega)$ of (1.1) with Δu makes appear a dissipative term of the form $\|u \wedge \Delta u\|_{L^2(\Omega)}$. This dissipation is not sufficient to obtain energy estimate in $H^2(\Omega)$.

We observe then, that for u regular enough and $|u| = 1$ in Ω, the system (1.1) is equivalent to:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \Delta u = |\nabla u|^2 u + u \wedge \Delta u + u \wedge H(u) - u \wedge (u \wedge H(u)) & \text{in } I\!R_t^+ \times \Omega, \\[2ex]
\dfrac{\partial u}{\partial \nu} = 0 & \text{on } I\!R_t^+ \times \partial\Omega, \\[2ex]
u(0, \cdot) = u_0 & \text{in } \Omega.
\end{cases}
\tag{3.1}
$$

This equation appears to be more convenient to build regular approximate solutions of (1.1), provide we can show *a posteriori* that $|u| \equiv 1$. This property results from the uniqueness of the following parabolic equation

$$\begin{cases} \dfrac{\partial a}{\partial t} - \Delta a - 2|\nabla u|^2(a-1) = 0, \\ \\ \dfrac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \\ a(0,\cdot) = a_0 = 1, \end{cases}$$

where $a = |u|^2$ for $u \in L^\infty\left(0, T; H^2(\Omega)\right)$.

3.1 Resolution of (3.1)

First step : approximate problem.

We denote by V_n the finite dimension space built on the n first eigen-functions of $-\Delta + Id$ with domain $D(A) = \left\{ u \in H^2(\Omega),\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$, and by P_n the orthogonal projection from $L^2(\Omega)$ on V_n.

So we seek a solution u_n in V_n of

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \Delta u_n - P_n \left[|\nabla u_n|^2 u_n + u_n \wedge \left(\Delta u_n + H\left(u_n\right)\right) - u_n \wedge \left(u_n \wedge H\left(u_n\right)\right) \right] = 0. \\ \\ u_n(0) = P_n(u_0) \end{cases} \tag{3.2}$$

Thanks to the Cauchy-Lipschitz Theorem, there exists an unique solution of (3.2) defined on $[0, T_n[$.

Second step : L^2 estimate for the approximate solution.

Taking the inner product in $L^2(\Omega)$ of (3.2) by u_n, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2_{L^2(\Omega)}\right) + \|\nabla u_n\|^2_{L^2(\Omega)} \leq \|u_n\|^2_{L^\infty(\Omega)} \|\nabla u_n\|^2_{L^2(\Omega)} \tag{3.3}$$

Third step : H^2 estimate for the approximate solution.

We take the inner product of (3.2) by $\Delta^2 u_n$, and we integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta u_n(t)\|^2_{L^2(\Omega)}\right) + \|\nabla \Delta u_n(t)\|^2_{L^2(\Omega)} = I_1 + I_2 + I_3 + I_4$$

with

$$I_1 = \int_\Omega \nabla\left(|\nabla u_n(t)|^2 u_n(t)\right) \nabla\Delta u_n(t) dx,$$

$$I_2 = \int_\Omega \nabla\Delta\left(u_n(t) \wedge \Delta u_n(t)\right) \nabla\Delta u_n(t) dx,$$

$$I_3 = \int_\Omega \nabla\left(u_n(t) \wedge H\left(u_n(t)\right)\right) \nabla\Delta u_n(t) dx,$$

$$I_4 = -\int_\Omega \nabla\left(u_n(t) \wedge \left(u_n(t) \wedge H\left(u_n(t)\right)\right)\right) \nabla\Delta u_n(t) dx.$$

We bound separately each term.

• Estimate on I_1

$$|I_1| \leq C_1 I_{11} + C_2 I_{12}$$

with

$$I_{11} = \int_\Omega |\nabla u_n|^3 |\nabla \Delta u_n| dx,$$

and

$$I_{12} = \int_\Omega |D^2 u_n||\nabla u_n||u_n||\nabla \Delta u_n| dx,$$

where $D^p u$ denotes the collection of all derivatives of order exactly p.
Using Lemma 2.2, we obtain

$$I_{11} \leq \|\nabla u_n\|^3_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

$$\leq C \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{3}{2}} \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

from (2.5).

Furthermore

$$|I_{12}| \leq \|u_n\|_{L^\infty(\Omega)} \|D^2 u_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

$$\leq \|u_n\|_{L^\infty(\Omega)} \left(\left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} + \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{1}{4}} \|\nabla \Delta u_n\|_{L^2(\Omega)} \right)$$
$$\times \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\nabla \Delta u_n\|_{L^2(\Omega)}$$

$$\leq C \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{3}{2}} \|\nabla \Delta u_n\|_{L^2(\Omega)} + C \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{5}{4}} \|\nabla \Delta u_n\|^{\frac{3}{2}}_{L^2(\Omega)},$$

with(2.4), (2.5) and (2.7).

• Estimate on I_2
By Sobolev embeddings and interpolation, I_2 is bounded as follow

$$|I_2| \leq \|\nabla u_n\|_{L^6(\Omega)} \|\Delta u_n\|_{L^3(\Omega)} \| \, \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

$$\leq C \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right) \|\nabla \Delta u_n\|_{L^2(\Omega)} + C \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right)^{\frac{3}{4}} \|\nabla \Delta u_n\|^{\frac{3}{2}}_{L^2(\Omega)},$$

from (2.5) and (2.7).

• Estimate on I_3

$$I_3 = \int_\Omega \left(\nabla u_n \wedge H(u_n) \right) \nabla \Delta u_n dx + \int_\Omega \left(u_n \wedge \nabla H(u_n) \right) \nabla \Delta u_n dx.$$

$$|I_3| \leq \left(\|\nabla u_n\|_{L^6(\Omega)} \|H(u_n)\|_{L^3(\Omega)} + \|u_n\|_{L^3(\Omega)} \|\nabla H(u_n)\|_{L^6(\Omega)} \right) \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

$$|I_3| \leq C \|\nabla \Delta u_n\|_{L^2(\Omega)} \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} \right),$$

from (2.8), (2.9) and (2.5).

• Estimate on I_4

$$I_4 = I_{41} + I_{42} + I_{43},$$

with

$$I_{41} = \int_\Omega \nabla u_n \wedge (u_n \wedge H(u_n))\, \nabla \Delta u_n dx,$$
$$I_{42} = \int_\Omega u_n \wedge (\nabla u_n \wedge H(u_n)) \nabla \Delta u_n dx,$$
$$I_{43} = \int_\Omega u_n \wedge (u_n \wedge \nabla H(u_n))\, \nabla \Delta u_n dx,$$

We bound separately each term.

$$|I_{41}| + |I_{42}| \leq 2 \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|H(u_n)\|_{L^3(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)}$$

$$\leq C \|u_n\|^2_{L^\infty(\Omega)} \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \|\nabla \Delta u_n\|_{L^2(\Omega)}$$

from (2.5), (2.8).

$$|I_{43}| \leq C \|u_n\|^2_{L^\infty(\Omega)} \|\nabla H(u_n)\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

$$\leq C \|u_n\|^2_{L^\infty(\Omega)} \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \|\nabla \Delta u_n\|_{L^2(\Omega)},$$

Summing the estimates on I_1, I_2, I_3 and I_4, and using (2.4), we obtain that there exists a constant C independent of u_0 and n such that

$$\begin{gathered}\frac{d}{dt}\left(\|\Delta u_n\|_{L^2(\Omega)}\right)^2 + 2\|\nabla \Delta u_n\|^2_{L^2(\Omega)} \leq \\ C\left(1 + \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^{\frac{5}{4}}\right) \|\nabla \Delta u_n\|^{\frac{3}{2}}_{L^2(\Omega)}.\end{gathered} \tag{3.4}$$

Fourth step : limit when n goes to $+\infty$.

Summing (3.3) and (3.4) and absorbing $\|\nabla \Delta u_n\|_{L^2(\Omega)}$, one finds a constant k_1 such that

$$\begin{gathered}\frac{d}{dt}\left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right) + \|\nabla \Delta u_n\|^2_{L^2(\Omega)} \leq \\ k_1\left(1 + \left(\|u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right)^5\right).\end{gathered} \tag{3.5}$$

Using the comparison Lemma, we obtain that there exist a time T^* and a constant C depending on the size of the initial data in $H^2(\Omega)$, but independent of n, such that for any $T < T^\star$

$$\sup_{t \leq T} \|u_n(t)\|^2_{H^2(\Omega)} \leq C,$$

$$\int_0^T \left(\|\nabla u_n(\tau)\|_{L^2}^2 + \|\nabla\Delta u_n(\tau)\|_{L^2}^2\right) d\tau \leq C,$$

and also, by the equation (3.2),

$$\sup_{t\leq T} \|\frac{\partial u_n}{\partial t}(t)\|_{L^2(\Omega)}^2 \leq C,$$

$$\int_0^T \|\frac{\partial}{\partial t}\nabla u_n(\tau)\|_{L^2(\Omega)}^2 d\tau \leq C.$$

Hence, we obtain the existence of a subsequence u_{n_k} and a function u such that

$$\begin{cases} u_{n_k} \rightharpoonup u \text{ in } L^2(0,T;H^3(\Omega)) & \text{weak,} \\ u_{n_k} \rightharpoonup u \text{ in } L^\infty(0,T;H^2(\Omega)) & \text{weak}\star, \\ \dfrac{\partial u_{n_k}}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \text{ in } L^2(0,T;H^1(\Omega)) & \text{weak,} \end{cases}$$

and according to Aubin's lemma we can assume that

$$u_{n_k} \longrightarrow u \text{ in } L^2(0,T;H^2(\Omega)) \text{ strong.}$$

And so,

$$u_{n_k} \longrightarrow u \text{ in } L^p(0,T;H^2(\Omega)) \text{ strong}, \quad 1 < p < \infty$$

Moreover, as H is a continuous map on $H^m(\Omega)$, for $m = 0, 1, 2$, one has

$$H(u_{n_k}) \longrightarrow H(u) \text{ in } L^p(0,T;H^2(\Omega)) \quad \text{strong}, \quad 1 < p < \infty$$

So, we can take the limit in (3.2), and we obtain that u satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = |\nabla u|^2 u + u \wedge (\Delta u + H(u)) - u \wedge (u \wedge H(u)) \text{ on } [0,T^\star[\times\Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } [0,T^\star[\times\partial\Omega, \\ u(0) = u_0 \end{cases}$$

3.2 Conservation of the ponctual norm

Taking the scalar product in $I\!R^3$ of (3.1) by u , we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 - (u\cdot\Delta u) - |\nabla u|^2 |u|^2 = 0 \text{ in } (0,T)\times\Omega. \tag{3.6}$$

As u belongs to $L^\infty((0,T);H^2(\Omega))$ the following identity is valid for $d \leq 3$:

$$\Delta|u|^2 = 2(u\cdot\Delta u) + 2|\nabla u|^2,$$

so (3.6) becomes

$$\frac{d}{dt}|u|^2 - \Delta|u|^2 - 2|\nabla u|^2(|u|^2 - 1) = 0.$$

Let us note by $b = |u|^2 - 1$. We have proved that b solves

$$\begin{cases} \dfrac{\partial b}{\partial t} - \Delta b - 2|\nabla u|^2 b = 0 \\ \\ \dfrac{\partial b}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \\ b(0) = |u_0|^2 - 1 = 0 \end{cases} \tag{3.7}$$

Now, we remark that $|\nabla u|^2$ belongs to $L^1(0,T;L^\infty(\Omega))$ since $H^2(\Omega) \subset L^\infty(\Omega)$. Hence the energy estimate associated to (3.7) gives

$$\frac{d}{dt}\|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \leq \|\nabla u\|_{L^\infty}^2 \|b\|_{L^2}^2$$

and we conclude that $\|b\|_{L^2}^2 = 0$ through Gronwall inequality.

So we have proved that $|u| = 1$ in $[0,T] \times \Omega$, as soon as $|u_0| = 1$ on Ω.

Now, if $|u| \equiv 1$, then (3.1) is equivalent to (1.1). Hence, the proof of Theorem 1.1 is fulfilled.

4 Stability Results

Let us denote by u_1 and u_2 two solutions of (3.1), $T^\star = \min(T_1^\star, T_2^\star)$, and $v = u_1 - u_2$. Then, we have the following proposition

Proposition 4.1 *For all $T < T^\star$, there exists a constant C such that*

$$\sup_{t \leq T} \|v(t)\|_{L^2}^2 \leq C\|v(0)\|_{L^2}^2.$$

Furthermore, we can prove the following H^2 stability result

Proposition 4.2 *For all $T < T^\star$ there exists a constant C such that*

$$\sup_{t \leq T} \left(\|v(t)\|_{L^2(\Omega)}^2 + \|\Delta v(t)\|_{L^2(\Omega)}^2\right) \leq C\left(\|v(0)\|_{L^2(\Omega)}^2 + \|\Delta v(0)\|_{L^2(\Omega)}^2\right).$$

and such that

$$\|v\|_{L^2(0,T;H^3(\Omega))} \leq C\left(\|v(0)\|_{L^2(\Omega)}^2 + \|\Delta v(0)\|_{L^2(\Omega)}^2\right).$$

4.1 Uniqueness and L^2 stability

Proof : the difference v satisfies the following equation

$$\begin{aligned} &\frac{\partial v}{\partial t} - \Delta v = v \wedge (\Delta u_1 + H(u_1)) + u_2 \wedge (\Delta v + H(v)) \\ &+|\nabla u_1|^2 v + \left(|\nabla u_1|^2 - |\nabla u_2|^2\right) u_2 - v \wedge (u_1 \wedge H(u_1)) \\ &-u_2 \wedge (v \wedge H(u_1) + u_2 \wedge H(v)). \end{aligned} \tag{4.1}$$

Taking the inner product in L^2 of (4.1) by v, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2_{L^2(\Omega)}\right)+\|\nabla v\|^2_{L^2(\Omega)}\leq\int_\Omega(u_2\wedge\Delta v)\,vdx+\int_\Omega(u_2\wedge H(v))\,vdx$$

$$+\int_\Omega|\nabla u_1|^2|v|^2dx+\int_\Omega(|\nabla u_1|+|\nabla u_2|)\,|\nabla v||u_2||v|dx$$

$$-\int_\Omega u_2\wedge(v\wedge H(u_1))\,vdx-\int_\Omega u_2\wedge(u_2\wedge H(v))\,vdx.$$

After an integration by parts of the first term of the right-hand side of the equation above, we obtain, as $|u_1|\equiv 1$ and $|u_2|\equiv 1$,

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2_{L^2(\Omega)}\right)+_2\|\nabla v\|^2_{L^2(\Omega)}\leq\|v\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|\nabla u_2\|_{L^\infty(\Omega)}$$

$$+C\left(1+\|\nabla u_1\|^2_{L^\infty(\Omega)}\right)\|v\|^2_{L^2(\Omega)}+\|H(u_1)\|_{L^\infty(\Omega)}\|v\|^2_{L^2(\Omega)}$$

$$+C\left(\|\nabla u_1\|_{L^\infty(\Omega)}+\|\nabla u_2\|_{L^\infty(\Omega)}\right)\|\nabla v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\,.$$

As u_1 and u_2 are bounded in $L^\infty(0,T;H^2(\Omega))\cap L^2(0,T;H^3(\Omega))$, we obtain that there exists a function f belonging to $L^1(0,T)$ such that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2}+\|\nabla v\|^2_{L^2}\leq f(t)\|v\|^2_{L^2}. \tag{4.2}$$

The end of the proof of Proposition 4.1 follows from Gronwall lemma.

4.2 H^2 stability

We go back to Galerkin approximation of (4.1). Taking the inner product of this approximation with $\Delta^2 v_n$, integrating by parts on Ω, integrating in time between 0 and t, and taking the limit when n tends to $+\infty$, we obtain the following inequality, using the lower semi-continuity of the norm under the weak topology

$$\frac{1}{2}\|\Delta v(t)\|^2_{L^2(\Omega)}+\int_0^t\|\nabla\Delta v\|^2_{L^2(\Omega)}\leq\frac{1}{2}\|\Delta v_0\|^2_{L^2(\Omega)}+\int_0^t(I_1+\ldots+I_8)(s)ds$$

where $I_1,\ldots,I_8$ are eight terms which we bound separately without details

- $I_1=\left|\int_\Omega\nabla(v\wedge\Delta u_1)\,\nabla\Delta vdx\right|\leq g_1(t)\left(\|v\|^2_{L^2(\Omega)}+\|\Delta v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\|\nabla\Delta v\|_{L^2(\Omega)}\,,$
  where $g_1\in L^2(0,T)$.

- $I_2=\left|\int_\Omega\nabla(v\wedge H(u_1))\,\nabla\Delta vdx\right|\leq g_2(t)\left(\|v\|^2_{L^2(\Omega)}+\|\Delta v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\|\nabla\Delta v\|_{L^2(\Omega)}\,,$
  where $g_2\in L^\infty(0,T)$.

- $I_3=\left|\int_\Omega\nabla(u_2\wedge\Delta v)\,\nabla\Delta vdx\right|\leq g_3(t)\left(\|v\|^2_{L^2(\Omega)}+\|\Delta v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\|\nabla\Delta v\|_{L^2(\Omega)}\,,$
  where $g_3\in L^4(0,T)$.

- $I_4 = \left| \int_\Omega \nabla \left(u_2 \wedge H(v) \right) \nabla \Delta v dx \right| \leq g_4(t) \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\nabla \Delta v\|_{L^2(\Omega)}$, where $g_4 \in L^\infty(0,T)$.

- $I_5 = \left| \int_\Omega \nabla \left(|\nabla u_1|^2 v \right) \nabla \Delta v dx \right| \leq g_5(t) \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\nabla \Delta v\|_{L^2(\Omega)}$, where $g_5 \in L^1(0,T)$.

- $I_6 = \left| \int_\Omega \nabla \left(\left(|\nabla u_1|^2 - |\nabla u_2|^2 \right) u_2 \right) \nabla \Delta v dx \right|$
$\leq g_6(t) \left(\left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right) + \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\nabla \Delta v\|_{L^2(\Omega)} \right)$
where $g_6 \in L^4(0,T)$.

- $I_7 = \left| \int_\Omega \nabla \left(v \wedge (u_1 \wedge H(u_1)) \right) \nabla \Delta v dx \right| \leq g_7(t) \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\nabla \Delta v\|_{L^2(\Omega)}$, where $g_7 \in L^\infty(0,T)$.

- $I_8 = \left| \int_\Omega \nabla \left(u_2 \wedge (v \wedge H(u_1) + u_2 \wedge H(v)) \right) \nabla \Delta v dx \right| \leq g_8(t) \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right)$, where $g_8 \in L^\infty(0,T)$.

Furthermore, using Young inequality, we get the existence of a function denoted f lying in $L^1(0,T)$ such that

$$\begin{aligned} \|\Delta v(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla \Delta v(s)\|^2_{L^2(\Omega)} \, ds \leq \|\Delta v_0\|^2_{L^2(\Omega)} \\ + \int_0^t f(s) \left(\|v(s)\|^2_{L^2(\Omega)} + \|\Delta v(s)\|^2_{L^2(\Omega)} \right) ds. \end{aligned} \tag{4.3}$$

Then, integrating (4.2) and summing with (4.3) we obtain

$$\left(\|v(t)\|^2_{L^2(\Omega)} + \|\Delta v(t)\|^2_{L^2(\Omega)} \right) + 2\int_0^t \left(\|\Delta v(s)\|^2_{L^2(\Omega)} + \|\nabla \Delta v(s)\|^2_{L^2(\Omega)} \, ds \right) \leq$$

$$\left(\|v_0\|^2_{L^2(\Omega)} + \|\Delta v_0\|^2_{L^2(\Omega)} \right) + \int_0^t f(s) \left(\|v(s)\|^2_{L^2(\Omega)} + \|\Delta v(s)\|^2_{L^2(\Omega)} \right) ds$$

Using Gronwall Lemma, we derive the proof of Proposition 4.2 and Theorem 1.2.

5 Proof of Theorem 1.4

We deal now with the problem (1.3)

$$
(1.3) \quad \begin{cases} \dfrac{\partial u}{\partial t} = u \wedge \Delta u - u \wedge (u \wedge \Delta u) \\[2ex] \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\[2ex] u(0) = u_0, \\[2ex] |u| \equiv 1. \end{cases}
$$

Under Assumption (H), the proof of local existence of regular solutions (see Theorem 1.3) is now straightforward : it is a simplification of the proof of Theorem 1.1.
Now, in the 2D case, we can improve the previous result by showing global existence for small data.

First step : energy estimate on ∇u.

We observe that as $|u| \equiv 1$, the first equation of (5.4) is equivalent to

$$
\frac{\partial u}{\partial t} - u \wedge \Delta u + u \wedge (u \wedge \Delta u) = 0 \tag{5.1}
$$

and to

$$
\frac{\partial u}{\partial t} + u \wedge \frac{\partial u}{\partial t} - 2u \wedge \Delta u = 0. \tag{5.2}
$$

Now multiplying (5.1) by $\dfrac{\partial u}{\partial t}$ and (5.2) by $-2\Delta u$ we get

$$
\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\frac{\partial u}{\partial t}(\tau)\|^2_{L^2(\Omega)} d\tau = \|\nabla u_0\|^2_{L^2(\Omega)}. \tag{5.3}
$$

Second step : estimate on Δu.

We know that for regular solutions, (1.3) is equivalent to the following problem

$$
\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = -|\nabla u|^2 u + u \wedge \Delta u \text{ on } [0,T] \times \Omega \\[2ex] \dfrac{\partial u}{\partial \nu} = 0 \text{ on } [0,T] \times \partial\Omega \\[2ex] u(0) = u_0, \\[2ex] |u| = 1 \text{ on } [0,T] \times \Omega. \end{cases} \tag{5.4}
$$

Taking the inner product in $L^2(\Omega)$ of the first equation of (5.4) by Δu, we obtain since $(u \cdot \Delta u) = -|\nabla u|^2$,

$$
\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^4_{L^4(\Omega)}. \tag{5.5}
$$

In two dimensional space, the following Sobolev estimate is valid :

$$\begin{aligned}
&\|\nabla u\|_{L^4(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)}^{1/2} \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2\right)^{1/4}, \\
&\|\nabla u\|_{L^6(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)}^{1/3} \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2\right)^{1/3}, \\
&\|\nabla u\|_{L^\infty(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)}^{1/2} \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla \Delta u\|_{L^2(\Omega)}^2\right)^{1/4}, \\
&\|\Delta u\|_{L^4(\Omega)} \leq C(\Omega)\|\Delta u\|_{L^2(\Omega)}^{1/2} \left(\|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla \Delta u\|_{L^2(\Omega)}^2\right)^{1/4}.
\end{aligned} \tag{5.6}$$

So inequality (5.5) gives

$$\frac{d}{dt}\|\nabla u\|_{L^2(\Omega)}^2 + \left(1 - C_1\|\nabla u\|_{L^2(\Omega)}^2\right)\|\Delta u\|_{L^2(\Omega)}^2 \leq C_2\|\nabla u\|_{L^2(\Omega)}^4. \tag{5.7}$$

Using (5.3), we obtain that

$$\frac{d}{dt}\|\nabla u\|_{L^2(\Omega)}^2 + \left(1 - C_1\|\nabla u_0\|_{L^2(\Omega)}^2\right)\|\Delta u\|_{L^2(\Omega)}^2 \leq C_2\|\nabla u_0\|_{L^2(\Omega)}^4. \tag{5.8}$$

Integrating (5.8) between 0 and t, using (5.8), we obtain that if $\|\nabla u_0\|_{L^2(\Omega)}^2 \leq \dfrac{1}{2C_1}$, then

$$\int_0^t \|\Delta u(\tau)\|_{L^2(\Omega)}^2 \, d\tau \leq \|\nabla u_0\|_{L^2(\Omega)}^2 + C_2 \|\nabla u_0\|_{L^2(\Omega)}^4 \; t. \tag{5.9}$$

Third step : estimate on $\nabla \Delta u$.

As in Section 3.1, we build regular solutions of (5.4) using a Galerkin approximation process. We seek a solution u_n in V_n to

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \Delta u_n = P_n\left(|\nabla u_n|^2 u_n + u_n \wedge \Delta u_n\right), \\ u_n(0) = P_n(u_0). \end{cases} \tag{5.10}$$

Taking the inner product of (5.10) by $\Delta^2 u_n$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta u_n\|_{L^2(\Omega)}^2\right) + \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 \leq \|\nabla u_n\|_{L^6(\Omega)}^3 \|\nabla \Delta u_n\|_{L^2(\Omega)} \\
&+ \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^\infty(\Omega)} \left(\|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2\right)^{1/2} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\
&+ \|\nabla u_n\|_{L^4(\Omega)} \|\Delta u_n\|_{L^4(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)}.
\end{aligned}$$

According to inequality (5.6), we get

$$\frac{d}{dt}\|\Delta u_n\|^2_{L^2(\Omega)} + \|\nabla\Delta u_n\|^2_{L^2(\Omega)} \leq C\,\|\nabla u_n\|_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\} \|\nabla\Delta u_n\|_{L^2(\Omega)}$$

$$+C\,\|u_n\|_{L^\infty(\Omega)}\,\|\nabla u_n\|^{1/2}_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)} + \|\nabla\Delta u_n\|^2_{L^2(\Omega)}\right\}^{1/4}$$

$$\times \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\}^{1/2} \|\nabla\Delta u_n\|_{L^2(\Omega)}$$

$$+C\,\|\nabla u_n\|^{1/2}_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\}^{1/4}$$

$$\times \left\{\|\Delta u_n\|^2_{L^2(\Omega)} + \|\nabla\Delta u_n\|^2_{L^2(\Omega)}\right\}^{1/4} \|\Delta u_n\|^{1/2}_{L^2(\Omega)}\,\|\nabla\Delta u_n\|_{L^2(\Omega)}\,.$$

Using Young inequality, and after absorbtion of the higher degree term, we get

$$\frac{d}{dt}\|\Delta u_n\|^2_{L^2(\Omega)} + \|\nabla\Delta u_n\|^2_{L^2(\Omega)} \leq C\,\|\nabla u_n\|^2_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\}^2$$

$$+C\,\|u_n\|^2_{L^\infty(\Omega)}\,\|\nabla u_n\|_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + C\,\|\Delta u_n\|^2_{L^2(\Omega)}\right\}^{3/2}$$

$$+C\,\|u_n\|^4_{L^\infty(\Omega)}\,\|\nabla u_n\|^2_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\}^2$$

$$+C\,\|\nabla u_n\|_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\}^{1/2} \|\Delta u_n\|^2_{L^2(\Omega)}$$

$$+C\,\|\nabla u_n\|^2_{L^2(\Omega)} \left\{\|\nabla u_n\|^2_{L^2(\Omega)} + \|\Delta u_n\|^2_{L^2(\Omega)}\right\} \|\Delta u_n\|^2_{L^2(\Omega)}\,.$$

We perform an integration in time of the previous equation. As $|u| \equiv 1$, $\|\nabla u(t)\|_{L^2} \leq \|\nabla u_0\|_{L^2}$ and as the norms are lower semi continuous for the weak topology we obtain
a constant k_4 such that

$$\|\Delta u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\Delta u(\tau)\|^2_{L^2(\Omega)} d\tau \leq \|\Delta u_0\|_{L^2(\Omega)} + k_4 \int_0^t (1 + \|\Delta u(\tau)\|^4_{L^2(\Omega)}) d\tau \qquad (5.11)$$

Fourth step : conclusion.

Now for $\|\nabla u_0\|^2_{L^2(\Omega)} \leq \dfrac{1}{2C_1}$, we have obtain inequality (5.9) in the second step.

So we can apply Gronwall lemma to (5.11) to obtain

$$\forall\, t \leq T < T^\star,\;\; \|\Delta u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\Delta u(\tau)\|^2_{L^2(\Omega)} d\tau \leq h(t)$$

where h is a nonnegative continuous function on $\mathbb{R}_+$. So the solution given by theorem 1.3 is global as soon as

$$\|\nabla u_0\|^2_{L^2(\Omega)} \leq \frac{1}{2C_1}.$$

This ends the proof of Theorem 1.4.

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