



HAL
open science

Linear family of Lie brackets on the space of matrices $Mat(n \times m,)$ and Ado's Theorem

Bechir Dali

► **To cite this version:**

Bechir Dali. Linear family of Lie brackets on the space of matrices $Mat(n \times m,)$ and Ado's Theorem. 2008. hal-00295135

HAL Id: hal-00295135

<https://hal.science/hal-00295135>

Preprint submitted on 11 Jul 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LINEAR FAMILY OF LIE BRACKETS ON THE SPACE OF MATRICES $Mat(n \times m, \mathbb{K})$ AND ADO'S THEOREM

BÉCHIR DALI

ABSTRACT. In this paper we classify a linear family of Lie brackets on the space of rectangular matrices $Mat(n \times m, \mathbb{K})$ and we give an analogue of the Ado's Theorem. We give also a similar classification on the algebra of the square matrices $Mat(n, \mathbb{K})$ and as a consequence, we prove that we can't built a faithful representation of the $(2n + 1)$ -dimensional Heisenberg Lie algebra \mathfrak{H}_n in a vector space V with $\dim V \leq n + 1$. Finally, we prove that in the case of the algebra of square matrices $Mat(n, \mathbb{K})$, the corresponding Lie algebras structures are a contraction of the canonical Lie algebra $\mathfrak{gl}(n, \mathbb{K})$.

Mathematics Subject Classification (2000): 17B05, 11C20.

Key words: Lie algebras, Matrices, Extension, Contraction.

0. INTRODUCTION

We begin by setting some notations which will be used throughout the paper. Let \mathbb{K} be a field with characteristic $p = 0$, $Mat(n \times m, \mathbb{K})$ be the linear space of $n \times m$ rectangular matrices with coefficients in \mathbb{K} and $Mat(n, \mathbb{K})$ is the associative algebra of square matrices with coefficients in \mathbb{K} .

When dealing with the problem of representation of finite dimensional Lie algebras two presentations leap into mind: a presentation by matrices and a presentation by an array of structure constants. In the first presentation the Lie algebra is given by a finite set of matrices $\{A_1, \dots, A_n\}$ that form a basis of the Lie algebra \mathfrak{g} . If A and B are two elements of the space spanned by the A_i , then their Lie product is defined as $[A, B] = A.B - B.A$ (where the $.$ denotes the ordinary matrix multiplication). The second approach is more abstract. The Lie algebra is a (abstract) vector space over a field \mathbb{K} with basis $\{x_1, \dots, x_n\}$ and the Lie multiplication is determined by

$$[x_i, x_j] = \sum_{k=1}^n c_{i,j}^k x_k.$$

Here $(c_{i,j}^k) \in \mathbb{K}^3$ is an array of n^3 structure constants that determines the Lie multiplication completely. By linear algebra it is seen that from a presentation of a Lie algebra by matrices, we easily can obtain a presentation of it by structure constants. Ado's theorem states that the opposite direction is also possible.

Theorem 0.1. (Ado) Every finite-dimensional Lie algebra \mathfrak{g} over a field of characteristic zero can be viewed as a Lie algebra of square matrices under the commutator bracket. More precisely \mathfrak{g} has a linear representation ρ over \mathbb{K} , on a finite dimensional vector space V , that is a faithful representation, making \mathfrak{g} isomorphic to a subalgebra of the endomorphisms of V .

If a such representation of \mathfrak{g} is built on a vector space V , we can request if we can built such representation in a lower dimensional vector space V .

In the other hand, let \mathcal{A} be a finite-dimensional associative algebra over a field \mathbb{K} of a characteristic $p = 0$, then \mathcal{A} is canonically equipped with Lie algebra structure given by the following bracket

$$[u, v] = u.v - v.u \quad u, v \in \mathcal{A}.$$

For instance, if $\mathcal{A} = \text{End}(V)$ the set of linear transformations on V where V is a finite dimensional vector space over \mathbb{K} . In equivalent way, if we consider $\text{Mat}(n, \mathbb{K})$ the set of square matrices with coefficients in \mathbb{K} , then we get a canonical Lie algebra structure with the above commutator bracket, this algebra is denoted by $\mathfrak{gl}(n, \mathbb{K})$.

However the structure of associative algebra over \mathcal{A} is not unique [4], indeed if we fix $w \in \mathcal{A}$ then we can define the product $(u \circ v)_w = u.w.v$ and with respect \mathcal{A} is again an associative algebra. This induces a new Lie algebra structure, defined by the bracket

$$[u, v]_w = u.w.v - v.w.u \quad u, v \in \mathcal{A} \quad (1).$$

Thus we obtain a family of Lie brackets, labelled by the element w . It is readily seen that we actually have a linear space of Lie brackets, since the sum of two such brackets is also a Lie bracket of the same type, and a natural question is to classify these structures. For instance if $w = 0$ then the Lie algebra $(\mathcal{A}, [,]_0)$ is just the abelian Lie algebra \mathbb{K}^{n^2} . The above construction can be applied to the algebra of square matrices $\text{Mat}(n, \mathbb{K})$. It can be applied also even if u, v, w are not $n \times n$ matrices, since (1) makes sense when $u, v \in \text{Mat}(n \times m, \mathbb{K})$ -the linear space of $n \times m$ matrices and $w \in \text{Mat}(m \times n, \mathbb{K})$ - the linear space of $m \times n$ matrices.

In this paper we will deal with the algebra of the endomorphisms of a finite-dimensional vector space V or in equivalent way, the algebra of square matrices $\text{Mat}(n, \mathbb{K})$ and we will deal also with $\text{Mat}(n \times m, \mathbb{K})$ -the linear space of $n \times m$ matrices.

Our first goal in this paper is to give a complete classification of these Lie algebra structures, on $\text{Mat}(n, \mathbb{K})$ (respectively on $\text{Mat}(n \times m, \mathbb{K})$), labelled by the Lie brackets $[,]_J$ with $J \in \text{Mat}(n, \mathbb{K})$ (respectively in $\text{Mat}(m \times n, \mathbb{K})$).

The paper is organized as follows. First we begin by recalling some classical results of linear algebra and matrix properties. Secondly, we classify the Lie algebra structures on $\text{Mat}(n, \mathbb{K})$ and $\text{Mat}(n \times m, \mathbb{K})$. As a consequence of these classifications, we prove an analogue of the Ado's Theorem, namely, we prove that any finite dimensional Lie algebra \mathfrak{g} can be viewed as a Lie sub-algebra $(\text{Hom}(\mathbb{K}^m, \mathbb{K}^n), [,]_j)$ for some

positive integers n, m and some matrix $J \in Mat(m \times n, \mathbb{K})$. we prove also that there is no faithful linear representation of the classical $(2n + 1)$ -Heisenberg Lie algebra \mathfrak{H}_{2n+1} in $\mathfrak{gl}(V)$ with $\dim V \leq n + 1$. Finally we prove that in the case of the algebra of square matrices, the Lie algebra structure labelled by the family of brackets $([\cdot, \cdot]_J)$ constructed on $Mat(n, \mathbb{K})$ are a contraction of the canonical structure $\mathfrak{gl}(n, \mathbb{K})$.

1. PRELIMINARIES

Let n, m be integers in $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and let $Mat(n \times m, \mathbb{K})$ be the linear vector space of $n \times m$ rectangular matrices. We denote its canonical basis $(E_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ with

$$E_{i,j} = (\delta_{p,i} \delta_{q,j})_{1 \leq p \leq n, 1 \leq q \leq m},$$

where $\delta_{p,i}$ is the Kronecker symbol.

Definition 1.1. *Let J and J' be in $Mat(m \times n, \mathbb{K})$. We say that J is equivalent to J' if and only if there exist two invertible matrices $P \in GL(n, \mathbb{K})$ and $Q \in GL(m, \mathbb{K})$ such that*

$$J = QJ'P.$$

Remark 1.2. (1) *The matrices Q and P are not unique.*

(2) *The above property of equivalent matrices in $Mat(m \times n, \mathbb{K})$ is an equivalence relation.*

(3) *J and J' are equivalent if and only if they represent the same endomorphism in different bases.*

The characterization of two equivalent matrices is given by the following

Lemma 1.3. *Let $J, J' \in Mat(m \times n, \mathbb{K})$. J and J' are equivalent if and only if they have the same rank.*

From the above Lemma, we can conclude the following

Lemma 1.4. *Let J a matrix in $Mat(m \times n, \mathbb{K})$ of rank $r \leq \min(m, n)$, then there exist two invertible matrices $Q \in GL(m, \mathbb{K})$ and $P \in GL(n, \mathbb{K})$ such that*

$$J = Q \begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix} P,$$

where $0_{p,q}$ is the zero matrix of $Mat(p \times q, \mathbb{K})$ and I_r is the identity matrix of $Mat(r, \mathbb{K})$.

We will denote the matrix $\begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$ by $J_{m,n,r}$ and when $n = m$ we simply denote it by $J_{n,r}$. Note that $J_{n,n} = I_n$ the identity matrix.

Consider now the linear space $Mat(n \times m, \mathbb{K})$ and for $J \in Mat(m \times n, \mathbb{K})$ put

$$[A, B]_J = AJB - BJA, \quad A, B \in Mat(n \times m, \mathbb{K}),$$

then we have

Lemma 1.5. (i) $[A, B]_{J+\alpha J'} = [A, B]_J + \alpha[A, B]_{J'}, \forall \alpha \in \mathbb{K}, J, J' \in \text{Mat}(n \times m, \mathbb{K})$.
(ii) $(\text{Mat}(n \times m, \mathbb{K}), [\cdot, \cdot]_J)$ is a Lie algebra.

The proof is a simple verification of the anti-symmetry of the bracket $[\cdot, \cdot]_J$ and the Jacobi identity.

Now we give the following

Proposition 1.6. *If J and J' are equivalent matrices in $\text{Mat}(m \times n, \mathbb{K})$, then the corresponding Lie algebra structures on $\text{Mat}(n \times m, \mathbb{K})$ are isomorphic.*

Proof. Since J and J' are equivalent in $\text{Mat}(m \times n, \mathbb{K})$, then there exist two invertible matrices $Q \in GL(m, \mathbb{K})$ and $P \in GL(n, \mathbb{K})$ such that $J = QJ'P$. Consider now the map

$$\begin{aligned} \varphi : \text{Mat}(n \times m, \mathbb{K}) &\longrightarrow \text{Mat}(n \times m, \mathbb{K}) \\ A &\longmapsto PAQ \end{aligned}$$

we verify, since Q and P are invertible, that φ is an isomorphism of vector space and

$$\begin{aligned} \varphi([A, B]_J) &= P(AJB - BJA)Q \\ &= P(AQJ'PB - BQJ'PA)Q \\ &= [PAQ, PBQ]_{J'} \\ &= [\varphi(A), \varphi(B)]_{J'} \end{aligned}$$

and thus φ is a Lie algebra isomorphism. \square

Let $J \in \text{Mat}(m \times n, \mathbb{K})$ with $rk(J) = r$, from Proposition 1.6, J is equivalent to the following matrix $\begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$, we shall denote the last matrix by $J_{m, n, r}$. In the case when $m = n$, we simply denote it by $J_{n, r}$. Recall that $J_{n, n} = I_n$, the identity matrix of $\text{Mat}(n, \mathbb{K})$.

For easy of notations, we shall denote the Lie algebra $(\text{Mat}(n \times m, \mathbb{K}), [\cdot, \cdot]_{J_{m, n, r}})$ by $\mathfrak{gl}(n, m, r, \mathbb{K})$ and in the case when $m = n$, we simply denotes the lie algebra $(\text{Mat}(n, \mathbb{K}), [\cdot, \cdot]_{J_{n, r}})$ by $\mathfrak{gl}(n, r, \mathbb{K})$. In order to give a complete classification of these Lie algebras structures labelled by the linear family of Lie brackets $([\cdot, \cdot]_J)$, we first give the following

Lemma 1.7. *Let $M \in \text{Mat}(n \times m, \mathbb{K})$ and put $M = \begin{pmatrix} M_1 & M_3 \\ M_2 & M_4 \end{pmatrix}$ with $M_1 \in \text{Mat}(r, \mathbb{K})$, $M_2 \in \text{Mat}((n-r) \times r, \mathbb{K})$, $M_3 \in \text{Mat}(r \times (m-r), \mathbb{K})$, $M_4 \in \text{Mat}((n-r) \times (m-r), \mathbb{K})$. Then the Lie bracket $[A, B]_{J_{m, n, r}}$ is given by*

$$[A, B]_{J_{n, r}} = \begin{pmatrix} [A_1, B_1] & A_1 B_3 - B_1 A_3 \\ A_2 B_1 - B_2 A_1 & A_2 B_3 - B_2 A_3 \end{pmatrix}.$$

Proof. The proof is based on the following result. Put $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}$, with $A_1 \in \text{Mat}(r, \mathbb{K})$, $A_2 \in \text{Mat}((n-r) \times r, \mathbb{K})$, $A_3 \in \text{Mat}((m-r) \times r, \mathbb{K})$, $A_4 \in \text{Mat}((n-r) \times$

$(m - r), \mathbb{K}$). Then we have

$$AJ_{m,n,r} = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} \text{ and } J_{m,n,r}A = \begin{pmatrix} A_1 & A_3 \\ 0 & 0 \end{pmatrix}.$$

□

In order to give a classification of the Lie algebras structures on $Mat(n \times m, \mathbb{K})$ labelled by the family of Lie brackets $([\cdot, \cdot]_J)_J$, we shall distinguish the two possible cases $n = m$ and $n \neq m$.

2. THE CASE OF $Mat(n, \mathbb{K})$

In this section we will deal with the linear vector space of square matrices $Mat(n, \mathbb{K})$. It is well known that it has a structure of associative algebra and thus a Lie algebra with the commutator

$$[A, B] = AB - BA, \quad A, B \in Mat(n, \mathbb{K}).$$

The commutator of two matrices $E_{i,j}$ and $E_{k,l}$ of the canonical basis ($1 \leq i, j, k, l \leq n$) is given by

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{l,j}E_{k,i}.$$

where δ denotes the Kronecker symbol.

Fix $J \in Mat(n, \mathbb{K})$ and consider the bracket

$$[A, B]_J = AJB - BJA, \quad A, B \in Mat(n, \mathbb{K}).$$

We first give the following

Lemma 2.1. *Let \mathfrak{h} be a Lie sub-algebra of $\mathfrak{gl}(n, \mathbb{K})$, then \mathfrak{h} is a Lie sub-algebra of $\mathfrak{gl}(p, n, \mathbb{K})$ for any integer $p \geq n$.*

Proof. For any subset $\mathfrak{h} \subset Mat(n, \mathbb{K})$, and $p \geq n$, we have

$$[\mathfrak{h}, \mathfrak{h}]_{J_{p,n}} = [\mathfrak{h}, \mathfrak{h}].$$

This completes the proof. □

In order to classify the Lie algebras structures on $Mat(n, \mathbb{K})$ labelled by the family of brackets $[\cdot, \cdot]_J$, we consider a vector space V over \mathbb{K} with $\dim V = n$ and put $V = V_1 \oplus V_2$. Next, define \mathfrak{n} to be the two step nilpotent Lie algebra constructed on the vector space $Hom(V_1, V_2) \oplus Hom(V_2, V_1) \oplus Hom(V_2, V_2)$ equipped with the bracket

$$[(A, B, C), (A', B', C')] = (0, 0, AB' - A'B).$$

Finally, we define the Lie algebra $S(V_1, V_2)$ to be the semi direct product of $End(V_1)$ with \mathfrak{n} . Then we have the following

Proposition 2.2. *(i) The Lie algebra $S(V_1, V_2)$ is isomorphic to $\mathfrak{gl}(n, r, \mathbb{K})$.
 (ii) $End(V_1)$ is a Lie subalgebra of $\mathfrak{gl}(n, s, \mathbb{K})$ for any $s \geq r$.
 (iii) The reductive part of $\mathfrak{gl}(n, r, \mathbb{K})$ is $End(V_1) \oplus End(V_2)$.*

Proof. (i) First, we can simply verify that \mathfrak{n} is a two-step nilpotent Lie algebra. Now for each $X \in \text{End}(V_1)$, consider

$$\begin{aligned} \pi_X : \mathfrak{n} &\longrightarrow \mathfrak{n} \\ (A, B, C) &\longmapsto (-AX, XB, 0). \end{aligned}$$

Then we easily verify that $X \longmapsto \pi_X$ is a Lie algebra homomorphism from $\mathfrak{gl}(n, \mathbb{K})$ into $\partial(\mathfrak{n})$ the Lie algebra of the derivations of \mathfrak{n} .

Now consider the mapping

$$\begin{aligned} \phi : S(V_1, V_2) &\longrightarrow \mathfrak{gl}(n, r, \mathbb{K}) \\ (X, A, B, C) &\longmapsto \begin{pmatrix} X & B \\ A & C \end{pmatrix}. \end{aligned}$$

ϕ is a vector space isomorphism.

Let $X, X' \in \text{End}(V_1)$ and $(A, B, C), (A', B', C') \in \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1) \oplus \text{Hom}(V_2, V_2)$. Put $n = (A, B, C), n' = (A', B', C')$ then

$$\begin{aligned} \phi([(X, n), (X', n')]) &= \phi([([X, X'], [n, n'] + \pi_{X'}(n) - \pi_X(n'))]) \\ &= [\phi((X, n)), \phi((X', n'))]_{J_{n,r}}. \end{aligned}$$

(ii) The statement is a simple consequence of Lemma 1.7.

(iii) It is easy to verify that $\text{End}(V_1)$ is reductive in $\mathfrak{gl}(n, r, \mathbb{K})$. Now, let \mathfrak{g} be a reductive subalgebra in $\mathfrak{gl}(n, r, \mathbb{K})$ which is maximal (for the inclusion). From [3] we can write $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where \mathfrak{z} is the center of \mathfrak{g} . Since $\text{End}(V_1) \subset \mathfrak{g}$ then we must have $[Z, X] = 0$, for any $Z \in \mathfrak{z}$ and $X \in \text{End}(V_1)$.

Put $Z = \begin{pmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{pmatrix}, X = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ then

$$[Z, X]_{J_{n,r}} = \begin{pmatrix} [Z_1, X] & -XZ_3 \\ Z_2X & 0 \end{pmatrix} = 0,$$

and thus $Z = \begin{pmatrix} \lambda I_r & 0 \\ 0 & Z_4 \end{pmatrix}$ with $\lambda \in \mathbb{K}$ and $Z_4 \in \text{End}(V_2)$.

Finally, let $Z = \begin{pmatrix} \lambda I_r & 0 \\ 0 & Z_4 \end{pmatrix} \in \mathfrak{z}$ and $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \in \mathfrak{g}$, then we have

$$[Z, A]_{J_{n,r}} = \begin{pmatrix} 0 & \lambda A_3 \\ -\lambda A_2 & 0 \end{pmatrix} = 0.$$

Thus $\mathfrak{g} = \begin{pmatrix} \text{End}(V_1) & 0_{r, n-r} \\ 0_{n-r, r} & \text{End}(V_2) \end{pmatrix} \simeq \text{End}(V_1) \oplus \text{End}(V_2)$, which completes the proof. \square

Now we shall determine the center of the Lie algebra $(\text{Mat}(n, \mathbb{K}), [,]_J)$. Let us denote it by \mathcal{Z}_J , then we have the following

Proposition 2.3. *Put $r = \text{rank}(J)$, if $r < n$ then $\dim \mathcal{Z}_J = (n - r)^2$ while if $r = n$, then $\dim \mathcal{Z}_J = 1$.*

Proof. Case 1: $\text{rank}(J) = n$

In this case, the mapping $\varphi : A \mapsto JA$ is a Lie algebra isomorphism from $(Mat(n, \mathbb{K}), [,]_J)$ onto $\mathfrak{gl}(n, \mathbb{K})$, and since the center \mathcal{Z} of $\mathfrak{gl}(n, \mathbb{K})$ is $\mathcal{Z} = \mathbb{K}I_n$ then $\mathcal{Z}_J = \varphi^{-1}(\mathcal{Z}) = \mathbb{K}J^{-1}$.

Case 2: $\text{rank}(J) = r < n$

Let Q, P in $GL(n, \mathbb{K})$ such that

$$J = QJ_{n,r}P$$

and let φ be the isomorphism mapping defined in Proposition 1.6, then

$$\mathcal{Z}_J = \varphi^{-1}(\mathcal{Z}_{J_{n,r}}),$$

then we shall compute $\mathcal{Z}_{J_{n,r}}$ the center of $\mathfrak{gl}(n, r, \mathbb{K})$.

$$\mathcal{Z}_{J_{n,r}} = \{A \in Mat(n, \mathbb{K}), | [A, B]_{J_{n,r}} = 0 \forall B \in Mat(n, \mathbb{K})\}.$$

Put $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}$, with $A_1 \in Mat(r, \mathbb{K})$, $A_2 \in Mat((n - r) \times r, \mathbb{K})$, $A_3 \in Mat(r \times (n - r), \mathbb{K})$, $A_4 \in Mat(n - r, \mathbb{K})$, using Lemma 1.7, we get

$$[A, B]_{J_{n,r}} = \begin{pmatrix} [A_1, B_1] & A_1B_3 - B_1A_3 \\ A_2B_1 - B_2A_1 & A_2B_3 - B_2A_3 \end{pmatrix}.$$

Then $A \in \mathcal{Z}_{J_{n,r}}$ if and only if the bracket $[A, B]_{J_{n,r}}$ is identically zero for any matrix B in $Mat(n, \mathbb{K})$ which implies in particular $[A_1, B_1] = 0, \forall B_1 \in Mat(r, \mathbb{K})$ and thus A_1 is in the center of $\mathfrak{gl}(r, \mathbb{K})$, that is

$$A_1 = \lambda I_r, \quad \lambda \in \mathbb{K}.$$

Now with the equations

$$A_1B_3 - B_1A_3 = A_2B_1 - B_2A_1 = A_2B_3 - B_2A_3 = 0, \quad \forall B \in Mat(n, \mathbb{K}),$$

we obtain

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}, \quad \text{with } A_4 \in Mat(n - r, \mathbb{K}).$$

Thus

$$\mathcal{Z}_{J_{n,r}} = \mathbb{K} \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} \simeq End(V_2), \quad \text{with } A_4 \in Mat(n - r, \mathbb{K}),$$

and

$$\mathcal{Z}_J = P^{-1} \mathcal{Z}_{J_{n,r}} Q^{-1}.$$

□

Now we are able to give the following

Theorem 2.4. *Let n be an integer with $n \geq 2$. The Lie algebras $(Mat(n, \mathbb{K}), [,]_J)$ and $(Mat(n, \mathbb{K}), [,]_{J'})$ are isomorphic if and only if the matrices J and J' are equivalent.*

Proof. Let $J, J' \in Mat(n, \mathbb{K})$. If J and J' are equivalent then from Proposition 1.6 the corresponding Lie algebras are isomorphic.

Now we shall prove the converse.

Let $J, J' \in Mat(n, \mathbb{K})$ with $rk(J) = r$ and $rk(J') = s$ with $r \neq s$.

Let V be a vector space over \mathbb{K} with $\dim V = n$ and put $V = V_1 \oplus V_2 = W_1 \oplus W_2$ with $\dim V_1 = r$ and $\dim W_1 = s$. From Proposition 1.6, we can assume that $J = J_{n,r}$ and $J' = J_{n,s}$ and from Proposition 2.2, the reductive part of $\mathfrak{gl}(n, r, \mathbb{K})$ is $End(V_1) \oplus End(V_2)$ while the reductive part of $\mathfrak{gl}(n, s, \mathbb{K})$ is $End(W_1) \oplus End(W_2)$. These reductive parts have the same dimension if and only if $r + s = n$ but in this case they don't have the same dimensionality of their center. Thus they are not isomorphic and this completes the proof. \square

As a consequence of this classification, we have the following

Theorem 2.5. *Let \mathfrak{H}_n be the $(2n + 1)$ -dimensional Heisenberg algebra. Then there is no faithful finite dimensional linear representation of \mathfrak{H}_n on a vector space V with $\dim V \leq n + 1$.*

Proof. Suppose that there exists a faithful linear representation ρ of \mathfrak{H}_n , on a vector space V with $\dim V \leq n + 1$. Let ρ such mapping, then we have $\rho : \mathfrak{H}_n \mapsto \mathfrak{gl}(r, \mathbb{K})$ which an injective Lie algebra morphism.

Put $\mathcal{H}_n = Span\{X_i = E_{1,i+1}, Y_i = E_{i+1,n+1}, Z = E_{1,n+2}, \quad i = 1, \dots, n\}$. We simply check that \mathcal{H}_n is a subalgebra of $\mathfrak{gl}(n + 2, n + 1, \mathbb{K})$, and $[X_i, Y_j]_{J_{n+1,n}} = \delta_{i,j}Z$, it is the $(2n + 1)$ -dimensional Heisenberg algebra. Then we can take \mathcal{H}_n instead of \mathfrak{H}_n . If such mapping ρ exists, then we must have

$$\begin{aligned} \rho([X_i, Y_i]_{J_{n+2,n+1}}) &= [\rho(X_i), \rho(Y_i)] \\ \rho(Z) &= [\rho(X_i), \rho(Y_i)], \end{aligned}$$

but since ρ is an injective Lie algebra morphism, then $\rho(Z) = \lambda I_r$ with $\lambda \neq 0$ wick is impossible because I_r is traceless matrix. \square

2.1. Example. Let $\mathfrak{H} = span\{Z, Y, X\}$ with the only non vanishing bracket is $[X, Y] = Z$ then the classical representation of \mathfrak{H} is

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the last corollary, we can't built a faithful homomorphism \mathfrak{H} in $\mathfrak{gl}(2, \mathbb{R})$ but we can built such homomorphism in $\mathfrak{gl}(2, 1, \mathbb{R})$ with the following realization

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. THE CASE OF $Mat(n \times m, \mathbb{K})$

In this section we will deal with the linear space of strict rectangular matrices $Mat(n \times m, \mathbb{K})$ which is not an associative algebra. Fix $J \in Mat(m \times n, \mathbb{K})$ and put

$$[A, B]_J = AJB - BJA, \quad A, B \in Mat(n \times m, \mathbb{K}).$$

Recall that we have denoted the matrix $\begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$ by $J_{m, n, r}$ and we will denotes the Lie algebra $(Mat(n \times m, \mathbb{K}), [,]_{J_{m, n, r}})$ by $\mathfrak{gl}(n, m, r, \mathbb{K})$. In the case when $n = m$, $\mathfrak{gl}(n, n, r, \mathbb{K})$ is simply $\mathfrak{gl}(n, r, \mathbb{K})$

Lemma 3.1. *Let \mathfrak{h} be a Lie sub-algebra of $\mathfrak{gl}(n, \mathbb{K})$, then \mathfrak{h} is a Lie sub-algebra of $\mathfrak{gl}(p, q, n, \mathbb{K})$ for any integers $p, q \geq n$.*

Proof. For any subset $\mathfrak{h} \subset Mat(n, \mathbb{K})$ and $q, p \geq n$, we have

$$[\mathfrak{h}, \mathfrak{h}]_{J_{q, p, n}} = [\mathfrak{h}, \mathfrak{h}].$$

And thus the conclusion holds. □

Now let $J \in Mat(m \times n, \mathbb{K})$, and let's denotes by \mathcal{Z}_J the center of the Lie algebra $(Mat(n \times m, \mathbb{K}), [,]_J)$, then we have

Proposition 3.2. *Let $J \in Mat(m \times n, \mathbb{K})$ with $rk(J) = r$. Then the center of $(Mat(n \times m, \mathbb{K}), [,]_J)$ is $(n-r)(m-r)$ dimensional. In particular, if $\mathcal{Z}_{J_{m, n, r}}$ denotes the center of $(Mat(n \times m, \mathbb{K}), [,]_{J_{m, n, r}})$, then $\mathcal{Z}_{J_{m, n, r}}$ is spanned by the matrices of the form*

$$A = \begin{pmatrix} 0_r & 0_{r, n-r} \\ 0_{m-r, r} & A_4 \end{pmatrix}, \quad \text{with } A_4 \in Mat((m-r) \times (n-r), \mathbb{K}).$$

Proof. The proof of this proposition is similar to the proof of the Proposition 2.3. □

Now we can give the following

Theorem 3.3. *Let $J, J' \in Mat(m \times n, \mathbb{K})$ with $\min(m, n) \geq 2$. Then the Lie algebras $(Mat(n \times m, \mathbb{K}), [,]_J)$ and $(Mat(n \times m, \mathbb{K}), [,]_{J'})$ are isomorphic if and only if J and J' are equivalent in $Mat(m \times n, \mathbb{K})$.*

Proof. Let J and J' two equivalent matrices in $Mat(m \times n, \mathbb{K})$ then from Proposition 1.6, the corresponding Lie algebras structures on $Mat(n \times m, \mathbb{K})$ are isomorphic.

Conversely, let J, J' be in $Mat(m \times n, \mathbb{K})$ with $rk(J) \neq rk(J')$, then from Proposition 2.3, we have $\dim \mathcal{Z}_J \neq \dim \mathcal{Z}_{J'}$, since

$$\dim \mathcal{Z}_J = (n - rk(J))(m - rk(J)) \quad \text{and} \quad \dim \mathcal{Z}_{J'} = (n - rk(J'))(m - rk(J')).$$

□

Corollary 3.4. *Let V be a vector space over \mathbb{K} with $\dim V \geq 2$. Then V can be equipped with a non trivial Lie algebra structure.*

Proof. Let n be the dimension of V . We identify V with $Mat(n \times 1, \mathbb{K})$ and put $J = (1, 0, \dots, 0)$, then $(V, [\ , \]_J)$ is a non trivial Lie algebra. □

Now we can give an analogue of the Ado's Theorem

Theorem 3.5. *Every finite-dimensional Lie algebra \mathfrak{g} over a field \mathbb{K} of characteristic zero can be viewed as a Lie sub-algebra of $\mathfrak{gl}(n, m, r, \mathbb{K})$. More precisely \mathfrak{g} has a linear representation ρ over \mathbb{K} on $\mathfrak{gl}(n, m, r, \mathbb{K})$, that is a faithful representation, making \mathfrak{g} isomorphic to a subalgebra of $(Hom(\mathbb{K}^n, \mathbb{K}^m), [\ , \]_{J_{m,n,r}})$.*

Proof. By Ado's Theorem, \mathfrak{g} can be viewed as a subalgebra of $\mathfrak{gl}(p, \mathbb{K})$ and the last algebra is also a subalgebra of $\mathfrak{gl}(n, m, q, \mathbb{K})$ for any $q \geq p$ and $n, m \geq q$. □

3.1. Examples. (a) Let $V = \mathbb{R}^2$, we identify V with $Mat(2 \times 1, \mathbb{R})$, we choose a basis (e_1, e_2) in V with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we can check that $[e_2, e_1]_J = e_1$ with $J = (1, 0)$. Then the two-dimensional affine Lie algebra can be viewed as $(Hom(\mathbb{R}^2, \mathbb{R}), [\ , \]_{J_{1,2,1}})$.

These constructions can be generalized for $V = \mathbb{R}^n$ with $n \geq 2$.

(b) Let $V = \mathbb{R}^3 \simeq Mat(3 \times 1, \mathbb{R})$. Put $e_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then

we verify that $[e_1, e_2]_J = e_2, [e_1, e_3]_J = e_3, [e_2, e_3]_J = 0$ where $J = (1, 0, 0)$. It is the Lie algebra $\mathfrak{g}_{3,2}(1)$ (see [1] for notations).

(c) Let $V = \mathbb{R}^4$, then we can identify V with $Mat(4 \times 1, \mathbb{R})$ and in this case we take $J = (1, 0, 0, 0)$ or with $Mat(2, \mathbb{R})$ and in this case we can choose $J = I_2$ or $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

4. CONTRACTIONS AND EXTENSIONS OF LIE ALGEBRAS

Let $n, r \in \mathbb{N}$ with $n \geq r, n \geq 2$ and put $J = J_{n,r}$, then we have the following

Proposition 4.1. *The Lie algebra $(Mat(n, \mathbb{K}), [\ , \]_{J_{n,r}})$ is a contraction of $\mathfrak{gl}(n, \mathbb{K})$.*

Proof. Let $(E_{i,j})$ the canonical basis of $Mat(n, \mathbb{K})$, and define

$$\mathfrak{g} = Span\{E'_{i,j}, \quad i, j = 1, \dots, n\},$$

with

$$E'_{i,j} = \begin{cases} E_{i,j} & \text{if } i \leq r \text{ and } j \leq r, \\ \varepsilon E_{i,j} & \text{if } i > r \text{ and } j \leq r \text{ or } j > r \text{ and } i \leq r, \\ \varepsilon^2 E_{i,j} & \text{if } i > r \text{ and } j > r. \end{cases}$$

where $\varepsilon \in \mathbb{K}$. Then we have

$$[E'_{i,j}, E'_{k,l}] = \begin{cases} \delta_{j,k}E'_{i,l} - \delta_{l,i}E'_{k,j} & \text{if } i, j, k, l \leq r, \\ \delta_{j,k}E'_{i,l} & \text{if } i, j, k \leq r \text{ and } l > r, \\ -\delta_{l,i}E'_{k,j} & \text{if } i, j, l \leq r \text{ and } k > r, \\ 0 & \text{if } i, j \leq r \text{ and } k, l > r, \\ 0 & \text{if } i, k \leq r \text{ and } l, j > r, \\ \delta_{j,k}\varepsilon^2 E'_{i,l} & \text{if } i \leq r \text{ and } j, k, l > r, \\ \delta_{j,k}\varepsilon^2 E'_{i,l} - \delta_{l,i}\varepsilon^2 E'_{k,j} & \text{if } i, j, k, l > r. \end{cases}$$

And when $\varepsilon \rightarrow 0$, then we obtain

$$\lim_{\varepsilon \rightarrow 0} [E'_{i,j}, E'_{k,l}] = [E_{i,j}, E_{k,l}]_{J_{n,r}}.$$

□

If we repeat again this contraction, we get a new algebra which can be directly obtained from the first Lie algebra since $J_{n,r}J_{n,s} = J_{n,t}$ with $t = \min(r, s)$.

4.1. Example. Let $\mathfrak{sl}(2, \mathbb{R})$ be the semi-simple (real) Lie algebra spanned by $\{H, X, Y\}$ where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The (usual) brackets are:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Pick $J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then we obtain the following brackets:

$$[H, X]_J = X, \quad [H, Y]_J = -Y, \quad [X, Y]_J = 0.$$

The new Lie algebra is completely solvable and thus non isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ since $\mathfrak{sl}(2, \mathbb{R})$ is semi-simple.

Let us restrict ourself to the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$. Then we have the following

Lemma 4.2. ([4], [5]) *The map $(A, B) \mapsto [A, B]_J$ is a two-coboundary for the adjoint representation of $\mathfrak{gl}(n, \mathbb{K})$.*

Proof. We can easily check that

$$[A, B]_J = ad_A \alpha(B) - ad_B \alpha(A) - \alpha([A, B]) = (d\alpha)(A, B).$$

where

$$\alpha(X) = \frac{1}{2}(XJ + JX).$$

□

Let $t \in [0, 1]$ and $J = J_{n,r}$ (with $r < n$), then we can write

$$[A, B]_{(1-t)I+tJ} = [A, B] + t[A, B]_{J-I},$$

thus since $[A, B]_{J-I}$ is a coboundary, $t \mapsto [A, B]_{(1-t)I+tJ}$ is a deformation infinitesimally trivial at $t = 0$ from $\mathfrak{gl}(n, \mathbb{K})$ to $\mathfrak{gl}(n, r, \mathbb{K})$. For any $t < 1$, the Lie algebra $(Mat(n, \mathbb{K}), [,]_{(1-t)I+tJ})$ is isomorphic to $\mathfrak{gl}(n, \mathbb{K})$ but for $t = 1$ we get a completely different structure which is labelled by the bracket $[,]_J$. The mapping

$$\Psi_t : (Mat(n, \mathbb{K}), [,]_{(1-t)I+tJ}) \longrightarrow \mathfrak{gl}(n, \mathbb{K}), \quad \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \mapsto \begin{pmatrix} X_1 & (1-t)X_2 \\ X_3 & (1-t)X_4 \end{pmatrix}$$

is for any $t \in [0, 1[$ invertible and $[X, Y]_{(1-t)I+tJ} = \Psi_t^{-1}([\Psi_t(X), \Psi_t(Y)])$, and we have $\frac{\partial}{\partial t}([,]_{(1-t)I+tJ})|_{t=0}$ is the 2-coboundary: $(A, B) \mapsto [A, B]_{J-I}$.

Lemma 4.3. *The vector space $Mat(n, \mathbb{K})$ is a $(Mat(n, \mathbb{K}), [,]_J)$ -module.*

Proof. We can verify that the map

$$\begin{aligned} \varphi : (Mat(n, \mathbb{K}), [,]_J) &\longrightarrow \mathfrak{gl}(Mat(n, \mathbb{K})) \\ A &\longmapsto ad_A^J, \end{aligned}$$

where

$$ad_A^J(B) = [A, B]_J,$$

is a Lie algebra homomorphism since $[,]_J$ is a Lie bracket on $Mat(n, \mathbb{K})$. □

Let \mathfrak{g} be a Lie algebra over \mathbb{K} and (X_1, \dots, X_r) with the following commutations roles $[X_i, X_j] = \sum_k c_{i,j}^k X_k$. By the Ado's theorem, there is a faithful representation ρ of \mathfrak{g} in $\mathfrak{gl}(n, \mathbb{K})$ for some integer n . Let us identify \mathfrak{g} with its image under ρ . Then we have the following

Proposition 4.4. *Let $\mathfrak{h} = Mat(r \times (n - r), \mathbb{K})$, $\mathfrak{h}' = Mat((n - r) \times r, \mathbb{K})$ and $\mathfrak{h}'' = Mat(n - r, \mathbb{K})$. Then we have the following subalgebras inclusions*

$$\begin{pmatrix} \mathfrak{g} & \mathfrak{h} \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} \mathfrak{g} & \mathfrak{h} \\ 0 & \mathfrak{h}'' \end{pmatrix} \subset \mathfrak{gl}(n, r, \mathbb{K})$$

and

$$\begin{pmatrix} \mathfrak{g} & 0 \\ \mathfrak{h}' & 0 \end{pmatrix} \subset \begin{pmatrix} \mathfrak{g} & 0 \\ \mathfrak{h}' & \mathfrak{h}'' \end{pmatrix} \subset \mathfrak{gl}(n, r, \mathbb{K}).$$

REFERENCES

- [1] BERNAT P. ET AL., *Représentations des groupes de Lie résolubles*, Monographes de la société mathématique de France, Dunod Paris (1972).
- [2] DIXMIER. J., *Algèbres enveloppantes*, Gautier-Villars (1974) ISBN 2-04-008055-4
- [3] ISE AND TAKEUCHI, *Lie groups I, Lie groups II*, Translations of Mathematical Monographs, Volume **85** ISBN 0-8218-4544-6
- [4] YANOVSKI A. B., *Linear Bundles of Lie Brackets and their Applications*, J. Math Phys. **41** (2000) 7869-7882.
- [5] YANOVSKI A. B., *Compatible Poisson Tensors Related To Bundles of Linear Algebras*, Proceedings of the 7th international conference on geometry, integrability and quantization, Sts. Constantine and Elena, Bulgaria, June 2–10, 2005. Sofia: Bulgarian Academy of Sciences. 307-319 (2006)
- [6] HOCHSCHILD G. AND SERRE J. P., *Cohomologie of Lie algebras*, Annales of Math. t. 57, 1953, p. 591-603.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF BIZERTE, 7021 ZARZOUNA, BIZERTE, TUNISIA

E-mail address: bechir.dali@yahoo.fr