

SOME BOUNDS FOR RAMIFICATION OF p^n -TORSION SEMI-STABLE REPRESENTATIONS

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ABSTRACT. Let p be an odd prime, K a finite extension of \mathbb{Q}_p , $G_K = \text{Gal}(\bar{K}/K)$ its absolute Galois group and $e = e(K/\mathbb{Q}_p)$ its absolute ramification index. Suppose that T is a p^n -torsion representation of G_K that is isomorphic to a quotient of G_K -stable \mathbb{Z}_p -lattices in a semi-stable representation with Hodge-Tate weights $\{0, \dots, r\}$. We prove that there exists a constant μ depending only on n , e and r such that the upper numbering ramification group $G_K^{(\mu)}$ acts on T trivially.

CONTENTS

1. Introduction	1
2. G_s -representation attached to a torsion Kisin module	4
2.1. Definitions and basic properties of Kisin modules	4
2.2. Functors to Galois representations	4
2.3. The modules $J_{n,c}(\mathfrak{M})$	5
2.4. Brief discussion about sharpness of N	7
2.5. Some quotients of $W_n(R)$	8
3. Torsion semi-stable Galois representations	10
3.1. Torsion (φ, \hat{G}) -modules	10
3.2. G_s -action on $\hat{T}(\hat{\mathcal{L}})$	14
3.3. Comparison between $\hat{J}_{n,c}(\hat{\mathfrak{M}})$ and $J_{n,c}(\mathfrak{M})$	15
4. Ramification bound	17
4.1. The sets $J_{n,c}^{(s),E}(\mathfrak{M})$	17
4.2. Proof of Theorem 1.1	21
4.3. Proof of Theorem 1.3	23
5. Some results and questions about lifts	24
References	28

1. INTRODUCTION

Let $p > 2$ be a prime number and k a perfect field of characteristic p . We denote by $W = W(k)$ the ring of Witt vectors with coefficients in k . Fix K a

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totally ramified extension of $W[1/p]$ of degree e and \bar{K} an algebraic closure of K . Fix $\pi \in \mathcal{O}_K$ a uniformizer and $(\pi_s)_{s \geq 0}$ a compatible system of p^s -th root of π . Set $G = \text{Gal}(\bar{K}/K)$ and for all non negative integer s , put $K_s = K(\pi_s)$ and $G_s = \text{Gal}(\bar{K}/K_s)$. Denote by $G^{(\mu)}$ and $G_s^{(\mu)}$ ($\mu \in \mathbb{R}$) the upper ramification filtration of G and G_s , as defined in §1.1 of [9]. Note that conventions of *loc. cit.* differ by some shift with definition of [22], Chap. IV. Finally, let v_K be the discrete valuation on K normalized by $v_K(\pi) = 1$. It extends uniquely to a (not discrete) valuation on \bar{K} , that we denote again v_K .

Consider r a positive integer and V a semi-stable representation of G with Hodge-Tate weights in $\{0, 1, \dots, r\}$. Let T be the quotient of two G -stable \mathbb{Z}_p -lattices in V . It is a representation of G , which is killed by p^n for some integer n . Denote by $\rho : G \rightarrow \text{Aut}_{\mathbb{Z}_p}(T)$ the associated group homomorphism and by L (resp. L_s) the finite extension of K (resp. K_s) defined by $\ker \rho$ (resp. $\ker \rho|_{G_s}$). We will prove:

Theorem 1.1. *Keeping previous notations, for any integer $s > n + \log_p(\frac{nr}{p-1})$ and for all real number $\mu > \frac{ernp^n}{p-1}$, $G_s^{(\mu)}$ acts trivially on T .*

Remark 1.2. Condition on s implies $\frac{ernp^n}{p-1} < ep^s$. Hence one may always choose $\mu = ep^s$.

We also obtain a bound for the ramification of L/K :

Theorem 1.3. *Write $\frac{nr}{p-1} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{p} < \beta \leq 1$. Then:*

- (1) *if $\mu > 1 + e(n + \alpha + \max(\beta, \frac{1}{p-1}))$, then $G^{(\mu)}$ acts trivially on T ;*
- (2) *$v_K(\mathcal{D}_{L/K}) < 1 + e(n + \alpha + \beta) - \frac{1}{pn+\alpha}$*

where $\mathcal{D}_{L/K}$ is the different of L/K .

Before this work, some partial results were already known in this direction. First, in [9] and [11], Fontaine uses Fontaine-Laffaille theory (developped in [8]) to get some bounds when $e = 1$, $n = 1$, $r < p - 1$ and V is crystalline. In [1], Abrashkin follows Fontaine's general ideas to extend the result to arbitrary n (other restrictions remain the same). Later, with the extension by Breuil of Fontaine-Laffaille theory to semi-stable case (see [3]), it has been possible to achieve some cases where V is not crystalline. Precisely in [4]¹, Breuil obtains bounds for semi-stable representations that satisfies Griffith transversality when $n = 1$ and $er < p - 1$. Very recently in [14] and [15], Hattori proves a bound for all semi-stable representations with $r < p - 1$ (e and n are arbitrary here). Unfortunately, bounds found by those authors have the same shape than ours but are in general slightly better (at least for $n > 1$) and, in fact, we conjecture that Theorem 1.3 could be improved as follows:

Conjecture 1.4. *Writing $\frac{r}{p-1} = p^{\alpha'} \beta'$ with $\alpha' \in \mathbb{N}$ and $\frac{1}{p} < \beta' \leq 1$, we have:*

- (1) *if $\mu > 1 + e(n + \alpha' + \max(\beta', \frac{1}{p-1}))$, then $G^{(\mu)}$ acts trivially on T ;*
- (2) *$v_K(\mathcal{D}_{L/K}) < 1 + e(n + \alpha' + \beta')$.*

We also wonder if, in bounds given by the previous Conjecture, β' can be replaced by $\frac{\beta'}{p}$ (remark then that $\max(\frac{\beta'}{p}, \frac{1}{p-1})$ is just $\frac{1}{p-1}$), since it is actually the case in Proposition 9.2.2.2 of [2].

¹See Proposition 9.2.2.2 of [2] for the statement

Note finally that the dependance in r of bounds of Theorem 1.3 is logarithmic, which is in fact quite surprising since, until now all bounds seem to depend linearly on r . (Of course, it does not mean anything since these bounds are valid under the assumption for $r < p - 1$, and certainly not for r going to infinity.) We finally wonder if better bounds exist when V is crystalline. It is actually the case when $e = 1$ and $r < p - 1$ by results of Fontaine and Abrashkin, but it is not clear to us how to extend this to a more general setting.

Let us now explain the general plan of our proof (and in the same time of the article). For this we introduce first further notations: let $K_\infty = \bigcup_{s=1}^{\infty} K_s$ and $G_\infty = \text{Gal}(\bar{K}/K_\infty)$. By some works of Fontaine, Breuil and Kisin, we know that the restriction of T to G_∞ is described by some data of (semi-)linear algebra that we will call in the sequel *Kisin modules*². Let's call it \mathfrak{M} . In the two following sections, we will show that the data of \mathfrak{M} is enough to recover the whole action of G_s on T for $s > s_{\min} := n - 1 + \log_p(nr)$.

More precisely, we first prove in section 2 (Theorem 2.5.5) that any Kisin module killed by p^n determines a canonical representation of G_s with $s > s_{\min}$ (and not only G_∞). Note that this first step does not use any assumption of semi-stability: our result is valid for all representations (killed by p^n) coming from a Kisin module; no matter if it can be realized as a quotient of two lattices in a semi-stable representation. Then, in section 3, we show that the G_s -representation attached to \mathfrak{M} coincide with $T|_{G_s}$. At this level, let us mention an interesting corollary of the theory developed in these two sections:

Theorem 1.5 (Corollary 3.3.5). *Let V and V' be two semi-stable representations of G . Let T (resp. T') a quotient of two G -lattices in V (resp. V') which is killed by p^n . Then any morphism G_∞ -equivariant $f : T \rightarrow T'$ is G_s -equivariant for all integer $s > n - 1 + \log_p(nr)$.*

Then, we conclude the proof of Theorem 1.1 using usual techniques developed by Fontaine in [9]. Using some kind of transitivity formulas, we then deduce Theorem 1.3. Finally, in the section 5, we begin a discussion about the possibility, given a torsion representation of G_K , to write it as a quotient of two lattices in a \mathbb{Q}_p -representation satisfying some properties (like being crystalline, semi-stable, with prescribed Hodge-Tate weights).

Conventions. For any \mathbb{Z} -module M , we always use M_n to denote $M/p^n M$. If A be a ring, then $M_d(A)$ will denote the ring of $d \times d$ -matrices with coefficients in A . We reserve φ to represent various Frobenius structures (except that σ stands for usual Frobenius on $W(k)$) and φ_M will denote the Frobenius on M . But we always drop the subscript if no confusion arises.

Finally, if A is a ring equipped with a valuation v_A we will often set:

$$\mathfrak{a}_A^{\geq v} = \{x \in A / v_A(x) \geq v\} \quad \text{and} \quad \mathfrak{a}_A^{> v} = \{x \in A / v_A(x) > v\}.$$

²In fact, these modules were first introduced by Breuil in [5] and [6]. However, we think that the terminology is not so bad since ‘‘Breuil modules’’ is already used for other things and ‘‘Kisin modules’’ were actually intensively studied by Kisin in [17] and [18].

2. G_s -REPRESENTATION ATTACHED TO A TORSION KISIN MODULE

In this section, we prove that G_∞ -representation $T_{\mathfrak{S}_n}(\mathfrak{M})$ attached to Kisin modules \mathfrak{M} killed by p^n can be naturally extended to a G_s -representation for all $s > n - 1 + \log_p(nr)$ (and sometimes better).

2.1. Definitions and basic properties of Kisin modules. Recall the following notations: k is a perfect field, $W = W(k)$, K is a totally ramified extension of $W[1/p]$ of degree e , π is a fixed uniformizer of K and $E(u)$ is the minimal polynomial of π . Recall also that we have fixed a positive integer r . Define $E(u)$ to be the minimal polynomial of π over $W[1/p]$.

The base ring for Kisin modules is $\mathfrak{S} = W[[u]]$. It is endowed with a Frobenius map $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ defined by:

$$\varphi\left(\sum_{i \geq 0} a_i u^i\right) = \sum_{i \geq 0} \sigma(a_i) u^{pi}$$

where σ stands for usual Frobenius on W . By definition, a *free Kisin module* (of height $\leq r$) is a \mathfrak{S} -module \mathfrak{M} free of finite rank equipped with a φ -semi-linear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ such that the following condition holds:

$$(2.1.1) \quad \text{the } \mathfrak{S}\text{-submodule of } \mathfrak{M} \text{ generated by } \varphi_{\mathfrak{M}}(\mathfrak{M}) \text{ contains } E(u)^r \mathfrak{M}.$$

We denote by $\text{Mod}_{\mathfrak{S}}^{\varphi, r}$ their category. Of course, a morphism in $\text{Mod}_{\mathfrak{S}}^{\varphi, r}$ is just a \mathfrak{S} -linear map that commutes with Frobenius actions. In the sequel, if there is no risk of confusion, we will often write φ instead of $\varphi_{\mathfrak{M}}$.

There is also a notion of *torsion Kisin modules* of height $\leq r$. They are modules \mathfrak{M} over \mathfrak{S} equipped with a φ -semi-linear map $\varphi : \mathfrak{M} \rightarrow \mathfrak{M}$ such that:

- \mathfrak{M} is killed by a power of p ;
- \mathfrak{M} has no u -torsion;
- condition (2.1.1) holds.

Let us call $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi, r}$ (resp. $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$, resp. $\text{Free}_{\mathfrak{S}_n}^{\varphi, r}$) the category of all torsion Kisin modules (resp. of torsion Kisin modules killed by p^n , resp. torsion Kisin modules killed by p^n and free over $\mathfrak{S}_n = \mathfrak{S}/p^n \mathfrak{S}$). Obviously $\text{Free}_{\mathfrak{S}_n}^{\varphi, r} \subset \text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ and $\bigcup_{n \geq 1} \text{Mod}_{\mathfrak{S}_n}^{\varphi, r} = \text{Mod}_{\mathfrak{S}_\infty}^{\varphi, r}$ (the union is increasing). It is proved in theorem proposition 2.3.2 of [19] that torsion Kisin modules are exactly quotients of two free Kisin modules of same rank. In particular every object in $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ is a quotient of an object in $\text{Free}_{\mathfrak{S}_n}^{\varphi, r}$. We finally note that *dévissages* with torsion Kisin modules are in general quite easy to achieve since if \mathfrak{M} is in $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ then $\mathfrak{M}_{(p)} = \ker p|_{\mathfrak{M}}$ and $\mathfrak{M}/\mathfrak{M}_{(p)}$ are respectively in $\text{Mod}_{\mathfrak{S}_1}^{\varphi, r}$ and $\text{Mod}_{\mathfrak{S}_{n-1}}^{\varphi, r}$ and we obviously have an exact sequence $0 \rightarrow \mathfrak{M}_{(p)} \rightarrow \mathfrak{M} \rightarrow \mathfrak{M}/\mathfrak{M}_{(p)} \rightarrow 0$ (see Proposition 2.3.2 in [19]).

2.2. Functors to Galois representations. We first need to define some period rings. Let $R = \varprojlim_s \mathcal{O}_{\bar{K}}/p$ where transition maps are Frobenius. By definition an element $x \in R$ in a sequence $(x^{(0)}, x^{(1)}, \dots)$ such that $(x^{(s+1)})^p = x^{(s)}$. Fontaine proves in [12] that R is equipped with a valuation defined by $v_R(x) = \lim_{s \rightarrow \infty} p^s v_K(x^{(s)})$ if $x \neq 0$. (In this case, $x^{(s)}$ does not vanish for s large enough and its valuation is then well defined; starting from this rank, the sequence $p^s v_K(x^{(s)})$ is constant.) Note that k embeds naturally in R via $\lambda \mapsto (\lambda^{(0)}, \lambda^{(1)}, \dots)$ where $\lambda^{(s)}$ is the unique p^s -th root of λ in k (recall that k is assumed to be perfect).

This embedding turns R into a k -algebra. Now, consider $W(R)$ (resp. $W_n(R)$) the ring of Witt vectors (resp. truncated Witt vectors) with coefficients in R . It is a W -algebra (resp. a $W_n(k)$ -algebra). Moreover, since Frobenius is bijective on R , $W_n(R) = W(R)/p^n W(R)$. Recall that we have fixed (π_s) a compatible sequence of p^s -roots of π . It defines an element $\underline{\pi} \in R$ whose Teichmüller representative is denoted by $[\underline{\pi}]$. We can then define an embedding $\mathfrak{S} \hookrightarrow W(R)$, $u \mapsto [\underline{\pi}]u$. For any positive integer n , reducing modulo p^n , we get a map $\mathfrak{S}_n \hookrightarrow W_n(R)$ which remains injective. In the sequel, we will often still denote by u its image in $W(R)$ and $W_n(R)$. Let $\mathcal{O}_{\mathcal{E}}$ be the closure in $W(\text{Frac } R)$ of $\mathfrak{S}[1/u]$ (for the p -adic topology). Define $\mathcal{E} = \text{Frac } \mathcal{O}_{\mathcal{E}}$ and $\widehat{\mathcal{E}}^{\text{ur}}$ the p -adic completion of the maximal (algebraic) unramified extension of \mathcal{E} in $W(\text{Frac } R)[1/p]$. Denote $\mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}}$ its ring of integers and put $\mathfrak{S}^{\text{ur}} = W(R) \cap \mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}}$. Clearly \mathfrak{S}^{ur} is subring of $W(R)$ and one can check (see Proposition 2.2.1 of [19]) that it induces an embedding $\mathfrak{S}_n^{\text{ur}} = \mathfrak{S}^{\text{ur}}/p^n \mathfrak{S}^{\text{ur}} \hookrightarrow W_n(R)$. Remark finally that all previous rings are endowed with a Frobenius action.

Recall that G (resp. G_s) is the absolute Galois group of K (resp. $K_s = K(\pi_s)$) and that G_{∞} is intersection of all G_s . Denote by $\text{Rep}_{\mathbb{Z}_p}^{\text{free}}(G_{\infty})$ (resp. $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(G_{\infty})$) the category of free (resp. torsion) \mathbb{Z}_p -representations of G_{∞} . We define functors $T_{\mathfrak{S}} : \text{Mod}_{\mathfrak{S}}^{\varphi, r} \rightarrow \text{Rep}_{\mathbb{Z}_p}^{\text{free}}(G_{\infty})$ and $T_{\mathfrak{S}_n} : \text{Mod}_{\mathfrak{S}_n}^{\varphi, r} \rightarrow \text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(G_{\infty})$ by:

$$T_{\mathfrak{S}}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\text{ur}}) \quad \text{and} \quad T_{\mathfrak{S}_n}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}_n^{\text{ur}})$$

where $\text{Hom}_{\mathfrak{S}, \varphi}$ means that we take all \mathfrak{S} -linear morphism that commutes with Frobenius. Note that $T_{\mathfrak{S}}(\mathfrak{M})$ and $T_{\mathfrak{S}_n}(\mathfrak{M})$ are *not* representations of G because this group does not act trivially on $\mathfrak{S} \subset W(R)$. If $n' \geq n$ then any object \mathfrak{M} of $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ is obviously also in $\text{Mod}_{\mathfrak{S}_{n'}}^{\varphi, r}$, and we have a canonical identification $T_{\mathfrak{S}_n}(\mathfrak{M}) \simeq T_{\mathfrak{S}_{n'}}(\mathfrak{M})$. This fact allows us to glue all functors $T_{\mathfrak{S}_n}$ and define $T_{\mathfrak{S}_{\infty}} : \text{Mod}_{\mathfrak{S}_{\infty}}^{\varphi, r} \rightarrow \text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(G_{\infty})$. An important result is the exactness of $T_{\mathfrak{S}_{\infty}}$ (see corollary 2.3.4 of [19]).

Lemma 2.2.1 (Fontaine). *Let n be an integer and \mathfrak{M} be an object of $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$. The embedding $\mathfrak{S}_n^{\text{ur}} \hookrightarrow W_n(R)$ induces an isomorphism $T_{\mathfrak{S}_n}(\mathfrak{M}) \xrightarrow{\sim} \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, W_n(R))$.*

Proof. See Proposition B.1.8.3 of [10]. \square

2.3. The modules $J_{n,c}(\mathfrak{M})$. Let n be an integer and \mathfrak{M} an object of $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$. For all non negative real number c , we define $\mathfrak{a}_R^{>c} = \{x \in R / v_R(x) > c\}$ and $[\mathfrak{a}_R^{>c}]$ the ideal of $W_n(R)$ generated by all $[x]$ with $x \in \mathfrak{a}_R^{>c}$ and, by the same way, $\mathfrak{a}_R^{\geq c}$ and $[\mathfrak{a}_R^{\geq c}]$. We have very explicit descriptions of these ideals:

Lemma 2.3.1. *Let $c \in \mathbb{R}^+$. Then:*

- (1) *for all $x_0, \dots, x_{n-1} \in R$, (x_0, \dots, x_{n-1}) is in $[\mathfrak{a}_R^{>c}]$ (resp $[\mathfrak{a}_R^{\geq c}]$) if and only if $v_R(x_i) > p^i c$ (resp. $v_R(x_i) \geq p^i c$) for all i ;*
- (2) *if $\gamma \in R$ has valuation c , then $[\mathfrak{a}_R^{\geq c}]$ is the principal ideal generated by $[\gamma]$.*

Proof. Easy with the formula $[z](x_0, \dots, x_{n-1}) = (zx_0, z^p x_1, \dots, z^{p^{n-1}} x_{n-1})$. \square

Since $[\mathfrak{a}_R^{>c}]$ is stable under φ and G -action, the quotient $W_n(R)/[\mathfrak{a}_R^{>c}]$ inherits a Frobenius action and it makes sense to define:

$$(2.3.1) \quad J_{n,c}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, W_n(R)/[\mathfrak{a}_R^{>c}]).$$

It is endowed with an action of G_∞ . Let's also denote $J_{n,\infty} = \text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W_n(R)) \simeq T_{\mathfrak{S}_n}(\mathfrak{M})$ (Lemma 2.2.1). Obviously, if $c \leq c' \leq \infty$, reduction modulo $[\mathfrak{a}_R^{>c}]$ defines a natural G_∞ -equivariant morphism $\rho_{c',c} : J_{n,c'}(\mathfrak{M}) \rightarrow J_{n,c}(\mathfrak{M})$. If $c \leq c' \leq c'' \leq \infty$, we have $\rho_{c'',c} = \rho_{c',c} \circ \rho_{c'',c'}$.

Lemma 2.3.2. *u is nilpotent in $W_n[u]/E(u)^r$.*

Proof. Since $E(u)$ is an Eisenstein polynomial, the congruence $E(u) \equiv u^e \pmod{p}$ holds in $W[u]$. Hence $E(u)^r \equiv u^{er} \pmod{p}$, which means that u^{er} is divisible by p in $W[u]/E(u)^r$. It follows that p^n divides u^{ern} in $W[u]/E(u)^r$, i.e. u^{ern} vanishes in $W_n[u]/E(u)^r$. \square

Fix N a positive integer such that $u^N = 0$ in $W_n[u]/E(u)^r$. By previous proof one can take $N = ern$, but in many situations this exponent can be improved. In the following subsection, we will examine several examples. *From now on, we put $b = \frac{N}{p-1}$ and $a = b + N = \frac{pN}{p-1}$.*

Proposition 2.3.3. *The morphism $\rho_{\infty,b} : T_{\mathfrak{S}_n}(\mathfrak{M}) \rightarrow J_{n,b}(\mathfrak{M})$ is injective and its image is $\rho_{a,b}(J_{n,a}(\mathfrak{M}))$.*

Proof. We first prove injectivity. Let $f : \mathfrak{M} \rightarrow [\mathfrak{a}_R^{>b}]$ be a φ -morphism. We want to show that $f = 0$. First, remark that since \mathfrak{M} is finitely generated, values of f are in $[\mathfrak{a}_R^{>b'}]$ for some $b' > b$. Let $x \in \mathfrak{M}$. By definition of N , $u^N x$ belongs to $E(u)^r \mathfrak{M}$. By condition (2.1.1) we can write $u^N x = \lambda_1 \varphi(x_1) + \dots + \lambda_k \varphi(x_k)$. Applying f , we get:

$$u^N f(x) = \lambda_1 \varphi(f(x_1)) + \dots + \lambda_k \varphi(f(x_k)) \in [\mathfrak{a}_R^{>pb'}]$$

and then $f(x) \in [\mathfrak{a}_R^{>pb'-N}]$ (since $u = [\pi]$). Repeating the argument again and again, we see that $f(\mathfrak{M}) \subset \bigcap_{i \geq 0} [\mathfrak{a}_R^{>b_i}] W_n(R)$ where (b_i) is the sequence defined by $b_0 = b'$ and $b_{i+1} = pb_i - N$. Now $b' > \frac{N}{p-1}$ implies $\lim_{i \rightarrow \infty} b_i = \infty$ and injectivity follows.

Let's prove the second part of the proposition. Since $\rho_{\infty,b}$ factors through $\rho_{a,b}$ we certainly have $\rho_{\infty,b}(J_{n,\infty}) \subset \rho_{a,b}(J_{n,a})$. Conversely, we want to prove that if $f : \mathfrak{M} \rightarrow W_n(R)/[\mathfrak{a}_R^{>a}]$ is a φ -morphism, then there exists a φ -morphism (necessarily unique) $g : \mathfrak{M} \rightarrow W_n(R)$ such that $g \equiv f \pmod{[\mathfrak{a}_R^{>b}]}$. Assume first $\mathfrak{M} \in \text{Free}_{\mathfrak{S}_n}^{e,r}$ and pick (e_1, \dots, e_d) a basis of \mathfrak{M} over \mathfrak{S}_n . Let A be a matrix with coefficients in \mathfrak{S}_n such that:

$$(\varphi(e_1), \dots, \varphi(e_d)) = (e_1, \dots, e_d)A$$

and let X be a line vector with coefficients in $W_n(R)$ that lifts $(f(e_1), \dots, f(e_d))$.

The commutation of f and φ implies the relation $XA = \varphi(X) \pmod{[\mathfrak{a}_R^{>a}]}$. Actually, the congruence holds in $[\mathfrak{a}_R^{>a'}]$ for some $a' > a$. For the rest of the proof, fix $\alpha \in R$ some element of valuation a' . By Lemma 2.3.1.(2), $[\mathfrak{a}_R^{>a'}]$ is the principal ideal generated by $[\alpha]$. Therefore, one have $XA - \varphi(X) = [\alpha]Q$ with coefficients of Q in $W_n(R)$. We want to prove that there exists a matrix Y with coefficients in $[\mathfrak{a}_R^{>b}]$ such that $(X + Y)A = \varphi(X + Y)$. Let us search Y of the shape $[\beta]Z$ with $\beta = \frac{\alpha}{u^N}$ (which belongs to R because of valuations) and coefficients of Z in $W_n(R)$. Our condition then becomes:

$$(2.3.2) \quad [\beta]ZA = [\beta^p]\varphi(Z) + [\alpha]Q$$

Using condition (2.1.1) and $u^N \in E(u)^r W_n[u]$, we find a matrix B (with coefficients in \mathfrak{S}_n) such that $BA = u^N$. Multiplying (2.3.2) by B on the left and simplifying by $[\alpha]$, we get the new equation:

$$(2.3.3) \quad Z = [\gamma]\varphi(Z)B + QB$$

with $\gamma = \alpha^{p-1}/u^{pN}$. Remark that $v_R(\gamma) = a'(p-1) - N > 0$; hence $\gamma \in R$. Now define a sequence (Z_i) by $Z_0 = 0$ and $Z_{i+1} = [\gamma]\varphi(Z_i)B + QB$. We have $Z_{i+1} - Z_i = [\gamma]\varphi(Z_i - Z_{i-1})B$. Since $v_R(\gamma) > 0$, $Z_{i+1} - Z_i$ goes to 0 for the u -adic topology (which is separate and complete on $W_n(R)$) when i goes to infinity. Hence (Z_i) converges to a limit Z which is solution of (2.3.3).

Finally, if \mathfrak{M} is just an object of $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ consider $\mathfrak{M}' \in \text{Free}_{\mathfrak{S}_n}^{\varphi, r}$ and a surjective map $f : \mathfrak{M}' \rightarrow \mathfrak{M}$. Then $\ker f$ is in $\text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ and sits in the following diagram:

$$\begin{array}{ccccc} 0 & & & & 0 \\ \downarrow & & & & \downarrow \\ T_{\mathfrak{S}_n}(\mathfrak{M}) & \longrightarrow & J_{n,a}(\mathfrak{M}) & \longrightarrow & J_{n,b}(\mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow \\ T_{\mathfrak{S}_n}(\mathfrak{M}') & \longrightarrow & J_{n,a}(\mathfrak{M}') & \longrightarrow & J_{n,b}(\mathfrak{M}') \\ \downarrow & & \downarrow & & \downarrow \\ T_{\mathfrak{S}_n}(\ker f) & \hookrightarrow & & \longrightarrow & J_{n,b}(\ker f) \end{array}$$

where all columns are exact (by left exactness of Hom) and the map on last line is injective (by first part of proposition). An easy diagram chase then ends the proof. \square

Remark 2.3.4. In general, $\rho_{a,b}$ is not surjective (nor injective) even for a and b big enough. Counter examples are very easy to produce: for instance, $\mathfrak{M} = \mathfrak{S}_1 \mathfrak{e}$ equipped with $\varphi(\mathfrak{e}) = E(u)^r \mathfrak{e}$ is convenient.

2.4. Brief discussion about sharpness of N . Here we are interested in finding integers N (as small as possible) such that $u^N = 0$ in $W_n[u]/E(u)^r$. As we have said before $N = ern$ is always convenient. If $n = 1$, it is obviously the best constant. However, it is not true anymore for bigger n : the three following lemmas could give better exponents in many cases. We do not know how to find the sharpest N in general.

In this paragraph, we will denote by $[x]$ the smallest integer not less than x .

Lemma 2.4.1. *We have $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = ep^{n-1} \lceil \frac{r}{p^{n-1}} \rceil$.*

Proof. Just remark that $E(u)^{p^{n-1}} \equiv u^{ep^{n-1}} \pmod{p^n}$. \square

Lemma 2.4.2. *Assume $E(u) = u^e - p$. Then $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = e(n+r-1)$.*

Remark 2.4.3. If $K/W[1/p]$ is tamely ramified, up to changing K by an unramified extension, we can always select a uniformizer whose minimal polynomial is $E(u) = u^e - p$.

Proof. Up to performing the variables change $v = u^e$, one may assume $e = 1$. We then have an isomorphism $f : K[u]/E(u)^r \rightarrow K^r$, $P \mapsto (P(p), P'(p), \dots, \frac{P^{(r-1)}(p)}{(r-1)!})$ whose inverse is given by $f^{-1}(x_0, \dots, x_{r-1}) = x_0 + x_1(u-p) + \dots + x_{r-1}(u-p)^{r-1}$. In particular $f(W[u]/E(u)^r) \supset W^r$. Moreover:

$$f(u^N) = \left(p^N, Np^{N-1}, \dots, \binom{N}{r-1} p^{N-r+1} \right) \in p^{N-r+1} W^r = p^n W^r.$$

Conclusion follows. \square

Lemma 2.4.4. *There exists a constant c depending only on K such that $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = en + c(r-1)$.*

Proof. The general plan of the proof is very similar to the previous one. We first consider the map $f : W[1/p][u]/E(u)^r \rightarrow K^r$, $P \mapsto (P(\pi), P'(\pi), \dots, \frac{P^{(r-1)}(\pi)}{(r-1)!})$. It is $W[1/p]$ -linear and injective. Since both sides are $W[1/p]$ -vector spaces of dimension er , f is an isomorphism. Denote by $\varpi \in W[1/p][u]/E(u)^r$ the preimage of $(\pi, 0, \dots, 0)$. The inverse of f is then given by the formula:

$$f^{-1}(x_0, \dots, x_{r-1}) = X_0(\varpi) + X_1(\varpi)(u - \varpi) + \dots + X_{r-1}(\varpi)(u - \varpi)^{r-1}$$

where X_i are polynomials with coefficients in K such that $X_i(\pi) = x_i$. Second, we would like to bound below the “ p -adic valuation” of $f^{-1}(x_0, \dots, x_{r-1})$ when all x_i lies in \mathcal{O}_K . For that, we remark that $E(\varpi)$ is mapped to 0 by f ; hence it vanishes. Solving this equation by successive approximations, we find that ϖ can be written $P_0(u) + P_1(u)E(u) + \dots + P_{r-1}(u)E(u)^{r-1}$ with $P_0(u) = u$ and:

$$E'(u)P_{i+1}(u) \equiv \frac{E(P_0(u) + P_1(u)E(u) + \dots + P_{i-1}(u)E(u)^{i-1})}{E(u)^i} \pmod{E(u)}$$

where P_i are uniquely determined modulo $E(u)^{r-i}$. Let $F(u) \in W[1/p][u]/E(u)$ be the inverse of $E'(u)$ and v an integer such that $p^v F(u) \in W[u]/E(u) \simeq \mathcal{O}_K$. (Note that $v = \lceil v_p(\mathcal{D}_{K/W[1/p]}) \rceil$ is convenient.) By induction we easily prove that $p^{iv} P_i(u) \in W[u]/E(u)^{r-i}$, and then that $Q(\varpi) \in W[u]/E(u)^r$ for all $Q \in p^{(r-1)v} W[u]$. Consequently $f(W[u]/E(u)^r) \supset p^{(r-1)v} \mathcal{O}_K^r$. Finally, defining $c = ev + 1$ and $N = en + c(r-1)$, we have:

$$f(u^N) = \left(\pi^N, N\pi^{N-1}, \dots, \binom{N}{r-1} \pi^{N-r+1} \right) \in \pi^{N-r+1} \cdot \mathcal{O}_K^r \subset p^{(r-1)v+n} \cdot \mathcal{O}_K^r$$

and we are done. \square

2.5. Some quotients of $W_n(R)$. The aim of this last subsection is to study the structure of quotients $W_n(R)/[\mathfrak{a}_R^{>c}]$ that appears in the definition of $J_{n,c}$ (see formula (2.3.1)). It will allow us to derive interesting corollaries about the prolongation to a finite index subgroup of G of the natural action of G_∞ on $T_{\mathfrak{S}_n}(\mathfrak{M})$.

For a non negative integer s , let us denote by θ_s the ring morphism $R \rightarrow \mathcal{O}_{\bar{K}}/p$, $x = (x^{(0)}, x^{(1)}, \dots) \mapsto x^{(s)}$. We emphasize that it is not k -linear: it induces a morphism of k -algebras between R and $k \otimes_{k, \sigma^s} \mathcal{O}_{\bar{K}}/p$. For a non negative real number c , define:

$$\mathfrak{a}_{\bar{K}}^{>c} = \{x \in \bar{K}/v_K(x) > c\} \subset \mathcal{O}_{\bar{K}}.$$

Lemma 2.5.1. *Let c be a positive real number. For any integer $s > \log_p(\frac{c}{e})$, the map θ_s induces a Galois equivariant isomorphism of k -algebras*

$$R/\mathfrak{a}_R^{>c} \rightarrow k \otimes_{k, \sigma^s} \mathcal{O}_{\bar{K}}/\mathfrak{a}_{\bar{K}}^{>c/p^s}.$$

Proof. The map is clearly surjective. It remains to show that $x = (x^{(0)}, x^{(1)}, \dots)$ has valuation greater than c if and only if $v_K(x^{(s)}) > \frac{c}{p^s}$, which follows directly from $\frac{c}{p^s} < e$. \square

Proposition 2.5.2. *Let c be a positive real number. For any $s > n - 1 + \log_p(\frac{c}{e})$, θ_s induces a Galois equivariant isomorphism of $W_n(k)$ -algebras:*

$$\frac{W_n(R)}{[\mathfrak{a}_R^{>c}]} \rightarrow W_n(k) \otimes_{W_n(k), \sigma^s} \frac{W_n(\mathcal{O}_{\bar{K}}/p)}{[\mathfrak{a}_{\bar{K}}^{>c/p^s}]}.$$

Proof. Since θ_s is surjective, the map considered in the lemma is also surjective. Let $x = (x_0, \dots, x_{n-1}) \in W_n(R)$ and assume that $x^{(s)} = (x_0^{(s)}, \dots, x_{n-1}^{(s)})$ lies in $[\mathfrak{a}_{\bar{K}}^{>c/p^s}]$. By an analogue of Lemma 2.3.1.(1), one obtain $v_K(x_i^{(s)}) > \frac{c}{p^{i-s}}$ for all i . Hence, $x_i^{(s)}$ is in $\mathfrak{a}_{\bar{K}}^{>cp^i/p^s}$. Since $\log_p(\frac{cp^i}{e}) = i + \log_p(\frac{c}{e}) \leq n - 1 + \log_p(\frac{c}{e}) < s$, we can apply Lemma 2.5.1 and deduce $x_i \in [\mathfrak{a}_R^{>cp^i}]$, i.e. $v_R(x_i) > cp^{n-1+i}$ for all i . By Lemma 2.3.1.(1), it follows that $x \in [\mathfrak{a}_R^{>c}]$. Thus, the map of the proposition is injective and we are done. \square

Define increasing functions s_0 and s_1 by $s_0(c) = n - 1 + \log_p(\frac{c}{e})$ and $s_1(c) = n - 1 + \log_p(\frac{c(p-1)}{ep}) = s_0(c) + \log_p(1 - \frac{1}{p})$. Recall that we have defined $a = \frac{pN}{p-1}$ (where N is an integer such that $u^N = 0$ in $W_n[u]/E(u)$) and set finally $s_{\min} = s_1(a) = n - 1 + \log_p(\frac{N}{e})$. If we choose $N = ern$, we just have $s_{\min} = n - 1 + \log_p(rn)$.

Proposition 2.5.3. *Let n be a positive integer and $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$. For any non negative integer $s > s_1(c)$, the natural action of G_s on $W_n(R)$ turns $J_{n,c}(\mathfrak{M})$ into a $\mathbb{Z}_p[G_s]$ -module. Furthermore, we have the following compatibilities:*

- the action of G_s is compatible with the usual action of G_∞ on $J_{n,c}(\mathfrak{M})$;
- if $s' \geq s \geq s_1(c)$, actions of $G_{s'}$ and G_s on $J_{n,c}$ are compatible each other;
- if $c' \geq c$ and $s \geq s_1(c')$, then $\rho_{c',c} : J_{n,c'}(\mathfrak{M}) \rightarrow J_{n,c}(\mathfrak{M})$ is G_s -equivariant.

Proof. For the first statement, it is enough to show that G_s acts trivially on $u \in W_n(R)/[\mathfrak{a}_R^{>c}]$ for $s = 1 + [s_1(c)]$ (where $[\cdot]$ denotes the integer part). Put $s' = 1 + [s_0(c)]$. Since $0 \leq \log_p(\frac{p}{p-1}) \leq 1$, we have $s' = s$ or $s' = s + 1$. By Proposition 2.5.2, $W_n(R)/[\mathfrak{a}_R^{>c}]$ is isomorphic to $W_n(\mathcal{O}_{\bar{K}}/p)/[\mathfrak{a}_{\bar{K}}^{>c/p^{s'}}]$. Hence we have to show that $g[\pi_{s'}] - [\pi_{s'}]$ belongs to $[\mathfrak{a}_R^{>c/p^{s'}}]$ for all $g \in G_s$. It is clear for $g \in G_{s'}$ (since the difference vanishes). It remains to consider the case where $s' = s + 1$ and $g \notin G_{s'} = G_{s+1}$. Then $g\pi_{s+1} = (1 + \eta)\pi_{s+1}$ where $(1 + \eta)$ is a primitive p -th root of unity. Let us compute $(g\pi_{s+1}, 0, \dots, 0) - (\pi_{s+1}, 0, \dots, 0) = (x_0, \dots, x_{n-1})$

in $W_n(\mathcal{O}_{\bar{K}})$. By writing phantom components, we get the following system:

$$\begin{cases} x_0 = \eta \pi_{s+1} \\ x_0^p + px_1 = 0 \\ \vdots \\ x_0^{p^{n-1}} + px_1^{p^{n-2}} + \cdots + p^{n-1}x_{n-1} = 0 \end{cases}$$

Using $v_K(\eta) = \frac{e}{p-1}$, we easily prove by induction on i that $v_K(x_i) = \frac{e}{p-1} + \frac{1}{p^{s+1-i}}$.

Thus $v_K(x_i) > \frac{c}{p^{s+1-i}}$ for all i and $(x_0, \dots, x_{n-1}) \in [\mathfrak{a}_{\bar{K}}^{>c/p^{s'}}]$ as expected.

Second part of proposition (*i.e.* compatibilities) is obvious. \square

Remark 2.5.4. If $c \geq \frac{p-1}{p-2}$, the bound $s_1(c)$ that appears in the Theorem can be replaced by $s_1(c-1)$. The proof is totally the same.

Theorem 2.5.5. *For any $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_n}^{\varphi, r}$ and any integer $s > s_{\min}$, $T_{\mathfrak{S}_n}(\mathfrak{M})$ is canonically endowed with an action of G_s (which prolongs the natural action of G_∞).*

Proof. Just combine Propositions 2.3.3 and Proposition 2.5.3. \square

Remark 2.5.6. Using Remark 2.5.4, it appears that we may replace $s_{\min} = s_1(a)$ by $s_1(a-1)$ in previous Theorem. However, it won't be useful in the sequel since s_{\min} is really needed in Theorem 3.3.4.

3. TORSION SEMI-STABLE GALOIS REPRESENTATIONS

In this section, we use the theory of (φ, \hat{G}) -modules to define $\hat{J}_{n,a}(\mathfrak{M})$ attached to p^n -torsion semi-stable representation T . After establishing isomorphism (of $\mathbb{Z}_p[G_\infty]$ -modules) between $\hat{J}_{n,a}(\mathfrak{M})$ and $J_{n,a}(\mathfrak{M})$, we will show that $J_{n,a}(\mathfrak{M}) \simeq T$ as G_s -modules with $s > s_{\min}$.

3.1. Torsion (φ, \hat{G}) -modules. We refer readers to [13] for the definition and standard facts on semi-stable representations.

We first review some facts on (φ, \hat{G}) -modules in [20] and extend them to p^n -torsion case. We denote by S the p -adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There is a unique continuous map (Frobenius) $\varphi : S \rightarrow S$ which extends the Frobenius on \mathfrak{S} . Define a continuous $W(k)$ -linear derivation $N : S \rightarrow S$ such that $N(u) = -u$.

Recall $R = \varprojlim_s \mathcal{O}_{\bar{K}}/p$. There is a unique surjective continuous map $\theta : W(R) \rightarrow \hat{\mathcal{O}}_{\bar{K}}$ which lifts the projection $R \rightarrow \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p -adic completion of the divided power envelope of $W(R)$ with respect to $\text{Ker}(\theta)$. Recall that $[\underline{\pi}] \in W(R)$ is the Teichmüller representative of $\underline{\pi} = (\pi_s)_{s \geq 0} \in R$ and we embed the $W(k)$ -algebra $W(k)[u]$ into $W(R)$ via $u \mapsto [\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$, this embedding extends to an embedding $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\text{cris}}$, and $\theta|_S$ is the $W(k)$ -linear map $s : S \rightarrow \mathcal{O}_{\bar{K}}$ defined by sending u to π . The embedding is compatible with Frobenius endomorphisms. As usual, we write $B_{\text{cris}}^+ := A_{\text{cris}}[1/p]$.

For any field extension F/\mathbb{Q}_p , set $F_{p^\infty} := \bigcup_{n=1}^{\infty} F(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n -th root of unity. Note that $K_{\infty, p^\infty} := \bigcup_{n=1}^{\infty} K(\pi_n, \zeta_{p^n})$ is Galois over K . Let $G_{p^\infty} :=$

$\text{Gal}(K_{\infty, p^\infty}/K_{p^\infty})$, $H_K := \text{Gal}(K_{\infty, p^\infty}/K_\infty)$ and $\hat{G} := \text{Gal}(K_{\infty, p^\infty}/K)$. By Lemma 5.1.2 in [21], we have $K_{p^\infty} \cap K_\infty = K$, $\hat{G} = G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$. For any $g \in G$, write $\underline{\epsilon}(g) = g(\underline{x})/\underline{x}$. Then $\underline{\epsilon}(g)$ is a cocycle from G to the group of units of R^* . In particular, fixing a topological generator τ of G_{p^∞} , the fact that $\hat{G} = G_{p^\infty} \rtimes H_K$ implies that $\underline{\epsilon}(\tau) = (\epsilon_s)_{s \geq 0} \in R^*$ with ϵ_s a primitive p^s -th root of unity. Therefore, $t := -\log([\underline{\epsilon}(\tau)]) \in A_{\text{cris}}$ is well defined and for any $g \in G$, $g(t) = \chi(g)t$ where χ is the cyclotomic character. We reserve $\underline{\epsilon}$ for $\underline{\epsilon}(\tau)$.

For any integer $n \geq 0$, let $t^{\{n\}} = t^{r(n)} \gamma_{\tilde{q}(n)}(t^{p-1}/p)$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. Define subrings \mathcal{R}_{K_0} and $\hat{\mathcal{R}}$ of B_{cris}^+ as in §2.2, [20]:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S[1/p] \text{ and } f_i \rightarrow 0 \text{ as } i \rightarrow +\infty \right\}$$

and $\hat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}$. Let $I_+R = \{x \in R / v_R(x) > 0\} = \mathfrak{a}_R^{>0}$ be the maximal ideal of R . We have exact sequences

$$0 \rightarrow W_n(I_+R) \rightarrow W_n(R) \xrightarrow{\nu_n} W_n(\bar{k}) \rightarrow 0 \text{ and } 0 \rightarrow W(I_+R) \rightarrow W(R) \xrightarrow{\nu} W(\bar{k}) \rightarrow 0.$$

One can naturally extend ν to $\nu : B_{\text{cris}}^+ \rightarrow W(\bar{k})[\frac{1}{p}]$ (see the proof of Lemma 2.2.1 in [20]). For any subring A of B_{cris}^+ (resp. $W_n(R)$), we write $I_+A = \text{Ker}(\nu) \cap A$ (resp. $I_+A = \text{Ker}(\nu_n) \cap A$) and $I_+ := I_+\hat{\mathcal{R}}$. Now recall M_n stands for $M/p^n M$.

Lemma 3.1.1. *We have the following commutative diagram :*

$$(3.1.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & W_n(I_+R) & \longrightarrow & W_n(R) & \xrightarrow{\nu_n} & W_n(\bar{k}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I_{+,n} & \longrightarrow & \hat{\mathcal{R}}_n & \longrightarrow & W_n(k) \longrightarrow 0 \end{array}$$

such that both rows are short exact and all vertical arrows are injective.

Proof. By Lemma 2.2.1 in [20], we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_+W(R) & \longrightarrow & W(R) & \xrightarrow{\nu} & W(\bar{k}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I_+ & \longrightarrow & \hat{\mathcal{R}} & \longrightarrow & W(k) \longrightarrow 0 \end{array}$$

Modulo p^n and noting that $I_+W(R) = I_+B_{\text{cris}}^+ \cap W(R) = W(I_+R)$, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_n(I_+R) & \longrightarrow & W_n(R) & \xrightarrow{\nu_n} & W_n(\bar{k}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & I_{+,n} & \longrightarrow & \hat{\mathcal{R}}_n & \longrightarrow & W_n(k) \longrightarrow 0 \end{array}$$

Now it suffices to show that the bottom arrow is left exact and the last two vertical arrows are injective. The last one is obvious. To see the middle arrow, it suffices to show that $(p^n W(R)) \cap \hat{\mathcal{R}} = p^n \hat{\mathcal{R}}$. Note that $\hat{\mathcal{R}} = \mathcal{R}_{K_0} \cap W(R)$. Let $x \in W(R)$ such that $p^n x \in \hat{\mathcal{R}} \subset \mathcal{R}_{K_0}$. Then $x \in W(R) \cap \mathcal{R}_{K_0} = \hat{\mathcal{R}}$. So $p^n x \in p^n \hat{\mathcal{R}}$ and

$(p^n W(R)) \cap \widehat{\mathcal{R}} = p^n \widehat{\mathcal{R}}$. To see the bottom is left exact, it suffices to show that $I_+ \cap p^n \widehat{\mathcal{R}} = p^n I_+$. But $I_+ = I_+ \mathcal{R}_{K_0} \cap \widehat{\mathcal{R}}$. Let $x \in \widehat{\mathcal{R}}$ such that $p^n x \in I_+ \cap p^n \widehat{\mathcal{R}}$. Then $x \in I_+ \mathcal{R}_{K_0}$. Thus $x \in I_+ \mathcal{R}_{K_0} \cap \widehat{\mathcal{R}} = I_+$ and $I_+ \cap p^n \widehat{\mathcal{R}} = p^n I_+$. \square

As in Lemma 2.2.1 in [20], we see that $\widehat{\mathcal{R}}$ (resp. $\widehat{\mathcal{R}}_n$) is φ -stable and G -stable subring of $W(R)$ (resp. $W_n(R)$), G -action on $\widehat{\mathcal{R}}$ factors through \hat{G} . Let (\mathfrak{M}, φ) be a finite free or p^n -torsion Kisin module of height $\leq r$, set $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and consider the following map

$$(3.1.2) \quad \mathfrak{M} \simeq \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \hat{\mathfrak{M}}.$$

We claim the it is an injective (thus \mathfrak{M} can be always regarded as a $\varphi(\mathfrak{S})$ -submodule of $\hat{\mathfrak{M}}$). Indeed, by Lemma 3.1.1, we have $\varphi(\mathfrak{S}_n) \hookrightarrow \widehat{\mathcal{R}}_n \hookrightarrow W_n(R)$. Thus the claim is clear if \mathfrak{M} is finite \mathfrak{S} -free or \mathfrak{M} is finite \mathfrak{S}_n -free. For a general \mathfrak{M} which is killed by p^n , by the discussion in the end of §2.1, \mathfrak{M} can be written as a successive extension of finite free \mathfrak{S}_1 -modules. Therefore one can reduce the proof of the claim to the following lemma.

Lemma 3.1.2. *The functor $\mathfrak{M} \mapsto \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ (resp. $\mathfrak{M} \mapsto W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$) is an exact functor from the category of Kisin modules to the category of $\widehat{\mathcal{R}}$ -modules (resp. $W(R)$ -modules).*

Proof. We only prove the exactness of the first functor, the proof for the second being totally the same. It suffices to prove that $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, \widehat{\mathcal{R}}) = 0$ for any Kisin module \mathfrak{M} . Note that there exists finite free Kisin modules $\mathfrak{L}_1 \subset \mathfrak{L}_2$ such that $\mathfrak{M} = \mathfrak{L}_2/\mathfrak{L}_1$ (cf discussion in the end of §2.1). Since $\widehat{\mathcal{R}} \hookrightarrow W(R)$ is an integral domain and $\varphi : W(R) \rightarrow W(R)$ is injective, we see $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{L}_1 \rightarrow \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{L}_2$ is injective. Thus $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, \widehat{\mathcal{R}}) = 0$. \square

Let (\mathfrak{M}, φ) be a Kisin module of height $\leq r$ and $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\mathfrak{S}, \varphi} \mathfrak{M}$. Frobenius φ on \mathfrak{M} can be extended to $\hat{\mathfrak{M}}$ semi-linearly by $\varphi_{\hat{\mathfrak{M}}}(a \otimes x) = \varphi_{\widehat{\mathcal{R}}}(a) \otimes \varphi_{\mathfrak{M}}(x)$.

Now we can make the following definition: a (φ, \hat{G}) -module of height $\leq r$ is a triple $(\mathfrak{M}, \varphi, \hat{G})$ where

- (1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module of height $\leq r$;
- (2) \hat{G} is a $\widehat{\mathcal{R}}$ -semi-linear \hat{G} -action on $\hat{\mathfrak{M}} = \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$;
- (3) \hat{G} commutes with $\varphi_{\hat{\mathfrak{M}}}$ on $\hat{\mathfrak{M}}$, i.e. for any $g \in \hat{G}$, $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}}g$;
- (4) regard \mathfrak{M} as a $\varphi(\mathfrak{S})$ -submodule in $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$;
- (5) \hat{G} acts on $W(k)$ -module $M := \hat{\mathfrak{M}}/I_+ \hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

A morphism $f : (\mathfrak{M}, \varphi, \hat{G}) \rightarrow (\mathfrak{M}', \varphi', \hat{G}')$ is a morphism $f : (\mathfrak{M}, \varphi) \rightarrow (\mathfrak{M}', \varphi')$ of Kisin modules such that $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} f : \hat{\mathfrak{M}} \rightarrow \hat{\mathfrak{M}}'$ is \hat{G} -equivariant. If $\mathfrak{M} = (\mathfrak{M}, \varphi, \hat{G})$ be a (φ, \hat{G}) -module, we will often abuse notations by denoting $\hat{\mathfrak{M}}$ the underline module $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. A (φ, \hat{G}) -module $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ is called *finite free* (resp. p^n -torsion) if \mathfrak{M} is finite \mathfrak{S} -free (resp. \mathfrak{M} is killed by p^n).

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be a (φ, \hat{G}) -module. We can associate $\mathbb{Z}_p[G]$ -modules:

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(R)) \text{ if } \mathfrak{M} \text{ is finite } \mathfrak{S}\text{-free.}$$

and

$$\hat{T}_n(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W_n(R)) \text{ if } \mathfrak{M} \text{ is of } p^n\text{-torsion.}$$

Here G acts on $\hat{T}(\hat{\mathfrak{M}})$ (resp. $\hat{T}_n(\hat{\mathfrak{M}})$) via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G$ and $f \in \hat{T}(\hat{\mathfrak{M}})$ (resp. $\hat{T}_n(\hat{\mathfrak{M}})$). For any $f \in T_{\mathfrak{S}}(\mathfrak{M})$ (resp. $T_{\mathfrak{S}_n}(\mathfrak{M})$), set $\theta(f) \in \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}}, W(R))$ (resp. $\theta_n(f) \in \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}}, W_n(R))$) via:

$$(3.1.3) \quad \theta(f)(a \otimes x) \text{ (resp. } \theta_n(f)(a \otimes x)) = a\varphi(f(x)) \text{ for any } a \in \hat{\mathcal{R}}, x \in \hat{\mathfrak{M}}.$$

It is routine to check that $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \rightarrow \hat{T}(\hat{\mathfrak{M}})$ (resp. $\theta_n : T_{\mathfrak{S}_n}(\mathfrak{M}) \rightarrow \hat{T}_n(\hat{\mathfrak{M}})$) is well-defined.

Denote by $\text{Rep}_{\text{tor}}(G)$ the category of G -representations on finite type \mathbb{Z}_p -modules which are killed by some p -power, and $\text{Rep}_{\text{tor}}^{\text{ss},r}(G)$ the full subcategory of *torsion semi-stable representations* with Hodge-Tate weights in $\{0, \dots, r\}$ in the sense that there exist G -stable \mathbb{Z}_p -lattices $L_1 \subset L_2 \subset V$ such that V is semi-stable with Hodge-Tate weights in $\{0, \dots, r\}$ and $T \simeq L_2/L_1$ as $\mathbb{Z}_p[G]$ -modules. The following is the main result of this subsection.

- Theorem 3.1.3.** (1) *Let $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ be a (φ, \hat{G}) -module. Then θ (resp. θ_n) induces a natural isomorphism of $\mathbb{Z}_p[G_{\infty}]$ -modules $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}})$ (resp. $\theta_n : T_{\mathfrak{S}_n}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}_n(\hat{\mathfrak{M}})$).*
- (2) *\hat{T} induces an anti-equivalence between the category of finite free (φ, \hat{G}) -modules of height $\leq r$ and the category of G -stable \mathbb{Z}_p -lattices in semi-stable representations with Hodge-Tate weights in $\{0, \dots, r\}$.*
- (3) *For any $T \in \text{Rep}_{\text{tor}}^{\text{ss},r}(G)$, there exists a torsion (φ, \hat{G}) -modules $\hat{\mathfrak{M}}$ such that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ as $\mathbb{Z}_p[G]$ -modules.*

Proof. (1) If \mathfrak{M} is finite \mathfrak{S} -free then it has been proved in Theorem 2.3.1 in [20]. The proof of the p^n -torsion case is almost the same, except one need to check that $\hat{\mathfrak{M}}$ is a $\varphi(\mathfrak{S})$ -submodule of $\hat{\mathfrak{M}}$ via (3.1.2), which has been proved below (3.1.2).

(2) See Theorem 2.3.1 in [20].

(3) Let $L_1 \subset L_2$ be G -stable \mathbb{Z}_p -lattices inside a semi-stable representation with Hodge-Tate weights in $\{0, \dots, r\}$ such that $T \simeq L_2/L_1$ as $\mathbb{Z}_p[G]$ -modules. By (2), there exists an injection of Kisin modules (resp. (φ, \hat{G}) -modules) $i : \mathfrak{L}_2 \hookrightarrow \mathfrak{L}_1$ (resp. $\hat{i} : \hat{\mathfrak{L}}_2 \hookrightarrow \hat{\mathfrak{L}}_1$) that corresponds the inclusion $L_1 \subset L_2$. Write $\mathfrak{M} := \mathfrak{L}_1/\mathfrak{L}_2$ (resp. $\hat{\mathfrak{M}} := \hat{\mathfrak{L}}_1/\hat{\mathfrak{L}}_2$). Apparently, there are a φ -action and a \hat{G} -action on \mathfrak{M} induced from $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$. We claim that $\hat{\mathfrak{M}} \simeq \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ as φ -modules and $(\hat{\mathfrak{M}}, \varphi, \hat{G})$ is a (φ, \hat{G}) -modules. To see these, tensor $\hat{\mathcal{R}}$ to the exact sequence $0 \rightarrow \mathfrak{L}_2 \rightarrow \mathfrak{L}_1 \rightarrow \mathfrak{M} \rightarrow 0$. By the proof of Lemma 3.1.2, we see that the sequence $0 \rightarrow \hat{\mathfrak{L}}_2 \rightarrow \hat{\mathfrak{L}}_1 \rightarrow \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow 0$ is still exact. Thus $\hat{\mathfrak{M}} \simeq \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ as φ -modules. Moreover, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{\mathfrak{L}}_2 & \longrightarrow & \hat{\mathfrak{L}}_1 & \longrightarrow & \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathfrak{L}_2 & \longrightarrow & \mathfrak{L}_1 & \longrightarrow & \mathfrak{M} \longrightarrow 0 \end{array}$$

So φ -action and \hat{G} -action on $\hat{\mathfrak{M}}$ commutes, H_K acts on \mathfrak{M} (as $\varphi(\mathfrak{S})$ -submodule in (3.1.2)) trivially, and \hat{G} acts on $\hat{\mathfrak{M}}/I_+ \hat{\mathfrak{M}}$ trivially. Thus $\hat{\mathfrak{M}} = (\hat{\mathfrak{M}}, \varphi, \hat{G})$ is a (φ, \hat{G}) -module. Finally, to see that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ as $\mathbb{Z}_p[G]$ -modules, it suffices to show that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq \hat{T}(\hat{\mathfrak{L}}_2)/\hat{T}(\hat{\mathfrak{L}}_1)$ and we reduce the proof to the following Lemma. \square

Lemma 3.1.4. *Let $0 \rightarrow \hat{\mathcal{L}}_2 \rightarrow \hat{\mathcal{L}}_1 \rightarrow \hat{\mathfrak{M}} \rightarrow 0$ be an exact sequence of (φ, G) -modules with $\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2$ finite free and $\hat{\mathfrak{M}}$ killed by p^n . Then we have an exact sequence of $\mathbb{Z}_p[G]$ -modules $0 \rightarrow \hat{T}(\hat{\mathcal{L}}_1) \rightarrow \hat{T}(\hat{\mathcal{L}}_2) \rightarrow \hat{T}_n(\hat{\mathfrak{M}}) \rightarrow 0$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & p^n \hat{\mathcal{L}}_2 & \longrightarrow & p^n \hat{\mathcal{L}}_1 & \longrightarrow & \hat{\mathfrak{M}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \hat{\mathcal{L}}_2 & \longrightarrow & \hat{\mathcal{L}}_1 & \longrightarrow & \hat{\mathfrak{M}} \longrightarrow 0 \end{array}$$

where the last vertical map is $p^n = 0$. By Snake lemma, we have an exact sequence

$$(3.1.4) \quad 0 \rightarrow \hat{\mathfrak{M}} \rightarrow (\hat{\mathcal{L}}_2)_n \rightarrow (\hat{\mathcal{L}}_1)_n \rightarrow \hat{\mathfrak{M}} \rightarrow 0.$$

Then we get a sequence of $\mathbb{Z}_p[G]$ -modules

$$(3.1.5) \quad 0 \rightarrow \hat{T}_n(\hat{\mathfrak{M}}) \rightarrow \hat{T}_n((\hat{\mathcal{L}}_1)_n) \rightarrow \hat{T}_n((\hat{\mathcal{L}}_2)_n) \rightarrow \hat{T}_n(\hat{\mathfrak{M}}) \rightarrow 0.$$

Since $\hat{T}((\hat{\mathcal{L}}_i)_n) \simeq (\hat{T}(\hat{\mathcal{L}}_i))_n$ for $i = 1, 2$, it suffices to show that the above sequence is exact. But the underline Kisin modules of the exact sequence (3.1.4) is exact. Since $T_{\mathfrak{S}}$ is exact, we get an exact sequence

$$0 \rightarrow T_{\mathfrak{S}_n}(\hat{\mathfrak{M}}) \rightarrow T_{\mathfrak{S}_n}((\hat{\mathcal{L}}_1)_n) \rightarrow T_{\mathfrak{S}_n}((\hat{\mathcal{L}}_2)_n) \rightarrow T_{\mathfrak{S}_n}(\hat{\mathfrak{M}}) \rightarrow 0.$$

Now the exactness of (3.1.5) follows from Theorem 3.1.3.(1). \square

Remark 3.1.5. For a fixed $T \in \text{Rep}_{\text{tor}}^{\text{ss},r}(G)$, it may exist two different (φ, \hat{G}) -modules $\hat{\mathfrak{M}}, \hat{\mathfrak{M}}'$ such that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq \hat{T}_n(\hat{\mathfrak{M}}') \simeq T$. The classical example of this is that $T = \mathbb{Z}/p\mathbb{Z}$ with the trivial G -action and $K = \mathbb{Q}_p(\zeta_p)$.

3.2. G_s -action on $\hat{T}(\hat{\mathcal{L}})$. Let $T \in \text{Rep}_{\text{tor}}^{\text{ss},r}(G)$ be a p^n -torsion representation, and $T \simeq L'/L$ where $L \subset L'$ are G -stable \mathbb{Z}_p -lattices in a semi-stable representation V with Hodge-Tate weights in $\{0, \dots, r\}$. By Theorem 3.1.3, there exists (φ, \hat{G}) -modules $\hat{\mathcal{L}}' \hookrightarrow \hat{\mathcal{L}}$ such that $\hat{T}(\hat{\mathcal{L}}) \hookrightarrow \hat{T}(\hat{\mathcal{L}}')$ corresponds to the injection $L \subset L'$ and $\hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ where $\hat{\mathfrak{M}} := \hat{\mathcal{L}}/\hat{\mathcal{L}}'$. Now write $\mathcal{L}, \mathcal{L}', \mathfrak{M}$ the underline Kisin modules for $\hat{\mathcal{L}}, \hat{\mathcal{L}}', \hat{\mathfrak{M}}$ respectively. Set $\mathcal{D} := S[1/p] \otimes_{\varphi, \mathfrak{S}} \mathcal{L}$ and recall $\underline{\epsilon}(g) := \frac{g(\underline{x})}{\underline{x}}$ for any $g \in G$. §3.2 of [20] explains that there exists a unique $W(k)$ -linear differential operator $N : \mathcal{D} \rightarrow \mathcal{D}$ over $N(u) = -u$ such that G acts on $B_{\text{cris}}^+ \otimes_S \mathcal{D} \simeq B_{\text{cris}}^+ \otimes_{\hat{\mathcal{R}}} \hat{\mathcal{L}}$ via

$$(3.2.1) \quad g(a \otimes x) = \sum_{i=0}^{\infty} g(a) \gamma_i(-\log([\underline{\epsilon}(g)])) \otimes N^i(x), \quad \text{for any } a \in B_{\text{cris}}^+, x \in \mathcal{D}.$$

In particular, recall $t := -\log([\underline{\epsilon}])$ with $\underline{\epsilon} = \underline{\epsilon}(\tau)$ and τ is a fixed generator in G_{p^∞} . For any $x \in \mathcal{L}$, we have $\tau(x) = \sum_{i=0}^{\infty} \gamma_i(t) \otimes N^i(x)$. Let $A \subset B_{\text{cris}}^+$ be a φ -stable subring. Set

$$I^{[m]}A = \{a \in A \mid \varphi^n(a) \in A \cap \text{Fil}^m B_{\text{cris}}^+, \text{ for all } n \geq 0\}.$$

By proposition 5.1.3 in [12], $I^{[m]}W(R)$ is generated by $([\underline{\epsilon}] - 1)^m$ and $v_R(\underline{\epsilon} - 1) = \frac{ep}{p-1}$.

Now, define $s_2(c) := n - 1 + \log_p\left(\frac{(p-1)c}{e}\right) = s_1(c) + 1$. We have the following lemma:

Lemma 3.2.1. *For any $s > s_2(c)$, $g \in G_s$, and $x \in \mathfrak{L}$*

$$g(x) - x \in ([\mathfrak{a}_R^{>pc}] + p^n W(R)) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{L}}$$

where, in a slight abuse of notations, we still denote by $[\mathfrak{a}_R^{>c}]$ the ideal of $W(R)$ generated by all $[x]$ with $x \in \mathfrak{a}_R^{>c}$.

Proof. Note that the G -action on $\widehat{\mathfrak{L}}$ factors through \widehat{G} . So it suffices to consider the action of \widehat{G}_s , which is the image of G_s in \widehat{G} . By Lemma 5.1.2 of [21] applied to K_s , we see that $\widehat{G}_s := G_{s,p^\infty} \rtimes H_K$, where $G_{s,p^\infty} = \text{Gal}(K_{\infty,p^\infty}/K_{s,p^\infty})$. Note that H_K acts on \mathfrak{L} trivially and G_{s,p^∞} is topologically generated by τ^{p^s} . Thus it suffices to prove the proposition for $g = \tau^{p^s}$. Writing

$$\tau^{p^s} - 1 = \sum_{i=1}^{p^s} \binom{p^s}{i} (\tau - 1)^i = \sum_{i=1}^{p^s} \frac{p^s}{i} \binom{p^s - 1}{i - 1} (\tau - 1)^i$$

we see that it is enough to show

$$(\tau - 1)^i(x) \in ([\mathfrak{a}_R^{>pc}] + p^{n-s+v_p(i)} W(R)) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{L}}$$

for all integer i such that $v_p(i) > s - n$, i.e. $v_p(i) \geq s - n + 1$.

Using formula (3.2.1), an easy induction on l shows that

$$(3.2.2) \quad (\tau - 1)^l(x) = \sum_{m=l}^{\infty} \left(\sum_{i_1 + \dots + i_l = m, i_j \geq 1} \frac{m!}{i_1! \dots i_l!} \right) \gamma_m(t) \otimes N^m(x)$$

for any $l \geq 0$ and $x \in \mathcal{D}$. In particular, $(\tau - 1)^l(x) \in (I^{[l]} B_{\text{cris}}^+)(B_{\text{cris}}^+ \otimes_S \mathcal{D})$. Since $x \in \mathfrak{L}$ and $W(R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{L}}$ is G -stable, we get $(\tau - 1)^l(x) \in (I^{[l]} W(R))(W(R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{L}})$. So it suffices to show that $([\underline{c}] - 1)^i \in [\mathfrak{a}_R^{>pc}] + p^{n-s+v_p(i)} W(R)$ for any i satisfying $v_p(i) \geq s - n + 1$. Write $i = p^{v+s-n+1} m$ with $v \geq 0$ and $p \nmid m$. From $v_R(\underline{c} - 1) = \frac{ep}{p-1}$, it follows that $([\underline{c}] - 1)^{p^{s-n+1}} \in [\mathfrak{a}_R^{>pc}] + pW(R)$. By induction (on v), we easily find $([\underline{c}] - 1)^{p^{v+s-n+1}} \in [\mathfrak{a}_R^{>pc}] + p^{v+1} W(R)$, which is exactly the expected result. \square

3.3. Comparison between $\widehat{J}_{n,c}(\widehat{\mathfrak{M}})$ and $J_{n,c}(\mathfrak{M})$. Let T be a p^n -torsion semi-stable representation and $\widehat{\mathfrak{M}}$ an attached (φ, \widehat{G}) -module via Theorem 3.1.3.(3). We have the following definitions and results similar to §2. For any non negative real number c , set

$$\widehat{J}_{n,c}(\widehat{\mathfrak{M}}) := \text{Hom}_{\widehat{\mathcal{R}}_n, \varphi}(\widehat{\mathfrak{M}}, W_n(R)/[\mathfrak{a}_R^{>pc}]).$$

and $\widehat{J}_{n,\infty}(\widehat{\mathfrak{M}}) = \widehat{T}_n(\widehat{\mathfrak{M}})$.

For any $c \leq \infty$, $\widehat{J}_{n,c}(\widehat{\mathfrak{M}})$ is a $\mathbb{Z}_p[G]$ -module and, for any $c \leq c' \leq \infty$, canonical projection induces a map $\widehat{\rho}_{c',c} : \widehat{J}_{n,c'}(\widehat{\mathfrak{M}}) \rightarrow \widehat{J}_{n,c}(\widehat{\mathfrak{M}})$. Moreover, we have a morphism $\theta_{n,c}(f) \in \text{Hom}_{\widehat{\mathcal{R}}_n}(\widehat{\mathfrak{M}}, W_n(R)/[\mathfrak{a}_R^{>pc}])$ defined by:

$$(3.3.1) \quad (\forall \alpha \in \widehat{\mathcal{R}}) (\forall x \in \widehat{\mathfrak{M}}) \quad \theta_{n,c}(f)(\alpha \otimes x) = \alpha \varphi(f(x)).$$

Proposition 3.3.1. *For any non negative integer $s > s_2(c) = n - 1 + \log_p\left(\frac{(p-1)c}{e}\right)$, $\theta_{n,c} : J_{n,c}(\mathfrak{M}) \rightarrow \widehat{J}_{n,c}(\widehat{\mathfrak{M}})$ is an isomorphism of $\mathbb{Z}_p[G_s]$ -modules.*

Remark 3.3.2. Since $s_2(c) = s_1(c) + 1 \geq s_1(c)$, Proposition 2.5.3 shows that $J_{n,c}(\mathfrak{M})$ is endowed with an action of G_s . Hence it makes sense to claim that $\theta_{n,c}$ is G_s -equivariant. We do not know if Proposition 3.3.1 remains true under the smaller assumption “ $s > s_1(c)$ ”: we conjecture that it is false but we do not know any counter-example.

Proof. It is routine to check that $\theta_{n,c}(f)$ is well defined and preserves Frobenius. Hence $\theta_{n,c}$ is also well defined. Let's first prove that it is bijective. Remark that $\varphi : W_n(R)/[\mathfrak{a}_R^{>c}] \xrightarrow{\sim} W_n(R)/[\mathfrak{a}_R^{>pc}]$ is an isomorphism. It follows easily that $\theta_{n,c}$ is injective. For any $\hat{f} \in \hat{J}_{n,c}(\hat{\mathfrak{M}})$, set $f' := \hat{f}|_{\mathfrak{M}}$ (recall that we regard \mathfrak{M} as a $\varphi(\mathfrak{S})$ -submodule of $\hat{\mathfrak{M}}$ via (3.1.2)). Then $f' : \mathfrak{M} \rightarrow W_n(R)/[\mathfrak{a}_R^{>pc}]$ is $\varphi(\mathfrak{S})$ -linear map and is compatible with Frobenius. Since $\varphi : W_n(R)/[\mathfrak{a}_R^{>c}] \simeq W_n(R)/[\mathfrak{a}_R^{>pc}]$, we can set $f = \varphi^{-1}(f') : \mathfrak{M} \rightarrow W_n(R)/[\mathfrak{a}_R^{>c}]$. It is finally easy to check that f belongs to $J_{n,c}(\mathfrak{M})$ and that $\theta_{n,c}(f) = \hat{f}$. Hence $\theta_{n,c}$ is surjective, as required.

It remains to prove that $\theta_{n,c}$ is G_s -equivariant. Let $g \in G_s$, $\alpha \in \hat{R}$ and $x \in \mathfrak{M}$. Expanding the definitions, we get $g(\theta_{n,c}(f))(\alpha \otimes x) = \alpha g(\theta_{n,c}(f)(g^{-1}(1 \otimes x)))$. Moreover Lemma 3.2.1 shows that $g^{-1}(1 \otimes x)$ is congruent to $1 \otimes x$ modulo $[\mathfrak{a}_R^{>pc}]$ and hence that these two terms have same image under $\theta_{n,c}(f)$. Thus:

$$\begin{aligned} g(\theta_{n,c}(f))(\alpha \otimes x) &= \alpha g(\theta_{n,c}(f)(g^{-1}(1 \otimes x))) = \alpha g(\theta_{n,c}(f)(1 \otimes x)) \\ &= \alpha g(\varphi(f(x))) = \alpha \varphi(g(f(x))) = \theta_{n,c}(g(f))(\alpha \otimes x) \end{aligned}$$

and equivariance is proved. \square

Recall that we have fixed an integer N such that $u^N = 0$ in $W[u]/E(u)^r$ and defined $b = \frac{N}{p-1}$ and $a = \frac{pN}{p-1}$. Combining Propositions 2.3.3 and 3.3.1, we directly get the following.

Corollary 3.3.3. *The morphism $\hat{\rho}_{\infty,b} : \hat{T}_{\mathfrak{S}_n}(\mathfrak{M}) \rightarrow \hat{J}_{n,b}(\hat{\mathfrak{M}})$ is injective and its image is $\hat{\rho}_{a,b}(\hat{J}_{n,a}(\hat{\mathfrak{M}}))$.*

Theorem 3.3.4. *With previous notations, the map $\theta_n : T_{\mathfrak{S}_n}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ is an isomorphism of $\mathbb{Z}_p[G_s]$ -modules for all integer $s > s_{\min} = n - 1 + \log_p(\frac{N}{c})$.*

Proof. We already know that θ_n is bijective (Theorem 3.1.3.(1)). Now, consider the following commutative diagram:

$$\begin{array}{ccc} T_{\mathfrak{S}_n}(\mathfrak{M}) & \xrightarrow[\sim]{\theta_n} & \hat{T}_n(\hat{\mathfrak{M}}) \\ \rho_{\infty,a} \downarrow & & \downarrow \hat{\rho}_{\infty,b} \\ J_{n,a}(\mathfrak{M}) & & \\ \rho_{a,b} \downarrow & & \\ J_{n,b}(\mathfrak{M}) & \xrightarrow[\sim]{\theta_{n,b}} & \hat{J}_{n,b}(\hat{\mathfrak{M}}) \end{array}$$

Note that $s_{\min} = s_1(a) = s_2(b)$. Thus by definition G_s -action on $T_{\mathfrak{S}_n}(\mathfrak{M})$ (resp. by Proposition 2.5.3, resp. by Proposition 3.3.1), $\rho_{\infty,a}$ (resp. $\rho_{a,b}$, resp. $\theta_{n,b}$) is G_s -equivariant. Since $\hat{\rho}_{\infty,b}$ is injective (Corollary 3.3.3) and G -equivariant, we deduce that θ_n is also G_s -equivariant as claimed. \square

We end this section by giving a proof of Theorem 1.5 of introduction. For convenience of the reader, we first recall its statement:

Corollary 3.3.5 (Theorem 1.5). *Let V and V' be two semi-stable representations of G . Let T (resp. T') a quotient of two G -lattices in V (resp. V') which is killed by p^n . Then any morphism G_∞ -equivariant $f : T \rightarrow T'$ is G_s -equivariant for all integer $s > n - 1 + \log_p(nr)$.*

Proof. Consider \mathfrak{M} (resp. \mathfrak{M}') some Kisin module such that $T_{\mathfrak{S}_n}(\mathfrak{M}) = T$ (resp. $T_{\mathfrak{S}_n}(\mathfrak{M}') = T'$). We may assume that \mathfrak{M} and \mathfrak{M}' are maximal in the sense of [7]. Then by Corollary 3.3.10 of *loc. cit.*, f comes from a morphism $g : \mathfrak{M}' \rightarrow \mathfrak{M}$. Using Theorem 3.3.4, one easily see that $T_{\mathfrak{S}_n}(g) = f$ is G_s -equivariant. \square

4. RAMIFICATION BOUND

In this section, we give proofs of Theorems 1.1 and 1.3 based on above preparations. Our strategy is similar to those in [1], [9], [14] and [15]. Let n be a positive integer. Recall that we have defined several constants, that are:

- N is an integer such that $u^N = 0$ in $W_n[u]/E(u)^r$ (recall also that one may choose $N = ern$);
- $b = \frac{N}{p-1}$ and $a = \frac{pN}{p-1}$;
- $s_0(a) = n - 1 + \log_p\left(\frac{a}{e}\right) = n + \log_p\left(\frac{N}{e(p-1)}\right)$;
- $s_{\min} = s_1(a) = s_2(b) = n - 1 + \log_p\left(\frac{N}{e}\right)$.

Note that if we have chosen $N = ern$, then $s_0(a)$ is nothing but the minority of s that appear in Theorem 1.1. Let $T = L/L'$ be a quotient of two lattices in a semi-stable representation and assume that T is killed by p^n . Since we have a surjective map $L/p^n L \rightarrow T$, it is enough to bound ramification for $L/p^n L$. Hence, without loss of generality, we will assume that T is free of $\mathbb{Z}/p^n \mathbb{Z}$. By Theorem 3.1.3, there exists a (φ, \hat{G}) -module $\hat{\mathfrak{M}}$ such that $\hat{T}_n(\hat{\mathfrak{M}}) = T$. With our extra assumption, \mathfrak{M} is finite free over \mathfrak{S}_n .

From now on, we fix an integer $s > s_0(a)$. Remark that $s_0(a) > s_{\min}$ so that we also have $s > s_{\min}$. Hence theory developed in previous sections applies. In particular, by Propositions 2.5.2 and 2.5.3, for all $c \in [0, ep^{s-n+1}[$, we have a G_s -equivariant isomorphism

$$(4.0.2) \quad J_{n,c}(\mathfrak{M}) \simeq J_{n,c}^{(s)}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi} \left(\mathfrak{M}, W_n(k) \otimes_{W_n(k), \sigma^s} \frac{W_n(\mathcal{O}_{\bar{K}}/p)}{[\mathfrak{a}_{\bar{K}}^{>c/p^s}]} \right)$$

where the structure of \mathfrak{S} -module on $W_n(\mathcal{O}_{\bar{K}}/p)$ is given by $u \mapsto 1 \otimes \pi_s$. Moreover, by Corollary 3.3.3 and Theorem 3.3.4

$$(4.0.3) \quad T|_{G_s} \simeq \text{im } \rho_{a,b} : J_{n,a}(\mathfrak{M}) \rightarrow J_{n,b}(\mathfrak{M}).$$

Define L to be the splitting field of T , that is, $L = (\bar{K})^{\text{Ker}(\rho)}$, where $\rho : G_K \rightarrow \text{GL}_{\mathbb{Z}_p}(T)$ the attached group homomorphism. Set $L_s = K_s L$.

4.1. The sets $J_{n,c}^{(s),E}(\mathfrak{M})$. Let E be an algebraic extension of K_s inside \bar{K} . By restriction, the valuation v_K induces a valuation on E and one may define, for all non negative real number c , $\mathfrak{a}_E^{\geq c} = \{x \in E / v_K(x) \geq c\}$ and $\mathfrak{a}_E^{>c} = \{x \in E / v_K(x) > c\}$. If c belongs to the interval $[0, ep^{s-n+1}[$, we put

$$J_{n,c}^{(s),E}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi} \left(\mathfrak{M}, W_n(k) \otimes_{W_n(k), \sigma^s} \frac{W_n(\mathcal{O}_E/p)}{[\mathfrak{a}_E^{>c/p^s}]} \right).$$

They are $\mathbb{Z}_p[G_s]$ -modules. As usual, if $0 \leq c \leq c' \leq ep^{s-n+1}$, we have a natural G_s -equivariant morphism $\rho_{c',c}^{(s),E} : J_{n,c'}^{(s),E}(\mathfrak{M}) \rightarrow J_{n,c}^{(s),E}(\mathfrak{M})$. Apparently $J_{n,c}^{(s),E}(\mathfrak{M})$ injects $J_{n,c}^{(s),\bar{K}}(\mathfrak{M}) = J_{n,c}^{(s)}(\mathfrak{M})$.

The aim of this subsection is to show the following theorem.

Theorem 4.1.1. *Notations as above. The natural injection $\rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M})) \subset \rho_{a,b}(J_{n,a}(\mathfrak{M}))$ is bijective if and only if $L_s \subset E$.*

Remark 4.1.2. By (4.0.3), $\rho_{a,b}(J_{n,a}(\mathfrak{M}))$ is canonically isomorphic to T as a $\mathbb{Z}_p[G_s]$ -module.

In order to achieve the proof, we will need to lift $J_{n,c}^{(s),E}(\mathfrak{M})$ at \mathcal{O}_E -level. We begin by defining a map $\varphi : W_n(\mathcal{O}_E) \rightarrow W_n(\mathcal{O}_E)$ by

$$\varphi(z) := (z_0^p, \dots, z_{n-1}^p).$$

Note that φ is *not* a ring homomorphism. Nevertheless one easily check that $\varphi([\lambda]z) = [\lambda^p]\varphi(z)$ for $\lambda \in \mathcal{O}_E$ and $z \in W_n(\mathcal{O}_E)$ and φ is G_s -equivariant.

Remark 4.1.3. If A is any ring, one can always define Frobenius $\phi : W(A) \rightarrow W(A)$ by $w_m(\phi(x)) = w_{m+1}(x)$, $\forall x \in W(A)$, where $w_n(x)$ is the m -th ghost component of x . Then ϕ can be proved to be a ring homomorphism (see p.14 in [16]). Unfortunately, such Frobenius does not preserve the kernel of natural projection $W(A) \rightarrow W_n(A)$ unless A has characteristic p . Hence it is not well-defined on $W_n(A)$ if A has characteristic 0.

Recall now that we have assumed that \mathfrak{M} is finite \mathfrak{S}_n -free. Select a basis (e_1, \dots, e_d) of \mathfrak{M} and write $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)A$ with $A \in M_d(\mathfrak{S}_n)$. As discussed in §2.3, there exists $B \in M_d(\mathfrak{S}_n)$ such that $AB = u^N I$. Let \tilde{A} and \tilde{B} be matrices in $M_d(W_n(\mathcal{O}_{K_s}))$ that respectively lifts images of A and B under the ring homomorphism $\mathfrak{S}_n \rightarrow W_n(\mathcal{O}_{K_s}/p)$, $u \mapsto \pi_s$, $\lambda \mapsto \sigma^{-s}(\lambda)$ ($\lambda \in W_n(k)$). Apparently, $\tilde{A}\tilde{B} \equiv [\pi_s]^N \pmod{W_n(p\mathcal{O}_{K_s})}$. Hence, using condition on s , one prove that there exists a matrix R with coefficients in $W_n([\mathfrak{a}_{K_s}^{>0}])$ such that $\tilde{A}\tilde{B} = [\pi_s]^N(I+R)$ (where I is the identity matrix). Noting that $I+R$ is invertible, one get $\tilde{A}\tilde{B}(I+R)^{-1} = [\pi_s]^N I$. Hence, up to replacing \tilde{B} by $\tilde{B}(I+R)^{-1}$, one may assume that $\tilde{A}\tilde{B} = [\pi_s]^N I$. Finally define a G_s -set

$$\tilde{J}_n^{(s),E}(\mathfrak{M}) := \{(\tilde{x}_1, \dots, \tilde{x}_d) \in W_n(\mathcal{O}_E)^d / (\varphi(\tilde{x}_1), \dots, \varphi(\tilde{x}_d)) = (\tilde{x}_1, \dots, \tilde{x}_d)\tilde{A}\}.$$

The natural projection $W_n(\mathcal{O}_E) \rightarrow W_n(\mathcal{O}_E/p) \rightarrow W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{>c/p^s}]$ induces a map $\tilde{\rho}_c^{(s),E} : \tilde{J}_n^{(s),E}(\mathfrak{M}) \rightarrow J_{n,c}^{(s),E}(\mathfrak{M})$.

Lemma 4.1.4. $\tilde{\rho}_b^{(s),E}$ is injective and its image is $\rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M}))$.

Proof. During the proof, if z is any element in $W_n(\mathcal{O}_E)$, we will denote by $z^{(i)} \in \mathcal{O}_E$ its i -th component. By the same way, we define $Z^{(i)}$ for a matrix Z with entries in $W_n(\mathcal{O}_E)$. Also, if Z is a matrix with entries in \mathcal{O}_E , we will denote by $v_K(Z)$ the smallest valuation of coefficients of Z .

We first show $\tilde{\rho}_b^{(s),E}$ is an injection. Assume that X and Y are in $\tilde{J}_n^{(s),E}(\mathfrak{M})$ such that $\tilde{\rho}_b^{(s),E}(X) - \tilde{\rho}_b^{(s),E}(Y) = 0$. Then $Z = X - Y \in [\mathfrak{a}_E^{>b/p^s}] + W_n(p\mathcal{O}_E) = [\mathfrak{a}_E^{>b/p^s}]$. We need to prove that $Z = 0$. Assume by contradiction that is it false and consider

m the smallest number such that $Z^{(m)} \neq 0$. Define $W := Z\tilde{A} = \varphi(X) - \varphi(Y) = \varphi(Y + Z) - \varphi(Y)$. Easy computations show that $W^{(i)} = 0$ for $i < m$ and

$$(4.1.1) \quad W^{(m)} = \sum_{i=1}^p \binom{p}{i} (Y^{(m)})^{p-i} (Z^{(m)})^i.$$

where the multiplication is computed component by component. If $1 \leq i < p$, we have

$$v_K \left(\binom{p}{i} (Y^{(m)})^{p-i} (Z^{(m)})^i \right) \geq e + v_K(Z^{(m)}) > Np^{m-1-s} + v_K(Z^{(m)})$$

and, using $v_K(Z^{(m)}) > bp^{m-1-s}$ (recall that $Z \in [\mathfrak{a}_E^{>b/p^s}]$), we find

$$v_K((Z^{(m)})^p) > (p-1)bp^{m-1-s} + v_K(Z^{(m)}) = Np^{m-1-s} + v_K(Z^{(m)}).$$

Hence each term in RHS of (4.1.1) has valuation greater than $Np^{m-1-s} + v_K(Z^{(m)})$. So $v_K(W^{(m)}) > Np^{m-1-s} + v_K(Z^{(m)})$. But, on the other hand, comparing the m -th component of $W\tilde{B} = [\pi_s]^N Z$, we get $v_K(W^{(m)}) \leq Np^{m-1-s} + v_K(Z^{(m)})$. This is a contradiction and injectivity follows.

Let us now prove the second statement. Remark first that for all $c \in [0, ep^{s-n+1}[$, we have $W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{>c/p^s}] \simeq W_n(\mathcal{O}_E)/[\mathfrak{a}_E^{>c/p^s}]$ and hence that $J_{n,c}^{(s),E}(\mathfrak{M})$ can be identified with

$$\{(\tilde{x}_1, \dots, \tilde{x}_d) \in W_n(\mathcal{O}_E)^d / (\varphi(\tilde{x}_1), \dots, \varphi(\tilde{x}_d)) \equiv (\tilde{x}_1, \dots, \tilde{x}_d)\tilde{A} \pmod{[\mathfrak{a}_E^{>c/p^s}]} \}.$$

Let $X = (\tilde{x}_1, \dots, \tilde{x}_d) \in W_n(\mathcal{O}_E)^d$ be an solution as above. We have equation $\varphi(X) = X\tilde{A} + Q'$ with coefficients of Q' in $[\mathfrak{a}_E^{>a'/p^s}]$. Actually, the congruence holds in $[\mathfrak{a}_E^{>a'}]$ for some a' satisfying $\frac{e}{p^{n-1}} \geq a' > \frac{a}{p^s}$. Note that $\frac{e}{p^{n-1}} \geq a'$ implies that $W_n(\mathcal{O}_E) \subset [\mathfrak{a}_E^{>a'}]$. For the rest of the proof, fix $\alpha \in \mathcal{O}_E$ some element of valuation a' . By the similar argument as in Lemma 2.3.1.(2), $[\mathfrak{a}_E^{>a'}]$ is the principal ideal generated by $[\alpha]$. Therefore, one have $\varphi(X) - X\tilde{A} = [\alpha]Q$ with the coefficients of Q in $W_n(\mathcal{O}_E)$. We want to prove that there exists a matrix Y with coefficients in $[\mathfrak{a}_E^{>b/p^s}]$ such that $(X + Y)\tilde{A} = \varphi(X + Y)$. Let us search Y of the shape $[\beta]Z$ with $\beta = \frac{\alpha}{\pi_s^N}$ (which belongs to \mathcal{O}_E because of valuations) and coefficients of Z in $W_n(\mathcal{O}_E)$. Our condition then becomes $(X + [\beta]Z)\tilde{A} = \varphi(X + [\beta]Z)$. Multiplying \tilde{B} on both sides and noting that $[\pi_s]$ is a non-zero divisor of $W_n(\mathcal{O}_E)$, we need to prove that the following equation has a (necessarily unique) solution:

$$(4.1.2) \quad [\pi_s]^N X + [\pi_s]^N [\beta]Z = \varphi(X + [\beta]Z)\tilde{B}.$$

Let us prove by induction on n . If $n = 1$, set $Z_0 = 0$ and $Z_{l+1} = \pi_s^{-N} \beta^{-1} (\varphi(X + \beta Z_l)\tilde{B} - \pi_s^N X)$. To see that Z_{l+1} is in \mathcal{O}_E , note that

$$\begin{aligned} \varphi(X + \beta Z_l)\tilde{B} - \pi_s^N X &= (X + \beta Z_l)^p \tilde{B} - \pi_s^N X \\ &= (\varphi(X)\tilde{B} - \pi_s^N X) + \sum_{i=1}^{p-1} \binom{p}{i} X^{p-i} (\beta Z_l)^i \tilde{B} + \beta^p (Z_l)^p \tilde{B}. \end{aligned}$$

Since $\varphi(X)\tilde{B} - \pi_s^N X = \alpha Q\tilde{B}$, $v_K(\pi_s^N \beta) \leq v_K(\alpha) \leq v_K(p)$ and $(p-1)v_K(\beta) \geq v_K(\pi_s^N)$, we see that Z_{l+1} is in \mathcal{O}_E . Note that

$$\begin{aligned} Z_{l+1} - Z_l &= \pi_s^{-N} \beta^{-1} (\varphi(X + \beta Z_l) - \varphi(X + \beta Z_{l-1})) \tilde{B} \\ &= \pi_s^{-N} \beta^{-1} \sum_{i=1}^p \binom{p}{i} X^{p-i} \beta^i (Z_l^i - Z_{l-1}^i) \tilde{B}. \end{aligned}$$

Since $v_K(p) \geq v_K(\pi_s^N \beta)$ and $(p-1)v_K(\beta) > v_K(\pi_s^N)$, we see that $v_K(Z_{l+1} - Z_l) \geq \gamma + v_K(Z_l - Z_{l-1})$, where $\gamma = \min(v_K(\beta), v_K(\beta^{p-1} \pi_s^{-N})) > 0$. Hence Z_l converge to Z and we solve the equation (4.1.2) for $n = 1$.

Now assume that equation (4.1.2) has a solution for $n \leq m-1$, consider the $n = m$ case. Recall that $z^{(i)} \in \mathcal{O}_E$ represents the i -th component of $z \in W_m(\mathcal{O}_E)$. Set $Z_0 = (Z_0^{(0)}, \dots, Z_0^{(m-1)})$ where $Z_0^{(m-1)} = 0$ and $(Z_0^{(0)}, \dots, Z_0^{(m-2)})$ is the solution of (4.1.2) in $n = m-1$ case. Now set $Z_{l+1} = [\pi_s]^{-N} [\beta]^{-1} (\varphi(X + [\beta]Z_l)\tilde{B} - [\pi_s]^N X)$. Since $(Z_0^{(0)}, \dots, Z_0^{(m-2)})$ is the solution of (4.1.2) in $n = m-1$ case, we see that $Z_l^{(i)} = Z_{l+1}^{(i)}$ for all l and $i = 0, \dots, m-2$. Now it suffices to check that Z_{l+1} has coefficients in $W_m(\mathcal{O}_E)$ and Z_l converges.

Since $\varphi(X + [\beta]Z_l) = \varphi(X) + \varphi([\beta]Z_l)$ in $W_m(\mathcal{O}_E/p)$, we have $\varphi(X + [\beta]Z_l) = \varphi(X) + \varphi([\beta]Z_l) + C'$ with coefficients of C' in $W_m(p\mathcal{O}_E)$. Since $W_m(p\mathcal{O}_E) \subset [\alpha]W_m(\mathcal{O}_E)$, we can write $C' = [\alpha]C$ with coefficients of C in $W_m(\mathcal{O}_E)$. Hence $\varphi(X + [\beta]Z_l)\tilde{B} - [\pi_s]^N X = (\varphi(X)\tilde{B} - [\pi_s]^N X) + [\beta]^p \varphi(Z_l)\tilde{B} + [\alpha]C\tilde{B} = [\alpha]Q\tilde{B} + [\beta]^p \varphi(Z_l)\tilde{B} + [\alpha]C\tilde{B}$. Since $(p-1)v_K(\beta) > v_K(\pi_s^N)$, we see that Z_{l+1} is well defined. Now $[\pi_s]^N [\beta] (Z_{l+1} - Z_l) = (\varphi(X + [\beta]Z_l) - \varphi(X + [\beta]Z_{l-1}))\tilde{B} = W\tilde{B}$, where

$$\begin{aligned} W &= \varphi(X + [\beta]Z_l) - \varphi(X + [\beta]Z_{l-1}) \\ &= \varphi(X + [\beta]Z_{l-1} + [\beta](Z_l - Z_{l-1})) - \varphi(X + [\beta]Z_{l-1}) \\ &= \varphi(V + [\beta](Z_l - Z_{l-1})) - \varphi(V) \end{aligned}$$

with $V = X + [\beta]Z_{l-1}$. Since $Z_l^{(i)} = Z_{l+1}^{(i)}$ for all l and $0 \leq i \leq m-2$, $W^{(i)} = 0$ for $i = 0, \dots, m-2$ and

$$W^{(m-1)} = \sum_{i=1}^p \binom{p}{i} (V^{(m-1)})^{p-i} (\beta^{p-i} (Z_l^{(m-1)} - Z_{l-1}^{(m-1)}))^i.$$

Hence $v_K((\pi_s^N \beta)^{p^{m-1}}) + v_K(Z_{l+1}^{(m-1)} - Z_l^{(m-1)}) \geq v_K(\beta^{p^m}) + v_K(Z_l^{(m-1)} - Z_{l-1}^{(m-1)})$ if $i = p$ and

$$v_K((\pi_s^N \beta)^{p^{m-1}}) + v_K(Z_{l+1}^{(m-1)} - Z_l^{(m-1)}) \geq v_K(p) + v_K(\beta) + v_K(Z_l^{(m-1)} - Z_{l-1}^{(m-1)})$$

if $1 \leq i \leq p-1$. Since $(p-1)v_K(\beta) > v_K(\pi_s^N)$ and $v_K((\pi_s^N \beta)^{p^{m-1}}) \leq v_K(p)$, we get

$$v_K(Z_{l+1}^{(m-1)} - Z_l^{(m-1)}) \geq \gamma + v_K(Z_l^{(m-1)} - Z_{l-1}^{(m-1)}),$$

where $\gamma = \min(v_K(\beta), v_K(\beta^{p-1} \pi_s^{-N}))$. Hence Z_l converges and we are done. \square

Proof of Theorem 4.1.1. We have G_s -equivariant bijections of sets:

$$\begin{aligned} \tilde{J}_n^{(s), \bar{K}}(\mathfrak{M}) &\simeq \rho_{a,b}^{(s)}(J_{n,b}^{(s)}(\mathfrak{M})) && \text{by Lemma 4.1.4 applied with } E = \bar{K} \\ &\simeq \rho_{a,b}(J_{n,b}(\mathfrak{M})) && \text{by Formula (4.0.2)} \\ &\simeq T|_{G_s} && \text{by Proposition 2.3.3 and Theorem 3.3.4.} \end{aligned}$$

Taking fixed points under $\text{Gal}(\bar{K}/E)$, we get a natural bijection $\tilde{J}_n^{(s),E}(\mathfrak{M}) \simeq T^{\text{Gal}(\bar{K}/E)}$. Hence again by Lemma 4.1.4, $T^{\text{Gal}(\bar{K}/E)} \simeq \rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M}))$, from what the theorem is easily deduced. \square

4.2. Proof of Theorem 1.1. Recall that $L_s = K_s L$ with L the splitting field of T . Now we are ready to bound the ramification of L . To do this, we need to recall the property $(P_m^{F/N})$ described by Fontaine (Proposition 1.5, [9]). First, in order to fix notations, we would like to recall some definitions about ramification filtration, although we refer to *loc. cit.* and [22], Chap. IV. for basic properties.

Let F/N be a Galois extension of p -adic fields, with Galois group G . For all non negative real number λ , we define a normal subgroup $G_{(\lambda)}$ of G by

$$G_{(\lambda)} = \{\sigma \in G / v_N(\sigma(x) - x) \geq \lambda, \forall x \in \mathcal{O}_N\}$$

where v_N is the valuation normalized by $v_N(N^*) = \mathbb{Z}$ and \mathcal{O}_N is the ring of integers of N . We underline that we use here conventions of [9] and that they differ by a shift with conventions of [22], Chap. IV. By definition $G_{(\lambda)}$ is called the *lower ramification filtration* of G . Now, let $\varphi_{N/K} : [0, +\infty[\rightarrow [0, +\infty[$ be the function defined by

$$\varphi_{N/K}(\lambda) := \int_0^\lambda \frac{\text{Card } G_{(t)}}{\text{Card } G_{(1)}} dt.$$

It is increasing, continuous, concave, piecewise affine and bijective. Let $\psi_{N/K}$ denote its inverse and set $G^{(\mu)} = G_{(\psi_{N/K}(\mu))}$: it is the *upper ramification filtration*. Finally call $\lambda_{F/N}$ (resp. $\mu_{F/N}$) the last break in the lower (resp. upper) ramification filtration of G , that is the infimum of λ (resp. μ) such that $G_{(\lambda)} = 1$ (resp. $G^{(\mu)} = 1$). Obviously $\mu_{F/N} = \varphi_{F/N}(\lambda_{F/N})$.

Proposition 4.2.1 (Fontaine). *Let F and N be finite extensions of K with $N \subset F \subset \bar{K}$ and F is Galois. For any positive real number m , consider the following property*

$$(P_m^{F/N}) \left\{ \begin{array}{l} \text{For any algebraic extension } E \text{ over } N. \text{ If there exists an } \mathcal{O}_N\text{-algebra} \\ \text{homomorphism } \mathcal{O}_F \rightarrow \mathcal{O}_E/\mathfrak{a}_E^{>m}, \text{ then there exists a } N\text{-injection } F \hookrightarrow E. \end{array} \right.$$

If $(P_m^{F/N})$ is true, then $\mu_{F/N} \leq e_{N/K}m + \frac{1}{e_{F/N}}$ where $e_{N/K}$ (resp. $e_{F/N}$) is the ramification index of N/K (resp. of F/N).

We will also need the following corollary:

Corollary 4.2.2. *If $(P_m^{F/N})$ holds for a positive real number m , then $v_K(\mathcal{D}_{F/N}) < m$.*

Proof. If F/N is unramified, $v_K(\mathcal{D}_{F/N}) = 0$ and the corollary is obvious. If not, by Proposition 1.3 of [9], we see that $e_{N/K}v_K(\mathcal{D}_{F/N}) = \mu_{F/N} - \lambda_{F/N}$. If $(P_m^{F/N})$ holds then $v_K(\mathcal{D}_{F/N}) \leq m - e_{N/K}^{-1}(\lambda_{F/N} - e_{F/N}^{-1})$. Conclusion then follows from that $e_{F/N} > 1$ and $\lambda_{F/N} \geq 1$ (both are true because F/N is assumed to be ramified). \square

We claim that $(P_m^{L_s/K_s})$ holds for $m = ap^{n-1-s}$. To see this, pick $f : \mathcal{O}_{L_s} \rightarrow \mathcal{O}_E/\mathfrak{a}_E^{>m}$ an \mathcal{O}_{K_s} -algebra homomorphism. Obviously, for any real number $c \in [0, m]$, f induces a map $f_c : \mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{>c} \rightarrow \mathcal{O}_E/\mathfrak{a}_E^{>c}$.

Lemma 4.2.3. (1) For any $c \leq m$, f_c is injective.
(2) For any $c \leq a$, $f_{cp^{n-1-s}}$ induces an injection

$$W_n(\mathcal{O}_{L_s}/p)/[\mathfrak{a}_{L_s}^{>c/p^s}] \hookrightarrow W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{>c/p^s}].$$

Proof. (1) It is the same as the proof of Lemma 4.4 of [14].

(2) Using an analogue of Lemma 2.3.1.(1), one easily prove that natural projections $\mathcal{O}_{L_s}/p \rightarrow \mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{>cp^{n-1-s}}$ and $\mathcal{O}_E/p \rightarrow \mathcal{O}_E/\mathfrak{a}_E^{>cp^{n-1-s}}$ induce isomorphisms

$$\begin{aligned} W_n(\mathcal{O}_{L_s}/p)/[\mathfrak{a}_{L_s}^{>c/p^s}] &\simeq W_n(\mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{>cp^{n-1-s}})/[\mathfrak{a}_{L_s}^{>c/p^s}] \\ W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{>c/p^s}] &\simeq W_n(\mathcal{O}_E/\mathfrak{a}_E^{>cp^{n-1-s}})/[\mathfrak{a}_E^{>c/p^s}]. \end{aligned}$$

Hence $f_{cp^{n-1-s}}$ indeed induces a map

$$W_n(\mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{>cp^{n-1-s}})/[\mathfrak{a}_{L_s}^{>c/p^s}] \rightarrow W_n(\mathcal{O}_E/\mathfrak{a}_E^{>cp^{n-1-s}})/[\mathfrak{a}_E^{>c/p^s}]$$

and checking injectivity is now straitforward using (1). \square

Thus, we get injections:

$$\rho_{b,a}^{(s),L_s}(J_{n,a}^{(s),L_s}(\mathfrak{M})) \hookrightarrow \rho_{b,a}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M})) \hookrightarrow \rho_{b,a}^{(s)}(J_{n,a}^{(s)}(\mathfrak{M})) = \rho_{b,a}(J_{n,a}(\mathfrak{M})) \simeq T$$

the first one being induced by f (which is obviously compatible with Frobenius since it is a ring homomorphism). By Theorem 4.1.1, LHS is isomorphic to T . The composite map is then an injective endomorphism of T . Consequently, it is an isomorphism because T is finite. It follows that $\rho_{b,a}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M})) \hookrightarrow \rho_{b,a}^{(s)}(J_{n,a}^{(s)}(\mathfrak{M}))$ is bijective and then, applying again Theorem 4.1.1, we get $L_s \subset E$. Property $(P_m^{L_s/K_s})$ is proved.

Corollary 4.2.4. For all integer $s > s_0(a) = n + \log_p(\frac{N}{e(p-1)})$ and all real number $\mu > \frac{Np^n}{p-1}$, $G_s^{(\mu)}$ acts trivially on T .

Proof. If L_s/K_s is tamely ramified, then $\mu_{L_s/K_s} \leq 1$ and we are done (note that $\frac{Np^n}{p-1} \geq 1$). Otherwise, by Proposition 4.2.1, we obtain:

$$\mu_{L_s/K_s} \leq e_{K_s/K} m + \frac{1}{e_{L_s/K_s}} = \frac{Np^n}{p-1} + \frac{1}{e_{L_s/K_s}}.$$

Since L_s/K_s is wildly ramified, we know that $e_{L_s/K_s} \mu_{L_s/K_s}$ lies in $p\mathbb{Z}$. From $e_{L_s/K_s} \mu_{L_s/K_s} (p-1) \leq e_{L_s/K_s} Np^n + (p-1)$ we then deduce $e_{L_s/K_s} \mu_{L_s/K_s} (p-1) \leq e_{L_s/K_s} Np^n$, i.e. $\mu_{L_s/K_s} \leq \frac{Np^n}{p-1}$. \square

Now taking $N = ern$, we get Theorem 1.1. (Recall that ern is not in general the best value one can choose (except for $n = 1$). See §2.4 for a discussion about this.)

As a conclusion, we would like to emphasize that previous Corollary is valid for all choice of π and compatible system (π_s) of p^s -roots of π . Therefore

$$L \subset \bigcap_{s,\mu,K_s} K_s^{(\mu)}$$

where $K_s^{(\mu)}$ is the extension of K_s cut by $G_s^{(\mu)}$ and where the intersection runs over all (s, μ) satisfying conditions of the Corollary and all extensions K_s over K generated by a p^s -root of some uniformizer of K . It should maybe be interesting to understand better this intersection.

4.3. Proof of Theorem 1.3. Consider α and β such that $\frac{N}{e(p-1)} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{p} < \beta \leq 1$. (If $N = ern$, then α and β are those of Theorem 1.3). From now on, we fix $s = n + \alpha$. One certainly have that $s \geq s_0(a) = n + \log_p(\frac{N}{e(p-1)})$ as it was assumed at the beginning of this section.

This is very easy now to bound valuation of $\mathcal{D}_{L/K}$. We just write:

$$\begin{aligned} v_K(\mathcal{D}_{L_s/K}) &= 1 + es - \frac{1}{p^s} + v_K(\mathcal{D}_{L_s/K_s}) < 1 + es - \frac{1}{p^s} + ap^{n-1-s} \\ &= 1 + es - \frac{1}{p^s} + ep^{\alpha+n-s}\beta = 1 + e(n + \alpha + \beta) - \frac{1}{p^{n+\alpha}} \end{aligned}$$

where the inequality $v_K(\mathcal{D}_{L_s/K}) < ap^{n-1-s}$ follows from Corollary 4.2.2 and the fact that $(P_{ap^{n-1-s}}^{L_s/K_s})$ holds as it was seen before. Now, since L is a subextension of L_s , we have $v_K(\mathcal{D}_{L/K}) \leq v_K(\mathcal{D}_{L_s/K})$ and the previous bound works also for $v_K(\mathcal{D}_{L/K})$. Taking $N = ern$, we get Theorem 1.3.(2).

To bound $u_{L/K}$, we first need a kind of transitivity formula:

Lemma 4.3.1. *Let $N \subset F$ two finite Galois extensions of K . Then $\mu_{F/K} = \max(\mu_{N/K}, \varphi_{N/K}(\mu_{F/N}))$.*

Proof. Let G (resp. H) denote the Galois group of F/K (resp. F/N). Since N is Galois over K , H is a normal subgroup in G . Using definition, one directly check $H^{(\lambda)} = G^{(\lambda)} \cap H$. Using $\varphi_{F/K} = \varphi_{N/K} \circ \varphi_{F/N}$ (see [22], Chap. IV, Proposition 15), and taking $\lambda = \psi_{F/K}(\mu)$, one obtain $H^{(\psi_{N/K}(\mu))} = G^{(\mu)} \cap H$. On the other hand, Proposition 14 of *loc. cit.* gives $(G/H)^{(\mu)} = G^{(\mu)}H/H$. Combining both results, we see that the sequence

$$0 \rightarrow H^{(\psi_{N/K}(\mu))} \rightarrow G^{(\mu)} \rightarrow (G/H)^{(\mu)} \rightarrow 0$$

is well defined and exact for all μ . Conclusion is then easy. \square

Let $N_s := K_s(\zeta_{p^s})$ and $F_s := LN_s$ where ζ_{p^s} is a p^s -th root of unity. Note that N_s and F_s are Galois extensions. We would like to apply Lemma 4.3.1 with $N = N_s$ and $F = F_s$. First we claim $(P_{ap^{n-1-s}}^{F_s/N_s})$ holds. Indeed, given an \mathcal{O}_{N_s} -algebra homomorphism $f : \mathcal{O}_{F_s} \rightarrow \mathcal{O}_E/\mathfrak{a}_E^{>m}$ (with $m = ap^{n-1-s}$), the restriction $f|_{\mathcal{O}_{L_s}} : \mathcal{O}_{L_s} \rightarrow \mathcal{O}_E/\mathfrak{a}_E^{>m}$ is an \mathcal{O}_{K_s} -algebra homomorphism. Then, by the same argument below Lemma 4.2.3, we have a K_s -injection $L_s \rightarrow E$. On the other hand, since $N_s \subset E$ and L are Galois, we see that there exists an N_s -injection $F_s = L_s N_s \rightarrow E$. So $(P_{ap^{n-1-s}}^{F_s/N_s})$ holds. By the same argument as in proof of Corollary 4.2.4, we show

$$\mu_{F_s/N_s} \leq e_{N_s/K} ap^{n-1-s} = e_{N_s/K} \frac{N}{p^\alpha(p-1)} = e_{N_s/K} e\beta$$

where $e_{N_s/K}$ is the ramification index of N_s/K . Furthermore, by Remark 5.5 of [15], we already know that $\mu_{N_s/K} = 1 + e(s + \frac{1}{p-1})$. Hence, it remains to give an estimation of $\varphi_{N_s/K}$. For this, note that there exists $\sigma \in \text{Gal}(N_s/K)$ such that $\sigma\pi_s = \zeta_p\pi_s$ where ζ_p is a primitive p -th root of unity. Hence $v_K(\sigma\pi_s - \pi_s) = \frac{e}{p-1} + \frac{1}{p^s}$ which implies

$$\lambda_{N_s/K} \geq e_{N_s/K} \left(\frac{e}{p-1} + \frac{1}{p^s} \right).$$

Moreover, using the definition, we see that $\varphi_{N_s/K}$ is affine with slope $\frac{1}{e_{N_s/K}}$ after $\lambda_{N_s/K}$. Thus, by concavity, one get

$$\begin{aligned} \varphi_{N_s/K}(\lambda) &\leq 1 + e \left(s + \frac{1}{p-1} \right) + \frac{1}{e_{N_s/K}} (\lambda - \lambda_{N_s/K}) \\ &\leq 1 + es + \frac{\lambda}{e_{N_s/K}} - \frac{1}{p^s}. \end{aligned}$$

Now apply Lemma 4.3.1:

$$\mu_{L/K} \leq \mu_{F_s/K} \leq 1 + e \left(s + \max \left(\beta, \frac{1}{p-1} \right) \right).$$

Since $s = n + \alpha$, Theorem 1.3.(1) is shown.

5. SOME RESULTS AND QUESTIONS ABOUT LIFTS

In this last section, we discuss some ideas about possible converses for Theorem 1.3. Precisely, we wonder when a given torsion representation of G_K can be realized as a quotient of two lattices in a semi-stable (or even crystalline) representation, eventually with prescribed Hodge-Tate weights. Denote by $\text{Rep}_{\mathbb{Z}_p}(G_K)$ (resp. $\text{Rep}_{\text{tor}}(G_K)$, resp. $\text{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)$) the category of all \mathbb{Z}_p -representations of G_K that are finitely generated and free (resp. killed by a power of p , resp. killed by p^n) as a \mathbb{Z}_p -module. For any full subcategory \mathcal{C} of $\text{Rep}_{\mathbb{Z}_p}(G_K)$, one can always raise the following question

Question 5.1. For any $T \in \text{Rep}_{\text{tor}}(G_K)$ (resp. $T \in \text{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)$), does there exist L and L' in \mathcal{C} such that $T \simeq L/L'$?

Obviously if \mathcal{C} is stable under subobject (which will in general be true in interesting examples), it is enough to find L together with a surjective G_K -equivariant morphism $L \rightarrow T$. In the sequel, we will call a *lift* such a morphism $L \rightarrow T$. If \mathcal{C} is moreover stable by direct sum, the problem can be further reduced as follows.

Proposition 5.2. *Assume that \mathcal{C} is stable under subobjects and direct sums. Assume also that any $T \in \text{Rep}_{\mathbb{Z}/p\mathbb{Z}}(G_K)$ admits a lift $L \in \mathcal{C}$. Then the answer to Question 5.1 is “yes”.*

Proof. We make an induction on n . The case $n = 1$ is obvious. Now assume the statement is valid for $m \leq n - 1$. Let T be a representation killed by p^n . Then we have an exact sequence $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$, where $T' = p^{n-1}T$ and $T'' = T/T'$. Since T'' is killed by p^{n-1} , by induction, there exists an $L \in \mathcal{C}$ that lifts T'' . Denote the surjections $L \rightarrow T''$ and $T \rightarrow T''$ by f and g respectively. Set $M := T \times_{T''} L = \{(x, y) \in T \times L \mid g(x) = f(y)\}$. Then we have an exact sequence $0 \rightarrow T' \rightarrow M \rightarrow L \rightarrow 0$. Since L is free over \mathbb{Z}_p , the sequence is split as \mathbb{Z}_p -module. In particular $pM \simeq pL \oplus pT' = pL$ is finite free over \mathbb{Z}_p . Now we have exact sequence $0 \rightarrow pM \rightarrow M \rightarrow M' \rightarrow 0$ with $M' = M/pM$. Since M/pM is killed by p , there exists an $L' \in \mathcal{C}$ such that L' lifts M/pM . Set $N := M \times_{M'} L'$. It sits in the exact sequence $0 \rightarrow pM \rightarrow N \rightarrow L' \rightarrow 0$, and since pM and L' are both finite free, N is also. Note that N is a lift of M hence a lift of T . Now it remains to show that N is in \mathcal{C} . To see this, note that $N := M \times_{M'} L' \subset M \times L'$. Then $pN \subset (pM) \times pL'$. But $pM \simeq pL \in \mathcal{C}$. Hence $pN \subset pL \times pL'$ belongs to \mathcal{C} . \square

We also have a kind of descent property:

Proposition 5.3. *Assume that \mathcal{C} is stable under subobjects and that the answer of question 5.1 is “yes”. Let L/K a finite extension.*

Then, for any $T \in \text{Rep}_{\text{tor}}(G_L)$ (resp. $T \in \text{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_L)$), there exist L and L' in \mathcal{C} such that $T \simeq L/L'$ as G_L -representations.

Proof. By a previous remark, it is enough to show that T admits a lift in \mathcal{C} . Let $T_0 := \text{Ind}_{G_K}^{G_L}(T)$. By assumption, there exists a lift $L_0 \rightarrow T_0$ with $L_0 \in \mathcal{C}$. But as G_L -representations, $T_0 \simeq T^{[K:L]}$. Composing the map $L_0 \rightarrow T^{[K:L]}$ with the first projection, we get the desired lift. \square

Nevertheless, of course, the answer to Question 5.1 is in general negative. For instance, we have the following theorem that can be seen as a consequence of ramification bounds obtained in this paper.

Theorem 5.4. *For any $r > 0$, answer to Question 5.1 is “no” if \mathcal{C} is the category of lattices in semi-stable representations with Hodge-Tate weights in $\{0, \dots, r\}$.*

Proof. There are several ways to prove this theorem. Below, we give two different methods.

The first one is based on results shown in this paper. Select a Galois extension F/K which has very large ramification and let T be the regular representation with $\mathbb{Z}/p^n\mathbb{Z}$ -coefficients of $\text{Gal}(F/K)$. Then the splitting field of T is F , and Theorem 1.3 shows that T cannot in general be lifted a semi-stable representation with Hodge-Tate weights in $\{0, \dots, r\}$.

The second proof we would like to give uses the main result of [19] which states that a finite free \mathbb{Z}_p -representation L of G_K is a lattice in a crystalline (resp. semi-stable) representation with Hodge-Tate weights $\{0, \dots, r\}$ if and only if $L/p^n L$ is a quotient of two such lattices. Therefore, starting from a representation L such that $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is not semi-stable, there must exist an integer n such that $L/p^n L$ gives a counter-example to Question 5.1 (with the category \mathcal{C} of the theorem). \square

Unfortunately, the above proof does not help us to solve the following more interesting question:

Question 5.5. Has Question 5.1 a positive answer when \mathcal{C} is the category of all lattices in semi-stable representations?

In fact, to check the above question, it suffices to look at representations killed by p (by Proposition 5.2) and we may assume that $K = \mathbb{Q}_p$ (by Proposition 5.3). Here are some partial results in favor of a positive answer to Question 5.5.

Proposition 5.6. *Let T be a torsion representation of G_∞ . Then T is a quotient of two representations arising from finite free Kisin modules.*

Proof. By a similar argument as in proof of Proposition 5.2, we may assume that T is killed by p . Let M be the étale φ -module over $k((u))$ attached to T (see for instance [10], A. 3). Since any torsion Kisin module can be written as a quotient of two free Kisin modules, it is enough to show that M admits a submodule \mathfrak{M} which

is a Kisin module for some integer r . Let (e_1, \dots, e_d) be a basis of M and A be the matrix with coefficients in $k((u))$ such that

$$(\varphi(e_1), \dots, \varphi(e_d)) = (e_1, \dots, e_d)A.$$

Since changing all e_i 's in ue_i changes A in $u^{p-1}A$, one may assume that A has coefficients in $k[[u]]$. Furthermore, the étaleness of Frobenius on M exactly means that A is invertible in $k((u))$. Hence $\det A$ does not vanish. Finally, we choose r such that $\det A$ divides u^{er} (that is $r \geq \frac{1}{e} \text{val}_u(\det A)$) and we are done. \square

Theorem 5.7. *Any tamely ramified \mathbb{F}_p -representation of G_K can be written as a quotient of two lattices in a crystalline representation with Hodge-Tate weights between 0 and $p-1$.*

Remark 5.8. In particular, the answer to Question 5.5 is yes if T is tamely ramified and killed by p .

Before giving the proof of Theorem 5.7, we need to achieve some computations with Kisin modules in the “unramified case” ($e = 1$). So from now, we assume $e = 1$ (that is $K = W[1/p]$) and we fix $\pi = -p$ to be the chosen uniformizer, so that $E(u) = u + p$. We denote by I the inertia subgroup of G_K and for all integer d , we recall the definition of the d -th fundamental tame inertia character θ_d :

$$\begin{aligned} \theta_d : I &\rightarrow \mu_{p^d-1}(\bar{K}) \simeq \mathbb{F}_{p^d}^* \\ g &\mapsto \frac{g(\pi^{1/(p^d-1)})}{\pi^{1/(p^d-1)}} \end{aligned}$$

where \mathbb{F}_{p^d} is the subfield of \bar{k} with p^d elements. For all $i \in \mathbb{Z}/d\mathbb{Z}$, set $\theta_{d,i} = \theta_d^{p^i}$. Let $(n_i)_{i \geq 0}$ be a sequence of non negative integers that is periodic of period d . Attached to it, define a filtered φ -module D and a free Kisin module \mathfrak{M} by

$$\left\{ \begin{array}{l} D = Ke_0 \oplus Ke_1 \oplus \dots \oplus Ke_{d-1} \\ \varphi(e_{i+1}) = p^{n_i} e_i \quad (i \in \mathbb{Z}/d\mathbb{Z}) \\ \text{Fil}^n D = \sum_{n_i \geq n} Ke_{i+1} \quad (i \in \mathbb{Z}/d\mathbb{Z}) \end{array} \right\} \quad \left\{ \begin{array}{l} \mathfrak{M} = \mathfrak{S}e_0 \oplus \mathfrak{S}e_1 \oplus \dots \oplus \mathfrak{S}e_{d-1} \\ \varphi(e_{i+1}) = (u+p)^{n_i} e_i \quad (i \in \mathbb{Z}/d\mathbb{Z}) \end{array} \right.$$

If ψ is a character of I with values in \mathbb{F}_{p^d} , we will denote by $\mathbb{F}_{p^d}(\psi)$ the one-dimensional representation of I given by ψ . Our technical lemma is the following.

Lemma 5.9. *We keep previous notations.*

- (1) *One have a canonical G_∞ -equivariant \mathbb{Q}_p -linear isomorphism*

$$T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \text{Hom}_{\varphi, \text{Fil}}(D, B_{\text{cris}}^+).$$

- (2) *Setting $I_\infty := I \cap G_\infty$, one have a canonical I_∞ -equivariant \mathbb{F}_p -linear isomorphism*

$$T_{\mathfrak{S}_1}(\mathfrak{M}/p\mathfrak{M}) \simeq \mathbb{F}_{p^d}(\theta_{d,0}^{n_0} \theta_{d,1}^{n_1} \dots \theta_{d,d-1}^{n_{d-1}}).$$

Proof. In what follows, we will make an intensive use of results of [18]. Before beginning the proof, we would like to emphasize that in the latter reference, K_0 is defined to be the maximal absolutely unramified subextension of K , whereas in this paper K_0 is just K . By chance, since we are assuming $e = 1$, the two definitions coincide.

We define a (φ, N_{∇}) -module over \mathcal{O} (see *loc. cit.*, §1.1 for the definition of all of this) by the following:

$$\begin{cases} \mathcal{M} = \mathcal{O}\mathcal{E}_0 \oplus \mathcal{O}\mathcal{E}_1 \oplus \cdots \oplus \mathcal{O}\mathcal{E}_{d-1} \\ \varphi(\mathcal{E}_{i+1}) = (u+p)^{n_i} \mathcal{E}_i \quad (i \in \mathbb{Z}/d\mathbb{Z}) \\ N_{\nabla}(\mathcal{E}_i) = \left(\sum_{j=0}^{\infty} n_{i+j} p^{j-1} u^{p^j} \lambda_j \right) \mathcal{E}_i \end{cases}$$

where $\lambda_j = \prod_{n \geq 0, n \neq j} \varphi^n \left(\frac{u}{p} + 1 \right)$. It is straitforward to check that the above (φ, N_{∇}) -module is well-defined (one only need to check that $N_{\nabla}\varphi = (u+p)\varphi N_{\nabla}$), and that $\mathcal{M} = \mathfrak{M} \otimes_{\mathcal{O}} \mathcal{O}$ as a φ -module. Now let's compute the filtered (φ, N) -module $D(\mathcal{M})$ where D is the functor defined in *loc. cit.*, §1.2.5. By definition $D(\mathcal{M}) = \mathcal{M}/u\mathcal{M}$ and it is equipped with Frobenius and monodromy operator induced respectively by φ and N_{∇} on the quotient. Computing the filtration is little more complicated. We first need to determine the unique φ -compatible section $\xi : D(\mathcal{M}) \rightarrow \mathcal{M}$ to the natural projection. Actually, one can easily check that it is given by

$$\xi(\bar{\mathcal{E}}_i) = \left(\prod_{j=0}^{\infty} \varphi^j \left(\frac{u}{p} + 1 \right)^{n_{i+j}} \right) \mathcal{E}_i \quad (i \in \mathbb{Z}/d\mathbb{Z}),$$

where $\bar{\mathcal{E}}_i = (\mathcal{E}_i \bmod u)$ is the image of \mathcal{E}_i in $D(\mathcal{M})$. Now applying recipe of *loc. cit.*, §1.2.7, one see that $\bar{\mathcal{E}}_{i+1} \in \text{Fil}^{n_i} D(\mathcal{M})$. Therefore, if D is equipped with $N = 0$, the map $f : D \rightarrow D(\mathcal{M})$, $e_i \mapsto \bar{\mathcal{E}}_i$ is compatible with φ , N and filtration on both side. Moreover, Theorem 1.3.8 of *loc. cit.* shows that $D(\mathcal{M})$ is admissible. Since the category of admissible filtered (φ, N) -modules is abelian, f has to be an isomorphism. Finally, we use Proposition 2.1.5 of *loc. cit.* to get (1).

The second part of the lemma is a simple computation left to the reader. □

Proof of Theorem 5.7. We first assume $e = 1$. Denote by I the inertia subgroup of G_K . Let T be a tamely ramified representation of G_K killed by p . Since the tame inertia group is procyclic or order prime to p , $T|_I$ splits as a direct sum of irreducible representations. By [22], §1.7, every irreducible representation of I is isomorphic to

$$\mathbb{F}_{p^d}(\theta_{d,0}^{n_0} \theta_{d,1}^{n_1} \cdots \theta_{d,d-1}^{n_{d-1}})$$

for one sequence of integers between 0 and $p-1$, periodic of period d . Then applying Lemma 5.9, we construct a I_{∞} -equivariant isomorphism $f : L/pL \simeq T$ where L is a lattice in a crystalline representation with Hodge-Tate weights in $\{0, \dots, p-1\}$. We already know that the wild inertia subgroup I_p does not act of T . On the other hand, using that L/pL is a direct sum of irreducible representations of G_{∞} and hence of G_K , a standard argument shows that I_p acts also trivially on L/pL . Thus f commutes with action of $I_p I_{\infty} = I$. Since L/pL and T are finite dimensional over \mathbb{F}_p , they are finite and f is $G_{K'}$ -equivariant for a finite unramified extension K' of K . Let g be the composite morphism $L \rightarrow L/pL \simeq T$. Consider the map

$$\text{Ind}_{G_{K'}}^{G_K} L = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_{K'}]} L \rightarrow T, \quad [\sigma] \otimes x \mapsto \sigma g(x).$$

It is apparently G_K -equivariant and surjective: it is a lift of T . Furthermore, the restriction of $\text{Ind}_{G_{K'}}^{G_K} L$ to $G_{K'}$ is a direct sum of copies of L , and hence is crystalline with Hodge-Tate weights in $\{0, \dots, p-1\}$. Since K'/K is unramified, also is $\text{Ind}_{G_{K'}}^{G_K} L$ and we are done in this case.

Next, we assume $K/W[1/p]$ to be tamely ramified. Let T be a p -torsion representation of G_K and set $T_0 = \text{Ind}_{G_K}^{G_{W[1/p]}} T$. Using that $K/W[1/p]$ is tamely ramified, we easily see that T_0 is still tamely ramified as a representation of $G_{W[1/p]}$. Hence, one can apply the first part of the proof and then find a lift $L \rightarrow T_0$ where L is a lattice in a crystalline representation of $G_{W[1/p]}$ with Hodge-Tate weights in $\{0, \dots, p-1\}$. Restricting actions to G_K , we get a surjective map $L \rightarrow T^{[K:W[1/p]]}$ and then composing with the first projection we get a lift of T , which is convenient since the restriction of a crystalline representation is again crystalline with same Hodge-Tate weights.

Finally we go to the general case (K arbitrary). Denote by K' the maximal tamely extension of $W[1/p]$ inside K . Let K^{tr} (resp. K'^{tr}) be the maximal tamely ramified extension of K (resp. K'). Using that K/K' is totally ramified of degree a power of p , it is easy to check that $K \cap K'^{\text{tr}} = K'$ and $KK'^{\text{tr}} = K^{\text{tr}}$. We then deduce the existence of a canonical isomorphism between $\text{Gal}(K^{\text{tr}}/K)$ and $\text{Gal}((K')^{\text{tr}}/K')$. Therefore, any tamely ramified representation of G_K has a natural prolongation to $G_{K'}$ which remains tamely ramified. Theorem follows easily from this remark. \square

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