On the vanishing of the Rokhlin invariant

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It is a natural consequence of fundamental properties of the Casson invariant that the Rokhlin invariant $\mu(M)$ of an amphichiral integral homology 3–sphere M vanishes. In this paper, we give a new direct proof of this vanishing property. For such an M, we construct a manifold pair (Y,Q) of dimensions 6 and 3 equipped with some additional structure (6–dimensional spin e-manifold), such that $Q \cong M \amalg M \amalg (-M)$, and $(Y,Q) \cong (-Y,-Q)$. We prove that (Y,Q) bounds a 7–dimensional spin e–manifold (Z,X) by studying the cobordism group of 6–dimensional spin e-manifolds and the $\mathbb{Z}/2$–actions on the two–point configuration space of $M\backslash\{pt\}$. For any such (Z,X), the signature of X vanishes, and this implies $\mu(M)=0$. The idea of the construction of (Y,Q) comes from the definition of the Kontsevich–Kuperberg–Thurston invariant for rational homology 3–spheres.

57M27; 57N70, 57R20, 55R80

1 Introduction and Main results

1.1 Introduction

The Rokhlin invariant $\mu(M)$ of a closed oriented spin 3–manifold M is defined by

$$\mu(M) = \operatorname{Sign} X \pmod{16},$$

where X is a smooth compact oriented spin 4–manifold bounded by M as a spin manifold, and $\operatorname{Sign} X$ is the signature of X. If M is a $\mathbb{Z}/2$–homology 3–sphere, then it admits a unique spin structure, and so $\mu(M)$ is a topological invariant of M. In 1980's, Casson defined an integer–valued invariant $\lambda(M)$, what is now called the Casson invariant, for oriented integral homology 3–spheres, and proved the following fundamental properties for λ (see [1]):

(1–1) $$\lambda(-M) = -\lambda(M)$$

(1–2) $$8\lambda(M) \equiv \mu(M) \pmod{16}$$

It is a natural consequence of (1–1) and (1–2) that, if M is amphichiral (namely, M admits a self–homeomorphism reversing the orientation), then its Rokhlin invariant vanishes:

$$M \cong -M \quad \Longrightarrow \quad \mu(M) = 0 \tag{1–3}$$

In this paper, we give a new proof of this vanishing property for integral homology 3–spheres (Corollary 3). We might say that our approach is more direct in the sense that we only consider the signature of 4–manifolds or related characteristic classes (Remark 1.6).

Remark 1.1 Walker [19] extended the Casson invariant to a rational–valued invariant $\lambda_W(M)$ for oriented rational homology 3–spheres, such that $\lambda_W(M) = 2\lambda(M)$ if M is an integral homology 3–sphere. He proved that $\lambda_W(-M) = -\lambda_W(M)$ holds for any M, and $4|H_1(M;\mathbb{Z})|^2\lambda_W(M) \equiv \mu(M) \pmod{16}$ holds for any $\mathbb{Z}/2$–homology 3–spheres, where $|A|$ denotes the number of elements in a set A. These two properties imply that the same statement (1–3) holds for all $\mathbb{Z}/2$–homology 3–spheres.

Remark 1.2 Some partial proofs of the vanishing property have been given by several authors (Galewski [2], Kawauchi [9] [8], Pao–Hsiang [7], Siebenman [17], etc.) before the Casson invariant was defined.

1.2 Outline of the proof

We outline our proof of (1–3) for integral homology 3–spheres (Corollary 3), without giving precise definitions and computations. See Section 1.3 and Section 1.4 for more details. Yet another proof is also given in Section 9 (see also Remark 1.5).

An invariant σ An n–dimensional e-manifold $\alpha = (W, V, e)$ is roughly a manifold pair (W, V) of dimensions n and $n-3$ equipped with a cohomology class $e \in H^2(W\backslash V;\mathbb{Q})$ called an e–class. In our previous paper [15], we defined a rational–valued invariant $\sigma(\alpha)$ for 6–dimensional closed e-manifolds such that $\sigma(-\alpha) = -\sigma(\alpha)$, and that $\sigma(\partial\beta) = \operatorname{Sign} X$ for a 7–dimensional e-manifold $\beta = (Z, X, e)$ (Theorem 1.4).

Outline of the proof For an oriented integral homology 3–sphere M, we construct a 6–dimensional closed spin e-manifold $\alpha_M = (Y, Q, e_M)$ (Y and Q are spin) such that $Q \cong M \amalg M \amalg (-M)$ and $\alpha_{-M} \cong -\alpha_M$. We can prove that α_M is spin null–cobordant (Theorem 2). Namely, there exists a spin e-manifold $\beta = (Z, X, e)$ such that $\partial\beta \cong \alpha_M$. Therefore,

$$\sigma(\alpha_M) = \operatorname{Sign} X \equiv \mu(M) \pmod{16}.$$

If $M \cong -M$, then $\alpha_M \cong -\alpha_M$ and $\sigma(\alpha_M) = 0$. Consequently, $\mu(M) \equiv 0$. □

1.3 e-classes and e-manifolds

In [15], we introduced the notion of e-class and e-manifold. Let (Z, X) be a pair of (smooth, oriented, and compact) manifold Z and a proper submanifold X ($\partial X \subset \partial Z$ and X is transverse to ∂Z) of codimension 3. Let $\rho_X\colon S(\nu_X) \to X$ be the unit sphere bundle associated with the normal bundle ν_X of X (identified with a tubular neighborhood of X), and $e(F_X) \in H^2(S(\nu_X); \mathbb{Z})$ the Euler class of the vertical tangent subbundle $F_X \subset TS(\nu_X)$ of $S(\nu_X)$ with respect to ρ_X.

Definition 1.3 ([15]) A cohomology class $e \in H^2(Z \setminus X; \mathbb{Q})$ is called an *e-class* of (Z, X) if $e|_{S(\nu_X)} = e(F_X)$ over $\mathbb{Q}$. The triple $\beta = (Z, X, e)$ is called an *e-manifold*. Set $\dim \beta = \dim Z$.

A spin structure of β will mean a pair of spin structures of Z and X. We call β a *spin* e-manifold if it has a spin structure. The boundary of β is defined as $\partial\beta = (\partial Z, \partial X, e|_{\partial Z \setminus \partial X})$, and the disjoint union of two e-manifolds $\beta_i = (Z_i, X_i, e_i)$ $(i = 1, 2)$ is defined as $\beta_1 \amalg \beta_2 = (Z_1 \amalg Z_2, X_1 \amalg X_2, e_3)$, where e_3 is the e-class such that $e_3|_{Z_i \setminus X_i} = e_i$. We also define $-\beta = (-Z, -X, e)$. We say β is *closed* if $\partial\beta$ is the *empty* e-manifold $\emptyset = (\emptyset, \emptyset, 0)$. If there exists an isomorphism $f\colon (Z_1, X_1) \to (Z_2, X_2)$ of pair of manifolds such that $f^* e_2 = e_1$, then we say β_1 and β_2 are *isomorphic* (denoted by $\beta_1 \cong \beta_2$). See [15, Section 2] for more details.

In [15], we defined the following invariant σ for 6–dimensional closed e-manifolds.

Theorem 1.4 ([15]) *There exists a unique rational–valued invariant $\sigma(\alpha)$ for 6–dimensional closed e-manifolds α satisfying the following properties*:

(a) $\sigma(-\alpha) = -\sigma(\alpha)$, $\sigma(\alpha \amalg \alpha') = \sigma(\alpha) + \sigma(\alpha')$.

(b) *For a 7–dimensional e-manifold $\beta = (Z, X, e)$, $\sigma(\partial\beta) = \operatorname{Sign} X$.*

This invariant σ is a generalization of Haefliger's invariant [6] for smooth 3–knots in S^6 [15, Theorem 5].

1.4 Main results

If a closed spin e-manifold α bounds, namely, if there exists a spin e-manifold β such that $\partial\beta \cong \alpha$ as a spin e-manifold, then we say α is *spin null–cobordant*. We define $\Omega_6^{e,spin}$ to be the cobordism group of 6–dimensional spin e-manifolds, namely, it is an abelian group consisting of the spin cobordism classes $[\alpha]$ of 6–dimensional closed spin e-manifolds α, with the group structure given by the disjoint sum.

In Section 3, for an oriented integral homology 3–sphere M, we construct a 6–dimensional closed spin e-manifold $\alpha_M = (Y, Q, e_M)$ such that $Q \cong M \amalg M \amalg (-M)$. The following theorem will be used to prove the vanishing of the spin cobordism class $[\alpha_M] \in \Omega_6^{e,spin}$ of α_M.

Theorem 1 *There is a unique isomorphism* $\Phi\colon \Omega_6^{e,spin} \to (\mathbb{Q}/16\mathbb{Z}) \oplus (\mathbb{Q}/4\mathbb{Z})$ *such that*

$$\Phi([W, \emptyset, e]) \equiv \left(\frac{1}{6}\int_W p_1(TW)e - e^3,\ \frac{1}{2}\int_W e^3\right) \mod 16\mathbb{Z} \oplus 4\mathbb{Z} \tag{1–4}$$

for any closed spin 6*–manifold* W *and* $e \in H^2(W; \mathbb{Q})$.

Here, $p_1(TW)$ is the first Pontryagin class of the tangent bundle TW of W. Any element in $\Omega_6^{e,spin}$ is represented by a closed spin e-manifold of the form $(W, \emptyset, e)$ (Proposition 5.5), and that is why Φ is uniquely determined by (1–4).

Theorem 2 *For an oriented integral homology* 3*–sphere* M*, the* 6*–dimensional closed spin* e*-manifold* α_M *satisfies the following properties.*

(1) $\alpha_{-M} \cong -\alpha_M$.

(2) $[\alpha_M] = 0$ *in* $\Omega_6^{e,spin}$.

As a corollary of Theorem 1.4 and Theorem 2, we obtain a new proof of the vanishing property (1–3) of the Rokhlin invariant for integral homology 3–spheres.

Corollary 3 ([1], [19] for $\mathbb{Z}/2$–homology 3–spheres) *If an oriented integral homology* 3*–sphere* M *is amphichiral, then* $\mu(M) = 0$.

Proof Assume $M \cong -M$. Theorem 2 (1) and Theorem 1.4 (a) implies $\sigma(\alpha_M) = 0$. By Theorem 2 (2), there exists a 7–dimensional spin e-manifold $\beta = (Z, X, e)$ such that $\partial\beta \cong \alpha_M$, and in particular, we have $\sigma(\alpha_M) = \operatorname{Sign} X$ by Theorem 1.4 (b). The manifold X is spin and $\partial X \cong Q$. Let us write $Q = M_1 \cup M_2 \cup (-M_3)$, $M_i \cong M$. Gluing

the boundary components M_2 and M_3 of X by a diffeomorphism, we obtain a compact oriented spin 4–manifold X' such that $\partial X' = M_1 \cong M$ and $\operatorname{Sign} X' = \operatorname{Sign} X$. By the definition of the Rokhlin invariant, we have

$$\mu(M) \underset{(\mathrm{mod}\ 16)}{\equiv} \operatorname{Sign} X' = \operatorname{Sign} X = \sigma(\alpha_M) = 0 \qquad \square$$

Remark 1.5 Yet another direct proof of Corollary 3 is given in Section 9, this is a shortcut to Corollary 3 without using Theorem 1.4. It follows from the properties of σ that, if a 7–dimensional e–manifold $\beta = (Z, X, e)$ if closed, then $\operatorname{Sign} X = 0$. We can also prove this directly by using Stokes' theorem, and this method is enough to prove Corollary 3. The proof given in Section 9 uses only Theorem 2 and Stokes' theorem.

1.5 Plan of the paper

Here is the plan of the paper.

Preliminaries In Section 2, we introduce notation and conventions. In Section 3, we construct a 6–dimensional closed spin e-manifold $\alpha_M = (Y, Q, e_M)$ such that $Y \cong (M \times M)\#(-S^3 \times S^3)$ and $Q \cong M \amalg M \amalg (-M)$.

An involution Let $G = \{1, \iota\}$ denote a multiplicative group of order 2. In Section 4, we define a G–action on (Y, Q) by using the permutation of coordinates on $M \times M$ and $S^3 \times S^3$. We can regard ι as an isomorphism between $-\alpha_M$ and α_{-M} (preserving the orientation), namely, Theorem 2 (1) holds.

Spin cobordism group of e-manifolds In Section 5, we prove Theorem 1, more precisely, we give a short exact sequence

$$0 \to \Omega_4^{spin}(BSpin(3)) \to \Omega_6^{spin}(K(\mathbb{Q}, 2)) \to \Omega_6^{e,spin} \to 0,$$

which is isomorphic to $0 \to 16\mathbb{Z} \oplus 4\mathbb{Z} \hookrightarrow \mathbb{Q} \oplus \mathbb{Q} \to (\mathbb{Q}/16\mathbb{Z}) \oplus (\mathbb{Q}/4\mathbb{Z}) \to 0$. Here, Ω_*^{spin} denotes the spin cobordism group. A pair (W, e) of a closed spin 6–manifold W and $e \in H^2(W; \mathbb{Q})$ represents an element $[W, e] \in \Omega_6^{spin}(K(\mathbb{Q}, 2))$, and the isomorphism (Lemma 5.3)

$$\Omega_6^{spin}(K(\mathbb{Q}, 2)) \to \mathbb{Q} \oplus \mathbb{Q}, \qquad [W, e] \mapsto \left(\frac{1}{6} \int_W p_1(TW)e - e^3,\ \frac{1}{2} \int_W e^3 \right)$$

induces the definition of Φ.

Signature modulo 32 In Section 6, we construct a certain closed spin e-manifold of the form $\alpha'_M = (Y', \emptyset, e'_M)$ such that $[\alpha'_M] = [\alpha_M]$ in $\Omega_6^{e,spin}$, and that Y' has an orientation reversing free G–action. We show that, if e'_M is the Poincaré dual of a 4–submanifold W of Y', then the following equivalence relation holds (Proposition 6.3):

$$[\alpha_M] = 0 \text{ (Theorem 2 (2))} \quad \Longleftrightarrow \quad \operatorname{Sign} W \equiv 0 \pmod{32}$$

G–vector bundle In Section 7, we prove Theorem 2 (2), by constructing such a W. This is done by assuming the existence of an oriented vector bundle F, of rank 2 over Y' with a G–action, such that

(i) $e(F) = e'_M$ over $\mathbb{Q}$,

(ii) $w_i(F/G) = w_i(TY'/G)$ in $H^i(Y'/G; \mathbb{Z}/2)$ for $i = 1, 2$,

where w_i denotes the i–th Stiefel–Whitney class. Fix a G–equivariant smooth section $s\colon Y' \to F$, and define $W = \{x \in Y' \mid s(x) = 0\}$. Then, the Poincaré dual of W is e'_M by (i). The second property (ii) implies that the quotient W/G is orientable and spinnable smooth manifold. By Rokhlin's theorem, we have $\operatorname{Sign} W = \pm 2 \operatorname{Sign} W/G \equiv 0 \pmod{32}$. Hence, Theorem 2 (2) holds. In Section 8, we prove the existence of F satisfying (i) and (ii).

In Section 9, we give yet another direct proof of Corollary 3.

1.6 Remarks

Remark 1.6 The Casson invariant $\lambda(M)$ is roughly defined by measuring the oriented number of irreducible representations of the fundamental group $\pi_1(M)$ in $SU(2)$, and so the geometric meaning is different from $\mu(M)$. The relation (1–2) is proved by showing that the Dehn surgery formula for $\lambda(M) \pmod 2$ coincides with that of $\mu(M)$. On the other hand, our proof does not require such formulas (or the fact that the Casson invariant is a finite type invariant) in any step including the proof of Theorem 1.4. Moreover, in this paper, we only need to consider the signature of 4–manifolds or the related characteristic classes to prove Corollary 3. Therefore, we might say that our proof is more direct.

Remark 1.7 The idea of the construction of α_M comes from the definition of the Kontsevich–Kuperberg–Thurston invariant $Z_{KKT}(M)$ for oriented rational homology 3–spheres [11] [10], which is a universal real finite type invariant for integral homology spheres in the sense of Ohtsuki [16], Habiro [5], and Goussarov [3]. A detailed review

and an elementary proof for the invariance of Z_{KKT} is given by Lescop [13]. The degree one part $Z_1(M)$ of $Z_{KKT}(M)$ is equal to $\lambda_W(M)/4$ (first proved by Kuperberg–Thurston [11] for integral homology 3–spheres, and later Lescop [14] extended this relation to all rational homology 3–spheres). By definition, $Z_1(M)$ is described as an integral over the configuration space $\mathrm{Conf}_2(M') = M' \times M' \setminus M'_\Delta$ of two points on $M' = M \setminus \{x_0\}$, where $M'_\Delta \subset M' \times M'$ is the diagonal submanifold.

Remark 1.8 By the construction of (Y, Q) (Section 3), the complement $Y \setminus Q$ is nothing but the union of the two configuration spaces $\mathrm{Conf}_2(M')$ and $-\mathrm{Conf}_2(\mathbb{R}^3)$, and the G–action on $Y \setminus Q$ corresponds to the permutation of coordinates on the configuration spaces. To be brief, the invariant $\sigma(\alpha_M)$ measures the difference between the manifolds $\mathrm{Conf}_2(M')$ and $\mathrm{Conf}_2(\mathbb{R}^3)$ (equipped with some second cohomology classes) by using the signature of 4–manifolds.

Remark 1.9 If M is an oriented rational homology 3–sphere, then we can define a 6–dimensional closed e-manifold $\alpha_M = (Y, Q, e_M)$ in exactly the same way as for integral homology 3–spheres. The isomorphism class of α_M is a topological invariant of M (this can be proved in the same way as the proof of Proposition 3.5), and therefore, the rational number $\sigma(\alpha_M) \in \mathbb{Q}$ is a topological invariant of M. In a future paper[1], we will prove that $\sigma(\alpha_M)$ is equal to the Casson–Walker invariant $\lambda_W(M)$ up to multiplication by a constant.

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2 Notation

We follow the notation introduced in [15]. All manifolds are assumed to be compact, smooth, and oriented unless otherwise stated, and we use the "outward normal first" convention for boundary orientation of manifolds.

For an oriented real vector bundle E of rank 3 over a manifold X, we denote the associated unit sphere bundle by $\rho_E\colon S(E) \to X$, and let $F_E \subset TS(E)$ denote the vertical tangent subbundle of $S(E)$ with respect to ρ_E. The orientations of F_E and $S(E)$ are given by the isomorphisms $\rho_E^* E \cong \mathbb{R}_E \oplus F_E$ and $TS(E) \cong \rho_E^* TX \oplus F_E$, where

[1] T. Moriyama, *Casson–Walker invariant the signature of spin* 4*–manifolds*, in preparation

$\mathbb{R}_E \subset \rho_E^* E$ is the tautological real line bundle of E over $S(E)$. Consequently, the Euler class

$$e(F_E) \in H^2(S(E); \mathbb{Z})$$

of F_E is defined.

Next, let (Z, X), $Z \supset X$, be a pair of manifolds, and we assume that X is properly embedded in Z and the codimension is 3. Throughout this paper, we always impose these assumptions for all pairs of manifolds. Denote by ν_X the normal bundle of X, which can be identified with a tubular neighborhood of X so that $X \subset \nu_X \subset Z$. For simplicity, we write

$$\hat{X} = S(\nu_X), \qquad \rho_X = \rho_{\nu_X} \colon \hat{X} \to X, \qquad F_X = F_{\nu_X}, \qquad Z_X = Z \setminus U_X,$$

where U_X is the total space of the open unit disk bundle of ν_X.

If we denote by $(W, V) = \partial(Z, X)$ the boundary pair of (Z, X), then we can define ν_V, F_V, $\hat{V}$, ρ_V, W_V, etc. in exactly the same way as above, and we have $\partial\hat{X} = \hat{V}$ and $e(F_X)|_{\hat{V}} = e(F_V)$.

In line with our orientation conventions, if $\dim Z = 7$ (and so $\dim W = 6$), then the oriented boundaries of Z_X and W_V are given as follows:

$$\partial Z_X = W_V \cup (-\hat{X}), \qquad \partial W_V = \hat{V}$$

Note that Z_X have the corner $\hat{V}$ which is empty when X is closed. By definition, $e \in H^2(Z \setminus X; \mathbb{Q})$ is an e-class (Z, X) if, and only if, $e|_{\hat{X}} = e(F_X)$ over $\mathbb{Q}$. See [15] for more detailed description.

3 Construction of α_M

Let M be an oriented integral homology 3–sphere. In this section, we give a precise construction of the e-manifold $\alpha_M = (Y, Q, e_M)$.

Identify the 3-sphere S^3 with the one-point compactification $\mathbb{R}^3 \amalg \{\infty\}$ of the Euclidean 3–space $\mathbb{R}^3$ by adding one point ∞ at infinity. We can regard $\mathbb{R}^3 \times \mathbb{R}^3$ as an open submanifold of $S^3 \times S^3$ such that $S^3 \times S^3 = (\mathbb{R}^3 \times \mathbb{R}^3) \amalg (S_1^3 \cup S_2^3)$, where

$$S_1^3 = S^3 \times \{\infty\}, \quad S_2^3 = \{\infty\} \times S^3. \tag{3–1}$$

Fix a base point $x_0 \in M$ and a smooth oriented local coordinates $\varphi \colon U \to \mathbb{R}^3$ such that $\varphi(x_0) = 0$. We shall assume that U is sufficiently small, so that, for any

such a local coordinates $\varphi'\colon U' \to \mathbb{R}^3$, there exists an orientation preserving smooth diffeomorphism $h\colon M \to M$ such that $h(U) = U'$ and $\varphi' h|_U = \varphi\colon U \to \mathbb{R}^3$. Set $P = \{(x_0, x_0)\}$. We define

$$Y = \big(M \times M \setminus P\big) \cup_{g_\varphi} \big(S^3 \times S^3 \setminus \{(0,0)\}\big)$$

to be the oriented closed 6-manifold obtained by gluing $U\times U\setminus P$ and $\mathbb{R}^3\times\mathbb{R}^3\setminus\{(0,0)\}$ by using the gluing map $g_\varphi\colon U \times U \setminus P \to \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(0,0)\}$ defined by

$$g_\varphi(x,y) = \frac{(\varphi(x), \varphi(y))}{\|(\varphi(y), \varphi(y))\|^2}, \quad (x,y) \in U \times U \setminus P, \tag{3–2}$$

where $\|\ \|$ is the standard norm of $\mathbb{R}^3\times\mathbb{R}^3 = \mathbb{R}^6$. By definition, $Y \cong (M\times M)\#(-S^3\times S^3)$.

Remark 3.1 We have to remember that we use g_φ to perform the gluing, so that we can define an involution on Y in Section 4.

We can regard $M \times M \setminus P$ and $-S^3 \times S^3 \setminus \{(0,0)\}$ as open submanifolds of Y. The closure of $M \times M \setminus P$ in Y is Y itself, and so this procedure to obtain Y from $M \times M$ is a kind of blow–up that replaces one point P to the bouquet $S^3_1 \cup S^3_2$, where note that $S^3_1 \cap S^3_2 = \{(\infty, \infty)\}$.

We have the following three 3–submanifolds M'_i $(i = 1,2,3)$ of Y:

$$M'_1 = (M \times \{x_0\}) \setminus P, \quad M'_2 = (\{x_0\} \times M) \setminus P, \quad M'_3 = M_\Delta \setminus P$$

Here, $M_\Delta \subset M \times M$ is the diagonal submanifold. The closure of M'_i in Y will be denote by M_i, which is smoothly embedded 3–submanifold of Y such that

$$M_1 = M'_1 \amalg \{(\infty, 0)\}, \quad M_2 = M'_2 \amalg \{(0,\infty)\}, \quad M_3 = M'_3 \amalg \{(\infty,\infty)\},$$
$$M_i \cong M, \quad M_i \cap M_j = \emptyset \quad (i \neq j),.$$

We then define

$$Q = M_1 \cup M_2 \cup (-M_3),$$

which is a 3–submanifold of Y, see Figure 1.

Notation 3.2 We will sometimes write $(Y(M), Q(M))$, instead of just (Y, Q), to emphasize that this is constructed from M.

Two (smooth oriented) manifold pairs (W, V) and (W', V') are said to be *isomorphic* if there exists an orientation preserving diffeomorphism $f\colon W \to W'$ such that $f(V) = V'$ as an oriented submanifold.

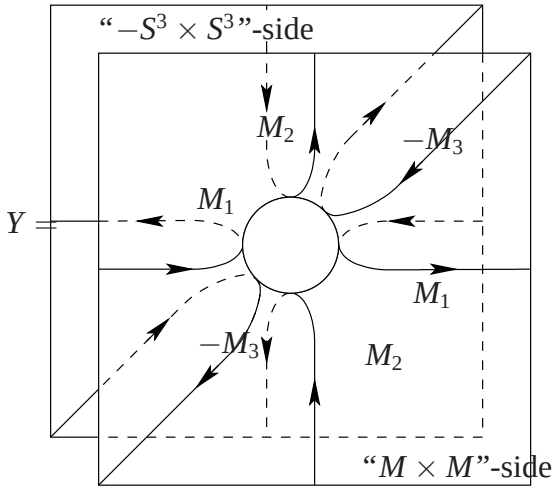

Figure 1: The manifold pair (Y, Q).

Lemma 3.3 *The isomorphism class of the pair (Y, Q) of manifolds depends only on the topological type and the orientation of M. In particular, it does not depend on x_0 or φ.*

Proof Let V_i be an oriented integral homology 3–sphere with a base point x_i and with an orientation preserving local coordinates $\varphi_i\colon U_i \to \mathbb{R}^3$ such that $\varphi_i(x_i) = 0$ $(i = 1, 2)$. Then, we can define the pair of manifolds

$$(Y_i, Q_i) = (Y(V_i), Q(V_i)),$$

by using the gluing map g_{φ_i} as in (3–2).

Assume $V_1 \cong V_2$ as an oriented topological manifold. Since the topological and the smooth categories are equivalent in dimension three, there exists an orientation preserving diffeomorphism $h\colon V_1 \to V_2$ such that $h(U_1) = U_2$ and $\varphi_1 = \varphi_2 h|_{U_1}$. Therefore, g_{φ_1} coincides with

$$g_{\varphi_2}(h \times h)|_{U_1 \times U_1 \setminus P_1}\colon U_1 \times U_1 \setminus P_1 \to \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(0,0)\},$$

where $P_i = \{(x_i, x_i)\}$. Hence, the diffeomorphism

$$h \times h\colon V_1 \times V_1 \setminus P_1 \to V_2 \times V_2 \setminus P_2$$

uniquely extends to an orientation preserving diffeomorphism $Y_1 \to Y_2$ which sends Q_1 onto Q_2. Hence, (Y_1, Q_1) and (Y_2, Q_2) are isomorphic. □

Lemma 3.4 *The pair (Y,Q) admits a unique e-class.*

Proof In general, a closed manifold pair (W,V) of dimensions 6 and 3 admits a unique e-class if it satisfies the following two conditions [15, Proposition 6.1 (5)]:

(1) The restriction $H^2(W;\mathbb{Q}) \to H^2(V;\mathbb{Q})$ is isomorphic.

(2) $[V] = 0$ in $H_3(W;\mathbb{Q})$, where $[V]$ is the fundamental homology class of V.

Since the first and the second betti–numbers of Y and Q vanish, (Y,Q) satisfies (1). By the same reason, we have $[M_1] + [M_2] = [M_3]$ in $H_3(Y;\mathbb{Q})$. Consequently, $[Q] = [M_1] + [M_2] - [M_3] = 0$, namely, (Y,Q) satisfies (2). Hence, (Y,Q) admits an unique e-class. □

We denote by $e_M \in H^2(Y \setminus Q;\mathbb{Q})$ the unique e-class of (Y,Q), and we define

$$\alpha_M = (Y, Q, e_M)$$

which is a 6–dimensional closed spin e-manifold.

Proposition 3.5 *The isomorphism class of α_M depends only on the topological type and the orientation of M.*

Proof In general, if there is an isomorphism $f\colon (W,V) \to (W',V')$ of pair of manifolds of codimension 3, then the pull–back $f^*\colon H^2(W' \setminus V';\mathbb{Q}) \to H^2(W \setminus V;\mathbb{Q})$ maps an e-class to an e-class. Thus, by Lemma 3.3 and Lemma 3.4, the isomorphism class of α_M depends only on the topological type and the orientation of M. □

4 An involution

Let $G = \{1, \iota\}$ be a multiplicative group of order two. Let M, g_φ, and $\alpha_M = (Y, Q, e_M)$ be as in Section 3. In this section, we prove Theorem 2 (1), by constructing a G–action on α_M which reverses the orientation of Y.

Remark 4.1 In this paper, G–actions we use may reverses the orientation of manifolds. Therefore, in this paper, a *G–manifold* (resp. *G–vector bundle*) will mean an oriented manifold (resp. vector bundle) with a smooth G–action which may reverses the orientation unless otherwise stated.

The group G acts on $M \times M$ and $S^3 \times S^3$ by permuting coordinates. Since the gluing map g_φ commutes with the G-action, Y has the induced smooth G-action. It is easy to check that $\iota(M_1) = M_2$, and that the fixed point set of the action on Y is M_3. Consequently, $\iota(Q) = Q$ as an oriented submanifold. Note that the involution ι reverses the orientation of Y and preserves that of Q. Thus, we can regard ι as an isomorphism

$$\iota\colon (-Y, -Q) \to (Y, -Q) \tag{4–1}$$

of pair of (oriented) manifolds.

Lemma 4.2 Theorem 2 (1) *holds, namely,* $\alpha_{-M} \cong -\alpha_M$.

Proof We shall identify $(Y(-M), Q(-M))$ with $(Y, -Q)$ which admits a unique e-class $-e_M$ by Lemma 3.4, and hence,

$$\alpha_{-M} = (Y, -Q, -e_M).$$

The homomorphism $\iota^*\colon H^2(Y \setminus Q; \mathbb{Q}) \to H^2(Y \setminus Q; \mathbb{Q})$ induced from (4–1) maps an e-class of $(Y, -Q)$ to an e-class of $(-Y, -Q)$, which means $\iota^*(-e_M) = e_M$. Thus, ι is an isomorphism from $-\alpha_M$ to α_{-M}. □

5 Spin cobordism group of e-manifolds

In [15], we proved that there is an isomorphism $\Omega_6^e \cong (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$, where Ω_6^e is the cobordism group of 6–dimensional e-manifolds. In this section, we prove that there is a similar isomorphism $\Omega_6^{e,spin} \cong (\mathbb{Q}/16\mathbb{Z}) \oplus (\mathbb{Q}/4\mathbb{Z})$. The only difference between the two proofs is that spin structures are not considered in [15], and the essential ideas behind the proofs are the same.

5.1 Preliminaries: $K(\mathbb{Q}, 2)$ and $BSpin(3)$

Let $K(\mathbb{Q}, 2)$ be the Eilenberg–MacLane space of type $(\mathbb{Q}, 2)$. The reduced homology group of $K(\mathbb{Q}, 2)$ is given as follows (cf. [4]):

$$\tilde{H}_k(K(\mathbb{Q}, 2); \mathbb{Z}) \cong \begin{cases} \mathbb{Q} & \text{if } k > 0 \text{ and } k \equiv 0 \pmod 2 \\ 0 & \text{otherwise} \end{cases} \tag{5–1}$$

The cohomology group $H^{2k}(K(\mathbb{Q}, 2); \mathbb{Q}) \cong \mathbb{Q}$ $(k \geq 0)$ is generated by the k–th power a_1^k of the dual element $a_1 \in H^2(K(\mathbb{Q}, 2); \mathbb{Q})$ of $1 \in \pi_2(K(\mathbb{Q}, 2))$.

Let $BSpin(3)$ be the classifying space of the Lie group $Spin(3)$. Since $BSpin(3)$ is homotopy equivalent to the infinite dimensional quaternionic projective space $\mathbb{H}P^\infty$, the following isomorphism holds:

$$H_k(BSpin(3);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \geq 0 \text{ and } k \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \tag{5–2}$$

We can assume that $K(\mathbb{Q},2)$ and $BSpin(3)$ have structures of CW–complexes. Let $\Omega^{spin}_*(V)$ denote the spin cobordism group of a CW–complex V. In low–dimensions, the spin cobordism group $\Omega^{spin}_* = \Omega^{spin}_*(pt)$ of one point pt is given as follows (cf. [12]):

(5–3)

k	0	1	2	3	4	5	6
Ω^{spin}_k	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0

In general, the Atiyah–Hirzebruch spectral sequence $E^n_{p,q}(Y)$ for $\Omega^{spin}_*(Y)$ converges (cf. [18]):

$$E^2_{p,q} = H_p(Y;\Omega^{spin}_q) \Longrightarrow \Omega^{spin}_{p+q}(Y)$$

The following lemma is an easy application of the Atiyah–Hirzebruch Spectral sequence.

Lemma 5.1 *The following isomorphisms hold:*

$$\Omega^{spin}_6(K(\mathbb{Q},2)) \cong \mathbb{Q}^{\oplus 2}, \quad \Omega^{spin}_4(BSpin(3)) \cong \mathbb{Z}^{\oplus 2}$$

Proof We use (5–1), (5–2), and (5–3) to prove this lemma. The Atiyah–Hirzebruch spectral sequence $E^n_{p,q} = E^n_{p,q}(K(\mathbb{Q},2))$ for $\Omega^{spin}_*(K(\mathbb{Q},2))$ converges on the E^2-stage within the range $p+q \leq 6$, and so $E^\infty_{p,q} \cong E^2_{p,q}$ in the same range. Consequently, we have

$$E^\infty_{p,6-p} \cong \begin{cases} \mathbb{Q} & \text{if } p = 6, 2, \\ 0 & \text{otherwise}, \end{cases}$$

and therefore, $\Omega^{spin}_6(K(\mathbb{Q},2)) \cong \mathbb{Q}^{\oplus 2}$.

Similarly, the spectral sequence $F^n_{p,q} = E^n_{p,q}(BSpin(3))$ converges on the F^2-stage in the range $p+q \leq 4$, and

$$F^\infty_{p,4-p} \cong \begin{cases} \mathbb{Z} & \text{if } p = 4, 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Omega^{spin}_4(BSpin(3)) \cong \mathbb{Z}^{\oplus 2}$. □

5.2 Spin cobordism groups of $BSpin(3)$ and $K(\mathbb{Q},2)$

We define three homomorphisms χ, ξ, and υ as follows. A pair (W,e) of a closed spin 6–manifold W and a cohomology class $e \in H^2(W;\mathbb{Q})$ represents an element $[W,e] \in \Omega_6^{spin}(K(\mathbb{Q},2))$. Here, we identify e with the homotopy class of a map $f\colon W \to K(\mathbb{Q},2)$ such that $f^*a_1 = e$. Define a homomorphism

$$\chi\colon \Omega_6^{spin}(K(\mathbb{Q},2)) \to \mathbb{Q}^{\oplus 2}$$

by $\chi([W,e]) = (\chi_1(W,e),\chi_2(W,e))$, where

$$\chi_1(W,e) = \frac{1}{6}\int_W p_1(TW)\,e - e^3 \in \mathbb{Q},$$
$$\chi_2(W,e) = \frac{1}{2}\int_W e^3 \in \mathbb{Q}.$$

Similarly, a pair (X,E) of a closed spin 4–manifold X and a spin vector bundle E of rank 3 over X represents an element $[X,E] \in \Omega_4^{spin}(BSpin(3))$. Here, we identify the isomorphism class of E with the homotopy class of the classifying map $X \to BSpin(3)$ of E. Note that $p_1(E) \equiv 0 \pmod 4$ (since E is spin), and that

$$\operatorname{Sign} X \equiv 0 \pmod{16} \tag{5–4}$$

by the Rokhlin's theorem. Define a homomorphism

$$\xi\colon \Omega_4^{spin}(BSpin(3)) \to 16\mathbb{Z} \oplus 4\mathbb{Z}$$

by
$$\xi([X,E]) = \left(\operatorname{Sign} X, \int_X p_1(E)\right).$$

We will see soon that χ and ξ are isomorphic (Lemma 5.3). We define a homomorphism

$$\upsilon\colon \Omega_4^{spin}(BSpin(3)) \to \Omega_6^{spin}(K(\mathbb{Q},2))$$

by $\upsilon([X,E]) = [S(E), e(F_E)]$.

Now, for a pair (X,E) representing an element in $\Omega_4^{spin}(BSpin(3))$, the characteristic classes of the vector bundles E, F_E, TX, and $TS(E)$ satisfy the following relations:

$$e(F_E)^2 = p_1(F_E) = \rho_E^* p_1(E) \tag{5–5}$$
$$\equiv p_1(TS(E)) - \rho_E^* p_1(TX) \quad \text{(modulo 2–torsion elements)}, \tag{5–6}$$
$$\rho_{E!}\, e(F_E) = 2 \tag{5–7}$$

Here, $\rho_{E!}\colon H^2(S(E);\mathbb{Z}) \to H^0(X;\mathbb{Z})$ is the Gysin homomorphism of ρ_E, and $2 \in H^0(X;\mathbb{Z})$ denotes the element given by the constant function on X with the value 2 (= Euler characteristic of S^2). The Hirzebruch signature theorem states that

$$\operatorname{Sign} X = \frac{1}{3}\int_X p_1(TX). \tag{5–8}$$

The next two lemmas are easy to prove.

Lemma 5.2 *$\chi\upsilon = \xi$. In other words, for any pair (X,E) of closed spin 4–manifold X and a spin vector bundle E of rank 3 over X, we have*

$$\chi([S(E), e(F_E)]) = \left(\operatorname{Sign} X,\ \int_X p_1(E)\right).$$

Proof This follows from the formulas (5–5), (5–6), (5–7), and (5–8). In fact, we have $p_1(TS(E))e(F_E) - e(F_E)^2 = \rho_E^* p_1(TX)e(F_E)$, and so

$$\chi_1(S(E), e(F_E)) = \frac{1}{6}\int_{S(E)} \rho_E^* p_1(TX)e(F_E) = \frac{1}{3}\int_X p_1(TX) = \operatorname{Sign} X.$$

Similarly, we have

$$\chi_2(S(E), e(F_E)) = \frac{1}{2}\int_{S(E)} \rho_X^* p_1(E)e(F_E) = \int_X p_1(E). \qquad \square$$

Lemma 5.3 *The homomorphisms χ and ξ are isomorphic.*

Proof The $K3$-manifold $K3$ is a closed spin 4–manifold with the signature -16. There exists an oriented spin vector bundle E of rank 3 over S^4 such that $p_1(E) = 4$ in $H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$. We define two elements $u_1, u_2 \in \Omega_4^{spin}(BSpin(3))$ as follows:

$$u_1 = [K3, K3 \times \mathbb{R}^3], \quad u_2 = [S^4, E]$$

Then, $\xi(u_1) = (-16, 0)$ and $\xi(u_2) = (0, 4)$ by definition. Therefore, $\operatorname{Im}\chi = (16\mathbb{Z}) \oplus (4\mathbb{Z})$. In particular, ξ is a surjective homomorphism from $\Omega_4^{spin}(BSpin(3)) \cong \mathbb{Z}^{\oplus 2}$ (Lemma 5.1) to $16\mathbb{Z} \oplus 4\mathbb{Z}$. This means that ξ is an isomorphism.

Similarly, we have $\chi(\upsilon(u_1)) = (-16, 0)$ and $\chi(\upsilon(u_1)) = (0, 4)$ by Lemma 5.2, and these two elements form a basis of the vector space $\mathbb{Q}^{\oplus 2}$ over $\mathbb{Q}$. Therefore, χ is a linear homomorphism from $\Omega_6^{spin}(K(\mathbb{Q}), 2) \cong \mathbb{Q}^{\oplus 2}$ (Lemma 5.1) to $\mathbb{Q}^{\oplus 2}$ of rank 2. This means that χ is an isomorphism. $\square$

Proposition 5.4 *The sequence of the homomorphisms*

$$0 \to \Omega_4^{spin}(BSpin(3)) \xrightarrow{\upsilon} \Omega_6^{spin}(K(\mathbb{Q},2)) \xrightarrow{\chi'} (\mathbb{Q}/16\mathbb{Z}) \oplus (\mathbb{Q}/4\mathbb{Z}) \to 0$$

is exact, where $\chi' = \chi \mod 16\mathbb{Z} \oplus 4\mathbb{Z}$.

Proof This follows from that, the diagram

$$\begin{array}{ccc} \Omega_4^{spin}(BSpin(3)) & \xrightarrow{\upsilon} & \Omega_4^{spin}(K(\mathbb{Q},2)) \\ \xi \downarrow \cong & & \chi \downarrow \cong \\ (16\mathbb{Z}) \oplus (4\mathbb{Z}) & \xrightarrow{\text{inclusion}} & \mathbb{Q}^{\oplus 2} \end{array}$$

commutes (Lemma 5.2) and the vertical arrows are isomorphic (Lemma 5.3). □

5.3 An extension

Let us consider the homomorphism

$$\pi\colon\ \Omega_6^{spin}(K(\mathbb{Q},2)) \to \Omega_6^{e,spin}$$

defined by $\pi([W,e]) = [W,\emptyset,e]$ for $[W,e] \in \Omega_6^{spin}(K(\mathbb{Q},2))$. We can prove that π is surjective as follows.

Let $\alpha = (W,V,e)$ be a 6–dimensional closed spin e-manifold. The normal bundle ν_V of V is trivial, because it is spin. We fix a trivialization of ν_V, so that a closed tubular neighborhood of V is identified with $V \times D^3$ such that $V \times S^2 = \hat{V}$. Let X be a spin 4–manifold such that $\partial X = V$.

Let $p\colon\ X \times S^2 \to S^2$ be the projection, and $e(TS^2)$ the Euler class of S^2. Two spin manifolds W_V and $X \times S^2$ have the common spin boundary $\partial W_V = \hat{V} = \partial(X \times S^2)$, and the cohomology classes e and $p^*e(TS^2)$ restrict to the same element $e(F_V)$ on $\hat{V}$ over $\mathbb{Q}$. Let us consider the closed oriented spin 6–manifold

$$W' = W_V \cup_{\hat{V}} (-X \times S^2) \tag{5–9}$$

obtained from W_V and $-X \times S^2$ by gluing along the common boundaries. There exists a cohomology class $e' \in H^2(W';\mathbb{Q})$ such that

$$e'|_{W_V} = e|_{W_V}, \quad e'|_{X\times S^2} = p^*e(TS^2). \tag{5–10}$$

We obtain a 6–dimensional closed spin e-manifold $\alpha' = (W',\emptyset,e')$ and a cobordism class $[W',e'] \in \Omega_6^{spin}(K(\mathbb{Q},2))$ such that $\pi([W',e']) = [\alpha']$.

Proposition 5.5 *Let α, X, and $\alpha' = (W', \emptyset, e')$ be as above. Then, there exists a 7–dimensional spin e-manifold of the form $\beta = (Z, X, \tilde{e})$ for some spin 7–manifold Z and $\tilde{e} \in H^2(Z \setminus X; \mathbb{Q})$ such that $\partial\beta \cong \alpha \amalg (-\alpha')$. In particular, $\pi([W', e']) = [\alpha]$ in $\Omega_6^{e,spin}$. Consequently, the homomorphism π is surjective.*

Proof Let $I = [0,1]$ be the interval. In this proof, for a subset $A \subset W$, we write $A_t = \{t\} \times A \subset I \times W$ for $t = 0, 1$.

Gluing the 7–manifolds $I \times W$ and $X \times D^3$ along $D(\nu_V)_0 \subset W_0$ and $V \times D^3 \subset \partial(X \times D^3)$ by using the identity map, we obtain a spin 7–manifold

$$Z = (X \times D^3) \cup_{(V \times D^3)_0} (I \times W)$$

with the boundary

$$\begin{aligned} \partial Z &= W_1 \amalg \left((X \times S^2) \cup_{\hat{V}_0} (-(W_V)_0)\right) \\ &\cong W \amalg (-W'), \end{aligned}$$

and we shall assume that ∂Z is smooth after the corner $\hat{V}_0$ is rounded. The spin 4–submanifold

$$(X \times \{0\}) \cup_{V_0} (I \times V) \subset Z$$

is properly embedded in Z, and is bounded by V_1. We will rewrite $X \cup_{V_0} (I \times V)$ as X and identify ∂Z with $W \amalg (-W')$, so that

$$\partial(Z, X) = (W, V) \amalg (-W', \emptyset)$$

as a spin manifold pair.

Now, all that is left to do is to show the existence of an e-class of (Z, X) restricting to e and e' on the boundary components. Since the inclusion $W' \hookrightarrow Z \setminus X$ is homotopy equivalence, there exists a cohomology class $\tilde{e} \in H^2(Z \setminus X; \mathbb{Q})$ of (Z, X) such that $\tilde{e}|_{W'} = e'$. By construction, $\tilde{e}$ is an e-class of (Z, X) such that $\tilde{e}|_{W \setminus V} = e$. Hence, we obtain a 7–dimensional spin e-manifold $\beta = (Z, X, \tilde{e})$ bounded by

$$\partial\beta = (W, V, \tilde{e}|_{W \setminus V}) \amalg (-W', \emptyset, \tilde{e}|_{W'}) = \alpha \amalg (-\alpha'). \qquad \square$$

5.4 Proof of Theorem 1

In this subsection, we prove Theorem 1. By Proposition 5.5, we can use the formula (1–4) to define the homomorphism $\Phi\colon \Omega_6^{e,spin} \to (\mathbb{Q}/16\mathbb{Z}) \oplus (\mathbb{Q}/4\mathbb{Z})$. The first thing we have to do is to show that Φ is well–defined.

Lemma 5.6 *The homomorphism* $\Phi\colon \Omega_6^{e,spin} \to (\mathbb{Q}/16\mathbb{Z}) \oplus (\mathbb{Q}/4\mathbb{Z})$ *is well–defined.*

Proof Let us consider two 6–dimensional closed spin e-manifolds of the forms $\alpha = (W, \emptyset, e)$ and $\alpha' = (W', \emptyset, e')$ such that $[W, \emptyset, e] = [W', \emptyset, e']$ in $\Omega_6^{e,spin}$. We only need to show that the difference $\chi([W, \emptyset, e]) - \chi([W', \emptyset, e'])$ belongs to $16\mathbb{Z} \oplus 4\mathbb{Z}$.

There exists a 7–dimensional spin e-manifold $\beta = (Z, X, \tilde{e})$ such that $\partial\beta = \alpha \amalg (-\alpha')$. The 4–submanifold X is closed, spin, and embedded in the interior of Z. Thus, the manifold Z_X has the smooth spin boundary

$$\partial Z_X = W \amalg (-W') \amalg (-\hat{X}).$$

Since $\tilde{e}|_{\hat{X}} = e(F_X)$, we have

$$\partial(Z_X, \tilde{e}|_{Z_X}) = (W, e) \amalg (-W', e') \amalg (-\hat{X}, e(F_X)),$$

and this implies $[W, e] - [W', e'] = [\hat{X}, e(F_X)]$ in $\Omega_6^{spin}(K(\mathbb{Q}, 2))$. By Lemma 5.2, we have

$$\chi([\hat{X}, e(F_X)]) = \chi(\upsilon([X, \nu_X])) = \xi([X, \nu_X]) \in 16\mathbb{Z} \oplus 4\mathbb{Z},$$

where ν_X is the normal bundle of X. □

Now, we can prove Theorem 1.

Proof of Theorem 1 Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_4^{spin}(BSpin(3)) & \xrightarrow{\upsilon} & \Omega_6^{spin}(K(\mathbb{Q},2)) & \xrightarrow{\pi} & \Omega_6^{e,spin} & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow{\scriptstyle \Phi} & & \\
0 & \longrightarrow & \Omega_4^{spin}(BSpin(3)) & \xrightarrow{\upsilon} & \Omega_6^{spin}(K(\mathbb{Q},2)) & \xrightarrow{\chi'} & \frac{\mathbb{Q}\oplus\mathbb{Q}}{16\mathbb{Z}\oplus 4\mathbb{Z}} & \longrightarrow & 0
\end{array}$$

The lower horizontal sequence is exact by Proposition 5.4, and the homomorphism π is surjective by Proposition 5.5. To complete the proof, we only have to show that the upper horizontal sequence is exact, more specifically,

$$\operatorname{Im}\upsilon = \operatorname{Ker}\pi.$$

We prove this in two steps as follows.

Claim 1: $\operatorname{Im}\upsilon \subset \operatorname{Ker}\pi$. Let $[X, E] \in \Omega_4^{spin}(BSpin(3))$ be any element, then

$$\pi(\upsilon([X, E])) = [S(E), \emptyset, e(F_E)]$$

by definition. We can regard X as the image of the zero–section of E so that $X \subset \operatorname{Int} D(E)$. The cohomology class $e(F_E)$ is an e-class of $(S(E), \emptyset) = \partial(D(E), X)$, and it uniquely extends to an e-class, say e_E, of $(D(E), X)$. The obtained spin e-manifold $(D(E), X, e_E)$ is bounded by $(S(E), \emptyset, e(F_E))$, and hence, we have $\pi(\upsilon([X, E])) = 0$.

Claim 2: $\operatorname{Im}\upsilon \supset \operatorname{Ker}\pi$. Next, we prove the opposite inclusion. Let $[W, e] \in \operatorname{Ker}\pi$ be any element, then $\alpha = (W, \emptyset, e)$ bounds a 7–dimensional spin e-manifold $\beta = (Z, X, \tilde{e})$, namely $\partial\beta = \alpha$. In particular, we have $\tilde{e}|_{\hat{X}} = e(F_X)$. Since

$$\partial(Z_X, \tilde{e}|_{Z_X}) = (W, e) \amalg (-\hat{X}, e(F_X)),$$

the cobordism class $[W, e] \in \Omega_6^{spin}(K(\mathbb{Q}, 2))$ satisfies

$$[W, e] = [\hat{X}, e(F_X)] = \upsilon([X, \nu_X]) \in \operatorname{Im}\upsilon,$$

where ν_X is the normal bundle of X. □

6 Signature modulo 32

Let M be an oriented integral homology 3–sphere, and $\alpha_M = (Y, Q, e_M)$ the 6–dimensional closed spin e-manifold constructed in Section 3. Let $[\alpha_M] \in \Omega_6^{e,spin}$ denote the spin cobordism class of α_M. In this section, by using the isomorphism Φ, we derive a necessary and sufficient condition for the vanishing $[\alpha_M] = 0$ in terms of the signature of a 4–manifold (Proposition 6.3).

Recall that we constructed a G–action on (Y, Q) in Section 4. The normal bundle ν_Q of Q has a G–equivariant trivialization $\nu_Q = Q \times \mathbb{R}^3$ such that

$$\iota(x, v) = (\iota(x), -v), \tag{6–1}$$

$$\hat{Q} = Q \times S^2, \tag{6–2}$$

where $(x, v) \in \nu_Q$.

Let X_0 be an oriented spin 4–manifold equipped with an identification $\partial X_0 = M$, and consider the union

$$X = X_1 \cup X_2 \cup X_3,$$

where X_i $(i = 1, 2, 3)$ are disjoint copies of X_0 such that $\partial X_i = M_i$, and so

$$\partial X = Q. \tag{6–3}$$

The G–action on Q naturally extends to an action on X such that $\iota(X_1) = \iota(X_2)$ and that ι restricts to the identity on X_3. We define a G–action on the trivial vector bundle

$X \times \mathbb{R}^3$ over X in the same way as (6–1). Then, the G–vector bundle $X \times \mathbb{R}^3$ restricts to ν_Q over Q. Consequently,

$$\partial X \times S^2 = \hat{Q}. \tag{6–4}$$

Note that (6–2), (6–3), and (6–4) hold as G–manifolds.

As in (5–9) and (5–10), let us consider the closed spin 6–dimensional G–manifold

$$Y' = Y_Q \cup_{\hat{Q}} (-X \times S^2) \tag{6–5}$$

obtained by gluing the common boundaries $\partial Y_Q = \hat{Q} = \partial(X \times S^2)$, and the cohomology class $e'_M \in H^2(Y'; \mathbb{Q})$ such that

$$e'_M|_{Y_Q} = e_M, \quad e'_M|_{X \times S^2} = f_X^* e(TS^2), \tag{6–6}$$

where

$$f_X \colon X \times S^2 \to S^2 \tag{6–7}$$

is the projection. We obtain a 6–dimensional closed spin e-manifold

$$\alpha'_M = (Y', \emptyset, e'_M).$$

Note that the G–action on Y' is free, and the quotient Y'/G is a smooth closed unoriented manifold.

Lemma 6.1 *For $k \leq 3$, the restriction homomorphisms*

$$H^k(Y'; \mathbb{Z}) \to H^k(X \times S^2; \mathbb{Z}),$$
$$H^k(Y'/G; \mathbb{Z}/2) \to H^k((X \times S^2)/G; \mathbb{Z}/2)$$

are injective.

Proof We identify the cohomology group $H^*(Y', X \times S^2; \mathbb{Z})$ with $H^*(Y_Q, \hat{Q}; \mathbb{Z})$, and $H^*(Y'/G, (X \times S^2)/G; \mathbb{Z}/2)$ with $H^*(Y_Q/G, \hat{Q}/G; \mathbb{Z}/2)$ by using the excision isomorphisms.

The homomorphism $\delta^* \colon H^{k-1}(X \times S^2; \mathbb{Z}) \to H^k(Y', X \times S^2; \mathbb{Z})$ given by the pair $(Y', X \times S^2)$ coincides with the composition of two homomorphisms

$$H^{k-1}(X; \mathbb{Z}) \to H^{k-1}(\hat{Q}; \mathbb{Z}) \to H^k(Y_Q, \hat{Q}; \mathbb{Z}), \tag{6–8}$$

where the first arrow is the restriction, and where the second arrow is the homomorphism given by $(Y_Q, \hat{Q})$. Note that the homomorphism $H^k(Y_Q, \hat{Q}; \mathbb{Z}) \to H^k(Y_Q; \mathbb{Z})$ is trivial. Both homomorphisms in (6–8) are surjective, and so is δ^*. Hence, $H^k(Y'; \mathbb{Z}) \to H^k(X \times S^2; \mathbb{Z})$ is injective.

Similarly, the homomorphism $H^{k-1}(X/G;\mathbb{Z}/2) \to H^k(Y'/G,(X\times S^2)/G;\mathbb{Z}/2)$ coincides with the composition of two surjective homomorphisms

$$H^{k-1}(X/G;\mathbb{Z}/2) \to H^{k-1}(\hat{Q}/G;\mathbb{Z}/2) \to H^k(Y_Q/G,\hat{Q}/G;\mathbb{Z}/2).$$

Note that the homomorphism $H^k(Y_Q/G,\hat{Q}/G;\mathbb{Z}/2) \to H^k(Y_Q/G;\mathbb{Z}/2)$ is trivial. Therefore, $H^k(Y'/G;\mathbb{Z}/2) \to H^k(Y'/G,(X\times S^2)/G;\mathbb{Z}/2)$ is injective. □

The following lemma is easy to prove.

Lemma 6.2 $e'_M \equiv 0 \pmod 2$.

Proof The Euler characteristic of S^2 is 2, which is even. Therefore, the cohomology class $e'_M \mod 2$ belongs to the kernel of the restriction

$$H^2(Y';\mathbb{Q}/2\mathbb{Z}) \to H^2(X\times S^2;\mathbb{Q}/2\mathbb{Z})$$

by (6–6). On the other hand, this homomorphism is injective by Lemma 6.1. Therefore, $e'_M \equiv 0 \pmod 2$. □

Proposition 6.3 *Assume that there is a* 4*–submanifold* W *of* Y' *which Poincaré dual is* e'_M*. Then, the* e*-manifold* α_M *is spin null–cobordant* (*namely,* Theorem 2 (2) *holds*) *if, and only if,*

$$\operatorname{Sign} W \equiv 0 \pmod{32}.$$

Proof By Proposition 5.5, $[\alpha_M] = [\alpha'_M]$ in $\Omega_6^{e,spin}$. By (1–4), we have

$$\Phi([\alpha'_M]) \equiv \left(\frac{1}{6}\int_{Y'} p_1(TY')e'_M - e'^3_M,\ \frac{1}{2}\int_{Y'} e'^3_M\right) \mod 16\mathbb{Z}\oplus 4\mathbb{Z}.$$

Since $p_1(TY')|_W = p_1(TW) + e'^2_M$, the first component on the right–hand side is equal to $\operatorname{Sign} W/2$. By Lemma 6.2, we have $e'^3_M/2 \equiv 0 \pmod 4$, and so $\Phi([\alpha_M]) \equiv (\operatorname{Sign} W/2, 0) \mod 16\mathbb{Z}\oplus 4\mathbb{Z}$. Since Φ is an injective by Theorem 1, $[\alpha_M] = 0$ if, and only if, $\operatorname{Sign} W/2 \equiv 0 \pmod{16}$. □

7 Proof of Theorem 2

In this section, we prove Theorem 2 (2), by constructing a 4–submanifold W of Y' as in Proposition 6.3.

Proposition 7.1 *There exists an oriented vector bundle F of rank 2 over Y' with a G–action satisfying the following two properties.*

(1) $e(F) = e'_M$ *over* $\mathbb{Q}$.

(2) $w_i(F/G) = w_i(TY'/G)$ *in* $H^i(Y'/G;\mathbb{Z}/2)$ *for* $i = 1,2$.

Here, F/G is the quotient of F, which is unoriented vector bundle of rank 2 over the unoriented manifold Y'/G, and here, w_i denotes the i–th Stiefel–Whitney class. The proof will be given in Section 8.

Since G acts freely on Y', there exits a G–equivariant smooth section $s\colon Y' \to F$ transverse to the zero section. We define $W = \{x \in Y' \mid s(x) = 0\}$, which is a smooth oriented closed 4–dimensional G–submanifold of Y'. By Proposition 7.1 (1), the Poincaré dual of W is e'_M. The quotient space W/G is a unoriented smooth submanifold of Y'/G.

Lemma 7.2 $w_i(TW/G) = 0$ *for* $i = 1,2$.

Proof There is an isomorphism $TY'/G|_{W/G} \cong TW/G \oplus F/G|_{W/G}$. Since TY'/G and F/G have the same Stiefel–Whitney classes w_i $(i = 1,2)$ by Proposition 7.1 (2), we have $w_i(TW/G) = 0$. □

We can prove Theorem 2 (2) as follows.

Proof of Theorem 2 (2) By Lemma 7.2, the closed smooth manifold W/G is orientable and spinnable. We fix an orientation of W/G, then $\operatorname{Sign} W = \pm 2 \operatorname{Sign} W/G$. By Rokhlin's theorem (5–4), we have $\operatorname{Sign} W/G \equiv 0 \pmod{16}$, and consequently, $\operatorname{Sign} W \equiv 0 \pmod{32}$. □

8 G–vector bundle

In this section, we prove Proposition 7.1. To construct the G–vector bundle F, we prove the existence of a G–equivariant classifying map $f_M\colon Y' \to \mathbb{C}P^3$ of F. Here, a G–action on $\mathbb{C}P^3$ is defined as follows.

Let $\mathbb{H}$ denote the quaternions spanned by $\{1,i,j,k\}$ over $\mathbb{R}$ such that $i^2 = j^2 = k^2 = ijk = -1$. By regarding $\mathbb{H}$ as the complex space $\mathbb{C} \oplus \mathbb{C}j$, we can identify the complex projective space $\mathbb{C}P(\mathbb{H}^{n+1})$ with $\mathbb{C}P^{2n+1}$ for $n \geq 0$ (our main interest is

when $n = 0, 1$). The multiplication by j on vectors on $\mathbb{H}^{n+1}$ from the left provides a free involution $\iota\colon \mathbb{C}P^{2n+1} \to \mathbb{C}P^{2n+1}$, and so $\mathbb{C}P^{2n+1}$ is a G–manifold. Note that the natural inclusion $S^2 = \mathbb{C}P^1 \to \mathbb{C}P^3$ commutes with the G–action. The unit 2–sphere $S^2 \subset \mathbb{R}^3$ has a free G–action given by the multiplication by a scalar -1. We shall identify $\mathbb{C}P^1$ with S^2 as a G–manifold.

Let $f_Q\colon \hat{Q} \to S^2$ be the projection onto the fiber given by the trivialization (6–2), and $f_X\colon X \times S^2 \to S^2$ be as in (6–7). Let S^3_i $(i = 1, 2)$ be as in (3–1). Note that $f_X|_{\hat{Q}} = f_Q$ and $\iota(S^3_1) = S^3_2$.

Let P_i $(i = 1, 2, 3)$ be 0–dimensional submanifolds of Y defined as follows:

$$P_1 = \{(0, \infty)\}, \qquad P_2 = \{(\infty, 0)\}, \qquad P_3 = \{(\infty, \infty)\},$$

then $S^3_1 \cap Q = P_1 \cup (-P_3)$ and $S^3_2 \cap Q = P_2 \cup (-P_3)$ as oriented manifolds. We define

$$C_i = S^3_i \setminus (Q \times \operatorname{Int} D^3) \quad (i = 1, 2),$$

which is a proper 3–submanifold of Y_Q. We shall assume that C_i is diffeomorphic to $S^2 \times [0, 1]$, by choosing a small tubular neighborhood $Q \times D^3$ of Q (so that $S^3_i \cap (Q \times D^3)$ is the disjoint union of two small 3–balls in S^3_i). In particular, the boundary ∂C_i is the disjoint union two 2–spheres $\partial C_i = (S^3_i \cap \hat{M}_i) \amalg (-S^3_i \cap \hat{M}_3)$. The involution $\iota\colon Y' \to Y'$ restricts to a diffeomorphism $\iota|_{C_1}\colon C_1 \to C_2$. We write

$$C = \hat{Q} \cup C_1 \cup C_2,$$

then $\iota(C) = C$.

Lemma 8.1 *The map $f_Q\colon \hat{Q} \to S^2$ extends to a G–equivariant map $f_C\colon C \to S^2$.*

Proof By the definition of f_Q, the two maps

$$f_Q|_{S^3_1 \cap \hat{M}_1}\colon S^3_1 \cap \hat{M}_1 \to S^2, \quad f_Q|_{S^3_1 \cap \hat{M}_3}\colon S^3_1 \cap \hat{M}_3 \to S^2$$

have the degree $+1$ and -1 respectively. Therefore $f_Q|_{\partial C_1}\colon \partial C_1 \to S^2$ extends to a map $f_{C_1}\colon C_1 \to S^2$. We define a map $f_C\colon C \to S^2$ by

$$f_C(x) = \begin{cases} f_Q(x) & \text{if } x \in \hat{Q} \\ f_{C_1}(x) & \text{if } x \in C_1 \\ f_{C_1}(\iota(x)) & \text{if } x \in C_2 \end{cases}$$

for $x \in C$. It is easy to check that this is well–defined and G–equivariant. □

To obtain a classifying map $f_M\colon Y_Q \to \mathbb{C}P^3$, we consider the obstruction classes to extending the map f_C to a G–equivariant map f_M. The primary obstruction class belongs to the cohomology group

$$H^3(Y_Q/G, C/G; \mathbb{Z}_-), \tag{8–1}$$

where $\mathbb{Z}_-$ is the local system on Y_Q/G given by the non–trivial characteristic homomorphism $G \to \mathrm{Aut}(\pi_2(\mathbb{C}P^3)) = \{id, -id\}$. In other words, (8–1) is the G–equivariant cohomology group with coefficients in the non–trivial G–module $\mathbb{Z}$ (such that $\iota 1 = -1$).

Lemma 8.2 *The obstruction group* (8–1) *vanishes.*

Proof The low–dimensional cohomology groups of (Y_Q, C) and $(Y_Q/G, C/G)$ are given as follows:

$$H^k(Y_Q, C; \mathbb{Z}) \cong H^k(Y_Q/G, C/G; \mathbb{Z}) \cong 0 \quad (k \leq 3) \tag{8–2}$$

There is a long exact sequence

$$\cdots \xrightarrow{\delta^*} H^k(Y_Q/G, C/G; \mathbb{Z}_-) \xrightarrow{q^*} H^k(Y_Q, C; \mathbb{Z}) \xrightarrow{q_!} H^k(Y_Q/G, C/G; \mathbb{Z}) \to \cdots,$$

where q^* is the pull–back of the covering map $q\colon Y_Q \to Y_Q/G$, and $q_!$ is the Gysin homomorphism. The vanishing (8–2) and the exact sequence implies

$$H^k(Y_Q/G, C/G; \mathbb{Z}_-) = 0 \quad (k \leq 3). \qquad \square$$

Proposition 8.3 *There exists a G–equivariant map $f_M\colon Y' \to \mathbb{C}P^3$ such that $f_M|_{\hat{Q}} = f_Q$.*

Proof By Lemma 8.2, the primary obstruction class vanishes. The higher obstruction groups vanishes, since $\pi_i(\mathbb{C}P^3) = 0$ for $3 \leq i \leq 6$. Hence, f_C extends to a G–equivariant map $f_M\colon Y_Q \to \mathbb{C}P^3$. $\square$

Now, let us consider the fiber bundle $\rho\colon \mathbb{C}P^3 \to \mathbb{H}P^1$ which maps a complex line l in $\mathbb{H}^2$ to the corresponding quaternionic line $\mathbb{H} \otimes_{\mathbb{C}} l$. The fiber of ρ is $\mathbb{C}P^1$, and the G–action preserves the fiber. Let $F_1 \subset T\mathbb{C}P^3$ be the tangent subbundle of $\mathbb{C}P^3$ with respect to ρ, which is an oriented vector bundle of rank 2 over $\mathbb{C}P^3$ with a G–action. We then define

$$F = f_M^* F_1 \tag{8–3}$$

to be the pull–back of F_1 under f_M. It is an oriented vector bundle of rank 2 over Y' with a G–action.

Proof of Proposition 7.1 By the construction of F, we have

$$e(F)|_{X\times S^2} = f_X^* e(TS^2) = e'_M|_{X\times S^2}.$$

By Lemma 6.1, the homomorphism $H^2(Y';\mathbb{Q}) \to H^2(X\times S^2;\mathbb{Q})$ is injective, and therefore, $e(F) = e'_M$, and Proposition 7.1 (1) holds.

The vector bundle F restricts to $f_X^*TS^2$ over $X\times S^2$. The quotient manifold $(X\times S^2)/G$ is diffeomorphic to the disjoint union of $X\times S^2$ and $X\times \mathbb{R}P^2$. Since X is oriented and spin, we have

$$w_i(TY'/G)|_{(X\times S^2)/G} = w_i(f_X^*TS^2/G) = w_i(F/G)|_{(X\times S^2)/G} \quad (i=1,2).$$

By Lemma 6.1, we have $w_i(TY'/G) = w_i(F/G)$. Namely, Proposition 7.1 (2) holds. □

9 Appendix: Yet another proof of Corollary 3

Let M be an oriented integral homology 3–sphere, and $\alpha_M = (Y,Q,e_M)$ the 6–dimensional closed spin e-manifold constructed from M. The aim of this section is to give yet another direct proof of Corollary 3 using Theorem 2 and without using Theorem 1.4.

Proof Let us assume $M \cong -M$, then $\alpha_M \cong -\alpha_M$ by Theorem 2 (1). Namely, there exists a diffeomorphism

$$h\colon (Y,Q) \to (Y,Q)$$

which reverses the orientations of Y and Q such that $h^*e_M = e_M$. By Theorem 2 (2), there exists a 7–dimensional spin e-manifold $\beta = (Z,X,\tilde{e})$ such that $\partial\beta = \alpha_M$.

Let us consider the 7–dimensional closed spin e-manifold

$$\beta' = \beta \cup_h \beta$$

obtained by gluing the boundaries of two disjoint copies of β by using h, more precisely, we can write

$$\beta' = (Z',X',\tilde{e}'), \quad Z' = Z\cup_h Z, \quad X' = X\cup_h X,$$

where $\tilde{e}' \in H^2(Z'\setminus X';\mathbb{Q})$ is the e-class of (Z',X') obtained by gluing two copies of $\tilde{e}$. Note that the manifolds Z' and X' are closed spin.

What we need to prove is $\operatorname{Sign} X \equiv 0 \pmod{16}$, or equivalently

$$\operatorname{Sign} X' \equiv 0 \pmod{32}.$$

It is easy to show that there is the following formula (see also Lemma 5.2):

$$\operatorname{Sign} X' = \frac{1}{6}\int_{\hat{X}'} p_1(T\hat{X}')e(F_{X'}) - e(F_{X'})^3$$

Since $\partial Z'_{X'} = -\hat{X}'$ and $\tilde{e}'$ is an e-class of (Z', X') (namely, $\tilde{e}'|_{\hat{X}'} = e(F_{X'})$ by definition), the right–hand side is equal to

$$-\frac{1}{6}\int_{\partial Z'_{X'}} p_1(TZ'_{X'})\tilde{e}' - \tilde{e}'^3 = 0$$

by Stokes' theorem. Consequently, $\operatorname{Sign} X' \equiv 0 \pmod{32}$. □

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