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► **To cite this version:**

Hossein Abbaspour, Thomas Tradler, Mahmoud Zeinalian. Algebraic string bracket as a Poisson bracket. *Journal of Noncommutative Geometry*, 2010, 2. hal-00296647v3

HAL Id: hal-00296647

<https://hal.science/hal-00296647v3>

Submitted on 3 Sep 2008

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ALGEBRAIC STRING BRACKET AS A POISSON BRACKET

HOSSEIN ABBASPOUR, THOMAS TRADLER, AND MAHMOUD ZEINALIAN

ABSTRACT. In this paper we construct a Lie algebra representation of the algebraic string bracket on negative cyclic cohomology of an associative algebra with appropriate duality. This is a generalized algebraic version of the main theorem of [AZ] which extends Goldman's results using string topology operations. The main result can be applied to the de Rham complex of a smooth manifold as well as the Dolbeault resolution of the endomorphisms of a holomorphic bundle on a Calabi-Yau manifold.

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1. INTRODUCTION

Goldman original work [Go] on the Lie algebra of free homotopy classes of oriented closed curves on an oriented surface was extensively generalized through the introduction of String Topology by Chas and Sullivan [CS]. In particular, they generalized this Lie bracket to one on the equivariant homology of the free loop space of a compact and oriented manifold M . From the beginning, it was clear that this bracket had a deep relation to the holonomy map on a vector bundle; see [Go, CFP, CCR, CR]. This relation was the subject of a paper, [AZ], by the first and third author. It was shown there that using Chen's iterated integral one obtains a map of Lie algebras from the equivariant homology of the free loop space to the space of functions on a space of generalized flat connections.

Algebraic analogues of string topology Lie algebra have also been considered in recent years. Jones [J] had shown that for a simply connected topological space X the equivariant homology of the free loop space is isomorphic to the negative cyclic cohomology of the algebra of cochains on X . Using this, and Connes long exact sequence relating negative cyclic cohomology and Hochschild cohomology, together with the BV algebra on Hochschild cohomology, Menichi [Men] deduced a

Key words and phrases. Free loop space, Cyclic homology, Maurer Cartan, Symplectic reduction.

Lie bracket on the negative cyclic cohomology in a way similar to the one in string topology [CS, Section 6].

The starting point for this work was to obtain a generalization of the the results in [AZ] and place it in a more algebraic setting where the equivariant homology of the loop space is replaced by negative cyclic cohomology. A suitable setting for this is to consider a unital differential graded algebra A over a field $k = \mathbb{R}$ or \mathbb{C} , with a reasonable trace $\text{Tr} : A \rightarrow k$. Using the results of [T], the above assumptions imply an isomorphism of the Hochschild cohomologies of A with values in A and its dual A^* , $HH^\bullet(A, A) \cong HH^\bullet(A, A^*)$, such that the cup product on $HH^\bullet(A, A)$ and the dual of Connes B -operator on $HH^\bullet(A, A^*)$ make these spaces into a BV algebra. This BV algebra, together with a Connes long exact sequence between the Hochschild cohomology $HH^\bullet(A, A^*)$ and negative cyclic cohomology $HC^\bullet(A)$, imply a Lie algebra structure on $HC^\bullet(A)$ by a theorem of Menichi's [Men, Proposition 7.1], which is based on a similar marking/erasing result of Chas and Sullivan [CS, Theorem 6.1].

Now, using work of Gan and Ginzburg in [GG], we may look at the moduli space of Maurer-Cartan solutions,

$$(1) \quad \mathcal{MC} = \{a \in A^{\text{odd}} \mid da + a \cdot a = 0\} / \sim$$

Since we only consider odd elements, the trace induces a symplectic structure ω on \mathcal{MC} , and thus one can define a Poisson bracket on the function ring $\mathcal{O}(\mathcal{MC})$ of \mathcal{MC} . More details of this construction will be given in Section 3.

We may connect the two sides of the above discussion via a canonical map $\{a \in A^{\text{odd}} \mid da + a \cdot a = 0\} \rightarrow HC_\bullet^-(A)$, $a \mapsto \sum_{n \geq 0} 1 \otimes a^{\otimes n}$, and dualizing this gives a map $\rho : HC_\bullet^-(A) \rightarrow \mathcal{O}(\mathcal{MC})$. We may now compare the two Lie algebras from above. Our main result then states, that the brackets are indeed preserved.

Theorem 1. $\rho : HC_\bullet^-(A) \rightarrow \mathcal{O}(\mathcal{MC})$ is a map of Lie algebras.

In a special case considered in [AZ] this map becomes the generalized holonomy map from the equivariant homology of the free loop space of M to the space of functions on the moduli space of generalized flat connections on a vector bundle $E \rightarrow M$. In fact one has a commutative diagram,

$$(2) \quad \begin{array}{ccc} HC_\bullet^-(A) & \xrightarrow{\rho} & \mathcal{O}(\mathcal{MC}) \\ & \swarrow \sigma & \searrow \Psi \\ & H_{2\bullet}^{S^1}(LM) & \end{array}$$

where Ψ is the generalized holonomy discussed in [AZ] and σ comes from Chen's iterated integral map, as described in Section 5. In particular, for $\dim M = 2$, this recovers Goldman's results on the space of flat connections on a surface.

Another motivation of this work is to study string topology in a holomorphic setting via the moduli stack of the holomorphic structure on a fixed complex bundle $E \rightarrow M$, where M is a complex manifold. Algebraically, this will correspond to the choice of the algebra $A = \Omega^{0,*}(M, \text{End}(E))$, with the Dolbeault differential $\bar{\partial}$. This discussion, once done at the chain level, relates to the algebraic structure of the B-model.

Finally, we remark, that the above discussion generalizes in a straight forward way to the case of a cyclic A_∞ algebra A . This will be the topic of the last Section 6.

In fact, by the same reasoning as above, we obtain the Lie bracket on the negative cyclic cohomology $HC_{\bullet}^{\bullet}(A)$. Also, by symmetrization we may associate an L_{∞} algebra to A , which induces a Maurer-Cartan space similar to (1). We find, that the canonical map ρ is still well-defined, such that Theorem 1 also remains valid in this generalized setting.

Notation: For a map F of complexes, F_{\bullet} (resp. F^{\bullet}) denotes the induced map in homology (resp. cohomology).

Acknowledgments: The authors would like to thank Victor Ginzburg and Luc Menichi for useful discussions and correspondence on this topic. The authors were partially supported by the Max-Planck Institute in Bonn and the second author warmly thanks the Laboratoire Jean Leray at the University of Nantes for their invitation throught the Matpyl program.

2. THE LIE ALGEBRA $HC_{\bullet}^{\bullet}(A)$

In this section, we recall the Lie algebra structure of the negative cyclic cohomology $HC_{\bullet}^{\bullet}(A)$, for a dga (A, d, \cdot) with a trace $\text{Tr} : A \rightarrow k$. The Lie bracket comes from the long exact sequence that relates negative cyclic (co-)homology to Hochschild (co-)homology. For simplicity, we will work in the normalized setting.

Definition 2. Let $(A = \bigoplus_{i \in \mathbb{Z}} A^i, d : A^i \rightarrow A^{i+1}, \cdot)$ be a differential graded associative algebra over a field k , and let $M = \bigoplus_{i \in \mathbb{Z}} M^i$ be a differential graded A -bimodule. The (normalized) Hochschild chain complex defined as,

$$(3) \quad \bar{C}_{\bullet}(A, M) := \prod_{n \geq 0} M \otimes \bar{A}^{\otimes n},$$

where $\bar{A} = A/k$, and s denotes shifting down by one. The boundary $\delta : \bar{C}_{\bullet}(A, M) \rightarrow \bar{C}_{\bullet+1}(A, M)$ is defined by,

$$\begin{aligned} \delta(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &:= \sum_{i=0}^n (-1)^{\epsilon_i} a_0 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n \\ &+ \sum_{i=0}^{n-1} (-1)^{\epsilon_i} a_0 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n - (-1)^{\epsilon'_n} (a_n \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned}$$

where $a_0 \in M$, $a_1, \dots, a_n \in A$, $\epsilon_0 = |a_0|$, $\epsilon_i = (|a_0| + \cdots + |a_{i-1}| + i - 1)$, and $\epsilon'_n = (|a_n| + 1) \cdot (|a_0| + \cdots + |a_{n-1}| + n - 1)$. Note that the differential is well defined; see [L]. Similarly, the (normalized) Hochschild cochain complex is defined by,

$$(4) \quad \bar{C}^n(A, M) := \left\{ f : s\bar{A}^{\otimes n} \rightarrow M \mid f(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n) = 0, \text{ if } a_i = 1 \right\},$$

where the differential $\delta^* : \bar{C}^{\bullet}(A, M) \rightarrow \bar{C}^{\bullet-1}(A, M)$ is given by,

$$\begin{aligned} (\delta^* f)(a_1 \otimes \cdots \otimes a_n) &:= \sum_{i=1}^n (-1)^{|f|+\epsilon_i} f(a_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_n) \\ &+ d(f(a_1 \otimes \cdots \otimes a_n)) + \sum_{i=1}^{n-1} (-1)^{|f|+\epsilon_i} f(a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n) \\ &+ (-1)^{|f|+(|a_1|+1)} a_1 \cdot f(a_1 \otimes \cdots \otimes a_n) + (-1)^{|f|+\epsilon_n} f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n. \end{aligned}$$

The respective (co-)homology theories are denoted by

$$HH_{\bullet}(A, M) = H(\bar{C}_{\bullet}(A, M), \delta), \quad HH^{\bullet}(A, M) = H(\bar{C}^{\bullet}(A, M), \delta^*).$$

Denoting by $A^* = \text{Hom}(A, k)$ the graded dual of A , we see that the dual of $\bar{C}_{\bullet}(A, A)$ is given by $\bar{C}^{\bullet}(A, A^*)$. Recall furthermore, that there is a cup product \cup on $\bar{C}^{\bullet}(A, A)$ defined by

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{m+n}) := f(a_1 \otimes \cdots \otimes a_m) \cdot g(a_{m+1} \otimes \cdots \otimes a_{m+n}).$$

Next, we define the (normalized) negative cyclic chains $\overline{CC}_{\bullet}(A)$ of A to be the vector space $\bar{C}_{\bullet}(A, A)[[u]]$, where u is of degree $+2$, and with differential $\delta + uB$, where $B : \bar{C}_{\bullet}(A, A) \rightarrow \bar{C}_{\bullet-1}(A, A)$ is Connes operator,

$$(5) \quad B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^n (-1)^{\epsilon_i} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1},$$

$$\text{where } \epsilon_i = (|a_i| + \cdots + |a_n| + n - i + 1)(|a_0| + \cdots + |a_{i-1}| + i - 1).$$

Thus, every element of $\overline{CC}_n(A)$ is an infinite sum $\sum_{i=0}^{\infty} a_i u^i \in \bar{C}_{\bullet}(A, A)[[u]]$, where $a_i \in \bar{C}_{n-2i}(A, A)$, δ acts on $a_i \in \bar{C}_{\bullet}(A, A)$, and uB acts as

$$(6) \quad \cdots \xleftarrow{uB} \bar{C}_{\bullet}(A, A) \cdot u^2 \xleftarrow{uB} \bar{C}_{\bullet}(A, A) \cdot u \xleftarrow{uB} \bar{C}_{\bullet}(A, A).$$

Dually, define the (normalized) negative cyclic cochains $\overline{CC}_{-}^{\bullet}(A)$ of A by taking $\overline{CC}_{-}^{\bullet}(A) = \bar{C}^{\bullet}(A, A^*) \otimes k[v, v^{-1}]/vk[v]$, where v is an element of degree -2 . Explicitly, the degree n part $\overline{CC}_{-}^n(A)$ is represented by finite sums $\sum_{i=0}^k a_i v^{-i}$ where $a_i \in \bar{C}^{n-2i}(A, A^*)$. The differential is given by $\delta^* + vB^*$, where δ^* acts on $\bar{C}^{\bullet}(A, A^*)$, and vB^* acts as follows.

$$\cdots \xrightarrow{vB^*} \bar{C}^{\bullet}(A, A^*) \cdot v^{-2} \xrightarrow{vB^*} \bar{C}^{\bullet}(A, A^*) \cdot v^{-1} \xrightarrow{vB^*} \bar{C}^{\bullet}(A, A^*).$$

Note, that if $C_{\bullet}(A, A)$ is finite dimensional in each degree, then the graded dual of $\overline{CC}_{-}^n(A)$ is isomorphic to the chain complex $\overline{CC}_n(A) = \text{Hom}(\overline{CC}_{-}^n(A), k)$, see also [HL, Lemma 3.7]. It is easy to see that $B^2 = \delta B + B\delta = 0$, and we define the associated (co-)homology theories by,

$$HC_{\bullet}^{-}(A) = H(\overline{CC}_{\bullet}^{-}(A), \delta + uB), \quad HC_{-}^{\bullet}(A) = H(\overline{CC}_{-}^{\bullet}(A), \delta^* + vB^*).$$

Lemma 3. *If $H_{\bullet}(A, A)$ is bounded from below, then both $\bar{C}_{\bullet}(A, A)[u]$ and $\bar{C}_{\bullet}(A, A)[[u]]$ with differential $\delta + uB$ calculate negative cyclic homology $HC_{\bullet}^{-}(A)$.*

This lemma follows from a spectral sequence argument for the inclusion $\bar{C}_{\bullet}(A, A)[u] \hookrightarrow \bar{C}_{\bullet}(A, A)[[u]]$, similarly to [HL, Lemma 3.6]. Note, that our sign convention is opposite to the one from [HL], but in agreement with [GJP], since our differential $\delta : \bar{C}_{\bullet}(A, A) \rightarrow \bar{C}_{\bullet+1}(A, A)$ is of degree $+1$.

From now on, we additionally assume, that we also have a suitable trace map.

Definition 4. Let $\text{Tr} : A \rightarrow k$ be a trace map, satisfying $\text{Tr}(da) = 0$ and $\text{Tr}(ab) = -(-1)^{|a| \cdot |b|} \text{Tr}(ba)$, for all $a, b \in A$. Assume furthermore that the map $\omega : A \rightarrow A^*$, $\omega(a)(b) := \text{Tr}(ab)$ is a bimodule map, which induces an isomorphism on homology $H(A) \rightarrow H(A^*)$. By abuse of language, we will also view ω as a map $\omega : A \otimes A \rightarrow k$, $\omega(a, b) = \text{Tr}(ab)$. In this case, A is also called a *symmetric algebra*.

Notice that $\omega : A \rightarrow A^*$ induces a morphism of the Hochschild complexes $\omega_{\sharp} : \bar{C}^{\bullet}(A, A) \rightarrow \bar{C}^{\bullet}(A, A^*)$ via composition $\omega_{\sharp}(f) := \omega \circ f$, which is an isomorphism on

homology $\omega_{\sharp}^{\bullet} : H^{\bullet}(A, A) \rightarrow H^{\bullet}(A, A^*)$. We may thus transfer the cup product \cup on $H^{\bullet}(A, A)$ to a product \sqcup on $HH^{\bullet}(A, A^*)$, by setting $f \sqcup g := \omega_{\sharp}^{\bullet}((\omega_{\sharp}^{\bullet})^{-1} f \cup (\omega_{\sharp}^{\bullet})^{-1} g)$. Define furthermore the operator $\Delta : HH^{\bullet}(A, A^*) \rightarrow HH^{\bullet}(A, A^*)$ as the dual of B on homology. Then we assume, that $(HH^{\bullet}(A, A^*), \sqcup, \Delta)$ is a BV-algebra, *i.e.* \sqcup is a graded associative, commutative product, $\Delta^2 = 0$, and the bracket $\{a, b\} := (-1)^{|a|} \Delta(a \sqcup b) - (-1)^{|a|} \Delta(a) \sqcup b - a \sqcup \Delta(b)$ is a derivation in each variable.

Recall from Menichi [Men] that this BV-algebra induces a Lie algebra on the negative cyclic cohomology $HC_{\bullet}^{\bullet}(A)$ using the long exact sequences of Hochschild and negative cyclic cohomology. The inclusion $\overline{CC}_{\bullet}^{\bullet}(A) \xrightarrow{\times u} \overline{CC}_{\bullet}^{\bullet}(A)$ given by multiplication by u has cokernel $\bar{C}_{\bullet}(A, A)$. We thus obtain a short exact sequence

$$(7) \quad 0 \rightarrow \overline{CC}_{\bullet}^{\bullet}(A) \xrightarrow{\times u} \overline{CC}_{\bullet}^{\bullet}(A) \rightarrow \bar{C}_{\bullet}(A, A) \rightarrow 0,$$

which induces Connes long exact sequence of homology groups.

$$(8) \quad \cdots \rightarrow HH_n(A, A) \xrightarrow{\mathcal{B}_{\bullet}} HC_{n-1}^{\bullet}(A) \rightarrow HC_{n+1}^{\bullet}(A) \xrightarrow{I_{\bullet}} HH_{n+1}(A, A) \xrightarrow{\mathcal{B}_{\bullet}} \cdots$$

Here, the projection to the u^0 term $I : \overline{CC}_{\bullet}^{\bullet}(A) \rightarrow \bar{C}_{\bullet}(A, A)$ induces the map I_{\bullet} , and the connecting map \mathcal{B}_{\bullet} , is induced by the composition $\bar{C}_{\bullet}(A, A) \xrightarrow{B} \bar{C}_{\bullet}(A, A) \xrightarrow{inc} \overline{CC}_{\bullet}^{\bullet}(A)$. Note, that unlike $inc \circ B : \bar{C}_{\bullet}(A, A) \rightarrow \overline{CC}_{\bullet}^{\bullet}(A)$, the inclusion $inc : \bar{C}_{\bullet}(A, A) \rightarrow \overline{CC}_{\bullet}^{\bullet}(A)$ is not a chain map.

Dually, we have the short exact sequence

$$0 \rightarrow \bar{C}^{\bullet}(A, A) \rightarrow \overline{CC}_{\bullet}^{\bullet}(A) \rightarrow \overline{CC}_{\bullet}^{\bullet}(A) \rightarrow 0,$$

inducing Connes long exact sequence of cohomology groups

$$(9) \quad \cdots \rightarrow HH^n(A, A^*) \xrightarrow{I^{\bullet}} HC_{n-1}^{\bullet}(A) \rightarrow HC_{n-2}^{\bullet}(A) \xrightarrow{\mathcal{B}^{\bullet}} HH^{n-1}(A, A^*) \xrightarrow{I^{\bullet}} \cdots$$

Notice that the composition

$$(10) \quad B^{\bullet} = \mathcal{B}^{\bullet} \circ I^{\bullet}$$

is exactly the Δ operator of our BV-algebra on $HH^{\bullet}(A, A^*)$, so that we may obtain an induced Lie algebra from [Men, Lemma 7.2], much like the marking/erasing situation in [CS].

Proposition 5 (L. Menichi [Men]). *The bracket $\{a, b\} := I^{\bullet}(\mathcal{B}^{\bullet}(a) \sqcup \mathcal{B}^{\bullet}(b))$ induces a Lie algebra structure on $HC_{\bullet}^{\bullet}(A)$.*

We end this section with some examples of the above definitions.

Examples 6. Let M be a smooth, compact and oriented Riemannian manifold.

- A first example is obtained by taking $A = \Omega^{\bullet}(M)$ the De Rham forms on M , $d = d_{DR}$ the exterior derivative on A , and $\text{Tr}(a) := \int_M a$.
- More generally, if $E \rightarrow M$ is a finite dimensional complex vector bundle over M , with a flat connection ∇ , then we may take $A = \Omega^{\bullet}(M, \text{End}(E))$ with the usual differential d_{∇} . Similarly, the trace is given by a combination of integration and trace in $\text{End}(E)$. The cyclic property of the trace guarantees that this induces an injective bimodule map $\omega : A \rightarrow A^*$ that is a quasi-isomorphism.
- Both of the above examples are special cases of elliptic Calabi-Yau space as defined in [C]. By definition, this means that we have a bundle of finite dimensional associative \mathbb{C} algebras over M , whose algebra of sections is

denoted by A . Furthermore, there is a differential operator $d : A \rightarrow A$, which is an odd derivative with $d^2 = 0$ making A into an elliptic complex, a \mathbb{C} linear trace $\text{Tr} : A \rightarrow \mathbb{C}$, a hermitian metric $A \otimes A \rightarrow \mathbb{C}$, and a complex antilinear, $C^\infty(M, \mathbb{R})$ linear operator $*$: $A \rightarrow A$, satisfying certain natural conditions. It can be seen that this example satisfies the above assumptions. The details and other examples of elliptic Calabi-Yau spaces can be found in [C] and [DT].

3. MAURER-CARTAN SOLUTIONS

In this section we define the moduli space of Maurer Cartan solutions for a symmetric algebra $A = \bigoplus_{i \geq 0} A^i$, and then explain its symplectic nature. The main reference for this section is the paper [GG] by Gan-Ginzburg, together with Section 4 of [AZ]. Let us assume $k = \mathbb{R}$ or \mathbb{C} .

For $a, b \in A$ define the Lie bracket $[a, b] := a \cdot b - (-1)^{|a| \cdot |b|} b \cdot a$ and the bilinear form $\omega(a, b) := \text{Tr}(ab)$. The first remark is that $(A = A^{odd} \oplus A^{even}, d, [\cdot, \cdot], \omega)$ is a *cyclic differential graded Lie algebra* as it is defined in Section 4 of [AZ], therefore all results in [GG] applies here to define the Maurer-Cartan solutions.

Definition 7. We define the Maurer-Cartan moduli stack as

$$\begin{aligned} MC &:= \{a \in A^{odd} \mid da + \frac{1}{2}[a, a] = da + a \cdot a = 0\}, \text{ and} \\ \mathcal{MC} &:= MC / \sim, \end{aligned}$$

where the equivalence is generated by the infinitesimal action of A^0 on A , where for $a \in A^0$, the vector field ξ_x on A is defined by,

$$\xi_x(a) = [x, a] - dx.$$

Recall that ω is a symplectic form and the infinitesimal action is Hamiltonian. Moreover, the map $\mu : a \mapsto \phi_a \in (A^{even})^*$, where

$$\phi(x) = \omega(da + \frac{1}{2}[a, a], x),$$

is the moment map corresponding to the Hamiltonian action above. One should think of the tangent space $T_{[a]}\mathcal{MC}$ at a class $[a]$ as the 3-term complex

$$(11) \quad T_{[a]}\mathcal{MC} : T_{[a]}^{-1}\mathcal{MC} := A^{even} \xrightarrow{\xi(a)} T_{[a]}^0\mathcal{MC} := T_a A^{odd} = A^{odd} \xrightarrow{\mu'_a} T_{[a]}^1\mathcal{MC} := A^{even*},$$

graded by -1, 0 and 1. Here $\xi(a)$ is the map $x \mapsto \xi_x(a)$. The $\ker \mu'$ is the Zarisky tangent space to MC and the image of $\xi(a)$ accounts for the tangent space of the action orbit. Ideally, when 0 is a regular value for μ and the infinitesimal action of A^{even} on $MC = \mu^{-1}(0)$ is free, this complex is concentrated in degree zero and the Zarisky tangent space to \mathcal{MC} at $[a]$ is the cohomology group $H^0(T_{[a]}\mathcal{MC}) = H^*(A^{odd}, d_a)$ where $d_a b = db + [a, b]$.

Note that 3-term complex (11) is self-dual where the self-duality at the middle term is given by the symplectic form

$$(12) \quad \omega(X_a, Y_a) := \text{Tr}(X_a \cdot Y_a) \in k.$$

By assumption from the previous section, ω is non-degenerate. This gives rise to an isomorphism $T_{[a]}\mathcal{MC} \xrightarrow{\cong} (T_{[a]}\mathcal{MC})^*$ and equips $(T_{[a]}\mathcal{MC})$ with a symplectic form

given by (12). In the case of a nonsingular point $[a]$ this is the usual pairing on $H^0(T_{[a]}\mathcal{MC}) = H(A^{odd}, d_a)$ induced by ω .

The function space $\mathcal{O}(\mathcal{MC})$ is defined to be the subspace of $\mathcal{O}(MC)$ invariant by the infinitesimal action. The symplectic form allows us to define the Hamiltonian vector field X^ψ of a function $\psi \in \mathcal{O}(\mathcal{MC})$ via

$$\omega(X_a^\psi, Y_a) = d\psi_a(Y_a) := \lim_{t \rightarrow 0} \frac{d}{dt} \psi(a + tY_a), \quad \forall Y_a \in T_{[a]}^1 \mathcal{MC}.$$

We then define the Poisson bracket on $\mathcal{O}(\mathcal{MC})$ by,

$$\{\psi, \chi\} := \omega(X^\psi, X^\chi) = \text{Tr}(X^\psi \cdot X^\chi).$$

4. THE INDUCED LIE MAP

In this section, we define a map $\rho : HC_-^{2\bullet}(A) \rightarrow \mathcal{O}(\mathcal{MC})$, and prove it respects the brackets. We start by defining a map $P : MC \rightarrow \bar{C}_\bullet(A, A)$, and in turn the map $R : MC \rightarrow \overline{CC}_\bullet(A)$ which factors through P . Dualizing R will induce the wanted map ρ .

Definition 8. Recall that $MC = \{a \in A^{odd} \mid da + a \cdot a = 0\}$ and $\bar{C}_\bullet(A, A) = \prod_{n \geq 0} A \otimes \bar{A}^{\otimes n}$. Then, let $P : MC \rightarrow \bar{C}_\bullet(A, A)$ be given by the expression,

$$P(a) := \sum_{i \geq 0} 1 \otimes a^{\otimes i} = (1 \otimes 1) + (1 \otimes a) + (1 \otimes a \otimes a) + \dots$$

Notice that for $a \in MC$, it is $\delta(P(a)) = \sum 1 \otimes a \otimes \dots \otimes da \otimes \dots \otimes a + \sum 1 \otimes a \otimes \dots \otimes (a \cdot a) \otimes \dots \otimes a = 0$, due to the relation $da + a \cdot a = 0$ in MC . Thus, we obtain in fact a Hochschild homology class $[P(a)] \in HH_\bullet(A, A)$.

Next, define the map $R := inc \circ P$ as the composition $R : MC \xrightarrow{P} \bar{C}_\bullet(A, A) \xrightarrow{inc} \overline{CC}_\bullet(A)$. Just as above, we have that $\delta(R(a)) = 0$, and since we are in the normalized setting, we see that $B(R(a)) = 0$, so that $(\delta + uB)(R(a)) = 0$. The induced negative cyclic homology class is again denoted by $[R(a)] \in HC_-^\bullet(A)$. It is immediate to see that under the long exact sequence (8), we have that $I(R(a)) = P(a)$.

Using the pairing between negative cyclic homology and negative cyclic cohomology, $\langle \cdot, \cdot \rangle : HC_-^\bullet(A) \otimes HC_-^\bullet(A) \rightarrow k$, we define the map ρ by

$$\begin{aligned} \rho : HC_-^\bullet(A) &\rightarrow \mathcal{O}(\mathcal{MC}), \\ \rho([\alpha])([a]) &:= \langle [\alpha], [R(a)] \rangle = \langle \alpha, R(a) \rangle, \quad \text{for } [\alpha] \in HC_-^\bullet(A), [a] \in \mathcal{MC}. \end{aligned}$$

To simplify notation, we will also write $\rho(\alpha)$ instead of $\rho([\alpha])$.

Lemma 9. ρ is well-defined.

Proof. We need to show that the value $\rho([\alpha])([a]) = \langle \alpha, R(a) \rangle$ is independent of the choice of the representative $[a] \in \{x \in A^{odd} \mid dx + x \cdot x = 0\} / \sim$. Infinitesimally, this amounts to showing that $L_{X(b)}\rho(\alpha)(a) = 0$, where $L_{X(b)}$ is the Lie derivative along a vector field in the direction $X(b)_a = db + [a, b] \in T_{[a]}MC$, for any $b \in A^{even}$. To see this, note that

$$\begin{aligned} L_{X(b)}\rho(\alpha)(a) &= (i_{X(b)} \circ d + d \circ i_{X(b)})\rho(\alpha)(a) \\ &= i_{X(b)} \circ d(\rho(\alpha))(a) \\ &= \left\langle \alpha, \frac{d}{dt} \Big|_{t=0} R(a + tX(b)_a) \right\rangle \end{aligned}$$

Now, for any $Y_a \in T_{[a]}MC$, we have

$$(13) \quad \begin{aligned} \left. \frac{d}{dt} \right|_{t=0} R(a + tY_a) &= 1 \otimes Y_a + 1 \otimes Y_a \otimes a + 1 \otimes a \otimes Y_a + \cdots \\ &= \mathcal{B}(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots), \end{aligned}$$

where we used Connes operator $\mathcal{B} : \bar{C}_\bullet(A, A) \rightarrow \overline{CC}_\bullet(A)$ from in the long exact sequence (8) applied to $Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) \in \bar{C}_\bullet(A, A)$. Thus, setting $Y_a = X(b)_a = db + [a, b]$ in the above expression, we obtain

$$\begin{aligned} L_{X(b)}\rho(\alpha)(a) &= \langle \alpha, \mathcal{B}(db + [a, b] + db \otimes a + [a, b] \otimes a \\ &\quad + db \otimes a \otimes a + [a, b] \otimes a \otimes a + \cdots) \rangle \\ &= \langle \alpha, \mathcal{B} \circ \delta(b + (b \otimes a) + (b \otimes a \otimes a) + \cdots) \rangle \\ &= \langle \alpha, \delta \circ \mathcal{B}(b + (b \otimes a) + (b \otimes a \otimes a) + \cdots) \rangle \\ &= \langle \delta^* \alpha, \mathcal{B}(b + (b \otimes a) + (b \otimes a \otimes a) + \cdots) \rangle \\ &= 0. \end{aligned}$$

□

We are now ready to prove our main theorem.

Theorem 1. $\rho : HC_-^{2\bullet}(A) \rightarrow \mathcal{O}(MC)$ is a map of Lie algebras.

Proof. We saw in (13) that $\left. \frac{d}{dt} \right|_{t=0} R(a + tY_a) = \mathcal{B}(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots) \in \overline{CC}_\bullet(A)$, where $(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots) \in \bar{C}_\bullet(A, A)$ for $Y_a \in T_{[a]}MC$. Therefore,

$$\begin{aligned} (d\rho(\alpha))_a(Y_a) &= \langle \alpha, \left. \frac{d}{dt} \right|_{t=0} R(a + tY_a) \rangle \\ &= \langle \alpha, \mathcal{B}(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots) \rangle \\ &= \langle \mathcal{B}^* \alpha, Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots \rangle \\ &= (\mathcal{B}^* \alpha)(1 + a + a \otimes a + \cdots)(Y_a), \end{aligned}$$

where $\alpha \in \overline{CC}_\bullet(A)$, $\mathcal{B}^* \alpha \in \bar{C}^\bullet(A, A^*)$, and thus $(\mathcal{B}^* \alpha)(\sum_{i \geq 0} a^{\otimes i}) \in A^*$. Now, using the isomorphism $\omega_\sharp^\bullet : HH^\bullet(A, A) \rightarrow HH^\bullet(A, A^*)$ from definition 4, we apply its inverse to obtain an element $[f_\alpha] := (\omega_\sharp^\bullet)^{-1} \mathcal{B}^* \alpha \in HH^\bullet(A, A)$. We then claim that the Hamiltonian vector field $X_a^{\rho(\alpha)}$ may be expressed as

$$(14) \quad X_a^{\rho(\alpha)} = f_\alpha \left(\sum_{i \geq 0} a^{\otimes i} \right) \in T_{[a]}MC.$$

This should be compared with [AZ, Lemma 7.2] and [Go, Proposition 3.7]. To this end, first note, that the relation $0 = (\delta^* f)(\sum_{i \geq 0} a^{\otimes i}) = d_a(f(\sum_{i \geq 0} a^{\otimes i}))$, for $f \in \bar{C}^\bullet(A, A)$, shows that $X_a^{\rho(\alpha)}$ given by equation (14), represents a well-defined class in $T_{[a]}MC$. We show (14), by applying the non-degeneracy of ω in the following equation, which is valid for any $Y_a \in T_{[a]}MC$,

$$\begin{aligned} \omega(f_\alpha(\sum a^{\otimes i}), Y_a) &= \text{Tr}(f_\alpha(\sum a^{\otimes i}) \cdot Y_a) = (\omega_\sharp f_\alpha)(\sum a^{\otimes i})(Y_a) \\ &= (\mathcal{B}^* \alpha)(\sum a^{\otimes i})(Y_a) = (d\rho(\alpha))_a(Y_a) = \omega(X_a^{\rho(\alpha)}, Y_a). \end{aligned}$$

Now, calculating the Lie bracket gives

$$\begin{aligned}
\rho(\{\alpha, \beta\})(a) &= \langle \{[\alpha], [\beta]\}, [R(a)] \rangle \\
&= \langle I^\bullet(\mathcal{B}^\bullet[\alpha] \sqcup \mathcal{B}^\bullet[\beta]), [R(a)] \rangle \\
&= \langle I^\bullet \omega_{\sharp}^\bullet((\omega_{\sharp}^\bullet)^{-1} \mathcal{B}^\bullet[\alpha] \cup (\omega_{\sharp}^\bullet)^{-1} \mathcal{B}^\bullet[\beta]), [R(a)] \rangle \\
&= \langle \omega_{\sharp}^\bullet([f_\alpha] \cup [f_\beta]), I_\bullet[R(a)] \rangle \\
&= \langle \omega_{\sharp}^\bullet([f_\alpha] \cup [f_\beta]), [P(a)] \rangle.
\end{aligned}$$

To evaluate this expression, note that for $f_\alpha : \bar{A}^{\otimes m} \rightarrow A$ and $f_\beta : \bar{A}^{\otimes n} \rightarrow A$, $\omega_{\sharp}^\bullet([f_\alpha] \cup [f_\beta])$ is represented by the composition

$$\bar{A}^{\otimes m+n} \xrightarrow{f_\alpha \otimes f_\beta} A \otimes A \xrightarrow{\cdot} A \xrightarrow{\omega} A^*.$$

The first arrow with $f_\alpha \otimes f_\beta$ applied to $P(a) = 1 + (1 \otimes a) + (1 \otimes a \otimes a) + \dots \in \prod_{i \geq 0} A \otimes \bar{A}^{\otimes i}$ then gives an expression, where we apply a to all possible inputs in $\bar{A}^{\otimes n+m}$. To this, we then apply the product in A , and apply ω with input $1 \in A$, since $P(a) = 1 \otimes (\dots)$. We thus obtain

$$\begin{aligned}
\rho(\{\alpha, \beta\})(a) &= \text{Tr}\left(f_\alpha(1 + a + a \otimes a + \dots) \cdot f_\beta(1 + a + a \otimes a + \dots) \cdot 1\right) \\
&\stackrel{(14)}{=} \text{Tr}(X_a^{\rho(\alpha)} \cdot X_a^{\rho(\beta)}) = \omega(X_a^{\rho(\alpha)}, X_a^{\rho(\beta)}) = \{\rho(\alpha), \rho(\beta)\}(a).
\end{aligned}$$

This is the claim of the theorem. \square

5. COMPARISON WITH GENERALIZED HOLONOMY

In this section we compare the map ρ with the generalized holonomy map Ψ studied in [AZ]. The relationship may be summarized in the diagram (2). This shows how a special case the result of this paper relates to the main theorem of [AZ]. The map $\text{Tr} : A \rightarrow \mathbb{C}$ is induced by the trace function on $\mathfrak{g} \subseteq \text{GL}(n, \mathbb{C})$ and integration of forms on M ; see Example 6.

Our model of S^1 -equivariant de Rham forms of LM is $(\Omega(LM)[u], d + u\Delta)$ where u is a generator of degree 2 and $\Delta : \Omega^\bullet(LM) \rightarrow \Omega^{\bullet-1}(LM)$ is the map induced by the S^1 action on LM ; see [GJP]. This model is quasi-isomorphic to the small Cartan model $(\Omega_{inv}(LM)[u], d + i_X u)$ for the S^1 action, where X is the fundamental vector field generated by the natural action of S^1 . The quasi-isomorphism is given by the averaging map $\Omega^\bullet(LM) \rightarrow \Omega_{inv}^\bullet(LM)$. More explicitly, for $\omega \in \Omega^\bullet(LM)$, $\Delta(\omega)$ is given by,

$$\Delta(\omega) = \int_{\text{fibre}} ev^*(\omega) \in \Omega^{\bullet-1}(LM)$$

$$(15) \quad \begin{array}{ccc} S^1 \times LM & \xrightarrow{ev} & LM \\ \downarrow \pi & & \\ LM & & \end{array}$$

Chen's iterated integral map and the trace map on \mathfrak{g} (see (6.3) [AZ], and Theorem A in [GJP]) yields a map, which we denote by,

$$S : (\bar{C}_\bullet(A, A), \delta) \rightarrow (\Omega^\bullet(LM), d).$$

S induces the map $S^{HH} : HH_{\bullet}(A, A) \longrightarrow H^{\bullet}(LM)$ on homology, and, after applying the pairing between homology and cohomology groups, we get,

$$H_{\bullet}(LM) \xrightarrow{\sigma^{HH}} HH^{\bullet}(A, A^*).$$

Extending S by u -linearity, we obtain a map, which we denote by abuse of notation by the same letter,

$$S : (\bar{C}_{\bullet}(A, A)[u], \delta + uB) \rightarrow (\Omega^{\bullet}(LM)[u], d + u\Delta).$$

Since, by Lemma 3, $(\bar{C}_{\bullet}(A, A)[u], \delta + uB)$ and $(\bar{C}_{\bullet}(A, A)[[u]], \delta + uB)$ are quasi-isomorphic in our setting, we obtain the induced map $S^{HC} : HC_{\bullet}^{-}(A) \longrightarrow H_{S^1}^{\bullet}(LM)$ on homology. Composing S^{HC} with the map $R : MC \rightarrow \overline{CC}_{\bullet}(A) = \bar{C}_{\bullet}(A, A)[u]$ from Section 4, we get,

$$MC \xrightarrow{R} HC_{\bullet}^{-}(A) \xrightarrow{S^{HC}} H_{S^1}^{\bullet}(LM).$$

Thus by duality, and using Lemma 9, we have,

$$H_{\bullet}^{S^1}(LM) \xrightarrow{\sigma = \sigma^{HC}} HC_{\bullet}^{\bullet}(A) \xrightarrow{\rho} \mathcal{O}(MC).$$

The composition $\rho \circ \sigma$ is the generalized holonomy map Ψ discussed in [AZ].

$$(16) \quad \begin{array}{ccc} HC_{\bullet}^{\bullet}(A) & \xrightarrow{\rho} & \mathcal{O}(MC) \\ & \searrow \sigma & \nearrow \Psi \\ & H_{\bullet}^{S^1}(LM) & \end{array}$$

It was proved in [AZ], that Ψ is the morphism of Lie algebras. We will shortly see how this is a consequence of Theorem 1. We first recall the following theorem.

Theorem 10 (S. Merkulov [Mer]). *The Chen integral induces a map of algebras $(H_{\bullet}(LM), \bullet) \rightarrow (HH^{\bullet}(A, A), \cup)$.*

Thus, by definition, $\sigma^{HH} : (H_{\bullet}(LM), \bullet) \rightarrow (HH^{\bullet}(A, A^*), \cup)$ is also a map of algebras. With this, we can now prove the following statement.

Theorem 11. *The map induced by the Chen iterated integrals $\sigma : (H_{\bullet}^{S^1}(LM), \{\cdot, \cdot\}) \rightarrow (HC_{\bullet}^{\bullet}(A), \{\cdot, \cdot\})$ is a map of Lie algebras. Here, the first bracket is the string bracket and the second one is defined in the statement of Proposition 5.*

Proof. The brackets on $H_{\bullet}^{S^1}(LM)$ and $HC_{\bullet}^{\bullet}(A)$ are determined by the products on $H_{\bullet}(LM)$ and $HC^{\bullet}(A, A^*)$, together with the maps in the corresponding Gysin long exact sequences. By Theorem 10, it thus remains to show that the long exact sequences correspond to each other, *i.e.* that the following diagrams commute,

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_{\bullet}^{S^1}(LM) & \xrightarrow{m_{\bullet}} & H_{\bullet+1}(LM) & \xrightarrow{e_{\bullet}} & H_{\bullet+1}^{S^1}(LM) & \longrightarrow & H_{\bullet-1}^{S^1}(LM) & \longrightarrow & \cdots \\ & & \downarrow \sigma & & \downarrow \sigma^{HH} & & \downarrow \sigma & & \downarrow \sigma & & \\ \cdots & \longrightarrow & HC_{\bullet}^{\bullet}(A) & \xrightarrow{B_{\bullet}} & HH^{\bullet+1}(A, A^*) & \xrightarrow{I_{\bullet}} & HC_{\bullet+1}^{\bullet}(A) & \longrightarrow & HC_{\bullet-1}^{\bullet}(A) & \longrightarrow & \cdots \end{array}$$

Equivalently, we need to show the commutativity of the following dual sequence,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & HC_{\bullet}^-(A) & \xrightarrow{I_{\bullet}} & HH_{\bullet}(A, A) & \xrightarrow{B_{\bullet}} & HC_{\bullet-1}^-(A) \longrightarrow HC_{\bullet-1}^-(A) \longrightarrow \cdots \\ & & \downarrow S^{HC} & & \downarrow S^{HC} & & \downarrow S^{HH} & & \downarrow S^{HC} \\ \cdots & \longrightarrow & H_{S^1}^{\bullet}(LM) & \xrightarrow{e_{\bullet}} & H^{\bullet}(LM) & \xrightarrow{m_{\bullet}} & H_{S^1}^{\bullet-1}(LM) \longrightarrow H_{S^1}^{\bullet-1}(LM) \longrightarrow \cdots \end{array}$$

The top long exact sequence is induced by the short exact sequence (7) while the bottom one is induced by the short exact sequence

$$(17) \quad 0 \rightarrow (\Omega^{\bullet}(M)[u], d + u\Delta) \xrightarrow{\times u} (\Omega^{\bullet}(M)[u], d + u\Delta) \xrightarrow{j} \Omega^{\bullet}(M) \rightarrow 0,$$

where $j(\sum a_i u^i) = a_0$, *cf.* [GS, Ma]. In this picture, m_{\bullet} corresponds to the connecting map of the long exact sequence (17). By a diagram chasing argument one finds that $m_{\bullet} = (i \circ \Delta)_{\bullet}$ where $i : \Omega^{\bullet}(M) \hookrightarrow \Omega^{\bullet}(M)[u]$ corresponds to $B_{\bullet} = (inc \circ B)_{\bullet}$ using Chen iterated integrals as corollary of Theorem A in [GJP]. Note that i is not a chain map, whereas $i \circ \Delta$ is a chain map, since $\Delta d = d\Delta$ and $\Delta^2 = 0$, (*cf.* [GJP]). \square

6. A_{∞} GENERALIZATION

The previous sections, given for the case of dgas (A, d, \cdot) with invariant inner product $\omega : A \otimes A \rightarrow k$, generalize in a straightforward way to the setting of cyclic A_{∞} algebras. In this section, we recall the relevant definitions (*cf.* [T]), and adopt the above to this situation.

Definition 12. An A_{∞} algebra on A consists of a sequence of maps $\{\mu_n\}_{n \geq 1}$, where $\mu_n : A^{\otimes n} \rightarrow A$ is of degree $(2 - n)$, such that

$$\forall n \geq 1 : \sum_{\substack{k+l=n+1 \\ r=0, \dots, n-l}} (-1)^{\epsilon_r^i} \cdot \mu_k(a_1 \otimes \cdots \otimes \mu_l(a_{r+1} \otimes \cdots \otimes a_{r+l}) \otimes \cdots \otimes a_n) = 0,$$

where $\epsilon_r^i = (l-1) \cdot (|a_1| + \cdots + |a_r| - r)$. A unit is an element $1 \in k \subset A^0$ such that $\mu_2(a, 1) = \mu_2(1, a) = a$, and $\mu_n(\cdots \otimes 1 \otimes \cdots) = 0$ for $n \neq 2$. Again, we write $\bar{A} = A/k$. We define the Hochschild chain complex of A with values in A or A^* to be the vector spaces $\bar{C}_{\bullet}(A, A)$ and $\bar{C}_{\bullet}(A, A^*)$ from equation (3) with the differentials modified as follows,

$$\begin{aligned} \delta : \bar{C}_{\bullet}(A, A) &\rightarrow \bar{C}_{\bullet}(A, A), \delta(a_0 \otimes \cdots \otimes a_n) = \sum \pm a_0 \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n \\ &\quad + \sum \pm \mu_k(a_s \otimes \cdots \otimes a_0 \otimes \cdots \otimes a_r) \otimes a_{r+1} \otimes \cdots \otimes a_{s-1}, \\ \delta : \bar{C}_{\bullet}(A, A^*) &\rightarrow \bar{C}_{\bullet}(A, A^*), \delta(a_0^* \otimes \cdots \otimes a_n) = \sum \pm a_0^* \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n \\ &\quad + \sum \pm \mu_k^*(a_s \otimes \cdots \otimes a_0^* \otimes \cdots \otimes a_r) \otimes a_{r+1} \otimes \cdots \otimes a_{s-1}, \end{aligned}$$

where $\mu_k^*(a_s \otimes \cdots \otimes a_0^* \otimes \cdots \otimes a_r) \in A^*$ is given by

$$\mu_k^*(a_s \otimes \cdots \otimes a_n \otimes a_0^* \otimes a_1 \otimes \cdots \otimes a_r)(a) := \pm a_0^*(\mu_k(a_1 \otimes \cdots \otimes a_r \otimes a \otimes a_s \otimes \cdots \otimes a_n)).$$

Here, the signs are given by the usual Koszul rule, where we a factor of $(-1)^{\epsilon\epsilon'}$ is introduced, whenever elements of degree ϵ and ϵ' are being commuted. For an

explicit discussion of the signs, see *e.g.* [T]. Similarly, $\bar{C}^\bullet(A, A)$ and $\bar{C}^\bullet(A, A^*)$ are defined by the spaces from (4) with the modified differentials

$$\begin{aligned} \delta^* : \bar{C}^\bullet(A, A) &\rightarrow \bar{C}^\bullet(A, A), & \delta^* f(a_1 \otimes \cdots \otimes a_n) \\ &= \sum \pm f(a_1 \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n) + \sum \pm \mu_k(a_1 \otimes \cdots \otimes f(\cdots) \otimes \cdots \otimes a_n), \end{aligned}$$

$$\begin{aligned} \delta^* : \bar{C}^\bullet(A, A^*) &\rightarrow \bar{C}^\bullet(A, A^*), & \delta^* f(a_1 \otimes \cdots \otimes a_n) \\ &= \sum \pm f(a_1 \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n) + \sum \pm \mu_k^*(a_1 \otimes \cdots \otimes f(\cdots) \otimes \cdots \otimes a_n). \end{aligned}$$

Since $\delta^2 = 0$, $(\delta^*)^2 = 0$ in all the above cases, we obtain the associated homologies and cohomologies $H_\bullet(A, A)$, $H_\bullet(A, A^*)$, $H^\bullet(A, A)$, and $H^\bullet(A, A^*)$.

There is a generalized cup product \cup on $H^\bullet(A, A)$ induced by,

$$(f \cup g)(a_1 \otimes \cdots \otimes a_n) := \sum_{k \geq 2} \pm \mu_k(a_1 \otimes \cdots \otimes f(\cdots) \otimes \cdots \otimes g(\cdots) \otimes \cdots \otimes a_n).$$

Furthermore, equation (5) defines an operator $B : \bar{C}_\bullet(A, A) \rightarrow \bar{C}_\bullet(A, A)$ with $B^2 = \delta B + B\delta = 0$. We define the negative cyclic chains $\overline{CC}_\bullet(A)$ of A to be the vector space $\bar{C}_\bullet(A, A)[[u]]$ with differential $\delta + uB$, and denote the negative cyclic homology by $HC_\bullet^-(A)$. Dualizing $\overline{CC}_\bullet(A)$, we obtain $\overline{CC}_\bullet^-(A)$ with dual differential and denote the negative cyclic cohomology by $HC_\bullet^+(A)$. For the same reasons as in Section 2, we obtain the long exact sequences (8) and (9).

Finally, assume we have a trace $\text{Tr} : A \rightarrow k$, such that the associated map $\omega : A \otimes A \rightarrow k$, $\omega(a, b) = \text{Tr}(\mu_2(a \otimes b))$ is a quasi-isomorphism, which satisfies for $n \geq 1$,

$$(18) \quad \omega(\mu_n(a_1 \otimes \cdots \otimes a_n), a_{n+1}) = \pm \omega(\mu_n(a_{n+1} \otimes a_1 \otimes \cdots \otimes a_{n-1}), a_n),$$

In this case, $\omega : A \rightarrow A^*$ induces a map of the Hochschild cohomologies $H^\bullet(A, A) \rightarrow H^\bullet(A, A^*)$, $\omega_\#^\bullet(f) = \omega \circ f$, which we assume to be an isomorphism. Thus, we may transfer the product \cup on $H^\bullet(A, A)$ to a product \sqcup on $H^\bullet(A, A^*)$. $(HH^\bullet(A, A^*), \sqcup, \Delta = B^*)$ is a BV-algebra, *cf.* [T], so that we obtain the Lie bracket $\{a, b\} := I^\bullet(\mathcal{B}^\bullet(a) \sqcup \mathcal{B}^\bullet(b))$ on $HC_\bullet^-(A)$ just as in Proposition 5.

Using this setup, we may now also generalize Section 3.

Definition 13. Recall that there are maps from the the n^{th} symmetric power of a vector space to the n^{th} tensor power $S^n : A^{\wedge n} \rightarrow A^{\otimes n}$, where $S^n(a_1 \wedge \cdots \wedge a_n) = \sum_{\sigma \in \Sigma_n} (-1)^{\epsilon_\sigma} (a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)})$. Defining $\nu_n : A^{\wedge n} \rightarrow A$ as $\nu_n := \mu_n \circ S^n$, we obtain an L_∞ algebra on A , *cf.* [LM, Theorem 3.1]. Furthermore, from (18), it is immediate to see that we have for $n \geq 1$,

$$\omega(\nu_n(a_1 \wedge \cdots \wedge a_n), a_{n+1}) = \pm \cdot \omega(\nu_n(a_{n+1} \wedge a_1 \wedge \cdots \wedge a_{n-1}), a_n).$$

For this L_∞ algebra, recall from [GG, Section 2] that the Maurer-Cartan solutions are defined by,

$$\begin{aligned} MC &:= \left\{ a \in A^1 \mid \nu_1(a) + \frac{1}{2!} \nu_2(a \wedge a) + \frac{1}{3!} \nu_3(a \wedge a \wedge a) + \cdots = 0 \right\}, \text{ and} \\ \mathcal{MC} &:= MC / \sim, \end{aligned}$$

where the equivalence is again generated by the infinitesimal action of A^0 on A^1 , where for $a \in A^0$, the vector field ξ_x on A^1 is defined by,

$$\xi_x(a) = \nu_1(x) + \nu_2(a \wedge x) + \frac{1}{2!}\nu_3(a \wedge a \wedge x) + \cdots.$$

Note, that under the above assumptions the tangent space to \mathcal{MC} at $[a]$ is the self-dual 3-term complex,

$$(19) \quad T_{[a]}\mathcal{MC} : \quad T_{[a]}^{-1}\mathcal{MC} := A^0 \xrightarrow{\xi(a)} T_{[a]}^0\mathcal{MC} := T_a A^1 = A^1 \xrightarrow{\mu'_a} T_{[a]}^1\mathcal{MC} := A^{0*},$$

where

$$\mu'_a(b) = \nu_1(b) + \nu_2(a \wedge b) + \frac{1}{2!}\nu_3(a \wedge a \wedge b) + \cdots.$$

The self-duality at the middle term is given by the symplectic form

$$\omega(X_a, Y_a) = \text{Tr}(\mu_2(X_a \otimes Y_a)) \in k.$$

This can be used to define the Hamiltonian vector field X^ψ associated to a function $\psi \in \mathcal{O}(\mathcal{MC})$, and thus the Lie bracket on $\mathcal{O}(\mathcal{MC})$ via the usual formula $\{\psi, \chi\} = \omega(X^\psi, X^\chi)$.

We may now define the map $P : MC \rightarrow \overline{C}_\bullet(A, A)$ by

$$P(a) := \sum_{i \geq 0} 1 \otimes a^{\otimes i} = (1 \otimes 1_{\overline{A}^{\otimes 0}}) + (1 \otimes a) + (1 \otimes a \otimes a) + \cdots,$$

and $R = inc \circ P : MC \rightarrow \overline{C}_\bullet(A)$. As in definition 8, we may again see, that $\delta(P(a)) = 0$, and $(\delta + uB)(R(a)) = 0$, and we define

$$\begin{aligned} \rho : HC_-^{2\bullet}(A) &\rightarrow \mathcal{O}(\mathcal{MC}), \\ \rho([\alpha])([a]) &:= \langle [\alpha], [R(a)] \rangle = \langle \alpha, R(a) \rangle, \quad \text{for } [\alpha] \in HC^\bullet(A), [a] \in \mathcal{MC}. \end{aligned}$$

With this, we have the same theorem as in the previous sections.

Theorem 14. *The map ρ is a well-defined map of Lie algebras.*

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