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# LINEAR FAMILY OF LIE BRACKETS ON THE SPACE OF MATRICES $Mat(n \times m, \mathbb{K})$ AND ADO'S THEOREM

BÉCHIR DALI

ABSTRACT. In this paper we classify a linear family of Lie brackets on the space of rectangular matrices  $Mat(n \times m, \mathbb{K})$  and we give an analogue of the Ado's Theorem. We give also a similar classification on the algebra of the square matrices  $Mat(n, \mathbb{K})$  and as a consequence, we prove that we can't built a faithful representation of the  $(2n + 1)$ -dimensional Heisenberg Lie algebra  $\mathfrak{H}_n$  in a vector space  $V$  with  $\dim V \leq n+1$ . Finally, we prove that in the case of the algebra of square matrices  $Mat(n, \mathbb{K})$ , the corresponding Lie algebras structures are a contraction of the canonical Lie algebra  $\mathfrak{gl}(n, \mathbb{K})$ .

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## 0. INTRODUCTION

We begin by setting some notations which will be used throughout the paper. Let  $\mathbb{K}$  be a field with characteristic  $p = 0$ ,  $Mat(n \times m, \mathbb{K})$  be the linear space of  $n \times m$  rectangular matrices with coefficients in  $\mathbb{K}$  and  $Mat(n, \mathbb{K})$  is the associative algebra of square matrices with coefficients in  $\mathbb{K}$ .

When dealing with the problem of representation of finite dimensional Lie algebras two presentations leap into mind: a presentation by matrices and a presentation by an array of structure constants. In the first presentation the Lie algebra is given by a finite set of matrices  $\{A_1, \dots, A_n\}$  that form a basis of the Lie algebra  $\mathfrak{g}$ . If  $A$  and  $B$  are two elements of the space spanned by the  $A_i$ , then their Lie product is defined as  $[A, B] = A.B - B.A$  (where the  $.$  denotes the ordinary matrix multiplication). The second approach is more abstract. The Lie algebra is a (abstract) vector space over a field  $\mathbb{K}$  with basis  $\{x_1, \dots, x_n\}$  and the Lie multiplication is determined by

$$[x_i, x_j] = \sum_{k=1}^n c_{i,j}^k x_k.$$

Here  $(c_{i,j}^k) \in \mathbb{K}^3$  is an array of  $n^3$  structure constants that determines the Lie multiplication completely. By linear algebra it is seen that from a presentation of a Lie algebra by matrices, we easily can obtain a presentation of it by structure constants. Ado's theorem states that the opposite direction is also possible.

**Theorem 0.1.** (Ado) Every finite-dimensional Lie algebra  $\mathfrak{g}$  over a field of characteristic zero can be viewed as a Lie algebra of square matrices under the commutator bracket. More precisely  $\mathfrak{g}$  has a linear representation  $\rho$  over  $\mathbb{K}$ , on a finite dimensional vector space  $V$ , that is a faithful representation, making  $\mathfrak{g}$  isomorphic to a subalgebra of the endomorphisms of  $V$ .

If a such representation of  $\mathfrak{g}$  is built on a vector space  $V$ , we can request if we can built such representation in a lower dimensional vector space  $V$ .

In the other hand, let  $\mathcal{A}$  be a finite-dimensional associative algebra over a field  $\mathbb{K}$  of a characteristic  $p = 0$ , then  $\mathcal{A}$  is canonically equipped with Lie algebra structure given by the following bracket

$$[u, v] = u.v - v.u \quad u, v \in \mathcal{A}.$$

For instance, if  $\mathcal{A} = \text{End}(V)$  the set of linear transformations on  $V$  where  $V$  is a finite dimensional vector space over  $\mathbb{K}$ . In equivalent way, if we consider  $\text{Mat}(n, \mathbb{K})$  the set of square matrices with coefficients in  $\mathbb{K}$ , then we get a canonical Lie algebra structure with the above commutator bracket, this algebra is denoted by  $\mathfrak{gl}(n, \mathbb{K})$ .

However the structure of associative algebra over  $\mathcal{A}$  is not unique [4], indeed if we fix  $w \in \mathcal{A}$  then we can define the product  $(u \circ v)_w = u.w.v$  and with respect  $\mathcal{A}$  is again an associative algebra. This induces a new Lie algebra structure, defined by the bracket

$$[u, v]_w = u.w.v - v.w.u \quad u, v \in \mathcal{A} \quad (1).$$

Thus we obtain a family of Lie brackets, labelled by the element  $w$ . It is readily seen that we actually have a linear space of Lie brackets, since the sum of two such brackets is also a Lie bracket of the same type, and a natural question is to classify these structures. For instance if  $w = 0$  then the Lie algebra  $(\mathcal{A}, [\ , \ ]_0)$  is just the abelian Lie algebra  $\mathbb{K}^{n^2}$ . The above construction can be applied to the algebra of square matrices  $\text{Mat}(n, \mathbb{K})$ . It can be applied also even if  $u, v, w$  are not  $n \times n$  matrices, since (1) makes sense when  $u, v \in \text{Mat}(n \times m, \mathbb{K})$ -the linear space of  $n \times m$  matrices and  $w \in \text{Mat}(m \times n, \mathbb{K})$ - the linear space of  $m \times n$  matrices.

In this paper we will deal with the algebra of the endomorphisms of a finite-dimensional vector space  $V$  or in equivalent way, the algebra of square matrices  $\text{Mat}(n, \mathbb{K})$  and we will deal also with  $\text{Mat}(n \times m, \mathbb{K})$ -the linear space of  $n \times m$  matrices.

Our first goal in this paper is to give a complete classification of these Lie algebra structures, on  $\text{Mat}(n, \mathbb{K})$  (respectively on  $\text{Mat}(n \times m, \mathbb{K})$ ), labelled by the Lie brackets  $[\ , \ ]_J$  with  $J \in \text{Mat}(n, \mathbb{K})$  (respectively in  $\text{Mat}(m \times n, \mathbb{K})$ ).

The paper is organized as follows. First we begin by recalling some classical results of linear algebra and matrix properties. Secondly, we classify the Lie algebra structures on  $\text{Mat}(n, \mathbb{K})$  and  $\text{Mat}(n \times m, \mathbb{K})$ . As a consequence of these classifications, we prove an analogue of the Ado's Theorem, namely, we prove that any finite dimensional Lie algebra  $\mathfrak{g}$  can be viewed as a Lie sub-algebra  $(\text{Hom}(\mathbb{K}^m, \mathbb{K}^n), [\ , \ ]_j)$  for some

positive integers  $n, m$  and some matrix  $J \in Mat(m \times n, \mathbb{K})$ . we prove also that there is no faithful linear representation of the classical  $(2n + 1)$ -Heisenberg Lie algebra  $\mathfrak{H}_{2n+1}$  in  $\mathfrak{gl}(V)$  with  $\dim V \leq n + 1$ . Finally we prove that in the case of the algebra of square matrices, the Lie algebra structure labelled by the family of brackets  $([\cdot, \cdot]_J)$  constructed on  $Mat(n, \mathbb{K})$  are a contraction of the canonical structure  $\mathfrak{gl}(n, \mathbb{K})$ .

## 1. PRELIMINARIES

Let  $n, m$  be integers in  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$  and let  $Mat(n \times m, \mathbb{K})$  be the linear vector space of  $n \times m$  rectangular matrices. We denote its canonical basis  $(E_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$  with

$$E_{i,j} = (\delta_{p,i} \delta_{q,j})_{1 \leq p \leq n, 1 \leq q \leq m},$$

where  $\delta_{p,i}$  is the Kronecker symbol.

**Definition 1.1.** *Let  $J$  and  $J'$  be in  $Mat(m \times n, \mathbb{K})$ . We say that  $J$  is equivalent to  $J'$  if and only if there exist two invertible matrices  $P \in GL(n, \mathbb{K})$  and  $Q \in GL(m, \mathbb{K})$  such that*

$$J = QJ'P.$$

**Remark 1.2.** (1) *The matrices  $Q$  and  $P$  are not unique.*

(2) *The above property of equivalent matrices in  $Mat(m \times n, \mathbb{K})$  is an equivalence relation.*

(3)  *$J$  and  $J'$  are equivalent if and only if they represent the same endomorphism in different bases.*

The characterization of two equivalent matrices is given by the following

**Lemma 1.3.** *Let  $J, J' \in Mat(m \times n, \mathbb{K})$ .  $J$  and  $J'$  are equivalent if and only if they have the same rank.*

From the above Lemma, we can conclude the following

**Lemma 1.4.** *Let  $J$  a matrix in  $Mat(m \times n, \mathbb{K})$  of rank  $r \leq \min(m, n)$ , then there exist two invertible matrices  $Q \in GL(m, \mathbb{K})$  and  $P \in GL(n, \mathbb{K})$  such that*

$$J = Q \begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix} P,$$

where  $0_{p,q}$  is the zero matrix of  $Mat(p \times q, \mathbb{K})$  and  $I_r$  is the identity matrix of  $Mat(r, \mathbb{K})$ .

We will denote the matrix  $\begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$  by  $J_{m,n,r}$  and when  $n = m$  we simply denote it by  $J_{n,r}$ . Note that  $J_{n,n} = I_n$  the identity matrix.

Consider now the linear space  $Mat(n \times m, \mathbb{K})$  and for  $J \in Mat(m \times n, \mathbb{K})$  put

$$[A, B]_J = AJB - BJA, \quad A, B \in Mat(n \times m, \mathbb{K}),$$

then we have

**Lemma 1.5.** (i)  $[A, B]_{J+\alpha J'} = [A, B]_J + \alpha[A, B]_{J'}, \forall \alpha \in \mathbb{K}, J, J' \in \text{Mat}(n \times m, \mathbb{K})$ .  
(ii)  $(\text{Mat}(n \times m, \mathbb{K}), [\cdot, \cdot]_J)$  is a Lie algebra.

The proof is a simple verification of the anti-symmetry of the bracket  $[\cdot, \cdot]_J$  and the Jacobi identity.

Now we give the following

**Proposition 1.6.** If  $J$  and  $J'$  are equivalent matrices in  $\text{Mat}(m \times n, \mathbb{K})$ , then the corresponding Lie algebra structures on  $\text{Mat}(n \times m, \mathbb{K})$  are isomorphic.

*Proof.* Since  $J$  and  $J'$  are equivalent in  $\text{Mat}(m \times n, \mathbb{K})$ , then there exist two invertible matrices  $Q \in GL(m, \mathbb{K})$  and  $P \in GL(n, \mathbb{K})$  such that  $J = QJ'P$ . Consider now the map

$$\begin{aligned} \varphi : \text{Mat}(n \times m, \mathbb{K}) &\longrightarrow \text{Mat}(n \times m, \mathbb{K}) \\ A &\longmapsto PAQ \end{aligned}$$

we verify, since  $Q$  and  $P$  are invertible, that  $\varphi$  is an isomorphism of vector space and

$$\begin{aligned} \varphi([A, B]_J) &= P(AJB - BJA)Q \\ &= P(AQJ'PB - BQJ'PA)Q \\ &= [PAQ, PBQ]_{J'} \\ &= [\varphi(A), \varphi(B)]_{J'} \end{aligned}$$

and thus  $\varphi$  is a Lie algebra isomorphism.  $\square$

Let  $J \in \text{Mat}(m \times n, \mathbb{K})$  with  $rk(J) = r$ , from Proposition 1.6,  $J$  is equivalent to the following matrix  $\begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$ , we shall denote the last matrix by  $J_{m, n, r}$ . In the case when  $m = n$ , we simply denote it by  $J_{n, r}$ . Recall that  $J_{n, n} = I_n$ , the identity matrix of  $\text{Mat}(n, \mathbb{K})$ .

For easy of notations, we shall denote the Lie algebra  $(\text{Mat}(n \times m, \mathbb{K}), [\cdot, \cdot]_{J_{m, n, r}})$  by  $\mathfrak{gl}(n, m, r, \mathbb{K})$  and in the case when  $m = n$ , we simply denotes the lie algebra  $(\text{Mat}(n, \mathbb{K}), [\cdot, \cdot]_{J_{n, r}})$  by  $\mathfrak{gl}(n, r, \mathbb{K})$ . In order to give a complete classification of these Lie algebras structures labelled by the linear family of Lie brackets  $([\cdot, \cdot]_J)$ , we first give the following

**Lemma 1.7.** Let  $M \in \text{Mat}(n \times m, \mathbb{K})$  and put  $M = \begin{pmatrix} M_1 & M_3 \\ M_2 & M_4 \end{pmatrix}$  with  $M_1 \in \text{Mat}(r, \mathbb{K})$ ,  $M_2 \in \text{Mat}((n-r) \times r, \mathbb{K})$ ,  $M_3 \in \text{Mat}(r \times (m-r), \mathbb{K})$ ,  $M_4 \in \text{Mat}((n-r) \times (m-r), \mathbb{K})$ . Then the Lie bracket  $[A, B]_{J_{m, n, r}}$  is given by

$$[A, B]_{J_{m, r}} = \begin{pmatrix} [A_1, B_1] & A_1 B_3 - B_1 A_3 \\ A_2 B_1 - B_2 A_1 & A_2 B_3 - B_2 A_3 \end{pmatrix}.$$

*Proof.* The proof is based on the following result. Put  $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}$ , with  $A_1 \in \text{Mat}(r, \mathbb{K})$ ,  $A_2 \in \text{Mat}((n-r) \times r, \mathbb{K})$ ,  $A_3 \in \text{Mat}(r \times (m-r), \mathbb{K})$ ,  $A_4 \in \text{Mat}((n-r) \times (m-r), \mathbb{K})$ .

$(m - r), \mathbb{K})$ . Then we have

$$AJ_{m,n,r} = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} \text{ and } J_{m,n,r}A = \begin{pmatrix} A_1 & A_3 \\ 0 & 0 \end{pmatrix}.$$

□

In order to give a classification of the Lie algebras structures on  $Mat(n \times m, \mathbb{K})$  labelled by the family of Lie brackets  $([\ , \ ]_J)_J$ , we shall distinguish the two possible cases  $n = m$  and  $n \neq m$ .

## 2. THE CASE OF $Mat(n, \mathbb{K})$

In this section we will deal with the linear vector space of square matrices  $Mat(n, \mathbb{K})$ . It is well known that it has a structure of associative algebra and thus a Lie algebra with the commutator

$$[A, B] = AB - BA, \quad A, B \in Mat(n, \mathbb{K}).$$

The commutator of two matrices  $E_{i,j}$  and  $E_{k,l}$  of the canonical basis ( $1 \leq i, j, k, l \leq n$ ) is given by

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{l,j}E_{k,i}.$$

where  $\delta$  denotes the Kronecker symbol.

Fix  $J \in Mat(n, \mathbb{K})$  and consider the bracket

$$[A, B]_J = AJB - BJA, \quad A, B \in Mat(n, \mathbb{K}).$$

We first give the following

**Lemma 2.1.** *Let  $\mathfrak{h}$  be a Lie sub-algebra of  $\mathfrak{gl}(n, \mathbb{K})$ , then  $\mathfrak{h}$  is a Lie sub-algebra of  $\mathfrak{gl}(p, n, \mathbb{K})$  for any integer  $p \geq n$ .*

*Proof.* For any subset  $\mathfrak{h} \subset Mat(n, \mathbb{K})$ , and  $p \geq n$ , we have

$$[\mathfrak{h}, \mathfrak{h}]_{J_{p,n}} = [\mathfrak{h}, \mathfrak{h}].$$

This completes the proof. □

In order to classify the Lie algebras structures on  $Mat(n, \mathbb{K})$  labelled by the family of brackets  $[\ , \ ]_J$ , we consider a vector space  $V$  over  $\mathbb{K}$  with  $\dim V = n$  and put  $V = V_1 \oplus V_2$ . Next, define  $\mathfrak{n}$  to be the two step nilpotent Lie algebra constructed on the vector space  $Hom(V_1, V_2) \oplus Hom(V_2, V_1) \oplus Hom(V_2, V_2)$  equipped with the bracket

$$[(A, B, C), (A', B', C')] = (0, 0, AB' - A'B).$$

Finally, we define the Lie algebra  $S(V_1, V_2)$  to be the semi direct product of  $End(V_1)$  with  $\mathfrak{n}$ . Then we have the following

**Proposition 2.2.** *(i) The Lie algebra  $S(V_1, V_2)$  is isomorphic to  $\mathfrak{gl}(n, r, \mathbb{K})$ .  
 (ii)  $End(V_1)$  is a Lie subalgebra of  $\mathfrak{gl}(n, s, \mathbb{K})$  for any  $s \geq r$ .  
 (iii) The reductive part of  $\mathfrak{gl}(n, r, \mathbb{K})$  is  $End(V_1) \oplus End(V_2)$ .*

*Proof.* (i) First, we can simply verify that  $\mathfrak{n}$  is a two-step nilpotent Lie algebra. Now for each  $X \in \text{End}(V_1)$ , consider

$$\begin{aligned}\pi_X : \mathfrak{n} &\longrightarrow \mathfrak{n} \\ (A, B, C) &\longmapsto (-AX, XB, 0).\end{aligned}$$

Then we easily verify that  $X \longmapsto \pi_X$  is a Lie algebra homomorphism from  $\mathfrak{gl}(n, \mathbb{K})$  into  $\partial(\mathfrak{n})$  the Lie algebra of the derivations of  $\mathfrak{n}$ .

Now consider the mapping

$$\begin{aligned}\phi : S(V_1, V_2) &\longrightarrow \mathfrak{gl}(n, r, \mathbb{K}) \\ (X, A, B, C) &\longmapsto \begin{pmatrix} X & B \\ A & C \end{pmatrix}.\end{aligned}$$

$\phi$  is a vector space isomorphism.

Let  $X, X' \in \text{End}(V_1)$  and  $(A, B, C), (A', B', C') \in \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1) \oplus \text{Hom}(V_2, V_2)$ . Put  $n = (A, B, C), n' = (A', B', C')$  then

$$\begin{aligned}\phi([(X, n), (X', n')]) &= \phi([([X, X'], [n, n'] + \pi_{X'}(n) - \pi_X(n'))]) \\ &= [\phi((X, n)), \phi((X', n'))]_{J_{n,r}}.\end{aligned}$$

(ii) The statement is a simple consequence of Lemma 1.7.

(iii) It is easy to verify that  $\text{End}(V_1)$  is reductive in  $\mathfrak{gl}(n, r, \mathbb{K})$ . Now, let  $\mathfrak{g}$  be a reductive subalgebra in  $\mathfrak{gl}(n, r, \mathbb{K})$  which is maximal (for the inclusion). From [3] we can write  $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ , where  $\mathfrak{z}$  is the center of  $\mathfrak{g}$ . Since  $\text{End}(V_1) \subset \mathfrak{g}$  then we must have  $[Z, X] = 0$ , for any  $Z \in \mathfrak{z}$  and  $X \in \text{End}(V_1)$ .

Put  $Z = \begin{pmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{pmatrix}, X = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$  then

$$[Z, X]_{J_{n,r}} = \begin{pmatrix} [Z_1, X] & -XZ_3 \\ Z_2X & 0 \end{pmatrix} = 0,$$

and thus  $Z = \begin{pmatrix} \lambda I_r & 0 \\ 0 & Z_4 \end{pmatrix}$  with  $\lambda \in \mathbb{K}$  and  $Z_4 \in \text{End}(V_2)$ .

Finally, let  $Z = \begin{pmatrix} \lambda I_r & 0 \\ 0 & Z_4 \end{pmatrix} \in \mathfrak{z}$  and  $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \in \mathfrak{g}$ , then we have

$$[Z, A]_{J_{n,r}} = \begin{pmatrix} 0 & \lambda A_3 \\ -\lambda A_2 & 0 \end{pmatrix} = 0.$$

Thus  $\mathfrak{g} = \begin{pmatrix} \text{End}(V_1) & 0_{r,n-r} \\ 0_{n-r,r} & \text{End}(V_2) \end{pmatrix} \simeq \text{End}(V_1) \oplus \text{End}(V_2)$ , which completes the proof.  $\square$

Now we shall determine the center of the Lie algebra  $(\text{Mat}(n, \mathbb{K}), [\cdot, \cdot]_J)$ . Let us denote it by  $\mathcal{Z}_J$ , then we have the following

**Proposition 2.3.** *Put  $r = \text{rank}(J)$ , if  $r < n$  then  $\dim \mathcal{Z}_J = (n - r)^2$  while if  $r = n$ , then  $\dim \mathcal{Z}_J = 1$ .*

*Proof. Case 1:  $\text{rank}(J) = n$*

In this case, the mapping  $\varphi : A \mapsto JA$  is a Lie algebra isomorphism from  $(Mat(n, \mathbb{K}), [\cdot, \cdot]_J)$  onto  $\mathfrak{gl}(n, \mathbb{K})$ , and since the center  $\mathcal{Z}$  of  $\mathfrak{gl}(n, \mathbb{K})$  is  $\mathcal{Z} = \mathbb{K}I_n$  then  $\mathcal{Z}_J = \varphi^{-1}(\mathcal{Z}) = \mathbb{K}J^{-1}$ .

**Case 2:  $\text{rank}(J) = r < n$**

Let  $Q, P$  in  $GL(n, \mathbb{K})$  such that

$$J = QJ_{n,r}P$$

and let  $\varphi$  be the isomorphism mapping defined in Proposition 1.6, then

$$\mathcal{Z}_J = \varphi^{-1}(\mathcal{Z}_{J_{n,r}}),$$

then we shall compute  $\mathcal{Z}_{J_{n,r}}$  the center of  $\mathfrak{gl}(n, r, \mathbb{K})$ .

$$\mathcal{Z}_{J_{n,r}} = \{A \in Mat(n, \mathbb{K}), \mid [A, B]_{J_{n,r}} = 0 \ \forall B \in Mat(n, \mathbb{K})\}.$$

Put  $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}$ , with  $A_1 \in Mat(r, \mathbb{K})$ ,  $A_2 \in Mat((n - r) \times r, \mathbb{K})$ ,  $A_3 \in Mat(r \times (n - r), \mathbb{K})$ ,  $A_4 \in Mat(n - r, \mathbb{K})$ , using Lemma 1.7, we get

$$[A, B]_{J_{n,r}} = \begin{pmatrix} [A_1, B_1] & A_1B_3 - B_1A_3 \\ A_2B_1 - B_2A_1 & A_2B_3 - B_2A_3 \end{pmatrix}.$$

Then  $A \in \mathcal{Z}_{J_{n,r}}$  if and only if the bracket  $[A, B]_{J_{n,r}}$  is identically zero for any matrix  $B$  in  $Mat(n, \mathbb{K})$  which implies in particular  $[A_1, B_1] = 0$ ,  $\forall B_1 \in Mat(r, \mathbb{K})$  and thus  $A_1$  is in the center of  $\mathfrak{gl}(r, \mathbb{K})$ , that is

$$A_1 = \lambda I_r, \quad \lambda \in \mathbb{K}.$$

Now with the equations

$$A_1B_3 - B_1A_3 = A_2B_1 - B_2A_1 = A_2B_3 - B_2A_3 = 0, \ \forall B \in Mat(n, \mathbb{K}),$$

we obtain

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}, \quad \text{with } A_4 \in Mat(n - r, \mathbb{K}).$$

Thus

$$\mathcal{Z}_{J_{n,r}} = \mathbb{K} \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} \simeq \text{End}(V_2), \quad \text{with } A_4 \in Mat(n - r, \mathbb{K}),$$

and

$$\mathcal{Z}_J = P^{-1} \mathcal{Z}_{J_{n,r}} Q^{-1}.$$

□

Now we are able to give the following



**Theorem 2.4.** *Let  $n$  be an integer with  $n \geq 2$ . The Lie algebras  $(Mat(n, \mathbb{K}), [\cdot, \cdot]_J)$  and  $(Mat(n, \mathbb{K}), [\cdot, \cdot]_{J'})$  are isomorphic if and only if the matrices  $J$  and  $J'$  are equivalent.*

*Proof.* Let  $J, J' \in Mat(n, \mathbb{K})$ . If  $J$  and  $J'$  are equivalent then from Proposition 1.6 the corresponding Lie algebras are isomorphic.

Now we shall prove the converse.

Let  $J, J' \in Mat(n, \mathbb{K})$  with  $rk(J) = r$  and  $rk(J') = s$  with  $r \neq s$ .

Let  $V$  be a vector space over  $\mathbb{K}$  with  $\dim V = n$  and put  $V = V_1 \oplus V_2 = W_1 \oplus W_2$  with  $\dim V_1 = r$  and  $\dim W_1 = s$ . From Proposition 1.6, we can assume that  $J = J_{n,r}$  and  $J' = J_{n,s}$  and from Proposition 2.2, the reductive part of  $\mathfrak{gl}(n, r, \mathbb{K})$  is  $End(V_1) \oplus End(V_2)$  while the reductive part of  $\mathfrak{gl}(n, s, \mathbb{K})$  is  $End(W_1) \oplus End(W_2)$ . These reductive parts have the same dimension if and only if  $r + s = n$  but in this case they don't have the same dimensionality of their center. Thus they are not isomorphic and this completes the proof.  $\square$

As a consequence of this classification, we have the following

**Theorem 2.5.** *Let  $\mathfrak{H}_n$  be the  $(2n + 1)$ -dimensional Heisenberg algebra. Then there is no faithful finite dimensional linear representation of  $\mathfrak{H}_n$  on a vector space  $V$  with  $\dim V \leq n + 1$ .*

*Proof.* Suppose that there exists a faithful linear representation  $\rho$  of  $\mathfrak{H}_n$ , on a vector space  $V$  with  $\dim V \leq n + 1$ . Let  $\rho$  such mapping, then we have  $\rho : \mathfrak{H}_n \longrightarrow \mathfrak{gl}(r, \mathbb{K})$  which an injective Lie algebra morphism.

Put  $\mathcal{H}_n = Span\{X_i = E_{1,i+1}, Y_i = E_{i+1,n+1}, Z = E_{1,n+2}, \quad i = 1, \dots, n\}$ . We simply check that  $\mathcal{H}_n$  is a subalgebra of  $\mathfrak{gl}(n + 2, n + 1, \mathbb{K})$ , and  $[X_i, Y_j]_{J_{n+1,n}} = \delta_{i,j}Z$ , it is the  $(2n + 1)$ -dimensional Heisenberg algebra. Then we can take  $\mathcal{H}_n$  instead of  $\mathfrak{H}_n$ . If such mapping  $\rho$  exists, then we must have

$$\begin{aligned} \rho([X_i, Y_i]_{J_{n+2,n+1}}) &= [\rho(X_i), \rho(Y_i)] \\ \rho(Z) &= [\rho(X_i), \rho(Y_i)], \end{aligned}$$

but since  $\rho$  is an injective Lie algebra morphism, then  $\rho(Z) = \lambda I_r$  with  $\lambda \neq 0$  wick is impossible because  $I_r$  is traceless matrix.  $\square$

**2.1. Example.** Let  $\mathfrak{H} = span\{Z, Y, X\}$  with the only non vanishing bracket is  $[X, Y] = Z$  then the classical representation of  $\mathfrak{H}$  is

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the last corollary, we can't built a faithful homomorphism  $\mathfrak{H}$  in  $\mathfrak{gl}(2, \mathbb{R})$  but we can built such homomorphism in  $\mathfrak{gl}(2, 1, \mathbb{R})$  with the following realization

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

### 3. THE CASE OF $Mat(n \times m, \mathbb{K})$

In this section we will deal with the linear space of strict rectangular matrices  $Mat(n \times m, \mathbb{K})$  which is not an associative algebra. Fix  $J \in Mat(m \times n, \mathbb{K})$  and put

$$[A, B]_J = AJB - BJA, \quad A, B \in Mat(n \times m, \mathbb{K}).$$

Recall that we have denoted the matrix  $\begin{pmatrix} I_r & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$  by  $J_{m, n, r}$  and we will denotes the Lie algebra  $(Mat(n \times m, \mathbb{K}), [\ , \ ]_{J_{m, n, r}})$  by  $\mathfrak{gl}(n, m, r, \mathbb{K})$ . In the case when  $n = m$ ,  $\mathfrak{gl}(n, n, r, \mathbb{K})$  is simply  $\mathfrak{gl}(n, r, \mathbb{K})$

**Lemma 3.1.** *Let  $\mathfrak{h}$  be a Lie sub-algebra of  $\mathfrak{gl}(n, \mathbb{K})$ , then  $\mathfrak{h}$  is a Lie sub-algebra of  $\mathfrak{gl}(p, q, n, \mathbb{K})$  for any integers  $p, q \geq n$ .*

*Proof.* For any subset  $\mathfrak{h} \subset Mat(n, \mathbb{K})$  and  $q, p \geq n$ , we have

$$[\mathfrak{h}, \mathfrak{h}]_{J_{q, p, n}} = [\mathfrak{h}, \mathfrak{h}].$$

And thus the conclusion holds. □

Now let  $J \in Mat(m \times n, \mathbb{K})$ , and let's denotes by  $\mathcal{Z}_J$  the center of the Lie algebra  $(Mat(n \times m, \mathbb{K}), [\ , \ ]_J)$ , then we have

**Proposition 3.2.** *Let  $J \in Mat(m \times n, \mathbb{K})$  with  $rk(J) = r$ . Then the center of  $(Mat(n \times m, \mathbb{K}), [\ , \ ]_J)$  is  $(n-r)(m-r)$  dimensional. In particular, if  $\mathcal{Z}_{J_{m, n, r}}$  denotes the center of  $(Mat(n \times m, \mathbb{K}), [\ , \ ]_{J_{m, n, r}})$ , then  $\mathcal{Z}_{J_{m, n, r}}$  is spanned by the matrices of the form*

$$A = \begin{pmatrix} 0_r & 0_{r, n-r} \\ 0_{m-r, r} & A_4 \end{pmatrix}, \quad \text{with } A_4 \in Mat((m-r) \times (n-r), \mathbb{K}).$$

*Proof.* The proof of this proposition is similar to the proof of the Proposition 2.3. □

Now we can give the following

**Theorem 3.3.** *Let  $J, J' \in Mat(m \times n, \mathbb{K})$  with  $\min(m, n) \geq 2$ . Then the Lie algebras  $(Mat(n \times m, \mathbb{K}), [\ , \ ]_J)$  and  $(Mat(n \times m, \mathbb{K}), [\ , \ ]_{J'})$  are isomorphic if and only if  $J$  and  $J'$  are equivalent in  $Mat(m \times n, \mathbb{K})$ .*

*Proof.* Let  $J$  and  $J'$  two equivalent matrices in  $Mat(m \times n, \mathbb{K})$  then from Proposition 1.6, the corresponding Lie algebras structures on  $Mat(n \times m, \mathbb{K})$  are isomorphic.

Conversely, let  $J, J'$  be in  $Mat(m \times n, \mathbb{K})$  with  $rk(J) \neq rk(J')$ , then from Proposition 2.3, we have  $\dim \mathcal{Z}_J \neq \dim \mathcal{Z}_{J'}$ , since

$$\dim \mathcal{Z}_J = (n - rk(J))(m - rk(J)) \quad \text{and} \quad \dim \mathcal{Z}_{J'} = (n - rk(J'))(m - rk(J')).$$

□

**Corollary 3.4.** *Let  $V$  be a vector space over  $\mathbb{K}$  with  $\dim V \geq 2$ . Then  $V$  can be equipped with a non trivial Lie algebra structure.*

*Proof.* Let  $n$  be the dimension of  $V$ . We identify  $V$  with  $Mat(n \times 1, \mathbb{K})$  and put  $J = (1, 0, \dots, 0)$ , then  $(V, [\cdot, \cdot]_J)$  is a non trivial Lie algebra. □

Now we can give an analogue of the Ado's Theorem

**Theorem 3.5.** *Every finite-dimensional Lie algebra  $\mathfrak{g}$  over a field  $\mathbb{K}$  of characteristic zero can be viewed as a Lie sub-algebra of  $\mathfrak{gl}(n, m, r, \mathbb{K})$ . More precisely  $\mathfrak{g}$  has a linear representation  $\rho$  over  $\mathbb{K}$  on  $\mathfrak{gl}(n, m, r, \mathbb{K})$ , that is a faithful representation, making  $\mathfrak{g}$  isomorphic to a subalgebra of  $(Hom(\mathbb{K}^n, \mathbb{K}^m), [\cdot, \cdot]_{J_{m,n,r}})$ .*

*Proof.* By Ado's Theorem,  $\mathfrak{g}$  can be viewed as a subalgebra of  $\mathfrak{gl}(p, \mathbb{K})$  and the last algebra is also a subalgebra of  $\mathfrak{gl}(n, m, q, \mathbb{K})$  for any  $q \geq p$  and  $n, m \geq q$ . □

**3.1. Examples.** (a) Let  $V = \mathbb{R}^2$ , we identify  $V$  with  $Mat(2 \times 1, \mathbb{R})$ , we choose a basis  $(e_1, e_2)$  in  $V$  with  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then we can check that  $[e_2, e_1]_J = e_1$  with  $J = (1, 0)$ . Then the two-dimensional affine Lie algebra can be viewed as  $(Hom(\mathbb{R}^2, \mathbb{R}), [\cdot, \cdot]_{J_{1,2,1}})$ .

These constructions can be generalized for  $V = \mathbb{R}^n$  with  $n \geq 2$ .

(b) Let  $V = \mathbb{R}^3 \simeq Mat(3 \times 1, \mathbb{R})$ . Put  $e_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . Then

we verify that  $[e_1, e_2]_J = e_2, [e_1, e_3]_J = e_3, [e_2, e_3]_J = 0$  where  $J = (1, 0, 0)$ . It is the Lie algebra  $\mathfrak{g}_{3,2}(1)$  (see [1] for notations).

(c) Let  $V = \mathbb{R}^4$ , then we can identify  $V$  with  $Mat(4 \times 1, \mathbb{R})$  and in this case we take  $J = (1, 0, 0, 0)$  or with  $Mat(2, \mathbb{R})$  and in this case we can choose  $J = I_2$  or  $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ .

#### 4. CONTRACTIONS AND EXTENSIONS OF LIE ALGEBRAS

Let  $n, r \in \mathbb{N}$  with  $n \geq r, n \geq 2$  and put  $J = J_{n,r}$ , then we have the following

**Proposition 4.1.** *The Lie algebra  $(Mat(n, \mathbb{K}), [\cdot, \cdot]_{J_{n,r}})$  is a contraction of  $\mathfrak{gl}(n, \mathbb{K})$ .*

*Proof.* Let  $(E_{i,j})$  the canonical basis of  $Mat(n, \mathbb{K})$ , and define

$$\mathfrak{g} = Span\{E'_{i,j}, \quad i, j = 1, \dots, n\},$$

with

$$E'_{i,j} = \begin{cases} E_{i,j} & \text{if } i \leq r \text{ and } j \leq r, \\ \varepsilon E_{i,j} & \text{if } i > r \text{ and } j \leq r \text{ or } j > r \text{ and } i \leq r, \\ \varepsilon^2 E_{i,j} & \text{if } i > r \text{ and } j > r. \end{cases}$$

where  $\varepsilon \in \mathbb{K}$ . Then we have

$$[E'_{i,j}, E'_{k,l}] = \begin{cases} \delta_{j,k} E'_{i,l} - \delta_{l,i} E'_{k,j} & \text{if } i, j, k, l \leq r, \\ \delta_{j,k} E'_{i,l} & \text{if } i, j, k \leq r \text{ and } l > r, \\ -\delta_{l,i} E'_{k,j} & \text{if } i, j, l \leq r \text{ and } k > r, \\ 0 & \text{if } i, j \leq r \text{ and } k, l > r, \\ 0 & \text{if } i, k \leq r \text{ and } l, j > r, \\ \delta_{j,k} \varepsilon^2 E'_{i,l} & \text{if } i \leq r \text{ and } j, k, l > r, \\ \delta_{j,k} \varepsilon^2 E'_{i,l} - \delta_{l,i} \varepsilon^2 E'_{k,j} & \text{if } i, j, k, l > r. \end{cases}$$

And when  $\varepsilon \rightarrow 0$ , then we obtain

$$\lim_{\varepsilon \rightarrow 0} [E'_{i,j}, E'_{k,l}] = [E_{i,j}, E_{k,l}]_{J_{n,r}}.$$

□

If we repeat again this contraction, we get a new algebra which can be directly obtained from the first Lie algebra since  $J_{n,r} J_{n,s} = J_{n,t}$  with  $t = \min(r, s)$ .

**4.1. Example.** Let  $\mathfrak{sl}(2, \mathbb{R})$  be the semi-simple (real) Lie algebra spanned by  $\{H, X, Y\}$  where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The (usual) brackets are:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Pick  $J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , then we obtain the following brackets:

$$[H, X]_J = X, \quad [H, Y]_J = -Y, \quad [X, Y]_J = 0.$$

The new Lie algebra is completely solvable and thus non isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$  since  $\mathfrak{sl}(2, \mathbb{R})$  is semi-simple.

Let us restrict ourself to the Lie algebra  $\mathfrak{gl}(n, \mathbb{K})$ . Then we have the following

**Lemma 4.2.** ([4], [5]) *The map  $(A, B) \mapsto [A, B]_J$  is a two-coboundary for the adjoint representation of  $\mathfrak{gl}(n, \mathbb{K})$ .*

*Proof.* We can easily check that

$$[A, B]_J = ad_A \alpha(B) - ad_B \alpha(A) - \alpha([A, B]) = (d\alpha)(A, B).$$

where

$$\alpha(X) = \frac{1}{2}(XJ + JX).$$

□

Let  $t \in [0, 1]$  and  $J = J_{n,r}$  (with  $r < n$ ), then we can write

$$[A, B]_{(1-t)I+tJ} = [A, B] + t[A, B]_{J-I},$$

thus since  $[A, B]_{J-I}$  is a coboundary,  $t \mapsto [A, B]_{(1-t)I+tJ}$  is a deformation infinitesimally trivial at  $t = 0$  from  $\mathfrak{gl}(n, \mathbb{K})$  to  $\mathfrak{gl}(n, r, \mathbb{K})$ . For any  $t < 1$ , the Lie algebra  $(Mat(n, \mathbb{K}), [\ , \ ]_{(1-t)I+tJ})$  is isomorphic to  $\mathfrak{gl}(n, \mathbb{K})$  but for  $t = 1$  we get a completely different structure which is labelled by the bracket  $[\ , \ ]_J$ . The mapping

$$\Psi_t : (Mat(n, \mathbb{K}), [\ , \ ]_{(1-t)I+tJ}) \longrightarrow \mathfrak{gl}(n, \mathbb{K}), \quad \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \mapsto \begin{pmatrix} X_1 & (1-t)X_2 \\ X_3 & (1-t)X_4 \end{pmatrix}$$

is for any  $t \in [0, 1[$  invertible and  $[X, Y]_{(1-t)I+tJ} = \Psi_t^{-1}([\Psi_t(X), \Psi_t(Y)])$ , and we have  $\frac{\partial}{\partial t}([\ , \ ]_{(1-t)I+tJ})|_{t=0}$  is the 2-coboundary:  $(A, B) \mapsto [A, B]_{J-I}$ .

**Lemma 4.3.** *The vector space  $Mat(n, \mathbb{K})$  is a  $(Mat(n, \mathbb{K}), [\ , \ ]_J)$ -module.*

*Proof.* We can verify that the map

$$\begin{aligned} \varphi : (Mat(n, \mathbb{K}), [\ , \ ]_J) &\longrightarrow \mathfrak{gl}(Mat(n, \mathbb{K})) \\ A &\longmapsto ad_A^J, \end{aligned}$$

where

$$ad_A^J(B) = [A, B]_J,$$

is a Lie algebra homomorphism since  $[\ , \ ]_J$  is a Lie bracket on  $Mat(n, \mathbb{K})$ . □

Let  $\mathfrak{g}$  be a Lie algebra over  $\mathbb{K}$  and  $(X_1, \dots, X_r)$  with the following commutations roles  $[X_i, X_j] = \sum_k c_{i,j}^k X_k$ . By the Ado's theorem, there is a faithful representation  $\rho$  of  $\mathfrak{g}$  in  $\mathfrak{gl}(n, \mathbb{K})$  for some integer  $n$ . Let us identify  $\mathfrak{g}$  with its image under  $\rho$ . Then we have the following

**Proposition 4.4.** *Let  $\mathfrak{h} = Mat(r \times (n - r), \mathbb{K})$ ,  $\mathfrak{h}' = Mat((n - r) \times r, \mathbb{K})$  and  $\mathfrak{h}'' = Mat(n - r, \mathbb{K})$ . Then we have the following subalgebras inclusions*

$$\begin{pmatrix} \mathfrak{g} & \mathfrak{h} \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} \mathfrak{g} & \mathfrak{h} \\ 0 & \mathfrak{h}'' \end{pmatrix} \subset \mathfrak{gl}(n, r, \mathbb{K})$$

and

$$\begin{pmatrix} \mathfrak{g} & 0 \\ \mathfrak{h}' & 0 \end{pmatrix} \subset \begin{pmatrix} \mathfrak{g} & 0 \\ \mathfrak{h}' & \mathfrak{h}'' \end{pmatrix} \subset \mathfrak{gl}(n, r, \mathbb{K}).$$

## REFERENCES

- [1] BERNAT P. ET AL., *Représentations des groupes de Lie résolubles*, Monographes de la société mathématique de France, Dunod Paris (1972).
- [2] DIXMIER. J., *Algèbres enveloppantes*, Gautier-Villars (1974) ISBN 2-04-008055-4
- [3] ISE AND TAKEUCHI, *Lie groups I, Lie groups II*, Translations of Mathematical Monographs, Volume **85** ISBN 0-8218-4544-6
- [4] YANOVSKI A. B., *Linear Bundles of Lie Brackets and their Applications*, J. Math Phys. **41** (2000) 7869-7882.
- [5] YANOVSKI A. B., *Compatible Poisson Tensors Related To Bundles of Linear Algebras*, Proceedings of the 7th international conference on geometry, integrability and quantization, Sts. Constantine and Elena, Bulgaria, June 2–10, 2005. Sofia: Bulgarian Academy of Sciences. 307-319 (2006)
- [6] HOCHSCHILD G. AND SERRE J. P., *Cohomologie of Lie algebras*, Annales of Math. t. 57, 1953, p. 591-603.

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