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COMPUTING VOLUME FUNCTION ON PROJECTIVE BUNDLE OVER A CURVE

Huayi Chen

Abstract. — We establish an explicit link between the volume function on a projective variety fibered on a curve and the asymptotic behaviour of the canonical filtration of direct images. As an application, we calculate explicitly the volume function on projective bundle over a curve.

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1. Introduction

Let k be a field and X be a projective variety of dimension d over $\operatorname{Spec} k$. For any line bundle L on X, the *volume* of L is defined as

$$\operatorname{vol}(L) := \limsup_{n\to\infty} \frac{\operatorname{rk}_k H^0(X, L^{\otimes n})}{n^d/d!}, \tag{1}$$

where the symbol $\limsup$ can be replaced by $\lim$, thanks to Fujita's approximation theorem [**9**] (see also [**15**] for positive characteristic case). One says that L is *big* if $\operatorname{vol}(L) > 0$. The volume function is invariant under numerical equivalence (see [**10**] Proposition 2.2.41), and therefore induces a mapping from the Néron-Severi group $N^1(X)$ to $\mathbb{R}_+$ which we shall still note vol.

If L is ample, then it follows from the asymptotic Riemann-Roch formula that $\operatorname{vol}(L) = c_1(L)^d$. In particular, the volume function is polynomial on the ample cone. However, on the larger cone of big divisors, the behaviour of the volume function is much more complicated, and the the volume function is more difficult to calculate.

It is know that when X is a surface or a toric variety, then the volume function is piecewise polynomial on the pesudo-effective cone, and can be computed through the Zariski decomposition (see [**13**] and [**8**]). For general case, some authors have proposed the *positive intersection product* for big classes which generalizes the classical intersection product and identifies $\mathrm{vol}(L)$ with the positive self-intersection product of the class of L (see [**1**] for complex analytic case and [**2**] for algebraic case). But the positive intersection product is only super-additive in each variable, which suggests that the volume function is much more complicated than a polynomial. Although the differentiability of the volume function has been proved in [**2**] (see also the Okounkov body approach [**11**] Corollary 4.25, where appears the *restricted volume* introduced in [**6**]), there does exist a counter-example where the volume function is not two-times differentiable (see [**10**] Example 2.2.46 with $n = 2$). Furthermore, the function vol may take irrational values (see [**5**]), which is not the case for self-intersection number.

In [**4**], the author has obtained an explicit link between the arithmetic volume function in sense of Moriwaki and the maximal value of the limit of Harder-Narasimhan polygons. This enables us to prove a weak form of a conjecture of Moriwaki ([**12**] Remark 5.7) asserting that the arithmetic volume function is actually a limit instead of a sup limit. Although the article was written in arithmetic framework, the similar idea can work in function field case. Let C be a smooth projective curve defined over k. Denote by $K := k(C)$ the field of meromorphic functions on C. If E is a non-zero vector bundle on C, the Harder-Narasimhan filtration of E is the unique flag

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0 \tag{2}$$

of E such that each sub-quotient E_i/E_{i+1} is semistable and that the successive slopes verify the following inequalities

$$\mu_0 < \mu_1 < \cdots < \mu_{d-1}, \tag{3}$$

where $\mu_i = \mu(E_i/E_{i+1})$. For any non-zero vector bundle E on C whose Harder-Narasimhan filtration is as (2), we define a probability measure ν_E on $\mathbb{R}$ as follows

$$\nu_E := \sum_{i=0}^{d-1} \frac{\mathrm{rk}\, E_{i+1} - \mathrm{rk}\, E_i}{\mathrm{rk}\, E} \delta_{\mu_i}, \tag{4}$$

where δ_x is the Dirac measure concentrated on $\{x\}$. For any real number $\varepsilon > 0$, we define an operator T_ε on the set of all Borel probability measure on $\mathbb{R}$: if ν is such a measure and if f is a continuous function of compact support on $\mathbb{R}$, then

$$\int f(x) T_\varepsilon \nu(\mathrm{d}x) = \int f(\varepsilon x) \nu(\mathrm{d}x).$$

The following is the main theorem of this article:

Theorem 1.1. — *Let $\pi : X \to C$ be a projective and flat morphism, and L be an arbitrary line bundle on X such that L_K is big. Then*

1) *the sequence of Borel probability measures $(T_{\frac{1}{n}} \nu_{\pi_*(L^{\otimes n})})_{n \geq 1}$ converges vaguely to a Borel probability measure ν_L^π.*

2) the following equality holds

$$\mathrm{vol}(L) = \dim(X)\,\mathrm{vol}(L_K)\int x_+\nu_L^\pi(\mathrm{d}x),$$

where $x_+ = \max\{x, 0\}$.

Recall that a sequence of Borel probability measure $(\nu_n)_{n\geq 1}$ is said to converge *vaguely* to a Borel measure ν if, for any continuous function of compact support f, one has $\displaystyle\int f(x)\nu(\mathrm{d}x) = \lim_{n\to\infty}\int f(x)\nu_n(\mathrm{d}x)$. We reminder that ν need not be a probability measure. However, it is the case when the supports of ν_n are uniformly bounded.

One of the ideas is to compare the rank of $\pi_*(L^{\otimes n})$ to the maximal value of its Harder-Narasimhan polygon, which coincides with the integral of x_+ with respect to the corresponding probability measure, multiplied by its rank.

Shortly after the first version of this article had been written, R. Lazarsfeld kindly communicated to the author Wolfe's Ph.D thesis. We observe that the idea of comparison mentioned above is quite similar to a result of Wolfe. In fact, for any non-zero vector bundle E on C, Wolfe has compared the rank of $H^0(C, E)$ to partial sums of $\mathrm{rk}\, H^0(C, \mathrm{sq}_i E)$, where $\mathrm{sq}_i E$ is the ith subquotient of the Harder-Narasimhan filtration of E. In view of some basic facts about vector bundles on C resumed in Lemma 2.1, Wolfe's result is more or less equivalent to Proposition 2.2 *infra*.

The more delicate part of the theorem is the convergence of dilated probability measures associated to direct images, which follows from a general convergence result on filtered graded algebra, established in a previous work [**4**] of the author (see also [**3**]).

Wolfe has studied in his thesis [**16**] (see also [**7**] page 9), by using his comparison result, the volume function on projective bundles over a curve. Let E be a non-zero vector bundle on C. One uses the symbol $\mathbb{P}(E)$ to denote the C-scheme which represents the functor

$$\begin{array}{ccc} \textbf{Scheme}/C & \longrightarrow & \textbf{Set} \\ (p : X \to C) & \longmapsto & \left\{\begin{array}{c}\text{locally free quotient}\\ \text{of rank 1 of } p^*E\end{array}\right\} \end{array} \tag{5}$$

Let $\pi : \mathbb{P}(E) \to C$ be the canonical morphism. Denote by $\mathcal{O}_E(1)$ the canonical line bundle on $\mathbb{P}(E)$, which is the universal quotient of π^*E corresponding to the representable functor (5). The Picard group of $\mathbb{P}(E)$ is then generated by $\mathcal{O}_E(1)$ and $\pi^*\,\mathrm{Pic}(C)$. He proved that the Néron-Severi group $N^1(\mathbb{P}(E))$ can be separated into several sectors determined by the Harder-Narasimhan filtration of E such that the volume function is polynomial on each piece.

In this article, we shall apply Theorem 1.1 to determine the volume function on $\mathbb{P}(E)$. It suffices to compute explicitly the limit measure $\nu_{\mathcal{O}_E(1)}$, which relies on the computation of the Harder-Narasimhan filtration of S^nE. Here we calculate firstly that of $E^{\otimes n}$, which is invariant by the action of the symmetric group $\mathfrak{S}_n$, thanks to the uniqueness of Harder-Narasimhan filtration. We then pass to the quotient by the

action of $\mathfrak{S}_n$ to obtain the Harder-Narasimhan filtration of S^nE. We shall actually establish the following result:

Theorem 1.2. — *Let r be the rank of E. Assume that the Harder-Narasimhan filtration of E is as* (2) *and that the successive slopes of E are as in* (3). *Let $\underline{s} = (s_1, \cdots, s_r)$ be a vector in $\mathbb{R}^r$ such that the value μ_i appears exactly* $\mathrm{rk}(E_i/E_{i+1})$ *times in its coordinates. Let $\Delta \subset \mathbb{R}^r$ be the simplex defined as*

$$\{(x_1, \cdots, x_r) \mid 0 \leq x_j \leq 1, \quad x_1 + \cdots + x_r = 1\}$$

and η be the Lebesgue measure on Δ normalized such that $\eta(\Delta) = 1$. Then one has $\nu^{\pi}_{\mathcal{O}_E(1)} = \varphi_{\underline{s}*}\eta$, where $\varphi_{\underline{s}} : \Delta \to \mathbb{R}$ sends $(x_1, \cdots, x_r)$ to $s_1x_1 + \cdots + s_rx_r$.

As a corollary of Theorems 1.1 and 1.2, we compute the volume of an arbitrary big line bundle on $\mathbb{P}(E)$:

Corollary 1.3. — *Keep the notation of Theorem 1.2. Let $m \geq 1$ be an integer and M be a line bundle of degree c on C, then*

$$\mathrm{vol}(\mathcal{O}_E(m) \otimes \pi^*M) = rm^r \int \big(x + c/m\big)_+ \varphi_{\underline{s}*}\eta(\mathrm{d}x). \tag{6}$$

where $y_+ := \max\{y, 0\}$ for any $y \in \mathbb{R}$.

Wolfe computed in his thesis the Harder-Narasimhan filtration of symmetric powers of E by using a more direct argument. He also expressed the volume $\mathrm{vol}(\mathcal{O}_E(m) \otimes \pi^*M)$ as an integral on the simplex Δ. The result of Wolfe is actually equivalent to the formula (19). This enabled him to prove that the volume function is piecewise polynomial on $N^1(\mathbb{P}(E))$. Here we propose a reformulation (20) of ν_{S^nE}, which leads to a simple form of the volume function (6). This permits us to recover Wolfe's result, and furthermore, to prove that on each sector, the volume function is a homogeneous polynomial of total degree $\mathrm{rk}(E)$ on m and c, and the degree of c does not exceed 3.

The article is organized as follows: in the second section, we introduce the notion of positive degree for vector bundles on C, and then prove that the volume function equals to the limit of normalized positive degree, which permits to establish Theorem 1.1. In the third section, we compute the Harder-Narasimhan filtrations of tensor powers and symmetric powers of a non-zero vector bundle on C, and then prove Theorem 1.2 and Corollary 1.3. Finally in the fourth section, we show that Corollary 1.3 permits to recover and slightly refine Wolfe's result on polynomial representation of the volume function.

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2. Volume function as limit of positive degrees

We reminder that all results in this section are valid in any characteristic.

2.1. Some reminders on vector bundles. — If E is a vector bundle on C, we denote respectively by $h^0(E)$ and $h^1(E)$ the rank of $H^0(C,E)$ and $H^1(C,E)$ over k. By Serre duality, $h^1(E) = h^0(E^\vee \otimes \omega_C)$. Recall that the Riemann-Roch theorem predicts

$$h^0(E) - h^1(E) = \deg(E) + \mathrm{rk}(E)(1-g). \tag{7}$$

This implies in particular that the degree of ω_C is $2(g-1)$. Suppose that E is non-zero. The *slope* of E is defined as the quotient $\deg(E)/\mathrm{rk}(E)$, denoted by $\mu(E)$, and the *maximal slope* $\mu_{\max}(E)$ is the maximal value of the slopes of all non-zero subbundles of E. By definition one has the inequality $\mu(E) \leq \mu_{\max}(E)$. We say that E is *semistable* if the equality $\mu(E) = \mu_{\max}(E)$ holds. This condition is also equivalent to the equality $\mu(E) = \mu_{\min}(E)$, where $\mu_{\min}(E)$ is the minimal value of slopes of all non-zero locally free quotient of E, called the *minimal slope* of E. As $\mu(E^\vee) = -\mu(E)$ and $\mu_{\max}(E^\vee) = -\mu_{\min}(E)$, we obtain that the semistability of E is equivalent to that of $E^\vee$.

Denote by $b(C)$ the positive integer

$$\{\deg(L) \mid L \in \mathrm{Pic}(C),\ L \text{ is ample}\}. \tag{8}$$

By definition, the degree of a line bundle on C is always a multiple of $b(C)$.

We recall in the following some basic facts about vector bundles on C.

Lemma 2.1. — *Let E be a non-zero vector bundle on C.*

1) If $\mu_{\max}(E) < 0$, then $h^0(E) = 0$.
2) If $\mu_{\min}(E) > 2g-2$, then $h^0(E) = \deg(E) + \mathrm{rk}(E)(1-g)$.
3) If $\mu_{\min}(E) \geq 0$, then $|h^0(E) - \deg(E)| \leq \mathrm{rk}(E)(g + b(C))$.

Proof. — 1) If $h^0(E) \neq 0$, then there is an injective homomorphism from $\mathcal{O}_C$ to E, therefore $0 = \mu(\mathcal{O}_C) \leq \mu_{\max}(E)$.

2) Since $\mu_{\min}(E) > 2g-2$, one has $\mu_{\max}(E^\vee \otimes \omega_C) = \deg(\omega_C) - \mu_{\min}(E) < 0$. After 1), $h^1(E) = h^0(E^\vee \otimes \omega_C) = 0$. So the Riemann-Roch formula implies $h^0(E) = \deg(E) + \mathrm{rk}(E)(1-g)$.

3) If $g = 0$, then, by 2), $h^0(E) = \deg(E) + \mathrm{rk}(E)$, so

$$|h^0(E) - \deg(E)| = \mathrm{rk}(E) \leq b(C)\,\mathrm{rk}\,E.$$

Assume in the following that $g \geq 1$. Let L be an ample invertible $\mathcal{O}_C$-module of degree $b(C)$ and let $a = \lfloor \frac{2g-2}{b(C)} \rfloor + 1$. Then the inequalities $2g-2 < \deg(L^{\otimes a}) \leq 2g-2+b(C)$ hold. As $\mu_{\min}(E \otimes L^{\otimes a}) = \mu_{\min}(E) + \deg(L^{\otimes a}) > 2g-2$, we obtain

$$h^0(E) \leq h^0(E \otimes L^{\otimes a}) = \deg(E \otimes L^{\otimes a}) + \mathrm{rk}(E)(1-g) \leq \deg(E) + \mathrm{rk}(E)(g-1+b(C)).$$

On the other hand, $h^0(E) = h^1(E) + \deg(E) + \mathrm{rk}(E)(1-g) \geq \deg(E) + \mathrm{rk}(E)(1-g)$. Therefore, $|h^0(E) - \deg(E)| \leq \mathrm{rk}(E)(g-1+b(C))$. □

2.2. Positive degree and rank of global section space. — Let E be a non-zero vector bundle on C and r be its rank. We introduce the notion of positive degree of E, which is the maximal value of the Harder-Narasimhan polygon of E. We then show that this value approximates $h^0(E)$.

Recall that the Harder-Narasimhan polygon $\widetilde{P}_E$ is by definition the concave function defined on $[0,r]$ whose graph is the convex hull of points of the form $(\operatorname{rk} F, \deg(F))$, where F runs over all subbundles of E. Denote by $\deg_+(E)$ the integer $\max\limits_{x\in[0,r]} \widetilde{P}_E(x)$, called the *positive degree* of E.

The Harder-Narasimhan polygon can be determined from the Harder-Narasimhan flag of E, which is the only flag

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_i \supsetneq E_{i+1} \supsetneq \cdots \supsetneq E_d = 0 \tag{9}$$

such that the sub-quotients E_i/E_{i+1} are semistable and satisfies

$$\mu(E_0/E_1) < \mu(E_1/E_2) < \cdots < \mu(E_{d-1}/E_d).$$

In fact, the vertices of the polygon $\widetilde{P}_E$ are just $(\operatorname{rk}(E_i), \deg(E_i))$.

The proposition below compares $h^0(E)$ and $\deg_+(E)$.

Proposition 2.2. — *The following inequality holds:*

$$|h^0(E) - \deg_+(E)| \leq \operatorname{rk}(E)(g + b(C)). \tag{10}$$

Proof. — Let the Harder-Narasimhan flag of E be as in (9). For any integer i, $0 \leq i \leq d-1$, let $\mu_i = \mu(E_i/E_{i+1})$. Let j be the first index in $\{0, \cdots, d-1\}$ such that $\mu_j \geq 0$ (if such index does not exist, let $j = d$). By definition, the positive degree $\deg_+(E)$ coincides with $\deg(E_j)$.

If $j > 0$, then $\mu_{\max}(E/E_j) = \alpha_{j-1} < 0$, so Lemma 2.1 1) predicts that $h^0(E/E_j) = 0$; otherwise $E = E_j$ and we still have $h^0(E/E_j) = 0$. Hence $h^0(E) = h^0(E_j)$. If $j = n$, then $h^0(E_j) = 0 = \deg_+(E)$; otherwise $\mu_{\min}(E_j) = \alpha_j \geq 0$, and by Lemma 2.1 3),

$$|h^0(E) - \deg_+(E)| = |h^0(E_j) - \deg(E_j)| \leq \operatorname{rk}(E_j)(g + b(C)) \leq \operatorname{rk}(E)(g + b(C)).$$

□

We define the *positive slope* of E to be the quotient $\mu_+(E) := \deg_+(E)/\operatorname{rk}(E)$. By definition $\max(\mu_{\max}(E), 0) \geq \mu_+(E) \geq 0$.

2.3. Volume function and positive degree of direct image. — Let X be an integral projective scheme of dimension r over $\operatorname{Spec} k$ and L is a line bundle on X. Assume that $\pi : X \to C$ is a flat k-morphism. Denote by K the field of all rational functions on C. Then X_K is an integral projective scheme of dimension $r-1$ over $\operatorname{Spec} K$.

Proposition 2.3. — *The equality*

$$\operatorname{vol}(L) = \limsup_{n\to+\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} \tag{11}$$

holds. Furthermore, if L is big, then also is L_K.

Proof. — After Proposition 2.2,

$$|\deg_+(\pi_*(L^{\otimes n})) - \mathrm{rk}_k H^0(X, L^{\otimes n})| \leq \mathrm{rk}_K H^0(X_K, L_K^{\otimes n})(4g + 2b(C)).$$

Since $\mathrm{rk}_K H^0(X_K, L_K^{\otimes n}) \ll n^{r-1}$, we obtain

$$\lim_{n\to\infty} \left| \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} - \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^r/r!} \right| = 0, \tag{12}$$

which implies (11). If L is big, then $\limsup_{n\to\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} > 0$. On the other hand,

$$\limsup_{n\to\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n\,\mathrm{rk}(\pi_*(L^{\otimes n}))} \leq \limsup_{n\to\infty} \frac{\mu_{\max}(\pi_*(L^{\otimes n}))}{n} < +\infty.$$

This implies $\limsup_{n\to\infty} \frac{\mathrm{rk}(\pi_*(L^{\otimes n}))}{n^{r-1}/r!} > 0$. Therefore L_K is big. □

The Fujita's approximation theorem (the Fujita's approximation theorem in positive characteristic case has been proved by Takagi [**15**]) implies that the volume function is in fact a limit. More precisely, one has

$$\mathrm{vol}(L) = \lim_{n\to\infty} \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^r/r!}. \tag{13}$$

Furthermore, (13) applied on L_K implies

$$\mathrm{vol}(L_K) = \lim_{n\to\infty} \frac{\mathrm{rk}_K H^0(X_K, L_K)}{n^{r-1}/(r-1)!} = \lim_{n\to\infty} \frac{\mathrm{rk}(\pi_*(L^{\otimes n}))}{n^{r-1}/(r-1)!}.$$

Therefore we obtain the following equalities:

Corollary 2.4. — *The following equalities hold:*

$$\mathrm{vol}(L) = \lim_{n\to\infty} \frac{\mathrm{rk}_k H^0(X, L^{\otimes n})}{n^r/r!} = \lim_{n\to\infty} \frac{\deg_+(\pi_*(L^{\otimes n}))}{n^r/r!} = r\mathrm{vol}(L_K) \lim_{n\to\infty} \frac{\mu_+(\pi_*(L^{\otimes n}))}{n}.$$

2.4. Proof of Theorem 1.1. — The assertion 1) is a direct consequence of [**4**] Theorem 4.2. In the following, we shall prove the second assertion.

From Corollary 2.4, we obtain

$$\mathrm{vol}(L) = \dim(X)\,\mathrm{vol}(L_K) \lim_{n\to\infty} \frac{\mu_+(\pi_*(L^{\otimes n}))}{n},$$

where we reminder that $\mu_+(\pi_*(L^{\otimes n}))$ was defined as the maximal value of the (normalized) Harder-Narasimhan polygon of $\pi_*(L^{\otimes n})$. The assertion 2) then follows from the following lemma.

Lemma 2.5. — *Let E be a non-zero vector bundle of rank r on C. The following equality holds:*

$$\mu_+(E) = \int x_+ \nu_E(\mathrm{d}x), \tag{14}$$

where the probability measure was defined in (4).

Proof. — Let r be the rank of E,

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0$$

be the Harder-Narasimhan flag of E and $\mu_0 < \cdots < \mu_{d-1}$ be its successive slopes. For any $i \in \{0, \cdots, d-1\}$, let r_i be the rank of E_i/E_{i+1}. Assume that j is the first index in $\{0, \cdots, d-1\}$ such that $\mu_j \geq 0$ (if such index does not exist, let $j = d$). Then

$$\deg_+(E) = \deg(E_j) = \sum_{i=0}^{j} r_i \mu_i.$$

Therefore, $\mu_+(E) = \sum_{i=0}^{j} \frac{r_i}{r} \mu_i$. Since $\mu_E = \sum_{i=0}^{d-1} \frac{r_i}{r} \delta_{\mu_i}$, the equality (14) holds. □

By the above lemma, one has

$$\begin{aligned}\mathrm{vol}(L) &= \dim(X)\,\mathrm{vol}(L_K) \lim_{n\to\infty} \frac{1}{n} \int x_+ \nu_{\pi_*(L^{\otimes n})}(\mathrm{d}x) \\ &= \dim(X)\,\mathrm{vol}(L_K) \lim_{n\to\infty} \int x_+ T_{\frac{1}{n}} \nu_{\pi_*(L^{\otimes n})}(\mathrm{d}x).\end{aligned}$$

From the first part of the theorem, the sequence of measures $(T_{\frac{1}{n}} \nu_{\pi_*(L^{\otimes n})})_{n\geq 1}$ converges vaguely to ν_L^π. Furthermore, [**4**] Proposition 4.1 1) shows that the supports of measures $(T_{\frac{1}{n}} \nu_{\pi_*(L^{\otimes n})})_{n\geq 1}$ are uniformly bounded from above. Therefore we obtain

$$\mathrm{vol}(L) = \dim(X)\,\mathrm{vol}(L_K) \int x_+ \nu_L^\pi(\mathrm{d}x),$$

which proves the theorem.

Remark 2.6. — 1) By definition, for any integer $n \geq 1$, $\nu_{L^{\otimes n}}^\pi = T_n \nu_L^\pi$. Hence we recover $\mathrm{vol}(L) = n^{\dim X}\,\mathrm{vol}(L)$.

2) Let M be a line bundle of degree a on C, and E is a non-zero vector bundle on C whose Harder-Narasimhan filtration is (2) and whose successive slopes are (3). Assume that M is a line bundle of degree a on E. Then the Harder-Narasimhan flag of $E \otimes M$ is just $E \otimes M = E_0 \otimes M \supsetneq E_1 \otimes M \supsetneq \cdots \supsetneq E_d \otimes M = 0$, and its successive slopes are $\mu_i + a$, $i \in \{0, \cdots, d-1\}$. Therefore, one has $\nu_{E\otimes M} = \tau_a \nu_E$, where for any Borel probability measure ν on $\mathbb{R}$, $\tau_a \nu$ is defined as

$$\int f(x) \tau_a \nu(\mathrm{d}x) = \int f(x+a) \nu(\mathrm{d}x).$$

One verifies that $T_\varepsilon \tau_a \nu = \tau_{a\varepsilon} T_\varepsilon \nu$. Thus we obtain that, for any integer $n \geq 1$, $\pi_*((L \otimes \pi^* M)^{\otimes n}) = \pi_*(L^{\otimes n}) \otimes M^{\otimes n}$. Therefore, one has $\nu_{\pi_*((L\otimes\pi^*M)^{\otimes n})} = \tau_{na} \nu_{\pi_*(L^{\otimes n})}$, which implies that $\nu_{L\otimes\pi^*M}^\pi = \tau_a \nu_L^\pi$. In particular,

$$\mathrm{vol}(L \otimes \pi^* M) = \dim(X)\,\mathrm{vol}(L_K) \int (x+a)_+ \nu_L^\pi(\mathrm{d}x).$$

3. Harder-Narasimhan filtrations of tensor powers

Let E be a non-zero vector bundle on C whose Harder-Narasimhan filtration is

$$E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0 \tag{15}$$

and whose successive slopes are $\mu_i = \mu(E_i/E_{i+1})$. For any $i \in \{0, \cdots, d\}$, let $\mathrm{sq}_i(E)$ be the sub-quotient E_i/E_{i+1}. In this section, we shall determine, for any integer $n \geq 1$, the Harder-Narasimhan filtration of $E^{\otimes n}$ and that of $S^n E$. Here we should suppose that the characteristic of the ground field k is zero, so that the following results of Ramanan and Ramanathan [**14**] hold:

1) The tensor product of two semistable vector bundles on C is still semistable.
2) Any symmetric power of a semistable vector bundle E on C is semistable. Furthermore, one has $\mu(S^n E) = n\mu(E)$.
3) If E_1 and E_2 are two non-zero vector bundles on C, then

$$\mu_{\max}(E_1 \otimes E_2) = \mu_{\max}(E_1) + \mu_{\max}(E_2), \quad \mu_{\min}(E_1 \otimes E_2) = \mu_{\min}(E_1) + \mu_{\min}(E_2).$$

3.1. The case of $E^{\otimes n}$. — Denote by Θ the indices set $\{0, \cdots, d-1\}^n$. We introduce a partial order "$\leq$" on Θ such that

$$(j_1, \cdots, j_n) \leq (l_1, \cdots, l_n) \text{ if and only if } j_1 \leq l_1, \cdots, j_n \leq l_n.$$

We say that a subset A of Θ is *saturated* if

$$\alpha \in A,\ \beta \in \Theta \text{ and } \beta \geq \alpha \implies \beta \in A.$$

Assume that A is an arbitrary subset of Θ. Then $\{\beta \mid \exists \alpha \in A,\ \beta \geq \alpha\}$ is the smallest saturated subset of Θ containing A. We denote it by $\overline{A}$.

For any $\alpha = (a_1, \cdots, a_n) \in \Theta$, denote by E_α the tensor product $E_{a_1} \otimes \cdots \otimes E_{a_n}$ and by $\mathrm{sq}_\alpha(E)$ the tensor product of sub-quotients $\mathrm{sq}_{a_1}(E) \otimes \cdots \otimes \mathrm{sq}_{a_n}(E)$. Since each sub-quotient $\mathrm{sq}_i(E)$ is semistable, also is the tensor product $\mathrm{sq}_\alpha(E)$. For any non-empty subset A of Θ, let $E_A := \sum_{\alpha \in A} E_\alpha$. Write by convention $E_\emptyset = 0$. In the following are some basic properties of vector bundles E_A:

1) $E_A = E_{\overline{A}}$.
2) If $A_1 \subset A_2 \subset \Theta$, then $E_{A_1} \subset E_{A_2}$.

Proposition 3.1. — *Assume that $A \subset \Theta$ is non-empty and saturated, and that A'' is a subset of A consisting of maximal elements. Then $A' := A \setminus A''$ is also saturated. Furthermore, one has an isomorphism*

$$E_A/E_{A'} \cong \bigoplus_{\alpha \in A''} \mathrm{sq}_\alpha(E). \tag{16}$$

Proof. — Locally for the Zariski topology, the flag (15) is split. Therefore a classical argument in linear algebra leads locally to a natural isomorphism of the form (16). These natural isomorphisms do not depend on the choice of the splitting and therefore glue together into a global isomorphism. □

For any $\alpha = (a_1, \cdots, a_n) \in \Theta$, let μ_α be the sum $\mu_{a_1} + \cdots + \mu_{a_n}$. With this notation, the slope of $\mathrm{sq}_\alpha(E)$ is just μ_α. Denote by Σ_n the set of real numbers of the form μ_α where α takes through vectors in Θ. From the inequalities $\mu_0 < \mu_1 < \cdots < \mu_{d-1}$ we obtain that $\alpha \leq \beta$ implies $\mu_\alpha \leq \mu_\beta$, and $\alpha \lneqq \beta$ implies $\mu_\alpha < \mu_\beta$.

Proposition 3.2. — *The set Σ_n identifies with that of successive slopes of $E^{\otimes n}$. Furthermore, suppose that the elements in Σ_n are ordered as*

$$v_0 < v_1 < \cdots < v_{m-1},$$

then $E_{A_0} \supsetneq E_{A_1} \supsetneq \cdots \supsetneq E_{A_m} = 0$ is the Harder-Narasimhan filtration of $E^{\otimes n}$, where $A_j = \{\alpha \mid \mu_\alpha \geq v_j\}$ for $j \in \{0, \cdots, m-1\}$, and $A_m = \emptyset$.

Proof. — By definition of Harder-Narasimhan filtration, it suffices to prove that, for each $j \in \{0, \cdots, m-1\}$, the sub-quotient $E_{A_j}/E_{A_{j+1}}$ is semistable of slope v_j. We have $A_j \setminus A_{j+1} = \{\alpha \,|\, \mu_\alpha = v_j\}$. Moreover, if $\mu_\alpha = v_j$, then α is a maximal element of A_j. Therefore, Proposition 3.1 implies that

$$E_{A_j}/E_{A_{j+1}} \cong \bigoplus_{\alpha \in A_j \setminus A_{j+1}} \mathrm{sq}_\alpha(E)$$

is semi-stable of slope v_j. □

Remark 3.3. — In [**3**], the author has introduced another interpretation of Harder-Narasimhan filtration: one defines, for any $t \in \mathbb{R}$,

$$\mathcal{F}_t^{\mathrm{HN}} := \sum_{\substack{0 \neq F \subset E \\ \mu_{\min}(F) \geq t}} F. \tag{17}$$

This is a decreasing $\mathbb{R}$-filtration of E. The subbundles of E appearing in the $\mathbb{R}$-filtration (17) are just E_i, $i \in \{0, \cdots, d\}$. Furthermore, the measure ν_E defined in (4) coincides with the first order derivative (in the sense of distribution) of the function $t \mapsto -\operatorname{rk} \mathcal{F}_t^{\mathrm{HN}} E$. Proposition 3.2 shows that the $\mathbb{R}$-filtration of $E^{\otimes n}$ is just the nth tensor power of the $\mathbb{R}$-filtration of E. More precisely, one has

$$\mathcal{F}_t^{\mathrm{HN}} E^{\otimes n} = \sum_{t_1 + \cdots + t_n = t} \mathcal{F}_{t_1}^{\mathrm{HN}} E \otimes \cdots \otimes \mathcal{F}_{t_n}^{\mathrm{HN}} E, \tag{18}$$

and therefore $\nu_{E^{\otimes n}} = \nu_E^{*n}$, where $*$ denotes the convolution of Borel probability measures on $\mathbb{R}$.

3.2. The case of $S^n E$. — The symmetric group $\mathfrak{S}_n$ acts on $E^{\otimes n}$ by permuting factors. The nth symmetric power of E is defined as the quotient of $E^{\otimes n}$ by the action of $\mathfrak{S}_n$. Let us keep the notation of § 3.1. The group $\mathfrak{S}_n$ acts naturally on $\Theta = \{0, \cdots, d-1\}^n$. Denote by $\widetilde{\Theta}$ the quotient space $\Theta/\mathfrak{S}_n$. Each class $[\alpha]$ in $\widetilde{\Theta}$ corresponds bijectively to a partition of n into sum of d non-negative integers $(a_i)_{i=0}^{d-1}$. For such a class $[\alpha]$, we use the symbol $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ to denote the tensor product $\bigotimes_{i=0}^{d-1} S^{a_i} \mathrm{sq}_i(E)$. One observes that the group $\mathfrak{S}_n$ acts naturally on $\bigoplus_{\beta \in [\alpha]} \mathrm{sq}_\beta(E)$ and its quotient space by the action of $\mathfrak{S}_n$ is just $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$. Moreover, $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ is also semi-stable of slope μ_α.

For any $A \subset \Theta$ which is invariant by the action of $\mathfrak{S}_n$, denote by $S_A E$ the image of E_A in $S^n E$. It is the quotient of E_A by the action of $\mathfrak{S}_n$. Let $A_j \subset \Theta$, $j \in \{0, \cdots, m\}$ be as in Proposition 3.2. By the uniqueness of Harder-Narasimhan filtration, each A_j is invariant by the action of $\mathfrak{S}_n$.

Proposition 3.4. — *The set of successive slopes of $S^n E$ is also Σ_n. Furthermore, $S_{A_0} E \supsetneq S_{A_1} E \supsetneq \cdots \supsetneq S_{A_m} E = 0$ is the Harder-Narasimhan filtration of $S^n E$.*

Proof. — As pointed out above, each vector bundle E_{A_j} is invariant by the action of $\mathfrak{S}_n$, and $S_{A_j} E$ is the quotient of E_{A_j} by the action. Therefore, we obtain that

$$S_{A_j} E / S_{A_{j+1}} E \cong \bigoplus_{[\alpha] \in (A_j \setminus A_{j+1})/\mathfrak{S}_n} \widetilde{\mathrm{sq}}_{[\alpha]}(E).$$

Since $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ is semi-stable of slope μ_α, $S_{A_j} E / S_{A_{j+1}} E$ is semi-stable of slope v_j. The proposition is then proved. □

In the following, we compute the measure $\nu_{S^n E}$. Take an arbitrary class $[\alpha]$ in $\widetilde{\Theta}$ which corresponds to the partition $\underline{a} = (a_i)_{i=0}^{d-1}$ of n. The value μ_α (which does not depend on the representing element α) equals $\sum_{i=0}^{d-1} a_i \mu_i$. For each $i \in \{0, \cdots, d-1\}$, let r_i be the rank of $\mathrm{sq}_i(E)$. The rank of $\widetilde{\mathrm{sq}}_{[\alpha]}(E)$ is just

$$r_{\underline{a}} := \prod_{i=0}^{d-1} \binom{a_i + r_i - 1}{r_i - 1}.$$

We then obtain that

$$R_j := \mathrm{rk}(S_{A_j} E / S_{A_{j+1}} E) = \sum_{\substack{\underline{a} = (a_i)_{i=0}^{d-1} \in \mathbb{N}^d \\ a_0 + \cdots + a_{d-1} = n \\ a_0 \mu_0 + \cdots + a_{d-1} \mu_{d-1} = v_j}} r_{\underline{a}}.$$

By definition, one has

$$\nu_{S^n E} = \frac{1}{\mathrm{rk}\, S^n E} \sum_{j=0}^{m-1} R_j \delta_{v_j}, \tag{19}$$

where δ_{v_j} is the Dirac measure concentrated at v_j.

The formula (19) gives an explicit description of the probability measure $\nu_{S^n E}$. However, it seems that in the case where the r_i ($i \in \{0, \cdots, d-1\}$) are different, the values of R_j, $j \in \{0, \cdots, m-1\}$ are rather tedious to calculate. In the following we propose another explicit form of $\nu_{S^n E}$ which is adapted to the proof of Theorem 1.2.

Proposition 3.5. — *Let $r = r_0 + \cdots + r_{d-1}$ and $\underline{s} = (s_1, \cdots, s_r)$ be a vector in $\mathbb{R}^n$ such that the value μ_i appears exactly r_i times in the coordinates on $\underline{s}$. Then*

$$\nu_{S^n E} = \frac{1}{\mathrm{rk}\, S^n E} \sum_{\substack{\underline{b} = (b_l)_{l=1}^{r} \in \mathbb{N}^r \\ b_1 + \cdots + b_r = n}} \delta_{b_1 s_1 + \cdots + b_r s_r}. \tag{20}$$

Proof. — The only thing to verify is that, for any $v \in \mathbb{R}$,

$$\sum_{\substack{\underline{a}=(a_i)_{i=0}^{d-1}\in\mathbb{N}^d\\ a_0+\cdots+a_{d-1}=n\\ a_0\mu_0+\cdots+a_{d-1}\mu_{d-1}=v}} r_{\underline{a}} = \sum_{\substack{\underline{b}=(b_l)_{l=1}^{r}\in\mathbb{N}^r\\ b_1+\cdots+b_r=n\\ b_1s_1+\cdots+b_rs_r=v}} 1.$$

This relies on the fact that $\binom{a_i+r_i-1}{r_i-1}$ equals the number of partitions of a_i into sum of r_i positive integers. □

3.3. Proofs of Theorem 1.2 and Corollary 1.3. — By definition, $\nu^{\pi}_{\mathcal{O}_E(1)}$ coincides with the vague limit of measures $T_{\frac{1}{n}}\nu_{S^nE}$ when $n\to\infty$. By Proposition 3.5,

$$\begin{aligned} T_{\frac{1}{n}}\nu_{S^nE} &= \frac{1}{\operatorname{rk} S^nE}\sum_{\substack{\underline{b}=(b_l)_{l=1}^{r}\in\frac{1}{n}\mathbb{N}^r\\ b_1+\cdots+b_r=1}}\delta_{b_1d_1+\cdots+b_rd_r}\\ &= \varphi_{\underline{s}*}\Big[\frac{1}{\operatorname{rk} S^nE}\sum_{\underline{b}\in\frac{1}{n}\mathbb{N}^r\cap\Delta}\delta_{\underline{b}}\Big], \end{aligned}$$

where in the last bracket is the measure of the nth Riemann sum on Δ. Therefore, $(T_{\frac{1}{n}}\nu_{S^nE})_{n\geq 1}$ converges vaguely to $\varphi_{\underline{s}*}\eta$. Theorem 1.2 is thus proved.

Finally, by remark 2.6, one has

$$\begin{aligned} \operatorname{vol}(\mathcal{O}_E(m)\otimes\pi^*M) &= r\operatorname{vol}(\mathcal{O}_E(m)_K)\int(x+c)_+\nu_{\mathcal{O}_E(m)^\pi}(\mathrm{d}x)\\ &= rm^{r-1}\int(mx+c)_+\nu^{\pi}_{\mathcal{O}_E(1)}(\mathrm{d}x)\\ &= rm^r\int(x+c/m)_+\varphi_{\underline{s}*}\eta(\mathrm{d}x), \end{aligned}$$

which proves Corollary 1.3.

4. Polynomial representation of the volume function

In this section, we show that Corollary 1.3 permits to get the subdivision of the Néron-Severi group $N^1(\mathbb{P}(E))$ into sectors such that the volume function is polynomial on each sector. We shall keep the notation introduced in Theorem 1.2 and Corollary 1.3. Note that the value in (6) depends only on m and c, we denote it by $\Phi(m,c)$.

Firstly, let us discuss some degenerate cases. Assume that E is semistable, i.e., $d=1$. Then the limit measure $\nu^{\pi}_{\mathcal{O}(E)}$ is the Dirac measure concentrated at $\mu(E)$. Therefore, Corollary 1.3 implies

$$\Phi(m,c) = rm^r(\mu(E)+c/m)_+ = \begin{cases}\deg(E)m^r + crm^{r-1}, & \text{if } c\geq -\mu(E)m,\\ 0, & \text{else.}\end{cases}$$

If $r = d = 2$, then the measure $\varphi_{\underline{s}*}\eta$ is the uniform distribution on the interval $[\mu_0, \mu_1]$. In this case

$$\Phi(m,c) = \frac{2m^2}{\mu_1 - \mu_0}\int_{\mu_0}^{\mu_1}(x+c/m)_+\,\mathrm{d}x = \begin{cases} m^2(\mu_1+\mu_0)+2cm, & c > -m\mu_0, \\ (m\mu_1+c)^2/(\mu_1-\mu_0), & -m\mu_1 \le c \le m\mu_0, \\ 0, & c < -m\mu_1. \end{cases}.$$

In the following, we always suppose $r \ge 3$ and $d \ge 2$. Without loss of generality, we assume that the coordinates of $\underline{s}$ are ordered such that $s_1 \le \cdots \le s_r$. For any $i \in \{0, \cdots, d-1\}$, let $j_i = \mathrm{rk}(E/E_{i+1})$. Let $j_{-1} = 0$. Note that $s_l = \mu_i$ if and only if $l \in]j_{i-1}, j_i]$. For any real number $t \in [d_1, d_r]$, let Δ_t be the intersection of Δ and the hyperplane $\{(x_1, \cdots, x_r) \mid s_1x_1 + \cdots + s_rx_r = t\}$. Denote by v_t the density of $\varphi_{\underline{s}*}\eta$ with respect to the Lebesgue measure. Since for any $i \in \{1, \cdots, d-1\}$, Δ_t ($t \in [\mu_{i-1}, \mu_i]$) are similar, the function v_t is affine on $[\mu_{i-1}, \mu_i]$, which means that, there exists real numbers a_i, b_i ($i \in \{1, \cdots, d-1\}$), such that $v_t = a_it + b_i$ on $[\mu_{i-1}, \mu_i]$. Thus we obtain

$$\Phi(m,c) = rm^r\sum_{i=1}^{d-1}\int_{\mu_{i-1}}^{\mu_i}(t+c/m)_+(a_it+b_i)\,\mathrm{d}t.$$

Therefore, $\Phi(m,c) = 0$ if $c \le -\mu_{d-1}m$. If $c \in [-\mu_jm, -\mu_{j-1}m]$ ($j \in \{1, \cdots, d-1\}$), one has

$$\begin{aligned}\Phi(m,c) = r\sum_{i=j+1}^{d-1}\frac{1}{3}(\mu_i^3-\mu_{i-1}^3)a_im^r + \frac{1}{2}(\mu_i^2-\mu_{i-1}^2)(b_im^r+a_icm^{r-1}) + b_i(\mu_i-\mu_{i-1})cm^{r-1} \\ + r\Big(\frac{1}{3}a_j(\mu_j^3m^r - c^3m^{r-3}) + \frac{1}{2}(b_jm+a_jc)(\mu_j^2m^{r-1}-c^2m^{r-3}) + cb_j(\mu_jm^{r-1}-cm^{r-2})\Big),\end{aligned}$$

which is a polynomial in m and c. Note that the degree of c is at most 3. Finally, when $c > -\mu_0m$, then

$$\Phi(m,c) = r\sum_{i=1}^{d-1}\frac{1}{3}(\mu_i^3-\mu_{i-1}^3)a_im^r + \frac{1}{2}(\mu_i^2-\mu_{i-1}^2)(b_im^r+a_icm^{r-1}) + b_i(\mu_i-\mu_{i-1})cm^{r-1}$$

has degree 1 in c.

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