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A Jacobson Radical Decomposition of the Fano-Snowflake Configuration

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Abstract

The Fano-Snowflake, a specific *non*-unimodular projective lattice configuration associated with the smallest ring of ternions R_\diamond (arXiv:0803.4436 and 0806.3153), admits an interesting partitioning with respect to the Jacobson radical of R_\diamond . The totality of 21 free cyclic submodules generated by non-unimodular vectors of the free left R_\diamond -module R_\diamond^3 are shown to split into three disjoint sets of cardinalities 9, 9 and 3 according as the number of Jacobson radical entries in the generating vector is 2, 1 or 0, respectively. The corresponding “ternion-induced” factorization of the lines of the Fano plane sitting in the middle of the Fano-Snowflake (6 – 7 – 3) is found to *differ fundamentally* from the natural one, i. e., from that with respect to the Jacobson radical of the Galois field of two elements (3 – 3 – 1).

MSC Codes: 51C05, 51Exx

Keywords: Non-Unimodular Geometry over Rings — Smallest Ring of Ternions — Fano Plane

Projective lattice geometries over unital associative rings R (see, e. g., [1] and references therein) represent a very important generalization of classical (field) projective spaces, being endowed with a number of remarkable features not exhibited by the latter. One of the most striking differences is, for certain R , the existence of *free* cyclic submodules generated by *non*-unimodular vectors of the free left R -module R^n , $n \geq 1$. In a couple of recent papers [2, 3], an in-depth analysis has been performed of such non-unimodular portions of the lattice geometries for R being the ring of ternions, i. e., a ring isomorphic to that of upper triangular 2×2 matrices with entries from an arbitrary commutative field F . It has been found that for any $n \geq 2$ these non-unimodular free cyclic submodules of R^n can be associated with the lines of $\text{PG}(n, F)$, the n -dimensional projective space over F sitting in the middle of such non-unimodular world. In the finite case, $F = \text{GF}(q)$, basic combinatorial properties of such configurations have been derived and illustrated in exhaustive detail on the simplest, $n = q = 2$ case — dubbed the Fano-Snowflake geometry. In the present paper we shall have another look at the Fano-Snowflake and show that this geometry admits an intriguing decomposition with respect to the Jacobson radical of the ring in question.

To this end in view, we first collect the necessary background information from [2, 3]. We consider an associative ring with unity $1 (\neq 0)$, R , and denote the free left R -module on $n + 1$ generators over R by R^{n+1} . The set $R(r_1, r_2, \dots, r_{n+1})$, defined as follows

$$R(r_1, r_2, \dots, r_{n+1}) := \{(\alpha r_1, \alpha r_2, \dots, \alpha r_{n+1}) \mid (r_1, r_2, \dots, r_{n+1}) \in R^{n+1}, \alpha \in R\}, \quad (1)$$

is a *left* cyclic submodule of R^{n+1} . Any such submodule is called *free* if the mapping $\alpha \mapsto (\alpha r_1, \alpha r_2, \dots, \alpha r_{n+1})$ is injective, i. e., if $(\alpha r_1, \alpha r_2, \dots, \alpha r_{n+1})$ are all *distinct*. Next, we shall call a vector $(r_1, r_2, \dots, r_{n+1}) \in R^{n+1}$ *unimodular* if there exist elements x_1, x_2, \dots, x_{n+1} in R such that

$$r_1 x_1 + r_2 x_2 + \dots + r_{n+1} x_{n+1} = 1. \quad (2)$$

Table 1: Addition (*left*) and multiplication (*right*) in R_\diamond .

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	6	7	5	4	2	3
2	2	6	0	4	3	7	1	5
3	3	7	4	0	2	6	5	1
4	4	5	3	2	0	1	7	6
5	5	4	7	6	1	0	3	2
6	6	2	1	5	7	3	0	4
7	7	3	5	1	6	2	4	0

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	1	3	7	5	6	4
3	0	3	5	3	6	5	6	0
4	0	4	4	0	4	0	0	4
5	0	5	3	3	0	5	6	6
6	0	6	6	0	6	0	0	6
7	0	7	7	0	7	0	0	7

It is a very well-known fact (see, e.g., [4]–[7]) that if $(r_1, r_2, \dots, r_{n+1})$ is unimodular, then $R(r_1, r_2, \dots, r_{n+1})$ is free; any such free cyclic submodule represents a point of the n -dimensional projective space defined over R [5]. The converse statement, however, is not generally true. That is, there exist rings which also give rise to free cyclic submodules featuring exclusively non-unimodular vectors. The first case when this occurs is the smallest (non-commutative) ring of ternions, R_\diamond :

$$R_\diamond \equiv \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in GF(2) \right\}. \quad (3)$$

From this definition it is readily seen that the ring contains two maximal (two-sided) ideals,

$$I_1 = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in GF(2) \right\} \quad (4)$$

and

$$I_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in GF(2) \right\}, \quad (5)$$

which give rise to a non-trivial (two-sided) Jacobson radical J ,

$$J = I_1 \cap I_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in GF(2) \right\}. \quad (6)$$

As for our further purposes it will be more convenient to work with numbers than matrices, we shall relabel the elements of R_\diamond as follows

$$\begin{aligned} 0 &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & 1 &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & 2 &\equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & 3 &\equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\ 4 &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & 5 &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & 6 &\equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & 7 &\equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (7)$$

The addition and multiplication in the ring is that of matrices over $GF(2)$, which in our compact notation reads as shown in Table 1. The two maximal ideals now acquire the form

$$I_1 := \{0, 4, 6, 7\} \quad (8)$$

and

$$I_2 := \{0, 3, 5, 6\}, \quad (9)$$

and the Jacobson radical reads,

$$J = I_1 \cap I_2 = \{0, 6\}. \quad (10)$$

There exist altogether 21 free cyclic submodules of R_\diamond^3 which are generated by non-unimodular vectors. Taking their complete list from [2] one sees that they can be separated into the following three disjoint sets

$$\begin{aligned}
R_\diamond(6, 6, 7) &= R_\diamond(6, 6, 4) = \{(0, 0, 0), (6, 6, 7), (6, 6, 4), (6, 6, 0), (0, 0, 4), (6, 6, 6), (0, 0, 6), (0, 0, 7)\}, \\
R_\diamond(6, 7, 6) &= R_\diamond(6, 4, 6) = \{(0, 0, 0), (6, 7, 6), (6, 4, 6), (6, 0, 6), (0, 4, 0), (6, 6, 6), (0, 6, 0), (0, 7, 0)\}, \\
R_\diamond(7, 6, 6) &= R_\diamond(4, 6, 6) = \{(0, 0, 0), (7, 6, 6), (4, 6, 6), (0, 6, 6), (4, 0, 0), (6, 6, 6), (6, 0, 0), (7, 0, 0)\}, \\
R_\diamond(0, 6, 7) &= R_\diamond(0, 6, 4) = \{(0, 0, 0), (0, 6, 7), (0, 6, 4), (0, 6, 0), (0, 0, 4), (0, 6, 6), (0, 0, 6), (0, 0, 7)\}, \\
R_\diamond(0, 7, 6) &= R_\diamond(0, 4, 6) = \{(0, 0, 0), (0, 7, 6), (0, 4, 6), (0, 0, 6), (0, 4, 0), (0, 6, 6), (0, 6, 0), (0, 7, 0)\}, \\
R_\diamond(6, 0, 7) &= R_\diamond(6, 0, 4) = \{(0, 0, 0), (6, 0, 7), (6, 0, 4), (6, 0, 0), (0, 0, 4), (6, 0, 6), (0, 0, 6), (0, 0, 7)\}, \\
R_\diamond(7, 0, 6) &= R_\diamond(4, 0, 6) = \{(0, 0, 0), (7, 0, 6), (4, 0, 6), (0, 0, 6), (4, 0, 0), (6, 0, 6), (6, 0, 0), (7, 0, 0)\}, \\
R_\diamond(6, 7, 0) &= R_\diamond(6, 4, 0) = \{(0, 0, 0), (6, 7, 0), (6, 4, 0), (6, 0, 0), (0, 4, 0), (6, 6, 0), (0, 6, 0), (0, 7, 0)\}, \\
R_\diamond(7, 6, 0) &= R_\diamond(4, 6, 0) = \{(0, 0, 0), (7, 6, 0), (4, 6, 0), (0, 6, 0), (4, 0, 0), (6, 6, 0), (6, 0, 0), (7, 0, 0)\}, \\
R_\diamond(4, 6, 7) &= R_\diamond(7, 6, 4) = \{(0, 0, 0), (4, 6, 7), (7, 6, 4), (6, 6, 0), (4, 0, 4), (0, 6, 6), (6, 0, 6), (7, 0, 7)\}, \\
R_\diamond(4, 7, 6) &= R_\diamond(7, 4, 6) = \{(0, 0, 0), (4, 7, 6), (7, 4, 6), (6, 0, 6), (4, 4, 0), (0, 6, 6), (6, 6, 0), (7, 7, 0)\}, \\
R_\diamond(6, 4, 7) &= R_\diamond(6, 7, 4) = \{(0, 0, 0), (6, 4, 7), (6, 7, 4), (6, 6, 0), (0, 4, 4), (6, 0, 6), (0, 6, 6), (0, 7, 7)\}, \\
R_\diamond(4, 4, 6) &= R_\diamond(7, 7, 6) = \{(0, 0, 0), (4, 4, 6), (7, 7, 6), (6, 6, 6), (4, 4, 0), (0, 0, 6), (6, 6, 0), (7, 7, 0)\}, \\
R_\diamond(4, 6, 4) &= R_\diamond(7, 6, 7) = \{(0, 0, 0), (4, 6, 4), (7, 6, 7), (6, 6, 6), (4, 0, 4), (0, 6, 0), (6, 0, 6), (7, 0, 7)\}, \\
R_\diamond(6, 4, 4) &= R_\diamond(6, 7, 7) = \{(0, 0, 0), (6, 4, 4), (6, 7, 7), (6, 6, 6), (0, 4, 4), (6, 0, 0), (0, 6, 6), (0, 7, 7)\}, \\
R_\diamond(0, 4, 7) &= R_\diamond(0, 7, 4) = \{(0, 0, 0), (0, 4, 7), (0, 7, 4), (0, 6, 0), (0, 4, 4), (0, 0, 6), (0, 6, 6), (0, 7, 7)\}, \\
R_\diamond(4, 0, 7) &= R_\diamond(7, 0, 4) = \{(0, 0, 0), (4, 0, 7), (7, 0, 4), (6, 0, 0), (4, 0, 4), (0, 0, 6), (6, 0, 6), (7, 0, 7)\}, \\
R_\diamond(4, 7, 0) &= R_\diamond(7, 4, 0) = \{(0, 0, 0), (4, 7, 0), (7, 4, 0), (6, 0, 0), (4, 4, 0), (0, 6, 0), (6, 6, 0), (7, 7, 0)\}, \\
R_\diamond(4, 4, 7) &= R_\diamond(7, 7, 4) = \{(0, 0, 0), (4, 4, 7), (7, 7, 4), (6, 6, 0), (4, 4, 4), (0, 0, 6), (6, 6, 6), (7, 7, 7)\}, \\
R_\diamond(4, 7, 4) &= R_\diamond(7, 4, 7) = \{(0, 0, 0), (4, 7, 4), (7, 4, 7), (6, 0, 6), (4, 4, 4), (0, 6, 0), (6, 6, 6), (7, 7, 7)\}, \\
R_\diamond(7, 4, 4) &= R_\diamond(4, 7, 7) = \{(0, 0, 0), (7, 4, 4), (4, 7, 7), (0, 6, 6), (4, 4, 4), (6, 0, 0), (6, 6, 6), (7, 7, 7)\},
\end{aligned}$$

according as the number of Jacobson radical entries in the generating vector(s) is two, one or zero, respectively. Employing the picture of the Fano-Snowflake given in [2] (reproduced, for convenience, in Figure 1), the structure of and relation between the three sets can be represented diagrammatically as shown in Figure 2. As each submodule answers to a single line of the associated core Fano plane, the decomposition of the Fano-Snowflake induces an intriguing factorization of the lines of the plane itself. This is depicted in Figure 3, bottom panel, and it is seen to fundamentally differ from the corresponding partitioning of the Fano plane with respect to the Jacobson radical of its ground field $\text{GF}(2)(\{0\})$, namely

$$\begin{aligned}
\text{GF}(2)(1, 0, 0) &= \{(0, 0, 0), (1, 0, 0)\}, \\
\text{GF}(2)(0, 1, 0) &= \{(0, 0, 0), (0, 1, 0)\}, \\
\text{GF}(2)(0, 0, 1) &= \{(0, 0, 0), (0, 0, 1)\}, \\
\text{GF}(2)(1, 1, 0) &= \{(0, 0, 0), (1, 1, 0)\}, \\
\text{GF}(2)(1, 0, 1) &= \{(0, 0, 0), (1, 0, 1)\}, \\
\text{GF}(2)(0, 1, 1) &= \{(0, 0, 0), (0, 1, 1)\}, \\
\text{GF}(2)(1, 1, 1) &= \{(0, 0, 0), (1, 1, 1)\},
\end{aligned}$$

as displayed in Figure 3, top panel.

The origin of the factorization of the Fano plane when related to its ground field $\text{GF}(2)$ is easy to understand: the number of the Jacobson radical entries (i. e., only zeros in this case) in the coordinates of a line (and, by duality, of a point as well) has a clear meaning with respect to the triangle of base points of the coordinate system. Something similar holds obviously for the factorization of the Fano-Snowflake with respect to its ternionic coordinates, but passing to the embedded Fano plane this link seems to be lost or substantially distorted. Figure 2 illustrates this fact quite nicely: in the top figure all the polygons pass through the three corners/vertices of the basic triangle, in the middle figure all branches go through the points on each side of the

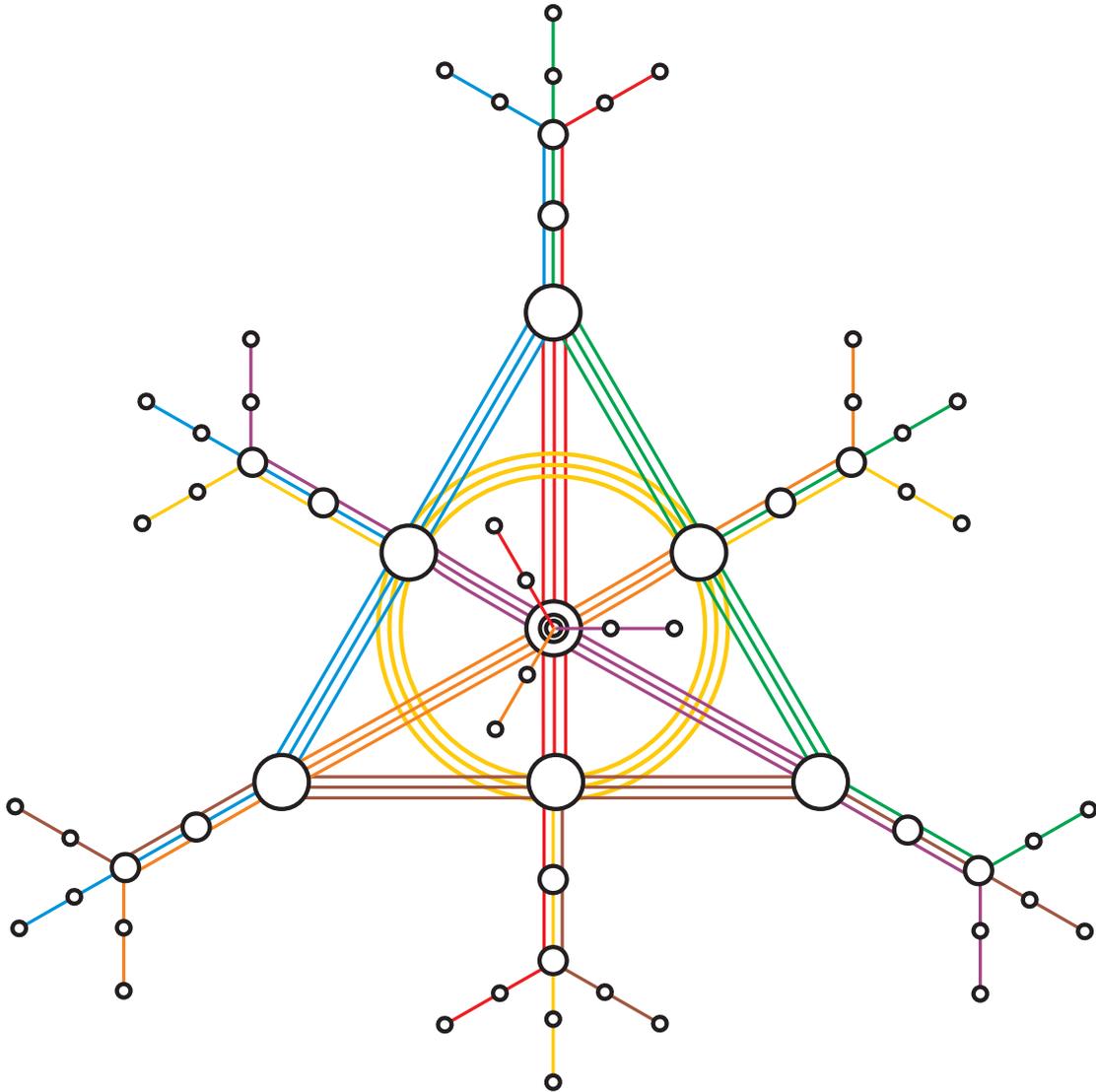


Figure 1: The Fano-Snowflake — a diagrammatic illustration of a very intricate relation between the 21 free left cyclic submodules generated by non-unimodular vectors of R_{\diamond}^3 . Each circle represents a vector of R_{\diamond}^3 (in fact, of I_1^3), its size being roughly proportional to the number of submodules passing through the given vector. As the $(0, 0, 0)$ triple is not shown, each submodule is represented by seven circles (three big, two medium-sized and two small) lying on a common polygon. The small circles stand for the vectors generating the submodules. The big circles represent the vectors whose all the three entries are from J ; these vectors correspond to the points of the Fano plane. The seven colors were chosen in such a way to make also the lines of the Fano plane, i. e., the intersections of the submodules with J^3 , readily discernible. See [2] and/or [3] for more details.

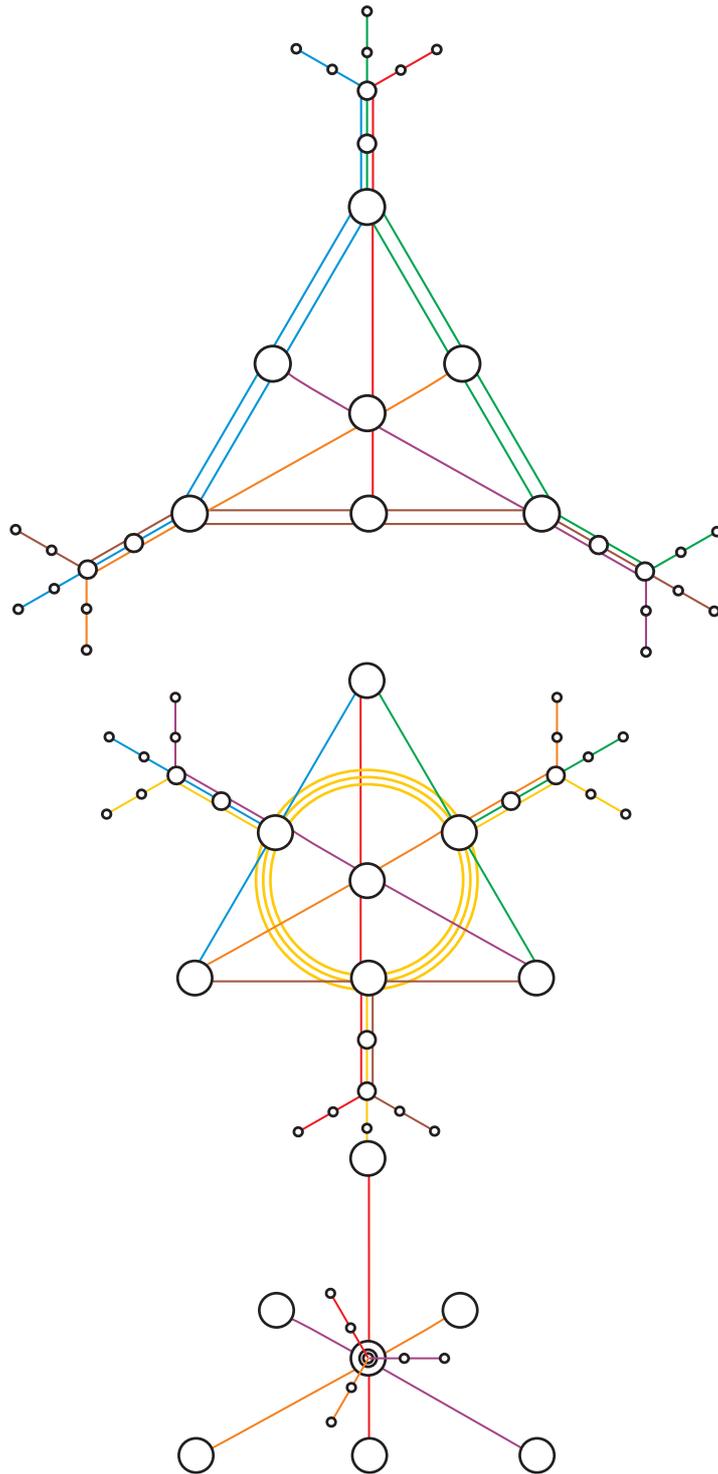


Figure 2: A sketchy illustration of the $9 - 9 - 3$ decomposition of the set of free cyclic submodules comprising the Fano-Snowflake with respect to the Jacobson radical of R_{\diamond} according as the number of radical entries in the submodule's generating vector(s) is two (top), one (middle) or zero (bottom), respectively.

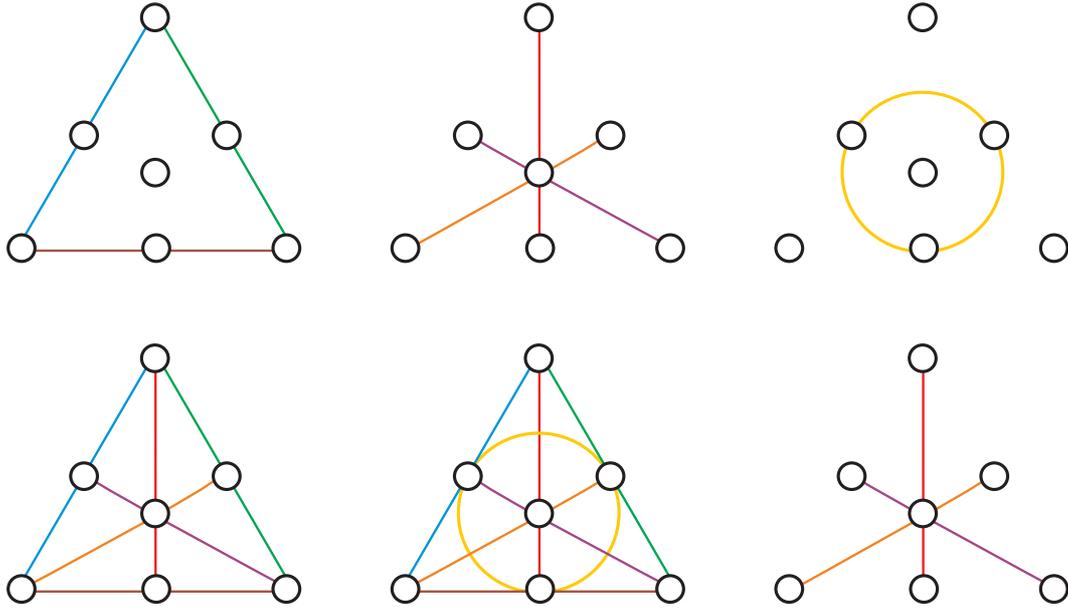


Figure 3: A comparison of the “ternion-induced” $6 - 7 - 3$ factorization of the lines of the Fano plane (bottom) with the ordinary $3 - 3 - 1$ one (top). Note a principal qualitative difference between the two factorizations as the three sets (factors) are pairwise disjoint in the latter case but not in the former one.

triangle which are not vertices, whereas in the bottom figure all the branches share the only point which is off the reference triangle. It is the intersections of the branches/polygons with the core Fano plane which behave “strange” and give rise to the fundamental difference between the two factorizations of the Fano plane shown in Figure 3.

The above-described observations clearly demonstrate that there is more to the algebraic structure of the Fano plane than meets the eye. The plane when considered of *its own* is found to “reveal” quite different aspects compared with the case when being *embedded into* a more general, non-unimodular projective lattice setting. This difference is likely to get more pronounced, and more intricate as well, as we pass to higher order rings giving rise to more complex forms of Fano-Snowflakes. A key question is to find out whether the Snowflakes’ decomposition patterns and their induced factorizations of the lines of the core Fano planes remain qualitatively the same as in the ternionic case; our preliminary analysis of such structures over a particular class of non-commutative rings of order sixteen and having twelve zero-divisors indicates that this might be so. Another worth-pursuing line of exploration is to stay with ternions but focus on higher-order ($q > 2$) and/or higher-dimensional ($n > 2$) “Snowflake” geometries and their core projective planes and/or spaces. Finding, however, a general build-up principle for these remarkable geometrical structures with respect to the properties of defining rings currently seems a truly difficult, yet extremely challenging task due to “ubiquity” of the Fano plane in various mathematical and physical contexts (see, e. g., [8]).

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References

- [1] Brehm, U., Greferath, and Schmidt, S. E.: 1995, Projective Geometry on Modular Lattices, in Handbook of Incidence Geometry, F. Buekenhout (ed.), Elsevier, Amsterdam, pp. 1115–1142.
- [2] Saniga, M., Havlicek, H., Planat, M., and Pracna, P.: 2008, Twin “Fano-Snowflakes” over the Smallest Ring of Ternions, Symmetry, Integrability and Geometry: Methods and Applications, Vol. 4, Paper 050, 7 pages (arXiv:0803.4436).
- [3] Havlicek, H., and Saniga, M.: 2008, Vectors, Cyclic Submodules and Projective Spaces Linked with Ternions, Journal of Geometry, submitted (arXiv:0806.3153).
- [4] Veldkamp, F. D.: 1981, Projective Planes over Rings of Stable Rang 2, Geometriae Dedicata, Vol. 11, pp. 285–308.
- [5] Veldkamp, F. D.: 1995, Geometry over Rings, in Handbook of Incidence Geometry, F. Buekenhout (ed.), Elsevier, Amsterdam, pp. 1033–1084.
- [6] Herzer, A.: 1995, Chain Geometries, in Handbook of Incidence Geometry, F. Buekenhout (ed.), Elsevier, Amsterdam, pp. 781–842.
- [7] Blunck, A., and Herzer, A.: 2005, Kettengeometrien – Eine Einführung, Berichte aus der Mathematik, Shaker Verlag, Aachen.
- [8] Brown, E.: 2002, The Many Names of $(7,3,1)$, Mathematics Magazine, Vol. 75, pp. 83–94.