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Energy image density property and local gradient for Poisson random measures

Nicolas BOULEAU and Laurent DENIS

Abstract

We introduce a new approach to absolute continuity of laws of Poisson functionals. It is based on the *energy image density* property for Dirichlet forms and on what we call *the lent particle method* which consists in adding a particle and taking it back after some calculation.

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1 Introduction

The aim of this article is to improve some tools provided by Dirichlet forms for studying the regularity of Poisson functionals. First, the *energy image density property* (EID) which guarantees the existence of a density for \mathbb{R}^d -valued random variables whose carré du champ matrix is almost surely regular. Second, the Lipschitz functional calculus for a *local gradient* satisfying the chain rule, which yields regularity results for functionals of Lévy processes.

For a local Dirichlet structure with carré du champ, the energy image density property is always true for real-valued functions in the domain of the form (Bouleau [5], Bouleau-Hirsch [9] Chap. I §7). It has been conjectured in 1986 (Bouleau-Hirsch [8] p251) that (EID) were true for any \mathbb{R}^d -valued function whose components are in the domain of the form for any local Dirichlet structure with carré du champ. This has been shown for the Wiener space equipped with the Ornstein-Uhlenbeck form and for some other structures by Bouleau-Hirsch (cf. [9] Chap. II §5 and Chap. V example 2.2.4) and also for the Poisson space by A. Coquio [11] when the intensity measure is the Lebesgue measure on an open set, but this conjecture being at present neither refuted nor proved in full generality, it has to be established in every particular setting. We will proceed in two steps : first (Part 2) we prove sufficient conditions for (EID) based mainly on a study of Shiqi Song [29] using a characterization of Albeverio-Röckner [2], then (Part 4) we show that the Dirichlet structure on the Poisson space obtained from a Dirichlet structure on the states space inherits from that one the (EID) property.

If we think a local Dirichlet structure with carré du champ $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ as a description of the Markovian movement of a particle on the space (X, \mathcal{X}) whose transition semi-group p_t is symmetric with respect to the measure ν and strongly continuous on $L^2(\nu)$, the construction of the Poisson measure allows to associate to this structure a

structure on the Poisson space $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ which describes similarly the movement of a family of independent identical particles whose initial law is the Poisson measure with intensity ν . This construction is ancient and may be performed in several ways.

The simplest one, from the point of view of Dirichlet forms, is based on products and follows faithfully the probabilistic construction (Bouleau [6], Denis [12], Bouleau [7] Chap. VI §3). The cuts that this method introduces are harmless for the functional calculus with the carré du champ Γ , but it does not clearly show what happens for the generator and its domain.

Another way consists in using the transition semi-groups (Martin-Löf [18], Wu [31], partially Bichteler-Gravereaux-Jacod [4], Surgailis [30]). It is supposed that there exists a Markov process x_t with values in X whose transition semi-group π_t is a version of p_t (cf. Ma-Röckner [20] Chap. IV §3), the process starting at the point z is denoted by $x_t(z)$ and a probability space (W, \mathcal{W}, Π) is considered where a family $(x_t(z))_{z \in X}$ of independent processes is realized. For a symmetric function F , the new semi-group P_t is directly defined by

$$(P_t F)(z_1, \dots, z_n, \dots) = \int F(x_t(z_1), \dots, x_t(z_n), \dots) d\Pi$$

Choosing as initial law the Poisson measure with intensity ν on (X, \mathcal{X}) , it is possible to show the symmetry and the strong continuity of P_t . This method, based on a deep physical intuition, often used in the study of infinite systems of particles, needs a careful formalization in order to prevent any drawback from the fact that the mapping $X \ni z \mapsto x_t(z)$ is not measurable in general due to the independence. For extensions of this method see [17].

In any case, the formulas involving the carré du champ and the gradient require computations and key results on *the configuration space* from which the construction may be performed as starting point. From this point of view the works are based either on the chaos decomposition (Nualart-Vives [23]) and provide tools in analogy with the Malliavin calculus on Wiener space, but non-local (Picard [24], Ishikawa-Kunita [15], Picard [25]) or on the expression of the generator on a sufficiently rich class and Friedrichs' argument (cf. what may be called the German school in spite of its cosmopolitanism, especially [1] and [21]).

We will follow a way close to this last one. Several representations of the gradient are possible (Privault [26]) and we will propose here a new one with the advantages of both locality (chain rule) and simplicity on usual functionals. It provides a new method of computing the carré du champ Γ — the lent particle method — whose efficiency is displayed on some examples. Applications to stochastic differential equations driven by Lévy processes will be gathered in an other article.

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2 The Energy Image Density property (EID)

In this part we give sufficient conditions for a Dirichlet structure to fulfill (EID) property. These conditions concern finite dimensional cases and will be extended to the infinite dimensional setting of Poisson measures in Part 4.

For each positive integer d , we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma-field on \mathbb{R}^d and by λ^d the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and as usually when no confusion is possible, we shall denote it by dx . For f measurable $f_*\nu$ denotes the image of the measure ν by f .

For a σ -finite measure Λ on some measurable space, a Dirichlet form on $L^2(\Lambda)$ with carré du champ γ is said to satisfy (EID) if for any d and for any \mathbb{R}^d -valued function U whose components are in the domain of the form

$$U_*[(\det\gamma[U, U^t]) \cdot \Lambda] \ll \lambda^d$$

where \det denotes the determinant.

2.1 A sufficient condition on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$

Given $r \in \mathbb{N}^*$, for any $\mathcal{B}(\mathbb{R}^r)$ -measurable function $u : \mathbb{R}^r \rightarrow \mathbb{R}$, all $i \in \{1, \dots, r\}$ and all $\bar{x} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r) \in \mathbb{R}^{r-1}$, we consider $u_{\bar{x}}^{(i)} : \mathbb{R} \rightarrow \mathbb{R}$ the function defined by

$$\forall s \in \mathbb{R}, u_{\bar{x}}^{(i)}(s) = u((\bar{x}, s)_i),$$

where $(\bar{x}, s)_i = (x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_r)$.

Conversely if $x = (x_1, \dots, x_r)$ belongs to \mathbb{R}^r we set $x^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r)$.

Then following standard notation, for any $\mathcal{B}(\mathbb{R})$ measurable function $\rho : \mathbb{R} \rightarrow \mathbb{R}^+$, we denote by $R(\rho)$ the largest open set on which ρ^{-1} is locally integrable.

Finally, we are given $k : \mathbb{R}^r \rightarrow \mathbb{R}^+$ a Borel function and $\xi = (\xi_{ij})_{1 \leq i, j \leq r}$ an $\mathbb{R}^{r \times r}$ -valued and symmetric Borel function.

We make the following assumptions which generalize Hamza's condition (cf. Fukushima-Oshima-Takeda [14] Chap. 3 §3.1 (3°), p105):

Hypotheses (HG):

1. For any $i \in \{1, \dots, r\}$ and λ^{r-1} -almost all $\bar{x} \in \{y \in \mathbb{R}^{r-1} : \int_{\mathbb{R}} k_y^{(i)}(s) ds > 0\}$, $k_{\bar{x}}^{(i)} = 0$, λ^1 -a.e. on $\mathbb{R} \setminus R(k_{\bar{x}}^{(i)})$.
2. There exists an open set $O \subset \mathbb{R}^r$ such that $\lambda^r(\mathbb{R}^r \setminus O) = 0$ and ξ is locally elliptic on O in the sense that for any compact subset K , in O , there exists a positive constant c_K such that

$$\forall x = (x_1, \dots, x_r) \in K, \sum_{i, j=1}^r \xi_{ij}(x) x_i x_j \geq c_K |x|^2.$$

Following Albeverio-Röckner, Theorems 3.2 and 5.3 in [2] and also Röckner-Wielens Section 4 in [27], we consider \mathbf{d} the set of $\mathcal{B}(\mathbb{R}^r)$ -measurable functions u in $L^2(kdx)$, such

that for any $i \in \{1, \dots, r\}$, and λ^{r-1} -almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolute continuous version $\tilde{u}_{\bar{x}}^{(i)}$ on $R(k_{\bar{x}}^{(i)})$ (defined λ^1 -a.e.) and such that $\sum_{i,j} \xi_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \in L^1(kdx)$, where

$$\frac{\partial u}{\partial x_i} = \frac{d\tilde{u}_{\bar{x}}^{(i)}}{ds}.$$

Sometimes, we will simply denote $\frac{\partial}{\partial x_i}$ by ∂_i .

And we consider the following bilinear form on \mathbf{d} :

$$\forall u, v \in \mathbf{d}, e[u, v] = \frac{1}{2} \int_{\mathbb{R}^r} \sum_{i,j} \xi_{ij}(x) \partial_i u(x) \partial_j v(x) k(x) dx.$$

As usual we shall simply denote $e[u, u]$ by $e[u]$. We have

Proposition 1. (\mathbf{d}, e) is a local Dirichlet form on $L^2(kdx)$ which admits a carré du champ operator γ given by

$$\forall u, v \in \mathbf{d}, \gamma[u, v] = \sum_{i,j} \xi_{ij} \partial_i u \partial_j v.$$

Proof. All is clear excepted the fact that e is a closed form on \mathbf{d} . To prove it, let us consider a sequence $(u_n)_{n \in \mathbb{N}^*}$ of elements in \mathbf{d} which converges to u in $L^2(kdx)$ and such that $\lim_{n,m \rightarrow +\infty} e[u_n - u_m] = 0$. Let $W \subset O$, an open subset which satisfies $\bar{W} \subset O$ and such that \bar{W} is compact.

Let \mathbf{d}_W be the set of $\mathcal{B}(\mathbb{R}^r)$ -measurable functions u in $L^2(\mathbf{1}_W \times kdx)$, such that for any $i \in \{1, \dots, r\}$, and λ^{r-1} -almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolute continuous version $\tilde{u}_{\bar{x}}^{(i)}$ on $R((\mathbf{1}_W \times k)_{\bar{x}}^{(i)})$ and such that $\sum_{i,j} \xi_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \in L^1(\mathbf{1}_W \times kdx)$, equipped with the bilinear form

$$\forall u, v \in \mathbf{d}_W, e_W[u, v] = \frac{1}{2} \int_W \sum_i \partial_i u(x) \partial_i v(x) k(x) dx = \frac{1}{2} \int_W \nabla u(x) \cdot \nabla v(x) k(x) dx.$$

One can easily verify, since W is an open set, that for all $\bar{x} \in \mathbb{R}^{r-1}$

$$S_{\bar{x}}^i(W) \cap R(k_{\bar{x}}^{(i)}) \subset R((\mathbf{1}_W \times k)_{\bar{x}}^{(i)}), \quad (1)$$

where $S_{\bar{x}}^i(W)$ is the open set $\{s \in \mathbb{R} : (\bar{x}, s)_i \in W\}$.

Then it is clear that the function $\mathbf{1}_W \times k$ satisfies property 1. of (HG), and as a consequence of Theorems 3.2 and 5.3 in [2], (\mathbf{d}_W, e_W) is a Dirichlet form on $L^2(\mathbf{1}_W \times kdx)$.

We have for all $n, m \in \mathbb{N}$

$$e_W(u_n - u_m) = \frac{1}{2} \int_W |\nabla u_n(x) - \nabla u_m(x)|^2 k(x) dx \leq \frac{1}{c_{\bar{W}}} e(u_n - u_m),$$

as (\mathbf{d}, e_W) is a closed form, we conclude that u belongs to \mathbf{d}_W .

Consider now an exhaustive sequence (W_m) , of relatively compact open sets in O such that for all $m \in \mathbb{N}$, $\bar{W}_m \subset W_{m+1} \subset O$. We have that for all m , u belongs to \mathbf{d}_{W_m} hence by Theorems 3.2 and 5.3 in [2], for all $i \in \{1, \dots, r\}$, and λ^{r-1} -almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolute continuous version on $\bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)_{\bar{x}}^{(i)})$. Using relation (1), we have

$$S_{\bar{x}}^i(O) \cap R(k_{\bar{x}}^{(i)}) = \bigcup_{m=1}^{+\infty} S_{\bar{x}}^i(W_m) \cap R(k_{\bar{x}}^{(i)}) \subset \bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)_{\bar{x}}^{(i)}).$$

As $\lambda^r(\mathbb{R}^r \setminus O) = 0$, we get that for almost all $\bar{x} \in \mathbb{R}^{r-1}$, $\bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)_{\bar{x}}^{(i)}) = R(k_{\bar{x}}^{(i)})$ λ^1 -a.e. Moreover, by a diagonal extraction, we have that a subsequence of (∇u_n) converges kdx -a.e. to ∇u , so by Fatou's Lemma, we conclude that $u \in \mathbf{d}$ and then $\lim_{n \rightarrow +\infty} e[u_n - u] = 0$, which is the desired result. \square

For any $d \in \mathbb{N}^*$, if $u = (u_1, \dots, u_d)$ belongs to \mathbf{d}^d , we shall denote by $\gamma[u]$ the matrix $(\gamma[u_i, u_j])_{1 \leq i, j \leq d}$.

Theorem 2. (EID) *property : the structure $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r), k dx, \mathbf{d}, \gamma)$ satisfies*

$$\forall d \in \mathbb{N}^* \quad \forall u \in \mathbf{d}^d \quad u_*[(\det \gamma[u]) \cdot kdx] \ll \lambda^d.$$

Proof. Let us mention that a proof was given by S. Song in [29], Theorem 16, in the more general case of *classical Dirichlet forms*. Following his ideas, we present here a shorter proof.

The proof is based on the *co-area formula* stated by H. Federer in [13], Theorems 3.2.5 and 3.2.12.

We first introduce the subset $A \subset \mathbb{R}^r$:

$$A = \{x \in \mathbb{R}^r : x_i \in R(k_{x_i}^{(i)}) \quad i = 1, \dots, r\}.$$

As a consequence of property 1. of (HG), $\int_{A^c} k(x) dx = 0$.

Let $u = (u_1, \dots, u_d) \in \mathbf{d}^d$. We follow the notation and definitions introduced by Bouleau-Hirsch in [9], Chap. II Section 5.1.

Thanks to Theorem 3.2 in [2] and Stepanoff's Theorem (see Theorem 3.1.9 in [13] or Remark 5.1.2 Chap. II in [9]), it is clear that for almost all $a \in A$, the *approximate derivatives*

$\text{ap} \frac{\partial u}{\partial x_i}$ exist for $i = 1, \dots, r$ and if we set: $Ju = \left[\det \left((\sum_{k=1}^r \partial_k u_i \partial_k u_j)_{1 \leq i, j \leq d} \right) \right]^{1/2}$, this is equal kdx a.e. to the determinant of the *approximate Jacobian matrix* of u . Then, by Theorem 3.1.4 in [13], u is *approximately differentiable* at almost all points a in A .

We denote by \mathcal{H}^{r-d} the $r - d$ dimensional Hausdorff measure on \mathbb{R}^r .

As a consequence of Theorems 3.1.8, 3.1.16 and Lemma 3.1.7 in [13], for all $n \in \mathbb{N}^*$, there exists a map $u^n : \mathbb{R}^r \rightarrow \mathbb{R}^d$ of class \mathcal{C}^1 such that

$$\lambda^r(A \setminus \{x : u(x) = u^n(x)\}) \leq \frac{1}{n}$$

and

$$\forall a \in \{x : u(x) = u^n(x)\}, \quad \text{ap} \frac{\partial u}{\partial x_i}(a) = \text{ap} \frac{\partial u^n}{\partial x_i}(a), \quad i = 1, \dots, r.$$

Assume first that $d \leq r$. Let B be a Borelian set in \mathbb{R}^d such that $\lambda^r(B) = 0$. Thanks to

the co-area formula we have

$$\begin{aligned}
\int_{\mathbb{R}^r} \mathbf{1}_B(u(x))Ju(x)k(x) dx &= \int_A \mathbf{1}_B(u(x))Ju(x)k(x) dx \\
&= \lim_{n \rightarrow +\infty} \int_{A \cap \{u=u^n\}} \mathbf{1}_B(u(x))Ju(x)k(x) dx \\
&= \lim_{n \rightarrow +\infty} \int_{A \cap \{u=u^n\}} \mathbf{1}_B(u^n(x))Ju^n(x)k(x) dx \\
&= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^r} \left(\int_{(u^n)^{-1}(y)} \mathbf{1}_{A \cap \{u=u^n\}}(x) \mathbf{1}_B(u^n(x))k(x) d\mathcal{H}^{r-d}(x) \right) d\lambda^r(y) \\
&= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^r} \mathbf{1}_B(y) \left(\int_{(u^n)^{-1}(y)} \mathbf{1}_{A \cap \{u=u^n\}}(x)k(x) d\mathcal{H}^{r-d}(x) \right) d\lambda^r(y) \\
&= 0
\end{aligned}$$

So that, $u_*(Ju \cdot k dx) \ll \lambda_d$.

We have

$$Ju = [\det(Du \cdot (Du)^t)]^{1/2} \text{ and } \gamma(u) = Du \cdot \xi \cdot Du^t,$$

where Du is the $d \times r$ matrix: $\left(\frac{\partial u_i}{\partial x_k} \right)_{1 \leq i \leq d; 1 \leq k \leq r}$.

As $\xi(x)$ is symmetric definite and positive on O and $\lambda^r(\mathbb{R}^r \setminus O) = 0$, we have

$$\{x \in A; Ju(x) > 0\} = \{x \in A; \det(\gamma(u)(x)) > 0\} \text{ a.e.,}$$

and this ends the proof in this case.

Now, if $d > r$, $\det(\gamma(u)) = 0$ and the result is trivial. \square

2.2 The case of a product structure

We consider a sequence of functions ξ^i and k_i , $i \in \mathbb{N}^*$, k_i being non-negative Borel functions such that $\int_{\mathbb{R}^r} k_i(x) dx = 1$. We assume that for all $i \in \mathbb{N}^*$, ξ^i and k_i satisfy hypothesis (HG) so that, we can construct, as for k in the previous subsection, the Dirichlet form (\mathbf{d}_i, e_i) on $L^2(\mathbb{R}^r, k_i dx)$ associated to the carré du champ operator γ_i given by:

$$\forall u, v \in \mathbf{d}_i, \gamma_i[u, v] = \sum_{k,l} \xi_{kl}^i \partial_k u \partial_l v.$$

We now consider the product Dirichlet form $(\tilde{\mathbf{d}}, \tilde{e}) = \prod_{i=1}^{+\infty} (\mathbf{d}_i, e_i)$ defined on the product space $((\mathbb{R}^r)^{\mathbb{N}^*}, (\mathcal{B}(\mathbb{R}^r))^{\mathbb{N}^*})$ equipped with the product probability $\Lambda = \prod_{i=1}^{+\infty} k_i dx$. We denote by $(X_n)_{n \in \mathbb{N}^*}$ the coordinates maps on $(\mathbb{R}^r)^{\mathbb{N}^*}$.

Let us recall that $U = F(X_1, X_2, \dots, X_n, \dots)$ belongs to $\tilde{\mathbf{d}}$ if and only if :

1. U belongs to $L^2((\mathbb{R}^r)^{\mathbb{N}^*}, (\mathcal{B}(\mathbb{R}^r))^{\mathbb{N}^*}, \Lambda)$.
2. For all $k \in \mathbb{N}^*$ and Λ -almost all $(x_1, \dots, x_{k-1}, x_{k+1}, \dots)$ in $(\mathbb{R}^r)^{\mathbb{N}^*}$, $F(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots)$ belongs to \mathbf{d}_k .

$$3. \tilde{e}(U) = \sum_k \int_{(\mathbb{R}^r)^{\mathbb{N}^*}} e_k(F(X_1(x), \dots, X_{k-1}(x), \cdot, X_{k+1}(x), \dots)) \Lambda(dx) < +\infty.$$

Where as usual, the form e_k acts only on the k -th coordinate. It is also well known that $(\tilde{\mathbf{d}}, \tilde{e})$ admits a carré du champ $\tilde{\gamma}$ given by

$$\tilde{\gamma}[U] = \sum_k \gamma_k[F(X_1, \dots, X_{k-1}, \cdot, X_{k+1}, \dots)](X_k).$$

To prove that (EID) is satisfied by this structure, we first prove that it is satisfied for a finite product. So, for all $n \in \mathbb{N}^*$, we consider $(\tilde{\mathbf{d}}_n, \tilde{e}_n) = \prod_{i=1}^n (\mathbf{d}_i, e_i)$ defined on the product space $((\mathbb{R}^r)^n, (\mathcal{B}(\mathbb{R}^r))^n)$ equipped with the product probability $\Lambda_n = \prod_{i=1}^n k_i dx$. By restriction, we keep the same notation as the one introduced for the infinite product. We know that this structure admits a carré du champ operator $\tilde{\gamma}_n$ given by $\tilde{\gamma}_n = \sum_{i=1}^n \gamma_i$.

Lemma 3. *For all $n \in \mathbb{N}^*$, the Dirichlet structure $(\tilde{\mathbf{d}}_n, \tilde{e}_n)$ satisfies (EID):*

$$\forall d \in \mathbb{N}^* \forall U \in (\tilde{\mathbf{d}}_n)^d \quad U_*[(\det \tilde{\gamma}_n[U]) \cdot \Lambda_n] \ll \lambda^d.$$

Proof. The proof consists in remarking that this is nothing but a particular case of Theorem 2 on \mathbb{R}^{nd} , ξ being replaced by Ξ , the diagonal matrix of the ξ^i , and the density being the product density. \square

As a consequence of Chapter V Proposition 2.2.3. in Bouleau-Hirsch [9], we have

Theorem 4. *The Dirichlet structure $(\tilde{\mathbf{d}}, \tilde{e})$ satisfies (EID):*

$$\forall d \in \mathbb{N}^* \forall U \in \tilde{\mathbf{d}}^d \quad U_*[(\det \tilde{\gamma}[U]) \cdot \Lambda] \ll \lambda^d.$$

2.3 The case of structures obtained by injective images

The following result could be extended to more general images (see Bouleau-Hirsch [9] Chapter V §1.3 p 196 *et seq.*). We give the statement in the most useful form for Poisson measures and processes with independent increments.

Let $(\mathbb{R}^p \setminus \{0\}, \mathcal{B}(\mathbb{R}^p \setminus \{0\}), \nu, \mathbf{d}, \gamma)$ be a Dirichlet structure on $\mathbb{R}^p \setminus \{0\}$ satisfying (EID). Thus ν is σ -finite, γ is the carré du champ operator and the Dirichlet form is $e[u] = 1/2 \int \gamma[u] d\nu$.

Let $U : \mathbb{R}^p \setminus \{0\} \mapsto \mathbb{R}^q \setminus \{0\}$ be an injective map such that $U \in \mathbf{d}^q$. Then $U_*\nu$ is σ -finite. If we put

$$\begin{aligned} \mathbf{d}_U &= \{\varphi \in L^2(U_*\nu) : \varphi \circ U \in \mathbf{d}\} \\ e_U[\varphi] &= e[\varphi \circ U] \\ \gamma_U[\varphi] &= \frac{dU_*(\gamma[\varphi \circ U] \cdot \nu)}{dU_*\nu} \end{aligned}$$

we have

Proposition 5. *The term $(\mathbb{R}^q \setminus \{0\}, \mathcal{B}(\mathbb{R}^q \setminus \{0\}), U_*\nu, \mathbf{d}_U, \gamma_U)$ is a Dirichlet structure satisfying (EID).*

Proof. a) That $(\mathbb{R}^q \setminus \{0\}, \mathcal{B}(\mathbb{R}^q \setminus \{0\}), U_*\nu, \mathbf{d}_U, \gamma_U)$ be a Dirichlet structure is general and does not use the injectivity of U (cf. the case ν finite in Bouleau-Hirsch [9] Chap. V §1 p. 186 *et seq.*).

b) By the injectivity of U , we see that for $\varphi \in \mathbf{d}_U$

$$(\gamma_U[\varphi]) \circ U = \gamma[\varphi \circ U] \quad \nu\text{-a.s.}$$

so that if $f \in (\mathbf{d}_U)^r$

$$f_*[\det \gamma_U[f].U_*\nu] = (f \circ U)_*[\det \gamma[f \circ U].\nu]$$

which proves (EID) for the image structure. \square

Remark 1. *Applying this result yields examples of Dirichlet structures on \mathbb{R}^n satisfying (EID) whose measures are carried by a (Lipschitzian) curve in \mathbb{R}^n or, under some hypotheses, a countable union of such curves, and therefore without density.*

3 Dirichlet structure on the Poisson space related to a Dirichlet structure on the states space

Let $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ be a local symmetric Dirichlet structure which admits a carré du champ operator i.e. (X, \mathcal{X}, ν) is a measured space called *the bottom space*, ν is σ -finite and the bilinear form

$$e[f, g] = \frac{1}{2} \int \gamma[f, g] d\nu,$$

is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and carré du champ operator γ (see Bouleau-Hirsch [9], Chap. I). We assume that for all $x \in X$, $\{x\}$ belongs to \mathcal{X} and that ν is diffuse ($\nu(\{x\}) = 0 \forall x$). The generator associated to this Dirichlet structure is denoted by a , its domain is $\mathcal{D}(a) \subset \mathbf{d}$ and it generates the Markovian strongly continuous semigroup $(p_t)_{t \geq 0}$ on $L^2(\nu)$.

Our aim is to study, thanks to Dirichlet forms methods, functionals of a Poisson measure N , associated to (X, \mathcal{X}, ν) . It is defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where Ω is the configuration space, the set of measures which are countable sum of Dirac measures on X , \mathcal{A} is the sigma-field generated by N and \mathbb{P} is the law of N (see Neveu [22]). The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is called the *upper space*.

3.1 Density lemmas

Let (F, \mathcal{F}, μ) be a probability space such that for all $x \in F$, $\{x\}$ belongs to \mathcal{F} and μ is diffuse. Let $n \in \mathbb{N}^*$, We denote by x_1, x_2, \dots, x_n the coordinates maps on $(F^n, \mathcal{F}^{\otimes n}, \mu^{\otimes n})$ and we consider the random measure $m = \sum_{i=1}^n \varepsilon_{x_i}$.

Lemma 6. *Let \mathcal{S} be the symmetric sub-sigma-field in $\mathcal{F}^{\otimes n}$ and $p \in [1, +\infty[$. Sets $\{m(g_1) \cdots m(g_n) : g_i \in L^\infty(\mu) \forall i = 1, \dots, n\}$ and $\{e^{m(g)} : g \in L^\infty(\mu)\}$ are both total in $L^p(F^n, \mathcal{S}, \mu^{\otimes n})$ and the set $\{e^{im(g)} : g \in L^\infty(\mu)\}$ is total in $L^p(F^n, \mathcal{S}, \mu^{\otimes n}; \mathbb{C})$.*

Proof. Because μ is diffuse, the set $\{g_1(x_1)\cdots g_n(x_n) : g_i \in L^\infty(\mu), g_i \text{ with disjoint supports } \forall i = 1, \dots, n\}$ is total in $L^p(\mu^{\otimes n})$. Let $G(x_1, \dots, x_n)$ be a linear combination of such functions. If $F(x_1, \dots, x_n)$ is symmetric and belongs to $L^p(\mu^{\otimes n})$ then the distance in $L^p(\mu^{\otimes n})$ between $F(x_1, \dots, x_n)$ and $G(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for $\sigma \in \mathbf{S}$ the set of permutations on $\{1, \dots, n\}$, does not depend on σ and as a consequence is larger than the distance between $F(x_1, \dots, x_n)$ and the barycenter $\frac{1}{n!} \sum_{\sigma \in \mathbf{S}} G(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. So, the set $\{\frac{1}{n!} \sum_{\sigma \in \mathbf{S}} G(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : g_i \in L^\infty(\mu), g_i \text{ with disjoint supports } \forall i = 1, \dots, n\}$ is total in $L^p(F^n, \mathcal{S}, \mu^{\otimes n})$. We conclude by using the following property : if $f_i, i = 1, \dots, n$, are \mathcal{F} -measurable functions with disjoint supports then: $m(f_1) \cdots m(f_n) = \sum_{\sigma \in \mathbf{S}} f_1(x_{\sigma(1)}) \cdots f_n(x_{\sigma(n)})$. \square

Lemma 7. *Let N_1 be a random Poisson measure on (F, \mathcal{F}, μ_1) where μ_1 is a finite and diffuse measure, defined on some probability space $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ where $\mathcal{A}_1 = \sigma(N_1)$. Then, for any $p \in [1, +\infty[$, the set $\{e^{-N_1(f)} : f \geq 0, f \in L^\infty(\mu_1)\}$ is total in $L^p(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and $\{e^{iN_1(f)} : f \in L^\infty(\mu_1)\}$ is total in $L^p(\Omega_1, \mathcal{A}_1, \mathbb{P}_1; \mathbb{C})$.*

Proof. Let us put $P = N_1(F)$, it is an integer valued random variable. As $\{e^{i\lambda P} : \lambda \in \mathbb{R}\}$ is total in $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathbb{P}_P)$ where \mathbb{P}_P is the law of P , for any $n \in \mathbb{N}^*$ and any $g \in L^\infty(\mu_1)$, one can approximate in $L^p(\Omega_1, \mathcal{A}_1, \mathbb{P}_1; \mathbb{C})$ the random variable $\mathbf{1}_{\{P=n\}} e^{iN_1(g)}$ by a sequence of variables of the form $\sum_{k=1}^K a_k e^{i\lambda_k P} e^{iN_1(g)}$ with $a_k, \lambda_k \in \mathbb{R}, k = 1 \cdots K$. But, as a consequence of the previous lemma, we know that $\{\mathbf{1}_{\{P=n\}} e^{iN_1(f)} : f \in L^\infty(\mu_1)\}$ is total in $L^p(\{P = n\}, \mathcal{A}_1|_{\{P=n\}}, \mathbb{P}_1|_{\{P=n\}}; \mathbb{C})$, which provides the result. \square

We now give the main lemma, with the notation introduced at the beginning of this section.

Lemma 8. *For $p \in [1, \infty[$, the set $\{e^{-N(f)} : f \geq 0, f \in L^1(\nu) \cap L^\infty(\nu)\}$ is total in $L^p(\Omega, \mathcal{A}, \mathbb{P})$ and $\{e^{iN(f)} : f \in L^1(\nu) \cap L^\infty(\nu)\}$ is total in $L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{C})$.*

Proof. Assume that ν is non finite. Let $(F_k)_{k \in \mathbb{N}}$ be a partition of Ω such that for all k , $\nu(F_k)$ be finite. By restriction of N to each set F_k , we construct a sequence of independent Poisson measures (N_k) such that $N = \sum_k N_k$. As any variable in L^p is the limit of variables which depend only on a finite number of N_k , we conclude thanks to the previous lemma. \square

3.2 Construction using the Friedrichs's argument

3.2.1. Basic formulas and pre-generator

We set $\tilde{N} = N - \nu$ then the identity $\mathbb{E}[(\tilde{N}(f))^2] = \int f^2 d\nu$, for $f \in L^1(\nu) \cap L^2(\nu)$ can be extended uniquely to $f \in L^2(\nu)$ and this permits to define $\tilde{N}(f)$ for $f \in L^2(\nu)$. The Laplace characteristic functional

$$\mathbb{E}[e^{i\tilde{N}(f)}] = e^{-\int (1 - e^{if} + if) d\nu} \quad f \in L^2(\nu). \quad (2)$$

yields:

Proposition 9. *For all $f \in \mathbf{d}$ and all $h \in \mathcal{D}(a)$,*

$$\mathbb{E} \left[e^{i\tilde{N}(f)} \left(\tilde{N}(a[h]) + \frac{i}{2} N(\gamma[f, h]) \right) \right] = 0. \quad (3)$$

Proof. Derivating in 0 the map $t \rightarrow \mathbb{E} \left[e^{i\tilde{N}(f+ta[h])} \right]$, we have thanks to (2),

$$\mathbb{E}[e^{i\tilde{N}(f)+f(1-e^{if}+if)d\nu} \tilde{N}(a[h])] = \int (e^{if} - 1)a[h] d\nu, \quad (4)$$

then using the fact that function $x \mapsto e^{ix} - 1$ is Lipschitz and vanishes in 0 and the functional calculus related to a local Dirichlet form (see Bouleau-Hirsch [9] Section I.6) we get that the member on the right hand side in (4) is equal to

$$-\frac{1}{2} \int \gamma[e^{if} - 1, h] d\nu = -\frac{i}{2} \int e^{if} \gamma[f, h] d\nu.$$

We conclude by applying once more (4) with $\gamma[f, h]$ instead of $a[h]$. \square

We now consider the vector space \mathcal{D}_0 whose elements are the linear combinations of variables of the form $e^{i\tilde{N}(f)}$ with $f \in \mathcal{D}(a) \cap L^1(\nu)$. Since any variable of the form $e^{iN(g)}$ with $g \in L^1(\nu) \cap L^2(\nu)$ is limit in L^2 of elements in \mathcal{D}_0 and thanks to Lemma 8 we obtain that \mathcal{D}_0 is dense in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{C})$.

If $U = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$ belongs to \mathcal{D}_0 , we put

$$A_0[U] = \sum_p \lambda_p e^{i\tilde{N}(f_p)} (i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p])). \quad (5)$$

This is a natural choice as candidate for the pregenerator of the upper structure, since, as easily seen using (5), it induces the relation $\Gamma[N(f)] = N(\gamma[f])$ between the carré du champ operators of the upper and the bottom structures, which is satisfied in the case $\nu(X) < \infty$.

One has to note that for the moment, A_0 is not uniquely determined since *a priori* $A_0[U]$ depends on the expression of U which is possibly non unique.

Proposition 10. *Let $U, V \in \mathcal{D}_0$, $U = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$ and $V = \sum_q \mu_q e^{i\tilde{N}(g_q)}$. One has*

$$-\mathbb{E}[A_0[U]\bar{V}] = -\frac{1}{2} \mathbb{E} \left[\sum_{p,q} \lambda_p \bar{\mu}_q e^{i\tilde{N}(f_p-g_q)} N(\gamma[f_p, g_q]) \right] \quad (6)$$

which is also equal to

$$\frac{1}{2} \mathbb{E} \left[\sum_{p,q} F'_p \bar{G}'_q N(\gamma[f_p, g_q]) \right], \quad (7)$$

where F and G are such that $U = F(\tilde{N}(f_1), \dots, \tilde{N}(f_n))$ and $V = F(\tilde{N}(g_1), \dots, \tilde{N}(g_m))$ and $F'_p = \frac{\partial F}{\partial x_p}(\tilde{N}(f_1), \dots, \tilde{N}(f_n))$, $G'_q = \frac{\partial G}{\partial x_q}(\tilde{N}(g_1), \dots, \tilde{N}(g_m))$.

Proof. We have

$$-\mathbb{E}[A_0[U]\bar{V}] = -\mathbb{E} \left[\sum_{p,q} \lambda_p \bar{\mu}_q e^{i\tilde{N}(f_p-g_q)} (i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p])) \right].$$

Thanks to Proposition 9,

$$\begin{aligned} -\mathbb{E} \left[\sum_{p,q} \lambda_p \overline{\mu}_q e^{i\tilde{N}(f_p - g_q)} i\tilde{N}(a[f_p]) \right] &= -\frac{1}{2} \mathbb{E} \left[\sum_{p,q} \lambda_p \overline{\mu}_q e^{i\tilde{N}(f_p - g_q)} N(\gamma[f_p, f_p + g_q]) \right] \\ &= -\frac{1}{2} \mathbb{E} \left[\sum_{p,q} \lambda_p \overline{\mu}_q e^{i\tilde{N}(f_p - g_q)} N(\gamma[f_p, g_q]) \right] - \frac{1}{2} \mathbb{E} \left[\sum_{p,q} \lambda_p \overline{\mu}_q e^{i\tilde{N}(f_p - g_q)} N(\gamma[f_p]) \right] \end{aligned}$$

what gives the statement. \square

It remains to prove that A_0 is uniquely determined and so is an operator acting on \mathcal{D}_0 . To this end, thanks to the previous proposition, we just have to prove that the quantity $\sum_{p,q} F'_p \overline{G}'_q N(\gamma[f_p, g_q])$ does not depend on the chosen representations for U and V . In the same spirit as Ma-Röckner (see [21]), the introduction of a gradient will yield this non-dependance. Let us mention that the gradient we introduce is different of the one considered by these authors and is based on a notion that we present now.

3.2.2. Particle-wise product of a Poisson measure and a probability

We are still considering N the random Poisson measure on (X, \mathcal{X}, ν) and we are given an auxiliary probability space (R, \mathcal{R}, ρ) . We construct a random Poisson measure $N \odot \rho$ on $(X \times R, \mathcal{X} \otimes \mathcal{R}, \nu \otimes \rho)$ such that if $N = \sum_i \varepsilon_{x_i}$ then $N \odot \rho = \sum_i \varepsilon_{(x_i, r_i)}$ where (r_i) is a sequence of i.i.d. random variables independent of N whose common law is ρ . Such a random Poisson measure $N \odot \rho$ is sometimes called a *marked* Poisson measure.

The construction of $N \odot \rho$ follows line by line the one of N . Let us recall it. We first study the case where ν is finite and we consider the probability space

$$(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_{\nu(X)}) \times (X, \mathcal{X}, \frac{\nu}{\nu(X)})^{\mathbb{N}^*},$$

where $P_{\nu(X)}$ denotes the Poisson law with intensity $\nu(X)$ and we put

$$N = \sum_{i=1}^Y \varepsilon_{x_i}, \quad (\text{with the convention } \sum_1^0 = 0)$$

where $Y, x_1, \dots, x_n, \dots$ denote the coordinates maps. We introduce the probability space

$$(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}}) = (R, \mathcal{R}, \rho)^{\mathbb{N}^*},$$

and the coordinates are denoted by r_1, \dots, r_n, \dots . On the probability space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_{\nu(X)}) \times (X, \mathcal{X}, \frac{\nu}{\nu(X)})^{\mathbb{N}^*} \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$, we define the random measure $N \odot \rho = \sum_{i=1}^Y \varepsilon_{(x_i, r_i)}$. It is a Poisson random measure on $X \times R$ with intensity measure $\nu \otimes \rho$. For $f \in L^1(\nu \otimes \rho)$

$$\hat{\mathbb{E}} \left[\int_{X \times R} f dN \odot \rho \right] = \int_X \left(\int_R f(x, r) d\rho(r) \right) N(dx) \quad \mathbb{P} - a.e. \quad (8)$$

and if $f \in L^2(\nu \otimes \rho)$

$$\hat{\mathbb{E}} \left[\left(\int_{X \times R} f dN \odot \rho \right)^2 \right] = \left(\int_X \int_R f d\rho dN \right)^2 - \int_X \left(\int_R f d\rho \right)^2 dN + \int_X \int_R f^2 d\rho dN, \quad (9)$$

where $\hat{\mathbb{E}}$ stands for the expectation under the probability $\hat{\mathbb{P}}$.

If ν is σ -finite, we extend this construction by a standard product argument. Eventually in all cases, we have constructed N on $(\Omega, \mathcal{A}, \mathbb{P})$ and $N \odot \rho$ on $(\Omega, \mathcal{A}, \mathbb{P}) \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$, it is a random Poisson measure on $X \times R$ with intensity measure $\nu \otimes \rho$.

We now are able to generalize identities (8) and (9):

Proposition 11. *Let F be a $\mathcal{A} \otimes \mathcal{X} \otimes \mathcal{R}$ measurable functions such that $\mathbb{E} \int_{X \times R} F^2 d\nu d\rho$ and $\mathbb{E} \int_R (\int_X |F| d\nu)^2 d\rho$ are both finite then the following relation holds*

$$\hat{\mathbb{E}}[(\int_{X \times R} F dN \odot \rho)^2] = (\int_X \int_R F d\rho dN)^2 - \int_X (\int_R F d\rho)^2 dN + \int_X \int_R F^2 d\rho dN, \quad (10)$$

Proof. Approximating first F by a sequence of elementary functions and then introducing a partition (B_k) of subsets of X of finite ν -measure, this identity is seen to be a consequence of (9). \square

We denote by \mathbb{P}_N the measure $\mathbb{P}_N = \mathbb{P}(dw)N_w(dx)$ on $(\Omega \times X, \mathcal{A} \otimes \mathcal{X})$. Let us remark that \mathbb{P}_N and $\mathbb{P} \otimes \nu$ are singular because ν is diffuse.

We will use the following consequence of the previous proposition :

Corollary 12. *Let F be a $\mathcal{A} \otimes \mathcal{X} \otimes \mathcal{R}$ measurable function. If F belongs to $L^2(\Omega \times X \times R, \mathbb{P}_N \otimes \rho)$ and $\int F(w, x, r)\rho(dr) = 0$ for \mathbb{P}_N -almost all (w, x) , then $\int F dN \odot \rho$ is well-defined and belongs to $L^2(\mathbb{P} \otimes \hat{\mathbb{P}})$, moreover*

$$\hat{\mathbb{E}}[(\int_{X \times R} F dN \odot \rho)^2] = \int F^2 dN d\rho \quad \mathbb{P}\text{-a.e.} \quad (11)$$

Proof. If F satisfies hypotheses of Proposition 11 then the result is clear. The general case is obtained by approximation. \square

3.2.3. Gradient and welldefinedness

From now on, we assume that the Hilbert space \mathbf{d} is separable so that (see Bouleau-Hirsch [9], ex.5.9 p. 242) the bottom Dirichlet structure admits a gradient operator in the sense that there exists a separable Hilbert space H and a continuous linear map D from \mathbf{d} into $L^2(X, \nu; H)$ such that

- $\forall u \in \mathbf{d}, \|D[u]\|_H^2 = \gamma[u]$.
- If $F : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz then

$$\forall u \in \mathbf{d}, D[F \circ u] = (F' \circ u)Du.$$

- If F is \mathcal{C}^1 (continuously differentiable) and Lipschitz from \mathbb{R}^d into \mathbb{R} (with $d \in \mathbb{N}$) then

$$\forall u = (u_1, \dots, u_d) \in \mathbf{d}^d, D[F \circ u] = \sum_{i=1}^d (F'_i \circ u)D[u_i].$$

As only the Hilbertian structure plays a role, we can choose for H the space $L^2(R, \mathcal{R}, \rho)$ where (R, \mathcal{R}, ρ) is a probability space such that the dimension of the vector space $L^2(R, \mathcal{R}, \rho)$ is infinite. As usual, we identify $L^2(\nu; H)$ and $L^2(X \times R, \mathcal{X} \otimes \mathcal{R}, \nu \otimes \rho)$ and we denote the gradient D by \flat :

$$\forall u \in \mathbf{d}, Du = u^\flat \in L^2(X \times R, \mathcal{X} \otimes \mathcal{R}, \nu \otimes \rho).$$

Without loss of generality, we assume moreover that operator \flat takes its values in the orthogonal space of 1 in $L^2(R, \mathcal{R}, \rho)$, in other words we take for H the orthogonal of 1. So that we have

$$\forall u \in \mathbf{d}, \int u^\flat d\rho = 0 \quad \nu\text{-a.e.} \quad (12)$$

Let us emphasize that hypothesis (12) although restriction-free, is a key property here (as in many applications to error calculus cf [7] Chap. V p225 et seq.) Thanks to corollary (12), it is the feature which will avoid non-local finite difference calculation on the upper space. Finally, although not necessary, we assume for simplicity that constants belong to \mathbf{d}_{loc} (see Bouleau-Hirsch [9] Chap. I Definition 7.1.3.) and that

$$1 \in \mathbf{d}_{loc} \text{ which implies } \gamma[1] = 0 \text{ and } 1^\flat = 0. \quad (13)$$

We now introduce a creation operator ε^+ well-known in quantic mechanics (see Meyer [19], Nualart-Vives [23], Picard [24] etc.) in the following way:

$$\forall x, w \in \Omega, \varepsilon_x^+(w) = w \mathbf{1}_{\{x \in \text{supp } w\}} + (w + \varepsilon_x) \mathbf{1}_{\{x \notin \text{supp } w\}}.$$

One can verify that for all $w \in \Omega$,

$$\varepsilon_x^+(w) = w \text{ for } N_w\text{-almost all } x \quad (14)$$

and

$$\varepsilon_x^+(w) = w + \varepsilon_x \text{ for } \nu\text{-almost all } x \quad (15)$$

We extend this operator to the functionals by setting:

$$\varepsilon^+ H(w, x) = H(\varepsilon_x^+ w, x).$$

The next lemma shows that the image of $\mathbb{P} \otimes \nu$ by ε^+ is nothing but \mathbb{P}_N :

Lemma 13. *Let H be $\mathcal{A} \otimes \mathcal{X}$ -measurable and non negative, then*

$$\mathbb{E} \int \varepsilon^+ H d\nu = \mathbb{E} \int H dN.$$

Proof. Let us assume first that $H = e^{-N(f)} g$ where f and g are non negative and belong to $L^1(\nu) \cap L^2(\nu)$. We have:

$$\mathbb{E} \int \varepsilon^+ H d\nu = \mathbb{E} \int e^{-N(f)} e^{-f(x)} g(x) d\nu(x),$$

and by standard calculations based on the properties of the Laplace functional we obtain that

$$\mathbb{E} \int e^{-N(f)} e^{-f(x)} g(x) d\nu(x) = \mathbb{E}[e^{-N(f)} N(g)] = \mathbb{E} \int H dN.$$

We conclude using a monotone class argument. □

Let us also remark that if $F \in L^2(\mathbb{P}_N \otimes \rho)$ satisfies $\int F d\rho = 0$ \mathbb{P}_N -a.e. then if we put $\varepsilon^+ F(w, x, r) = F(\varepsilon_x^+(w), x, r)$ we have

$$\int \varepsilon^+ F dN \odot \rho = \int F dN \odot \rho \quad \mathbb{P}\text{-a.e.} \quad (16)$$

Indeed $\int (\varepsilon^+ F - F)^2 dN d\rho = 0$ \mathbb{P} -a.e. because $\varepsilon_x^+(w) = w$ for N_w -almost all x .

Definition 14. For all $F \in \mathcal{D}_0$, we put

$$F^\sharp = \int (\varepsilon^+ F)^\flat dN \odot \rho.$$

Thanks to hypothesis (13) we have the following representation of F^\sharp :

$$F^\sharp(w, \hat{w}) = \int_{X \times R} (F(\varepsilon_x^+(w)) - F(w))^\flat(x, r) N \odot \rho(dx dr).$$

Let us also remark that definition (14) makes sense because for all $F \in \mathcal{D}_0$ and \mathbb{P} -almost all $w \in \Omega$, the map $y \mapsto F(\varepsilon_y^+(w)) - F(w)$ belongs to \mathbf{d} . To see it, take $F = e^{i\tilde{N}(f)}$ with $f \in \mathcal{D}(a) \cap L^1(\nu)$, then

$$F(\varepsilon_y^+(w)) - F(w) = e^{i\tilde{N}(f)}(e^{if(y)} - 1),$$

and we know that $e^{if} - 1 \in \mathbf{d}$. We now proceed and obtain

$$\begin{aligned} (e^{i\tilde{N}(f)})^\sharp &= \int e^{i\tilde{N}(f)}(e^{if} - 1)^\flat dN \odot \rho = \int e^{i\tilde{N}(f)+if} (if)^\flat dN \odot \rho \\ &= \int \varepsilon^+(e^{i\tilde{N}(f)})(if)^\flat dN \odot \rho, \end{aligned}$$

and by (16)

$$(e^{i\tilde{N}(f)})^\sharp = \int e^{i\tilde{N}(f)}(if)^\flat dN \odot \rho.$$

So, if $F, G \in \mathcal{D}_0$, $F = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$, $G = \sum_q \mu_q e^{i\tilde{N}(g_q)}$, as $\int f_p^\flat d\rho = \int g_q^\flat d\rho = 0$ and thanks to Corollary 12, we have

$$\hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p - g_q)} \int (if_p)^\flat \overline{(ig_q)^\flat} dN d\rho,$$

and so

$$\hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p - g_q)} N(\gamma(f_p, g_q)) \quad (17)$$

But, by Definition 14, it is clear that F^\sharp does not depend on the representation of F in \mathcal{D}_0 so as a consequence of the previous identity $\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p - g_q)} N(\gamma(f_p, g_q))$ depends only on F and G and thanks to (6), we conclude that A_0 is well defined and is a linear operator from \mathcal{D}_0 into $L^2(\mathbb{P})$.

3.2.4. Upper structure and first properties

As a consequence of Proposition 10, it is clear that A_0 is symmetric, non positive on \mathcal{D}_0 therefore (see Bouleau-Hirsch [9] p.4) it is closable and we can consider its Friedrichs extension $(A, \mathcal{D}(A))$ which generates a closed Hermitian form \mathcal{E} with domain $\mathbb{D} \supset \mathcal{D}(A)$ such that

$$\forall U \in \mathcal{D}(A) \forall V \in \mathbb{D}, \mathcal{E}(U, V) = -\mathbb{E}[A[U]\overline{V}].$$

Moreover, thanks to Proposition 10, it is clear that contractions operate, so (see Bouleau-Hirsch [9] ex. 3.6 p.16) $(\mathbb{D}, \mathcal{E})$ is a local Dirichlet form which admits a carr du champ operator Γ . The upper structure that we have obtained $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies the following properties :

- $\forall f \in \mathbf{d}, \tilde{N}(f) \in \mathbb{D}$ and

$$\Gamma[\tilde{N}(f)] = N(\gamma[f]), \quad (18)$$

moreover the map $f \mapsto \tilde{N}(f)$ is an isometry from \mathbf{d} into \mathbb{D} .

- $\forall f \in \mathcal{D}(a), e^{i\tilde{N}(f)} \in \mathcal{D}(A)$, and

$$A[e^{i\tilde{N}(f)}] = e^{i\tilde{N}(f)}(i\tilde{N}(a[f]) - \frac{1}{2}N(\gamma[f])). \quad (19)$$

- The operator \sharp (defined on \mathcal{D}_0) admits an extension on \mathbb{D} , still denoted \sharp , it is a gradient associated to Γ and for all $f \in \mathbf{d}$:

$$(\tilde{N}(f))^\sharp = \int_{X \times R} f^\flat dN \odot \rho. \quad (20)$$

As a gradient for the Dirichlet structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$, \sharp is a closed operator from $L^2(\mathbb{P})$ into $L^2(\mathbb{P} \otimes \hat{\mathbb{P}})$. It satisfies the chain rule and operates on the functionals of the form $\Phi(\tilde{N}(f))$, Φ Lipschitz $f \in \mathbf{d}$, or more generally $\Psi(\tilde{N}(f_1), \dots, \tilde{N}(f_n))$ with Ψ Lipschitz and \mathcal{C}^1 and f_1, \dots, f_n in \mathbf{d} .

Let us also remark that if F belongs to \mathcal{D}_0 ,

$$A[F] = N(a[\varepsilon^+ F]). \quad (21)$$

3.2.5. Link with the Fock space

The aim of this subsection is to make the link with other existing works and to present another approach based on the Fock space. It is independent of the rest of this article.

Let $g \in \mathcal{D}(a) \cap L^1(\nu)$ such that $-\frac{1}{2} \leq g \leq 0$ and $a[g] \in L^1(\nu)$. Clearly, $f = -\log(1+g)$ is non-negative and belongs to \mathbf{d} . We have for all $v \in \mathbf{d} \cap L^1(\nu)$

$$\begin{aligned} \mathcal{E}[e^{-N(f)}, e^{-N(v)}] &= \frac{1}{2} \mathbb{E} \left[e^{-N(f)} e^{-N(v)} \Gamma[N(f), N(v)] \right] \\ &= \frac{1}{2} \mathbb{E} \left[e^{-N(f)} e^{-N(v)} N(\gamma[f, v]) \right] \\ &= \frac{1}{2} \mathbb{E} \left[e^{\int_X (1-e^{-f-v}) d\nu} \int_X \gamma[f, v] e^{-f-v} d\nu \right]. \end{aligned}$$

As a consequence of the functional calculus

$$\int_X \gamma[f, v] e^{-f-v} d\nu = \int_X \gamma[g, e^{-v}] d\nu = -2 \int_X a[g] e^{-v} d\nu,$$

this yields

$$\mathcal{E}[e^{-N(f)}, e^{-N(v)}] = -\mathbb{E}[e^{-N(f)} e^{-N(v)} N(\frac{a[g]}{1+g})]. \quad (22)$$

Thus by Lemma 8, we obtain

Proposition 15. *Let $g \in \mathcal{D}(a) \cap L^1(\nu)$ such that $-\frac{1}{2} \leq g \leq 0$ and $a[g] \in L^1(\nu)$ then*

$$e^{N(\log(1+g))} \in \mathcal{D}(A) \text{ and } A[e^{N(\log(1+g))}] = e^{N(\log(1+g))} N(\frac{a[g]}{1+g}). \quad (23)$$

Let us recall that (p_t) is the semigroup associated to the bottom structure. If g satisfies the hypotheses of the previous proposition, $p_t g$ also satisfies them. The map $\Psi : t \mapsto e^{N(\log(1+p_t g))}$ is differentiable and $\frac{d\Psi}{dt} = A\Psi$ with $\Psi(0) = e^{N(\log(1+g))}$ hence $\Psi(t) = P_t[e^{N(\log(1+g))}]$ where (P_t) is the strongly continuous semigroup generated by A . So, we have proved

Proposition 16. *Let g be a measurable function with $-\frac{1}{2} \leq g \leq 0$, then*

$$\forall t \geq 0, P_t[e^{N(\log(1+g))}] = e^{N(\log(1+p_t g))}.$$

For any $m \in \mathbb{N}^*$, we denote by $L_{sym}^2(X^m, \mathcal{X}^{\otimes m}, \nu^{\otimes m})$ the set of symmetric functions in $L^2(X^m, \mathcal{X}^{\otimes m}, \nu^{\otimes m})$ and we recall that ν is diffuse.

For all $F \in L_{sym}^2(X^m, \mathcal{X}^{\otimes m}, \nu^{\otimes m})$, we put

$$I_m(F) = \int_{X^m} F(x_1, \dots, x_m) \mathbf{1}_{\{\forall i \neq j, x_i \neq x_j\}} \tilde{N}(dx_1) \cdots \tilde{N}(dx_m).$$

One can easily verify that for all $F, G \in L_{sym}^2(X^m, \mathcal{X}^{\otimes m}, \nu^{\otimes m})$ and all $n, m \in \mathbb{N}^*$, $\mathbb{E}[I_m(F)I_n(G)] = 0$ if $n \neq m$ and

$$\mathbb{E}[I_n(F)I_n(G)] = n! \langle F, G \rangle_{L_{sym}^2(X^n, \mathcal{X}^{\otimes n}, \nu^{\otimes n})},$$

where $\langle \cdot, \cdot \rangle_{L_{sym}^2(X^n, \mathcal{X}^{\otimes n}, \nu^{\otimes n})}$ denotes the scalar product in $L_{sym}^2(X^n, \mathcal{X}^{\otimes n}, \nu^{\otimes n})$. For all $n \in \mathbb{N}^*$, we consider C_n , the Poisson chaos of order n , i.e. the sub-vector space of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ generated by the variables $I_n(F)$, $F \in L_{sym}^2(X^n, \mathcal{X}^{\otimes n}, \nu^{\otimes n})$. The fact that

$$L^2(\Omega, \mathcal{A}, \mathbb{P}) = \mathbb{R} \otimes_{n=1}^{+\infty} C_n.$$

has been proved by K. Ito (see [16]) in 1956. This proof is based on the fact that the set $\{N(E_1) \cdots N(E_k), (E_i) \text{ disjoint sets in } \mathcal{X}\}$ is total in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

Another approach, quite natural, consists in studying carefully what has to be subtracted from the integral with respect to the product measure

$$\int_{X^n} g(x_1) \cdots g(x_n) \tilde{N}(dx_1) \cdots \tilde{N}(dx_n)$$

to obtain the Poisson stochastic integral

$$I_n(g^{\otimes n}) = \int_{X^n} g(x_1) \cdots g(x_n) \mathbf{1}_{\{v_i \neq j, x_i \neq x_j\}} \tilde{N}(dx_1) \cdots \tilde{N}(dx_n).$$

This can be done in an elegant way by the use of lattices of partitions and the Möbius inversion formula (see Comtet [10], Rota-Wallstrom [28]). This leads to the following formula

$$I_n(g^{\otimes n}) = \sum_{k=1}^n B_{n,k}(N(g), -1!N(g^2), 2!N(g^3), \dots, (-1)^{n-k}(n-k)!N(g^{n-k+1})),$$

where the $B_{n,k}$ are the exponential Bell polynomials given by

$$B_{n,k} = \sum \frac{n!}{c_1!c_2! \cdots (1!)^{c_1}(2!)^{c_2} \cdots} x_1^{c_1} x_2^{c_2} \cdots$$

the sum being taken over all the non-negative integers c_1, c_2, \dots such that

$$\begin{aligned} c_1 + 2c_2 + 3c_3 + \cdots &= n \\ c_1 + c_2 + \cdots &= k. \end{aligned}$$

$I_n(g^{\otimes n})$ is a homogeneous function of order n with respect to g . If we express the Taylor expansion of $e^{N(\log(1+tg))}$ and compute the n -th derivate with respect to t thanks to the formula of the composed functions (see Comtet [10]) we obtain

$$e^{N(\log(1+tg))} = 1 + \sum_{n=1}^{+\infty} \frac{t^n}{n!} \sum_{k=1}^n B_{n,k}(N(g), -1!N(g^2), 2!N(g^3), \dots, (-1)^{n-k}(n-k)!N(g^{n-k+1}))$$

this yields

$$e^{N(\log(1+g))} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} I_n(g^{\otimes n}). \quad (24)$$

The density of the chaos is now consequence of Lemma 8.

Conversely, one can prove formula (24) thanks to the density of the chaos, see for instance Surgailis [30]. A short recent proof of the density of the chaos is presented in Applebaum [3] Theorems 4.1 and 4.3.

3.3 Extension of the representation of the gradient and the lent particle method

3.3.1. Extension of the representation of the gradient

The goal of this subsection is to extend formula of Definition 14 to any $F \in \mathbb{D}$.

To this aim, we introduce an auxiliary vector space $\underline{\mathbb{D}}$ which is the completion of the algebraic tensor product $\mathcal{D}_0 \otimes \mathbf{d}$ with respect to the norm $\| \cdot \|_{\underline{\mathbb{D}}}$ which is defined as follows. Considering η , a fixed strictly positive function on X such that $N(\eta)$ belongs to $L^2(\mathbb{P})$, we set for all $H \in \mathcal{D}_0 \otimes \mathbf{d}$:

$$\|H\|_{\underline{\mathbb{D}}} = \left(\mathbb{E} \int_X \gamma[H(w, \cdot)](x) N(dx) \right)^{\frac{1}{2}} + \mathbb{E} \int |H(w, x)| \eta(x) N(dx).$$

One has to note that if $F \in \mathcal{D}_0$ then $\varepsilon^+ F \in \mathcal{D}_0 \otimes \mathbf{d}$ and if $F = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$, we have

$$\gamma[\varepsilon^+ F] = \gamma[\varepsilon^+ F - F] = \sum_{p,q} \lambda_p \bar{\lambda}_q e^{i\tilde{N}(f_p - f_q)} e^{i(f_p - f_q)} \gamma[f_p, f_q],$$

so that by (14)

$$\int_X \gamma[\varepsilon^+ F] dN = \int \sum_{p,q} \lambda_p \bar{\lambda}_q e^{i\tilde{N}(f_p - f_q)} \gamma[f_p, f_q] dN,$$

by the construction of Proposition 10, this last term is nothing but $\Gamma[F]$. Thus, if $F \in \mathcal{D}_0$ then $\varepsilon^+ F \in \underline{\mathbb{D}}$ and

$$\begin{aligned} \|\varepsilon^+ F\|_{\underline{\mathbb{D}}} &= (\mathbb{E}\Gamma[F])^{\frac{1}{2}} + \mathbb{E}\left[\int |\varepsilon^+ F| \eta dN\right] \\ &\leq (2\mathcal{E}[F])^{\frac{1}{2}} + \|F\|_{L^2(\mathbb{P})} \|N(\eta)\|_{L^2(\mathbb{P})} \end{aligned}$$

As a consequence, ε^+ admits a unique extension on \mathbb{D} that we still denote by ε^+ . It is a continuous linear map from \mathbb{D} into $\underline{\mathbb{D}}$ and this leads to the following theorem.

Theorem 17. *The formula*

$$\forall F \in \mathbb{D}, F^\sharp = \int_{X \times R} (\varepsilon^+ F)^\flat dN \odot \rho, \quad (25)$$

is justified by the following decomposition:

$$F \in \mathbb{D} \xrightarrow{\varepsilon^+} \varepsilon^+ F \in \underline{\mathbb{D}} \xrightarrow{\flat} (\varepsilon^+ F)^\flat \in L_0^2(\mathbb{P}_N \times \rho) \xrightarrow{d(N \odot \rho)} F^\sharp \in L^2(\mathbb{P} \times \hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one and where $L_0^2(\mathbb{P}_N \otimes \rho)$ is the closed set of elements G in $L^2(\mathbb{P}_N \otimes \rho)$ such that $\int_R G d\rho = 0$ \mathbb{P}_N -a.e.

Moreover, we have for all $F \in \mathbb{D}$

$$\Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_X \gamma[\varepsilon^+ F] dN.$$

Proof. Let $H \in \underline{\mathbb{D}}$, there exists a sequence (H_n) of elements in $\mathcal{D}_0 \otimes \mathbf{d}$ which converges to H in $\underline{\mathbb{D}}$ and we have for all $n \in \mathbb{N}$

$$\int (H_n^\flat)^2 d\mathbb{P}_N d\rho = \int \gamma[H_n(x, \cdot)] dN d\rho \leq \|H_n\|_{\underline{\mathbb{D}}}^2,$$

therefore (H_n^\flat) is a Cauchy sequence in $L_0^2(\mathbb{P}_N \otimes \rho)$ hence converges to an element in $L_0^2(\mathbb{P}_N \otimes \rho)$ that we denote by H^\flat .

Moreover, if $K \in L_0^2(\mathbb{P}_N \otimes \rho)$, we have

$$\mathbb{E}\hat{\mathbb{E}}\left(\int_{X \times R} K(w, x, r) N \odot \rho(dx dr)\right)^2 = \mathbb{E}\int_{X \times R} K^2 dN \odot \rho = \|K\|_{L^2(\mathbb{P}_N \otimes \rho)}^2.$$

This provides the assertion of the statement. \square

Remark 2. At this stage, it is important to note that an element in \mathbb{D} can not be viewed as a function H depending on the variables w and x and such that $H(w, \cdot) \in \mathbf{d}$ for \mathbb{P} -almost every w . Indeed, if it were the case we would get a function H^\flat defined $\mathbb{P} \otimes \nu$ -a.e. but not \mathbb{P}_N -a.e.

Actually, elements of \mathbb{D} are abstract objects on which \flat operates by considering each elements of \mathbb{D} as the limit of a sequence in $\mathcal{D}_0 \otimes \mathbf{d}$.

Nevertheless, we have the following result

Proposition 18. Let \mathcal{D}_1 be the set of random variables F in \mathbb{D} such that

1. for \mathbb{P} almost all $w \in \Omega$, $\varepsilon^+ F(w, \cdot)$ belongs to \mathbf{d} ,
2. $\gamma[\varepsilon^+ F]$ admits an \mathbb{R}^+ -valued version g such that
 - for \mathbb{P} -almost all $w \in \Omega$, $g(w, x) = \gamma[\varepsilon^+ F(w, \cdot)](x)$ for ν -almost all $x \in X$,
 - $\int_X g(\cdot, x) N(dx) = \Gamma[F]$ \mathbb{P} -a.e.
3. $(\varepsilon^+ F)^\flat$ admits an $L_0^2(\rho)$ -valued version f , such that
 - for \mathbb{P} -almost all $w \in \Omega$, $f(w, \cdot) = (\varepsilon^+ F)^\flat(w, \cdot)$ ν -a.e.
 - $\hat{\mathbb{E}}[(\int_{X \times R} f dN \odot \rho)^2] = \Gamma[F]$ \mathbb{P} -a.e.

Then clearly $\mathcal{D}_0 \subset \mathcal{D}_1$, moreover if $F \in \mathcal{D}_1$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz then $\Phi(F) \in \mathcal{D}_1$, more generally if $F_1, \dots, F_n \in \mathcal{D}_1$ and $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz and C^1 then $\Psi(F_1, \dots, F_n) \in \mathcal{D}_1$.

Proof. Let $n \in \mathbb{N}^*$, $F_1, \dots, F_n \in \mathcal{D}_1$ and $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ Lipschitz and C^1 . Let us denote by f_i the associated version of $(\varepsilon^+ F_i)^\flat$. Let us show that $V = \Psi(F_1, \dots, F_n)$ belongs to \mathcal{D}_1 . First of all, it is clear that for \mathbb{P} -almost all $w \in \Omega$, the map

$$x \rightarrow (\varepsilon^+ V)(w, x) = \Psi(\varepsilon^+ F_1(w, x), \dots, \varepsilon^+ F_n(w, x))$$

belongs to \mathbf{d} .

In addition, by the chain rule $h = \sum_i \Psi'_i(\varepsilon^+ F_1, \dots, \varepsilon^+ F_n) f_i$ is a version of $(\varepsilon^+ V)^\flat$ defined $\mathbb{P} \otimes \nu$ -a.e. Besides it satisfies

$$\begin{aligned} \int_{X \times R} \sum_i \Psi'_i(\varepsilon^+ F_1, \dots, \varepsilon^+ F_n) f_i dN \odot \rho &= \int_{X \times R} \sum_i \Psi'_i(F_1, \dots, F_n) f_i dN \odot \rho \\ &= \sum_i \Psi'_i(F_1, \dots, F_n) \int_{X \times R} f_i dN \odot \rho \end{aligned}$$

thus

$$\begin{aligned} \hat{\mathbb{E}}[(\int_{X \times R} h dN \odot \rho)^2] &= \sum_{i,j} \Psi'_i(F_1, \dots, F_n) \Psi'_j(F_1, \dots, F_n) \int_{X \times R} f_i dN \odot \rho \int_{X \times R} f_j dN \odot \rho \\ &= \sum_{i,j} \Psi'_i(F_1, \dots, F_n) \Psi'_j(F_1, \dots, F_n) \Gamma[F_i, F_j] \\ &= \Gamma[V]. \end{aligned}$$

□

3.3.2. The lent particle method: first application

The preceding theorem provides a new method to study the regularity of Poisson functionals, that we present on an example.

Let us consider, for instance, a real process Y_t with independent increments and Lévy measure σ integrating x^2 , Y_t being supposed centered without Gaussian part. We assume that σ has a density satisfying Hamza's condition (Fukushima-Oshima-Takeda [14] p105) so that a local Dirichlet structure may be constructed on $\mathbb{R} \setminus \{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$. If N is the random Poisson measure with intensity $dt \times \sigma$ we have $\int_0^t h(s) dY_s = \int \mathbf{1}_{[0,t]}(s) h(s) x \tilde{N}(ds dx)$ and the choice done for γ gives $\Gamma[\int_0^t h(s) dY_s] = \int_0^t h^2(s) d[Y, Y]_s$ for $h \in L^2_{loc}(dt)$. In order to study the regularity of the random variable $V = \int_0^t \varphi(Y_{s-}) dY_s$ where φ is Lipschitz and \mathcal{C}^1 , we have two ways:

a) We may represent the gradient \sharp as $Y_t^\sharp = B_{[Y, Y]_t}$ where B is a standard auxiliary independent Brownian motion. Then by the chain rule

$$V^\sharp = \int_0^t \varphi'(Y_{s-})(Y_{s-})^\sharp dY_s + \int_0^t \varphi(Y_{s-}) dB_{[Y]_s}$$

now using $(Y_{s-})^\sharp = (Y_s^\sharp)_-$, a classical but rather tedious stochastic calculus yields

$$\Gamma[V] = \hat{\mathbb{E}}[V^{\sharp 2}] = \sum_{\alpha \leq t} \Delta Y_\alpha^2 \left(\int_{j_\alpha}^t \varphi'(Y_{s-}) dY_s + \varphi(Y_{\alpha-}) \right)^2. \quad (26)$$

where $\Delta Y_\alpha = Y_\alpha - Y_{\alpha-}$. Since V has real values the *energy image density property* holds for V , and V has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (26).

b) An other more direct way consists in applying the theorem. For this we define \flat by choosing ξ such that $\int_0^1 \xi(r) dr = 0$ and $\int_0^1 \xi^2(r) dr = 1$ and putting $f^\flat = x f'(x) \xi(r)$.

1°. First step. We add a particle (α, x) i.e. a jump to Y at time α with size x what gives

$$\varepsilon^+ V - V = \varphi(Y_{\alpha-})x + \int_{j_\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-})) dY_s$$

2°. $V^\flat = 0$ since V does not depend on x , and

$$(\varepsilon^+ V)^\flat = (\varphi(Y_{\alpha-})x + \int_{j_\alpha}^t \varphi'(Y_{s-} + x) x dY_s) \xi(r) \quad \text{because } x^\flat = x \xi(r).$$

3°. We compute $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^\flat{}^2 dr = (\varphi(Y_{\alpha-})x + \int_{j_\alpha}^t \varphi'(Y_{s-} + x) x dY_s)^2$

4°. We take back the particle we gave, because in order to compute $\int \gamma[\varepsilon^+ V] dN$ the integral in N confuses $\varepsilon^+ \omega$ and ω . That gives

$$\int \gamma[\varepsilon^+ V] dN = \int \left(\varphi(Y_{\alpha-}) + \int_{j_\alpha}^t \varphi'(Y_{s-}) dY_s \right)^2 x^2 N(d\alpha dx)$$

and (26).

We remark that both operators $F \mapsto \varepsilon^+ F$, $F \mapsto (\varepsilon^+ F)^\flat$ are non-local, but instead $F \mapsto \int (\varepsilon^+ F)^\flat d(N \odot \rho)$ and $F \mapsto \int \gamma[\varepsilon^+ F] dN$ are local : taking back the lent particle gives the locality. We will deepen this example in dimension p in Part 5.

4 EID property on the upper space from EID property on the bottom space and the domain \mathbb{D}_{loc}

From now on, we make additional hypotheses on the bottom structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ which are stronger but satisfied in most of the examples.

Hypothesis (H1): X admits a partition of the form: $X = B \cup (\bigcup_{k=1}^{+\infty} A_k)$ where for all k , $A_k \in \mathcal{X}$ with $\nu(A_k) < +\infty$ and $\nu(B) = 0$, in such a way that for any $k \in \mathbb{N}^*$ may be defined a local Dirichlet structure with carré du champ:

$$\mathcal{S}_k = (A_k, \mathcal{X}|_{A_k}, \nu|_{A_k}, \mathbf{d}_k, \gamma_k),$$

with

$$\forall f \in \mathbf{d}, f|_{A_k} \in \mathbf{d}_k \text{ and } \gamma[f]|_{A_k} = \gamma_k[f|_{A_k}].$$

Hypothesis (H2): Any finite product of structures \mathcal{S}_k satisfies (EID).

Remark 3. *In many examples where X is a topological space, (H1) is satisfied by choosing for (A_k) $k \in \mathbb{N}^*$ a regular open set.*

Let us remark that (H2) is satisfied for the structures studied in Part 2.

The main result of this section is the following:

Proposition 19. *If the bottom structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ satisfies (H1) and (H2) then the upper structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies (EID).*

Proof. For all $k \in \mathbb{N}^*$, since $\nu(A_k) < +\infty$, we consider an upper structure $S_k = (\Omega_k, \mathcal{A}_k, \mathbb{P}_k, \mathbb{D}_k, \Gamma_k)$ associated to \mathcal{S}_k as a direct application of the construction by product (see §3.3.2 above or Bouleau [7] Chap. VI.3).

Let $k \in \mathbb{N}^*$, we denote by N_k the corresponding random Poisson measure on A_k with intensity $\nu|_{A_k}$ and we consider N^* the random Poisson measure on X with intensity ν , defined on the product probability space

$$(\Omega^*, \mathcal{A}^*, \mathbb{P}^*) = \prod_{k=1}^{+\infty} (\Omega_k, \mathcal{A}_k, \mathbb{P}_k),$$

by

$$N^* = \sum_{k=1}^{+\infty} N_k.$$

In a natural way, we consider the product Dirichlet structure

$$S^* = (\Omega^*, \mathcal{A}^*, \mathbb{P}^*, \mathbb{D}^*, \Gamma^*) = \prod_{k=1}^{+\infty} S_k.$$

In the third Part, we have built using the Friedrichs argument, the Dirichlet structure

$$S = (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma),$$

let us now make the link between those structures.

First of all, thanks to Theorem 2.2.1 and Proposition 2.2.2. of Chap. V in Bouleau-Hirsch [9], we know that a function φ in $L^2(\mathbb{P}^*)$ belongs to \mathbb{D}^* if and only if

1. For all $k \in \mathbb{N}^*$ and $\prod_{n \neq k} \mathbb{P}_n$ -almost all $\xi_1, \dots, \xi_{k_1}, \xi_{k+1}, \dots$, the map

$$\xi \mapsto \varphi(\xi_1, \dots, \xi_{k_1}, \xi, \xi_{k+1}, \dots)$$

belongs to \mathbb{D}_k .

2. $\sum_k \Gamma_k[\varphi] \in L^1(\mathbb{P}^*)$ and we have $\Gamma^*[\varphi] = \sum_k \gamma_k[\varphi]$.

Consider $f \in \mathbf{d} \cap L^1(\gamma)$ then clearly

$$N(f) = \sum_k N_k(f|_{A_k})$$

belongs to \mathbb{D}^* and in the same way

$$e^{i\tilde{N}(f)} = \prod_k e^{i\tilde{N}_k(f|_{A_k})} \in \mathbb{D}^*.$$

Moreover, by hypothesis (H1):

$$\begin{aligned} \Gamma^*[e^{i\tilde{N}(f)}] &= \sum_k \left| \prod_{l \neq k} e^{i\tilde{N}_l(f|_{A_l})} \right|^2 \Gamma_k[e^{i\tilde{N}_k(f|_{A_k})}] = \sum_k N_k(\gamma[f]|_{A_k}) \\ &= N(\gamma[f]) = \Gamma[e^{i\tilde{N}(f)}]. \end{aligned}$$

Thus as \mathcal{D}_0 is dense in \mathbb{D} , we conclude that $\mathbb{D} \subset \mathbb{D}^*$ and $\Gamma = \Gamma^*$ on \mathbb{D} .

As for all k , S_k is a product structure, thanks to hypothesis (H2) and Proposition 2.2.3 in Bouleau-Hirsch [9] Chapter V, we conclude that S^* satisfies (EID) hence S too. \square

We end this section by a few remarks on the localization of this structure which permits to extend the functional calculus related to Γ or \sharp to bigger spaces than \mathbb{D} , which is often convenient from a practical point of view.

Following Bouleau-Hirsch (see [9] p. 44-45) we recall that \mathbb{D}_{loc} denotes the set of functions $F : \Omega \rightarrow \mathbb{R}$ such that there exists a sequence $(E_n)_{n \in \mathbb{N}^*}$ in \mathcal{A} such that

$$\Omega = \bigcup_n E_n \text{ and } \forall n \in \mathbb{N}^*, \exists F_n \in \mathbb{D} \ F_n = F \text{ on } E_n.$$

Moreover if $F \in \mathbb{D}_{loc}$, $\Gamma[F]$ is well-defined and satisfies (EID) in the sense that

$$F_*(\Gamma[F].\mathbb{P}) \ll \lambda^1.$$

More generally, if $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies (EID),

$$\forall F \in (\mathbb{D}_{loc})^n, F_*(\det \Gamma[F] \cdot \mathbb{P}) \ll \lambda_n.$$

We can consider another space bigger than \mathbb{D}_{loc} by considering a partition of Ω consisting in a sequence of sets with negligible boundary. More precisely, we denote by \mathbb{D}_{LOC} the set of functions $F : \Omega \rightarrow \mathbb{R}$ such that there exists a sequence of disjoint sets $(A_n)_{n \in \mathbb{N}^*}$ in \mathcal{A} such that $\mathbb{P}(\Omega \setminus \bigcup_n A_n) = 0$ and

$$\forall n \in \mathbb{N}^*, \exists F_n \in \mathbb{D} \ F_n = F \text{ on } A_n.$$

One can easily verify that it contains the localized domain of any structure S^* as considered in the proof of Proposition 19, that Γ is well-defined on \mathbb{D}_{LOC} , that the functional calculus related to Γ or \sharp remains valid and that it satisfies (EID) i.e. if $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies (EID),

$$\forall F \in (\mathbb{D}_{LOC})^n, F_*(\det \Gamma[F] \cdot \mathbb{P}) \ll \lambda^n.$$

5 Two examples

5.1 Upper bound of a process on $[0, t]$

Let Y be a real process with stationary independent increments satisfying the hypotheses of example 3.3.2.

We consider a real càdlàg process K independent of Y and put $H_s = Y_s + K_s$.

Proposition 20. *If $\sigma(\mathbb{R} \setminus \{0\}) = +\infty$ and if $\mathbb{P}[\sup_{s \leq t} H_s = H_0] = 0$, the random variable $\sup_{s \leq t} H_s$ possesses a density.*

Proof. a) We may suppose that K satisfies $\sup_{s \leq t} |K_s| \in L^2$. Indeed, if random variables X_n have densities and $\mathbb{P}[X_n \neq X] \rightarrow 0$, then X has a density. Hence the assertion is obtained by considering $(K_s \wedge k) \vee (-k)$.

b) Let us put $M = \sup_{s \leq t} H_s$. Applying the lent particle method gives

$$\begin{aligned} (\varepsilon^+ M)(\alpha, x) &= \sup_{s \leq t} ((Y_s + K_s) \mathbf{1}_{\{s < \alpha\}} + (Y_s + x + K_s) \mathbf{1}_{\{s \geq \alpha\}}) \\ &= \max(\sup_{s < \alpha} (Y_s + K_s), \sup_{s \geq \alpha} (Y_s + x + K_s)) \\ \gamma[\varepsilon^+ M](\alpha, x) &= \mathbf{1}_{\{\sup_{s \geq \alpha} (Y_s + x + K_s) \geq \sup_{s < \alpha} (Y_s + K_s)\}} \gamma[j](x) \end{aligned}$$

where j is the identity map $j(x) = x$.

We take back the lent particle before integrating with respect to N and obtain, since $\gamma[j](x) = x^2$,

$$\Gamma[M] = \int \gamma[\varepsilon^+ M] N(d\alpha dx) = \sum_{\alpha \leq t} \Delta Y_\alpha^2 \mathbf{1}_{\{\sup_{s \geq \alpha} (Y_s + K_s) \geq \sup_{s < \alpha} (Y_s + K_s)\}}.$$

As $\sigma(\mathbb{R} \setminus \{0\}) = +\infty$, Y has infinitely many jumps on every time interval, so that

$$\Gamma[M] = 0 \quad \Rightarrow \quad \forall \alpha \leq t \quad \sup_{s \geq \alpha} (Y_s + K_s) < \sup_{s < \alpha} (Y_s + K_s)$$

and choosing α decreasing to zero, we obtain

$$\Gamma[M] = 0 \quad \Rightarrow \quad \sup_{t \geq s \geq 0} H_s = H_0$$

and the proposition. \square

It follows that any real Lévy process X starting at zero and immediately entering \mathbb{R}_+^* , whose Lévy measure dominates a measure σ satisfying Hamza's condition and infinite, is such that $\sup_{s \leq t} X_s$ has a density.

5.2 Regularizing properties of Lévy processes

Let Y be again a real process with stationary independent increments satisfying the hypotheses of example 3.3.2. By Hamza's condition, hypothesis (H1) is fulfilled and hypothesis (H2) ensues from Theorem 2, so that the upper structure verifies (EID).

Let S be an \mathbb{R}^p -valued semi-martingale independent of Y . We will say that S is pathwise p -dimensional on $[0, t]$ if almost every sample path of S on $[0, t]$ spans a p -dimensional vector space.

We consider the \mathbb{R}^p -valued process Z whose components are given by

$$Z_t^1 = S_t^1 + Y_t^1 \quad \text{and} \quad Z_t^i = S_t^i \quad \forall i \geq 2$$

and the stochastic integral

$$R = \int_0^t \psi(Z_{s-}) dZ_s$$

where ψ is a Lipschitz and \mathcal{C}^1 mapping from \mathbb{R}^p into $\mathbb{R}^{p \times p}$.

Proposition 21. *If $\sigma(\mathbb{R} \setminus \{0\}) = +\infty$, if the Jacobian determinant of the column vector $\psi_{\cdot 1}$ does not vanish and if S is pathwise p -dimensional on $[0, t]$, then the law of R is absolutely continuous with respect to λ^p .*

Proof. We apply the lent particle method. Putting $\bar{x} = (x, 0, \dots, 0)$ and $R^i = \sum_j \int_0^t \psi_{ij}(Z_{s-}) dZ_s^j$, we have

$$\varepsilon^+ R^i - R^i = \psi_{i1}(Z_{\alpha-})x + \int_{[\alpha]}^t (\psi_{i1}(Z_{s-} + \bar{x}) - \psi_{i1}(Z_{s-})) dY_s$$

as in example 3.3.2,

$$(\varepsilon^+ R^i)^b = (\psi_{i1}(Z_{\alpha-})x + \int_{[\alpha]}^t \partial_1 \psi_{i1}(Z_{s-} + \bar{x})x dY_s) \xi(r)$$

and

$$\gamma[\varepsilon^+ R^i, \varepsilon^+ R^j] =$$

$$\left(\psi_{i1}(Z_{\alpha-}) + \int_{[\alpha]}^t \partial_1 \psi_{i1}(Z_{s-} + \bar{x}) dY_s \right) \left(\psi_{j1}(Z_{\alpha-}) + \int_{[\alpha]}^t \partial_1 \psi_{j1}(Z_{s-} + \bar{x}) dY_s \right) x^2.$$

We take back the lent particle before integrating in N :

$$\Gamma[R^i, R^j] = \int \gamma[\varepsilon^+ R^i, \varepsilon^+ R^j] dN = \sum_{\alpha \leq t} \Delta Y_\alpha^2 U_\alpha U_\alpha^t$$

where U_α is the column vector $\psi_{\cdot 1}(Z_{\alpha-}) + \int_{[\alpha]}^t \partial_1 \psi_{\cdot 1}(Z_{s-}) dY_s$.

Let JT be the set of jump times of Y on $[0, t]$, we conclude that

$$\det \Gamma[R, R^t] = 0 \quad \Leftrightarrow \quad \dim \mathcal{L}(U_\alpha; \alpha \in JT) < p.$$

Let $A = \{\omega : \dim \mathcal{L}(U_\alpha; \alpha \in JT) < p\}$. Reasoning on A , there exists $\lambda_1, \dots, \lambda_p$ such that

$$\sum_{k=1}^p \lambda_k \left(\psi_{k1}(Z_{\alpha-}) + \int_{[\alpha]}^t \partial_1 \psi_{k1}(Z_{s-}) dY_s \right) = 0 \quad \forall \alpha \in JT, \quad (27)$$

now, since $\|\sigma\| = +\infty$, JT is a dense countable subset of $[0, t]$, so that taking left limits in (27), using (27) anew and the fact that ψ is \mathcal{C}^1 , we obtain

$$\sum_{k=1}^p \lambda_k \psi_{k1}(Z_{\alpha-}) = 0 \quad \forall \alpha \in JT \quad \text{hence} \quad \forall \alpha \in]0, t]$$

thus, on A , we have $\dim \mathcal{L}(\psi_{\cdot 1}(Z_{s-}); s \in]0, t]) < p$.

Then EID property yields the conclusion if we assume Z to be pathwise p -dimensional, and the proposition follows from the following lemma : \square

Lemma 22. *If S is pathwise p -dimensional on $[0, t]$ then Z is pathwise p -dimensional on $[0, t]$.*

Proof. By the independence of S and Y we may suppose these processes be defined on a product of probability spaces $(\Omega_1 \times \Omega_2, \mathbb{P}_1 \times \mathbb{P}_2)$.

If $\dim \mathcal{L}(S_{t_1}(\omega_1), \dots, S_{t_p}(\omega_1)) = p$, it is not possible that $\dim \mathcal{L}(S_{t_1}(\omega_1) + Y_{t_1}(\omega_2), \dots, S_{t_p}(\omega_1) + Y_{t_p}(\omega_2)) < p$ with positive probability because Y being a centered process, the points S_{t_i} are the barycenters of the $S_{t_i} + Y_{t_i}(\omega_2)$ for the expectation with respect to \mathbb{P}_2 . \square

The lent particle method and (EID) property may be applied to density results for solutions of stochastic differential equations driven by Lévy processes or random measures under Lipschitz hypotheses. Let us mention also that the gradient \sharp defined in §3.2 has the property to be easily iterated, this allows to obtain conclusions on C^∞ -regularity in the case of smooth coefficients. These applications will be investigated in forthcoming articles.

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