Asymptotic study of soft thin layer: the non convex case

Frédéric Lebon, Raffaella Rizzoni

Asymptotic Study on a Soft Thin Layer: The Non-Convex Case

F. Lebon[1] and R. Rizzoni[2]

[1] *Laboratoire de Mécanique et d'Acoustique, Université Aix-Marseille 1, France*

[2] *Dipartimento di Ingegneria, Università di Ferrara, Italy*

It is proposed to model the adhesive bonding of elastic bodies when the adhesive is a phase-transforming material. For this purpose, the (isothermal) Frémond model is adopted, including only two variants of martensite. In the first part of this paper, asymptotic expansions are used to study the asymptotic behavior of the adhesive as its thickness and elastic coefficients tend toward zero. In the second part, the energy minimization approach is used and the equilibrium of a one-dimensional bar is studied in detail. The simplified one-dimensional context adopted here makes it possible to compute contact laws taking nucleation and the kinetics of the phase transformation explicitly into account.

Keywords soft thin layers, asymptotic studies, non-convex energy

1. INTRODUCTION

Modeling adhesive bonding between elastic bodies often requires taking several parameters, typically the thickness and the rigidity of the adhesive layer, into account. An interesting case arises when both these parameters tend to zero. In the limit case, the thin layer is replaced by a sharp interface and an asymptotic contact law is obtained, linking the stress to the jump in the displacement occuring at the interface. The behaviour of several types of materials forming the adherent and adhesive bodies has been previously investigated [1–6, 25].

The aim of this study was to determine the contact law for an adhesive having a non-monotonous stress-strain relation. In particular, we consider the constitutive equation proposed by Frémond, which corresponds to a piecewise quadratic (but globally non convex) energy [7]. Since the pioneering study by Ericksen [8], non-convex energy densities have been used to model phase changes, and each domain of convexity is taken to correspond to a different phase of the material. Is is well known that as the result of the non-convexity of the energy, solutions to the equilibrium problem may either not exist or not have the properties of uniqueness or regularity [22].

To obtain the contact law in present case, we first adopt a classical approach based on matched asymptotic expansions. We obtain a non-monotonous contact law linking the stress vector to the jump in the displacement occuring at the interface.

In studies on the equilibrium of phase transforming materials, it is customary, however, to adopt the energy minimization approach. A method of computing the Γ-limit of the energy when the thickness and the rigidity of the adhesive tend towards zero has been presented in [9]. It has recently been pointed out however, that local minimizers play a key role in modeling equilibrium problems [9–11]. It is not possible to perform asymptotic studies directly on the behavior of the local energy minimizers, due to the complexity and non-uniqueness of the micro-structures involved.

To explore this issue, we adopted a highly simplified context: that of one-dimensional elasticity. Here we study the equilibrium problem in the case of an elastic bar subjected to Dirichlet boundary conditions, which is composed of two adherent linearly elastic parts separated by a thin adhesive layer having a non-monotonous stress-strain relation. Since a complete description of the equilibrium deformations is available in this case, it is possible to discuss in detail how they act as of global and (weak) local total energy minimizers. The corresponding contact laws can then be obtained quite logically by taking the thickness and the elastic constant of the adhesive to tend to zero. The constant law corresponding to a global minimizer is determined, but the local minimizers yield a whole set of contact laws because of their non-uniqueness in the original problem. The approach described by Abeyaratne and Knowles [9, 12, 13] is then adopted and a criterion to select preferred equilibrium configurations among local minimizers is introduced. This enables us to obtain various contact laws taking nucleation and the kinetics of the phase transformation into account.

This paper is organized as follows. In Section 2, the main notations and the equilibrium problem are presented. In Section 3,

Address correspondence to F. Lebon, Acoustic and Mechanical Laboratory, University of Provence, 31 Chemin Joseph Aiguier, 1302 Marseille Cedex 20, France. E-mail: lebon@lma.cnrs-mrs.fr

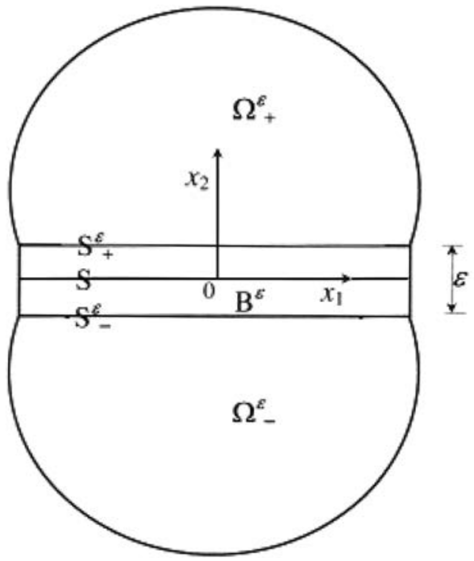

FIG. 1. A sketch of the composite structure modeled in this paper.

the contact law is determined using the matched asymptotic expansions method. The problem of the total energy minimization in the one-dimensional framework is addressed is Section 4, and the conclusions are presented in Section 5.

2. THE EQUILIBRIUM PROBLEM

In this section, we present the mechanical problem and give the notations used in this study. We consider a body occupying an open bounded set Ω of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$, which consists of three parts, $\Omega^\varepsilon_\pm$ and B^ε as shown in Figure 1. B^ε is the part occupied by the adhesive and $\Omega^\varepsilon_\pm$ are the parts occupied by the adherents. $S^\varepsilon_\pm$ are the (plane) interfaces between the adhesive and the adherents, and $\varepsilon > 0$ is the thickness of the adhesive. Introducing an orthonormal frame (O, x_1, x_2) with its origin at the center of the adhesive midplane and its x_2-axis running perpendicular to the interfaces $S^\varepsilon_\pm$, we have

$$\begin{aligned} B^\varepsilon &= \left\{(x_1, x_2) \in \Omega : |x_2| < \frac{\varepsilon}{2}\right\}, \\ \Omega^\varepsilon_\pm &= \left\{(x_1, x_2) \in \Omega : \pm x_2 > \frac{\varepsilon}{2}\right\}, \\ \Omega^\varepsilon &= \Omega^\varepsilon_+ \cup \Omega^\varepsilon_-, \\ S^\varepsilon_\pm &= \left\{(x_1, x_2) \in \Omega : \pm x_2 = \frac{\varepsilon}{2}\right\}. \end{aligned} \tag{1}$$

As ε tends to zero, we take Ω^0 to denote the geometrical limit of Ω^ε and S to denote the surface to which the adhesive and $S^{\varepsilon\pm}$ tend geometrically:

$$\begin{aligned} \Omega_\pm &= \{(x_1, x_2) \in \Omega : \pm x_2 > 0\}, \\ \Omega_0 &= \Omega_+ \cup \Omega_-, \\ S &= \{(x_1, x_2) \in \Omega : x_2 = 0\}. \end{aligned} \tag{2}$$

Let $u^\varepsilon : \Omega^\varepsilon \mapsto R^2$ denote a displacement field starting from Ω^ε. The adhesion between the adherents and the adhesive is assumed to be perfect, that is

$$[u^\varepsilon]_\pm = 0 \quad \text{or} \quad S^\varepsilon_\pm. \tag{3}$$

where

$$[u^\varepsilon]_\pm = u^\varepsilon\left(x_1, \left(\pm\frac{\varepsilon}{2}\right)^+\right) - u^\varepsilon\left(x_1, \left(\pm\frac{\varepsilon}{2}\right)^-\right). \tag{4}$$

are the jumps in the displacements on the interfaces $S^\varepsilon_\pm$.[1]

The adherents are assumed to show linearly elastic behavior starting from Ω^ε, and we take a_{ijkl} to denote the components of the elastic tensor, which are assumed to satisfy the usual symmetry and ellipticity conditions. Taking σ_ε to denote the stress tensor and defining the strain tensor $e = (e_{ij})$ as follows

$$e_{ij}(u^\varepsilon) = \frac{1}{2}\left(u^\varepsilon{}_{i,j} + u^\varepsilon{}_{j,i}\right) \tag{5}$$

we obtain

$$\sigma_{\varepsilon ij} = a_{ijkh} e_{kh} \quad \text{in } \Omega^\varepsilon_\pm \tag{6}$$

To model the constitutive behavior of the adhesive, we adopt the Frémond modelling procedure, which involves three phases, the austenite and two variants of martensite [7]:

$$\begin{cases} \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \alpha\chi_{ij}, & \text{if } \alpha\chi_{ij}e_{ij} \le -c \\ \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, & \text{if } |\alpha\chi_{ij}e_{ij}| \le c \\ \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \alpha\chi_{ij}, & \text{if } \alpha\chi_{ij}e_{ij} \ge c. \end{cases} \quad \text{in } B^\varepsilon \tag{7}$$

For the sake of simplicity, the phases are taken to consist of isotropic materials with the same elastic constants. The constants α and c are two material parameters. The coefficient α, which is non-negative depends the thermal dilatation, and $2c$ gives the amplitude of the austenite phase. The tensor χ is a given symmetric orientation tensor associated with the austenite-martensite transformation. The stress-strain diagrams for an extension test are shown in Figure 2. In case (a), corresponding to $c \le 0$, two variants of martensite are present, one undergoes positive strains and the other, negative strains. In case (b), where $c > 0$, the austenite exists in the strain range $(-c(\alpha\chi_{11})^{-1}, c(\alpha\chi_{11})^{-1})$ and the martensite variants act at strains outside this interval.

A body force density ϕ is applied to Ω^ε and a surface force density g to $\Gamma_g \subset \partial\Omega^\varepsilon$. On the complementary part

[1]Here $u^\varepsilon(x_1, (\frac{\varepsilon}{2})^+)$(resp. $u^\varepsilon(x_1, (\frac{\varepsilon}{2})^-)$) indicates the limit of $u^\varepsilon(x_1, x_2)$ when x_2 tends to $(\frac{\varepsilon}{2})$, $x_2 \ge (\frac{\varepsilon}{2})$(resp. $x_2 \le (\frac{\varepsilon}{2})$).

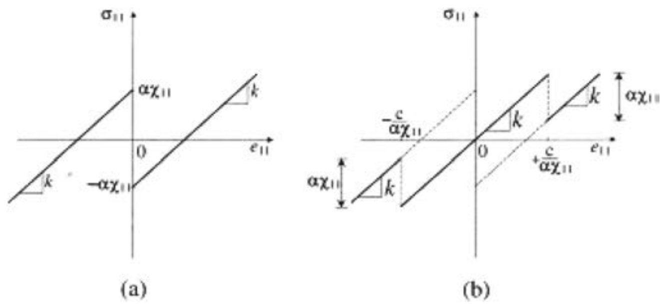

FIG. 2. Stress-strain diagrams obtained in an extension test on the adhesive: (a) $c \le 0$, (b) $c > 0$.

$\Gamma_u = \partial\Omega^\varepsilon \backslash \Gamma_g$, the following homogeneous boundary conditions are prescribed:

$$u^\varepsilon = 0 \quad \text{or} \quad \Gamma_u. \tag{8}$$

The equilibrium problem as regards the composite can be stated as follows:

$$(P_\varepsilon)\begin{cases} \text{Find } (u^\varepsilon, \sigma^\varepsilon) \text{ such that :} & \\ \sigma^\varepsilon_{ij} = -\varphi_i & \text{in } \Omega \\ \sigma^\varepsilon_{ij} = a_{ijkh} e_{kh}(u^\varepsilon) & \text{in } \Omega^\varepsilon \\ \begin{cases} \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \alpha\chi_{ij}, & \\ \qquad\qquad \text{if } \alpha\chi_{ij} e_{ij} \le -c & \\ \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \quad \text{if } |\alpha\chi_{ij} e_{ij}| \le c & \text{in } B^\varepsilon \\ \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \alpha\chi_{ij}, & \\ \qquad\qquad \text{if } \alpha\chi_{ij} e_{ij} \ge c, & \end{cases} & \\ u^\varepsilon = 0 & \text{on } \Gamma_u \\ \sigma^\varepsilon n = g & \text{on } \Gamma_g \\ [u^\varepsilon]_\pm = [\sigma^\varepsilon n]_\pm = 0 & \text{on } S^\varepsilon_\pm \end{cases} \tag{9}$$

Due to the non-monotony of the constitutive equation of the adhesive, the existence of equilibrium solutions is still an open problem.

3. MATCHED ASYMPTOTIC EXPANSIONS APPROACH

3.1. General Comments

In order to develop a two-dimensional approach, the following matched asymptotic expansions method was used. This method consists in finding two expansions of the displacement u^ε and the stress σ^ε in terms of power of ε (an external expansion in the bodies and an internal one in the joint), and to combine these two expansions in order to obtain the same limit [14, 22].

3.2. External Expansions

The external expansion is a classical expansion:

$$\begin{aligned} u^\varepsilon(x_1, x_2) &= u^0(x_1, x_2) + \varepsilon u^1(x_1, x_2) + \cdots \\ \sigma^\varepsilon_{ij}(x_1, x_2) &= \sigma^0_{ij}(x_1, x_2) + \varepsilon \sigma^1_{ij}(x_1, x_2) + \cdots \\ e_{ij}(u^\varepsilon)(x_1, x_2) &= e^0_{ij} + \varepsilon e^1_{ij} + \cdots \\ e^l_{ij} &= \frac{1}{2}\left(\frac{\partial u^i_j}{\partial x_j} + \frac{\partial u^i_j}{\partial x_i}\right) \end{aligned} \tag{10}$$

3.3. Internal Expansions

In the internal expansion, a blow-up of the second variable is performed. Let $y_2 = \frac{x_2}{\varepsilon}$. The internal expansion gives

$$\begin{aligned} u^\varepsilon(x_1, x_2) &= v^0(x_1, y_2) + \varepsilon v^1(x_1, y_2) + \cdots \\ \sigma^\varepsilon_{ij}(x_1, y_2) &= \varepsilon^{-1}\tau^{-1}_{ij}(x_1, y_2) + \tau^0_{ij}(x_1, y_2) + \varepsilon\tau^1_{ij}(x_1, y_2) + \cdots \\ e_{ij}(u^\varepsilon)(x_1, y_2) &= \varepsilon^{-1} e^{-1}_{ij} + e^0_{ij} + \varepsilon e^1_{ij} + \cdots \\ e^l_{11} &= \frac{\partial v^l_1}{\partial x_1} \\ e^l_{22} &= \frac{\partial v^{l+1}_2}{\partial y_2} \\ e^l_{12} &= \frac{1}{2}\left(\frac{\partial v^l_2}{\partial x_1} + \frac{\partial v^{l+1}_1}{\partial y_2}\right) \end{aligned} \tag{11}$$

3.4. Continuity Conditions

The third step in the method consists of combining the two expansions. In particular, we observe that when ε tends to zero, x_2 tends to $0^\pm$ and y_2 tends to $\pm\infty$. Combining the two expansions gives

$$\begin{aligned} v^0(x_1, \pm\infty) &= u^0(x_1, 0^\pm) \\ \tau^{-1}(x_1, \pm\infty) &= 0 \\ \tau^0(x_1, \pm\infty) &= \sigma^0(x_1, 0^\pm) \end{aligned} \tag{12}$$

3.5. Behavior in the Thin Layer

In the thin layer, the constitutive equation gives

- $c < 0$

$$\begin{aligned} &\varepsilon^{-1}\tau^{-1}_{ij} + \tau^0_{ij} + \varepsilon\tau^1_{ij} + \cdots \\ &\quad = \lambda\big(\varepsilon^{-1} e^{-1}_{kk} + e^0_{kk} + \varepsilon e^1_{kk} + \cdots\big)\delta_{ij} \\ &\qquad + 2\mu\big(\varepsilon^{-1} e^{-1}_{ij} + e^0_{ij} + \varepsilon e^1_{ij} + \cdots\big) + \alpha\chi_{ij} \\ &\qquad \text{if } \alpha\chi : e \le 0 \\ &\varepsilon^{-1}\tau^{-1}_{ij} + \tau^0_{ij} + \varepsilon\tau^1_{ij} + \cdots \\ &\quad = \lambda\big(\varepsilon^{-1} e^{-1}_{kk} + e^0_{kk} + \varepsilon e^1_{kk} + \cdots\big)\delta_{ij} \\ &\qquad + 2\mu\big(\varepsilon^{-1} e^{-1}_{ij} + e^0_{ij} + \varepsilon e^1_{ij} + \cdots\big) - \alpha\chi_{ij} \\ &\qquad \text{if } \alpha\chi : e \ge 0, \end{aligned} \tag{13}$$

- $c \geq 0$

$$
\begin{aligned}
&\varepsilon^{-1}\tau_{ij}^{-1} + \tau_{ij}^{0} + \varepsilon\tau_{ij}^{1} + \cdots \\
&\quad = \lambda\left(\varepsilon^{-1}e_{kk}^{-1} + e_{kk}^{0} + \varepsilon e_{kk}^{1} + \cdots\right)\delta_{ij} \\
&\qquad + 2\mu\left(\varepsilon^{-1}e_{ij}^{-1} + e_{ij}^{0} + \varepsilon e_{ij}^{1} + \cdots\right) + \alpha\chi_{ij} \\
&\qquad\ \text{if } \alpha\chi : e \leq -c, \\
&\varepsilon^{-1}\tau_{ij}^{-1} + \tau_{ij}^{0} + \varepsilon\tau_{ij}^{1} + \cdots \\
&\quad = \lambda\left(\varepsilon^{-1}e_{kk}^{-1} + e_{kk}^{0} + \varepsilon e_{kk}^{1} + \cdots\right)\delta_{ij} \\
&\qquad + 2\mu\left(\varepsilon^{-1}e_{ij}^{-1} + e_{ij}^{0} + \varepsilon e_{ij}^{1} + \cdots\right) \\
&\qquad\ \text{if } |\alpha\chi : e| \leq c, \\
&\varepsilon^{-1}\tau_{ij}^{-1} + \tau_{ij}^{0} + \varepsilon\tau_{ij}^{1} + \cdots \\
&\quad = \lambda\left(\varepsilon^{-1}e_{kk}^{-1} + e_{kk}^{0} + \varepsilon e_{kk}^{1} + \cdots\right)\delta_{ij} \\
&\qquad + 2\mu\left(\varepsilon^{-1}e_{ij}^{-1} + e_{ij}^{0} + \varepsilon e_{ij}^{1} + \cdots\right) - \alpha\chi_{ij} \\
&\qquad\ \text{if } \alpha\chi : e \geq c, .
\end{aligned}
\tag{14}
$$

In what follows, we assume that χ does not depend on ε. The identification of the various orders depends on the scaling of coefficients λ, μ and α with ε. In what follows, we take $\bar{\lambda}$ (resp. $\bar{\mu}$, $\bar{\alpha}$) to denote the limits of the ratios between λ (resp. μ, α) and ε. We thus obtain 18 cases corresponding to the three different limits of $\frac{\lambda}{\varepsilon}$ and $\frac{\mu}{\varepsilon}$, i.e., zero, a non-zero bounded value and infinity and the two different limits of $\frac{\alpha}{\varepsilon}$, i.e., infinity or a constant limit not equal to zero. Since the sign of c determines two different material responses, we will obtain 36 different contact laws. In what follows, only the case $\bar{\alpha} = \infty$ will be examined. In the other cases, a previously established result [1] is obtained. To simplify, we examine only 1 of the 18 possible values of the coefficients. The other cases can be obtained in the same way. Therefore, taking

$$
\lambda = \varepsilon\bar{\lambda}, \quad \mu = \varepsilon\bar{\mu}, \quad \alpha = \varepsilon^{\eta}\bar{\alpha}\ (\eta < 1) \tag{15}
$$

we have

$$
\begin{array}{ll}
\tau_{ij}^{-1} = 0, & \\
\tau_{ij}^{0} = \bar{\lambda}e_{kk}^{-1}\delta_{ij} + 2\bar{\mu}e_{ij}^{-1} + \bar{\alpha}\chi_{ij}, & \text{if } \alpha\chi_{ij}e_{ij} \leq -c, \\
\tau_{ij}^{0} = \bar{\lambda}e_{kk}^{-1}\delta_{ij} + 2\bar{\mu}e_{ij}^{-1}, & \text{if } |\alpha\chi_{ij}e_{ij}| \leq c, \\
\tau_{ij}^{0} = \bar{\lambda}e_{kk}^{-1}\delta_{ij} + 2\bar{\mu}e_{ij}^{-1} - \bar{\alpha}\chi_{ij}, & \text{if } \alpha\chi_{ij}e_{ij} \geq c,
\end{array}
\tag{16}
$$

3.5.1. Equilibrium Order −2

In this paragraph, we deal with the equilibium equation at order −2. We obtain

$$
\frac{\partial\tau_{i2}^{-1}}{\partial y_2} = 0 \tag{17}
$$

Therefore, τ_{i2}^{-1} does not depend on y_2. Due to the limit of τ_{i2}^{-1} in $\pm\infty$, which is zero, we have

$$
\tau_{i2}^{-1} = 0 \tag{18}
$$

In the thin layer and in the bodies, τ_{ij}^{-1} depends only on e_{i2}^{-1}. In the bodies, we have

$$
\tau_{i2}^{-1}(x_1, y_2) = a_{i2f2}\frac{\partial\upsilon_f^0}{\partial y_2}, \quad \tau_{11}^{-1}(x_1, y_2) = a_{11j1}\frac{\partial\upsilon_f^0}{\partial y_2} \tag{19}
$$

Therefore, for $|y_2| > \frac{1}{2}$, we have

$$
\frac{\partial\upsilon_i^0}{\partial y_2} = 0, \quad \upsilon_j^0(x_2, y_2) = \upsilon_j^0(x_1) \tag{20}
$$

We obtain $\tau_{ij}^{-1} = 0$ in the bodies, and due to the conditions involved in the combination, $\upsilon_j^0(x_1, y_2) = u_j^0(x_1, 0^{\pm})$. The same arguments give $\tau_{ij}^{-1} = 0$ in the adhesive.

3.5.2. Equilibrium Order −1

In this paragraph, we consider the equilibium equation at order −1, and since τ_{ij}^{-1} is equal to zero, we obtain

$$
\frac{\partial\tau_{i2}^{0}}{\partial y_2} = 0 \tag{21}
$$

Therefore, $\tau_{i2}^{0}(x_1, y_2)$ does not depend on y_2. Using the conditions involved in the combination, we have

$$
\tau_{i2}^{0} = \sigma_{i2}^{0}(x_1, 0) \tag{22}
$$

We have

- If $\alpha\chi_{ij}e_{ij} \leq -c$

$$
\begin{aligned}
\tau_{11}^{0} &= \bar{\lambda}\frac{\partial\upsilon_2^0}{\partial y_2} + \bar{\alpha}\chi_{11}, \\
\tau_{12}^{0} &= \bar{\mu}\frac{\partial\upsilon_1^0}{\partial y_2} + \bar{\alpha}\chi_{12}, \\
\tau_{22}^{0} &= (\bar{\lambda} + 2\bar{\mu})\frac{\partial\upsilon_2^0}{\partial y_2} + \bar{\alpha}\chi_{22}
\end{aligned}
\tag{23}
$$

- If $\left|\alpha\chi_{ij}e_{ij}\right| \leq c$

$$
\begin{aligned}
\tau_{11}^{0} &= \bar{\lambda}\frac{\partial\upsilon_2^0}{\partial y_2}, \\
\tau_{12}^{0} &= \bar{\mu}\frac{\partial\upsilon_1^0}{\partial y_2}, \\
\tau_{22}^{0} &= (\bar{\lambda} + 2\bar{\mu})\frac{\partial\upsilon_2^0}{\partial y_2}
\end{aligned}
\tag{24}
$$

- If $\alpha\chi_{ij}e_{ij} \geq c$

$$\tau_{11}^0 = \bar{\lambda}\frac{\partial \upsilon_2^0}{\partial y_2} - \bar{\alpha}\chi_{11},$$
$$\tau_{12}^0 = \bar{\mu}\frac{\partial \upsilon_1^0}{\partial y_2} - \bar{\alpha}\chi_{12}, \tag{25}$$
$$\tau_{22}^0 = (\bar{\lambda} + 2\bar{\mu})\frac{\partial \upsilon_2^0}{\partial y_2} + \bar{\alpha}\chi_{22}$$

Because of the perfect adhesion to $S_e^{\pm}$, we have

- If $\alpha\chi_{ij}e_{ij} \leq -c$

$$\begin{aligned} \sigma_{12}^0(x_1, 0) &= K_T\left[u_1^0\right] + K_{PT}\chi_{12} \\ \sigma_{22}^0(x_1, 0) &= K_N\left[u_2^0\right] + K_{PT}\chi_{22} \end{aligned} \tag{26}$$

- If $\left|\alpha\chi_{ij}e_{ij}\right| \leq c$

$$\sigma_{12}^0(x_1, 0) = K_T\left[u_1^0\right]$$
$$\sigma_{22}^0(x_1, 0) = K_N\left[u_2^0\right] \tag{27}$$

- If $\alpha\chi_{ij}e_{ij} \geq c$

$$\begin{aligned} \sigma_{12}^0(x_1, 0) &= K_T\left[u_1^0\right] - K_{PT}\chi_{12} \\ \sigma_{22}^0(x_1, 0) &= K_N\left[u_2^0\right] - K_{PT}\chi_{22} \end{aligned} \tag{28}$$

where $K_T = \bar{\mu}$, $K_N = \bar{\lambda} + 2\bar{\mu}$ and $K_{PT} = \bar{\alpha}$.

3.5.3. *Computation of $\alpha\chi_{ij}e_{ij}$*

To complete the analysis, we need to compute $\alpha\chi_{ij}e_{ij}$.

$$\alpha\chi_{ij}e_{ij} = \alpha\left(\chi_{12}\frac{\partial \upsilon_1^0}{\partial y_2} + \chi_{22}\frac{\partial \upsilon_2^0}{\partial y_2}\right) \tag{29}$$

Integrating along the thickness of the adhesive, the term $\alpha\chi_{ij}e_{ij}$ becomes $\alpha(\chi_{12}[u_1^0] + \chi_{22}[u_2^0])$. Note, that this term is of the order -1. If we assume that $c = \bar{c}\varepsilon^{-1}$, we have to compare $\alpha(\chi_{12}[u_1^0] + \chi_{22}[u_2^0])$ with $\bar{c}$.

3.6. Two-Dimensional Contact Laws

Let S, T denote the diagonal 2×2 matrices with $S_{11} = K_T$, $S_{22} = K_N$ and $T_{11} = T_{22} = K_{PT}$. Let x be the vector with $x_1 = \chi_{12}$ and $x_2 = \chi_{22}$. Note that the coefficients of S (resp. T) can be equal to infinity, zero or a non-zero bounded value (resp. infinity or a non-zero bounded value). Using these notations, we obtain the following family of contact laws

- $\bar{c} \leq 0$

$$\begin{aligned} \sigma n &= S[u] + Tx \quad \text{if } \alpha x_i[u_i] \leq 0, \\ \sigma n &= S[u] + Tx \quad \text{if } \alpha x_i[u_i] \geq 0, \end{aligned} \tag{30}$$

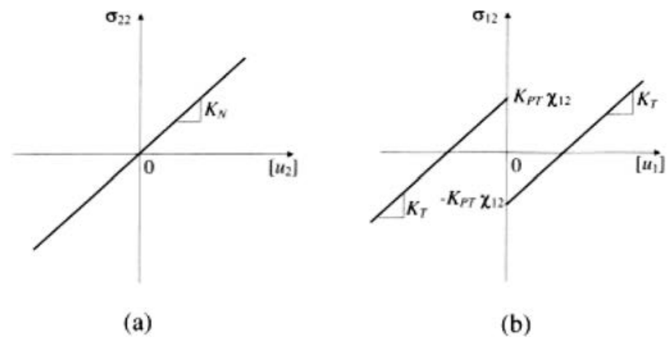

FIG. 3. Contact laws for $c \leq 0$: (a) normal contact law for $\chi_{22} = 0$; (b) tangential contact law.

- $\bar{c} > 0$

$$\begin{aligned} \sigma n &= S[u] + Tx \quad \text{if } \alpha x_i[u_i] \leq -\bar{c}, \\ \sigma n &= S[u] \quad \text{if } |\alpha x_i[u_i]| \leq \bar{c}, \\ \sigma n &= S[u] - Tx \quad \text{if } \alpha x_i[u_i] \geq \bar{c}, \end{aligned} \tag{31}$$

An interesting case arises when $\chi_{22} = 0$. We obtain a normal compliance law and a non monotonous friction law, which is given in Figure 3 for $c \leq 0$ and in Figure 4 for $c > 0$. Reviews of laws of this kind can be found in [15–17].

4. CONTACT LAWS ARISING FROM ENERGY MINIMIZATION

In this Section, it is proposed to study the contact laws that can be found by considering the asymptotic behavior of (global or local) minimizers of the total energy. Since it was established in [18] that the minimum energy principle is not equivalent to the mechanical problem (9), we do not expect to find the same contact laws as those obtained in Section 3. In studying the minimum problem, we restrict ourselves to the one-dimensional case, because this is the case where it is possible to give a complete description of the minimizers. We begin by studying the equilibrium problem, in terms of energy minimization, of a one-dimensional composite bar with Dirichlet boundary conditions.

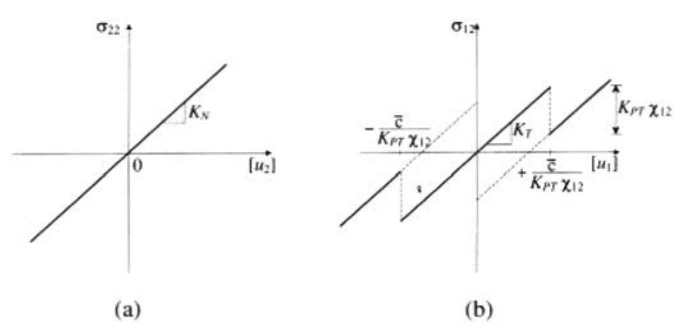

FIG. 4. Contact laws for $c > 0$: (a) normal contact law for $\chi_{22} = 0$; (b) tangential contact law.

4.1. Equilibrium Problem in the Case of a One-Dimensional Bar

Let us take a one-dimensional bar occupying a reference unstressed space configuration $\Omega = (0, l)$, starting from which there are displacement fields $u = u(x), x \in (0, l)$ which are continuous, with the piecewise continuous derivative $u' \geq 0$. The bar is fixed at the extremity $x = 0$, and is subjected to a prescribed displacement $\delta > 0$ at $x = L$.

The bar is assumed to be made of two different elastic materials. In the range $0 \leq x \leq \varepsilon L$, the bar is composed of an adhesive layer characterized by the piecewise quadratic stored energy density

$$w_a(e) = \min_{e>0} \left\{ \frac{k}{2}e^2; \frac{k}{2}e^2 - \alpha\chi e + c \right\}, \tag{32}$$

where $k > 0$ is the elasticity of the adhesive. The stress-strain diagram which was discussed in Section 3, is given in Figure 2 (where $e = e_{11}$, etc.).

In the range $\varepsilon L \leq x \leq L$, the bar is composed of a material with quadratic stored energy density having an elastic modulus K

$$w_b(e) = \frac{K}{2}e^2. \tag{33}$$

The total energy of the bar is given by

$$E(u) = \int_0^{eL} w_a(u')dx + \int_{\varepsilon L}^{L} w_b(u')dx, \tag{34}$$

where $u \in \mathcal{U}$ is the class of displacements continuous with the piecewise continuous derivative satisfying the boundary conditions. In the energy minimization problem (34), we are interested in finding the global minimizer $\bar{u} \in \mathcal{U}$:

$$E(\bar{u}) \leq E(u) \quad \forall u \in \mathcal{U} \tag{35}$$

and in finding the weak local minimizers $\bar{u} \in \mathcal{U}$ for which Eq. (35) holds at all the values of $u \in \mathcal{U}$ such that

$$\sup_{x\in(0,L)} |u(x) - \bar{u}(x)| + \sup_{x\in(0,L)} |u'(x) - \bar{u}'(x)| < \eta \tag{36}$$

for some $\eta > 0$ [19]. For $\bar{u} \in \mathcal{U}$ to be a global or relative minimizer, it is necessary that the first variation of $E(u)$ should vanish at $\bar{u}$:

$$\left[\frac{d}{dh}E(\bar{u} + hV)\right]_{h=0} = 0, \tag{37}$$

under all admissible perturbations $v : [0, L] \to \mathbb{R}$ which are continuous with a piecewise continuous derivative and such that $v(L) = 0 = v(0)$. This leads to the system of equilibrium equations:

$$\frac{\partial w_a}{\partial e}(\bar{u}') = \sigma_\varepsilon, \quad x \in (0, \varepsilon L), \qquad \frac{\partial w_b}{\partial e}(\bar{u}') = \sigma_\varepsilon, \quad x \in (\varepsilon L, L), \tag{38}$$

where σ_ε is the (constant) stress in the bar. In view of Eqs. (32) and (33), this system admits the following possible solutions [20].

i) If $0 \leq \sigma_\varepsilon < kc(\alpha\chi)^{-1} - \alpha\chi$, then, Eqs. (38) are satisfied by

$$\bar{u} = \begin{cases} \dfrac{\sigma_e}{k}x & x \in [0, \varepsilon L], \\ \dfrac{\sigma_\varepsilon}{K}x + \sigma_\varepsilon \varepsilon L \left(\dfrac{1}{k} - \dfrac{1}{K}\right) & x \in (\varepsilon L, L], \end{cases} \tag{39}$$

with $\sigma_\varepsilon = \hat{k}_\varepsilon \delta L^{-1}$, and $\hat{k}_\varepsilon^{-1} := \varepsilon k^{-1} + (1 - \varepsilon)K^{-1}$. Using the expression for σ_ε, it is easy to see that this solution is possible whenever $0 \leq \delta < (kc(\alpha\chi)^{-1} - \alpha\chi)Lk_\varepsilon^{-1}$. Since δ is positive, this solution exists only if $c > 0$ and corresponds to the case of adhesive consisting of only austenite.

ii) If $c > 0$ and $kc(\alpha\chi)^{-1} - \alpha\chi \leq \sigma_\varepsilon < kc(\alpha\chi)^{-1}$, then the adhesive is a mixture of austenite and martensite. Let $\lambda \in (0, 1)$ denote the austenite volume fraction. As λ varies in $[0, 1]$, we obtain a set of equilibrium solutions:

$$\bar{u} = \begin{cases} \frac{\sigma_e}{k}x & x \in [0, \lambda\varepsilon L], \\ \frac{\sigma_e + \alpha\chi}{k}x - \frac{\alpha\chi\varepsilon\lambda L}{k} & x \in (\lambda\varepsilon L, \varepsilon L], \\ \frac{\sigma_\varepsilon}{K}x + \varepsilon L\left(\dfrac{\sigma_\varepsilon + (1-\lambda)\alpha\chi}{k} - \dfrac{\sigma_\varepsilon}{K}\right) & x \in (\varepsilon L, L], \end{cases} \tag{40}$$

with

$$\sigma_\varepsilon = \hat{k}_\varepsilon \left(\frac{\delta}{L} - \varepsilon(1 - \lambda)\frac{\sigma\chi}{k}\right). \tag{41}$$

Using this expression for σ_ε, it turns out that the two-phase solution exists whenever

$$\begin{aligned} ((kc(\alpha\chi)^{-1} - \alpha\chi)\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi(1 - \lambda)k^{-1})L \leq \delta \\ \leq (kc(\alpha\chi)^{-1}\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi(1 - \lambda)k^{-1})L. \end{aligned} \tag{42}$$

Since the value of the volume fraction λ is within the $0 < \lambda < 1$ range, this condition can be further extended as follows

$$(kc(\alpha\chi)^{-1} - \alpha\chi)\, \hat{k}_\varepsilon^{-1}L \leq \delta \leq (kc\,(\alpha\chi)^{-1} + \varepsilon\alpha\chi k^{-1})L. \tag{43}$$

Therefore, given any $\lambda \in (0, 1)$, if the prescribed elongation δ lies in the above range, then there will exist in the adhesive a two-phase solution involving a mixture of both phases.

iii) If $\sigma_\varepsilon \geq kc(\alpha\chi)^{-1}$, then the solution is

$$\bar{u} = \begin{cases} \dfrac{\sigma_\varepsilon + \alpha\chi}{k} x & x \in [0, \varepsilon L], \\ \dfrac{\sigma_\varepsilon}{K} x + \varepsilon L \left(\dfrac{\sigma_\varepsilon + \alpha\chi}{k} - \dfrac{\sigma_\varepsilon}{K} \right) & x \in (\varepsilon L, L], \end{cases} \tag{44}$$

with $\sigma_\varepsilon = \hat{k}_\varepsilon(\delta L^{-1} - \alpha\chi\varepsilon k^{-1})$. This solution which is possible for $\delta > kc(\alpha\chi)^{-1} L k_\varepsilon^{-1} + \alpha\chi\varepsilon L k^{-1}$ when $c > 0$ and for any positive δ when $c \leq 0$, describes the case of adhesive consisting entirely of martensite.

In [20] it is established that the configurations described by the solutions (39), (40) and (44) correspond to weak local minimizers. It is also established that the following equilibrium configurations correspond to a global minimizer:

a) if $0 \leq \delta < (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1}$, then solution (39) is a global minimizer;
b) if $(kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} \leq \delta < (kc(\alpha\chi)^{-1} - \alpha\chi/2) L\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi L k^{-1}$, then the solution (40) with σ_ε given by the Maxwell stress $\sigma_M = kc(\alpha\chi)^{-1} - \alpha\chi/2$ is a global minimizer, Substituting σ_M into Eq. (41), we find the austenite volume fraction determined solely for the given δ:

$$\lambda_{glo} = 1 + \frac{K}{\varepsilon\alpha\chi} \left(\frac{k}{\hat{k}_\varepsilon} c(\alpha\chi)^{-1} - \frac{\alpha\chi}{2\hat{k}_\varepsilon} - \frac{\delta}{L} \right). \tag{45}$$

Note that when $\delta = (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1}$ we have $\lambda_{glo} = 1$, i.e. the adhesive is still in the austenite phase. If δ is continuously increased, the austenite volume fraction decreases and the stress in the bar remains constant and equal to the Maxwel stress. When $\delta = (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi L k^{-1}$, we have $\lambda_{glo} = 0$, and the adhesive has completed the transformation from austenite to martensite.
c) if $\delta \geq (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi L k^{-1}$, then the solution (44) is a global minimizer.

4.2. Contact Laws

To obtain the contact law we study the asymptotic behavior of the above equilibrium solutions when both the parameters (ε, k) tend to zero and the thin adhesive layer is replaced by a point. To be able to compare the result we are going to get with the contact laws obtained in Section 3, we set

$$k = \bar{k}\varepsilon. \tag{46}$$

Our aim here is to study the relations between the limits

$$\sigma = \lim_{\varepsilon \to 0} \sigma_\varepsilon, \quad [u] = \lim_{\varepsilon \to 0} (\bar{u}(\varepsilon L) - \bar{u}(0)), \tag{47}$$

where σ_ε and $\bar{u}$ correspond to the equilibrium configurations determined in the previous subsection. This study leads to defining an asymptotic contact law linking the limit stress in the bar, σ_ε to the jump in the displacement occuring at the adhesive interface, $[u]$. This contact law describes the limit behavior of the adhesive.

In addition to Eq. (46), we need to specify the scaling to the material parameters with ε. To make the limits (47) finite, we take

$$c = \bar{c}\varepsilon^{-1}, \quad \alpha = \bar{\alpha}, \quad \chi = \bar{\chi}, \tag{48}$$

where $\bar{k}$, $\bar{c}$, $\bar{\alpha}$ and $\bar{\chi}$ are independent of ε. Note that the sealing of α is different to that assumed in Section 3 (Eq. (15)).

Substituting Eqs. (46) and (48) into the expressions for σ_ε and $\bar{u}$ listed in Section 2 in (a), (b) and (c), taking the limit $\varepsilon \to 0^+$ and eliminating δ between σ_ε and $[u]$, we obtain the following contact law:

$$\sigma_{glo} = \begin{cases} \bar{k}\dfrac{[u]}{L}, & 0 \leq [u] < L\left(\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right), \\ \left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \dfrac{\bar{\alpha}\bar{\chi}}{2}\right), & L\left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right) \leq [u] \\ & \quad < L\left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right), \\ \bar{k}\dfrac{[u]}{L} - \bar{\alpha}\bar{\chi}, & [u] \geq L\left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right), \end{cases} \tag{49}$$

corresponding to global minimizers of the original equilibrium problem. In the same way, taking σ_ε and $\bar{u}$ as in (i), (ii) and (iii), we can calculate the limit contact law corresponding to local minimizers. Note that this law turns out to be undefined, because of the lack of information available due to the non uniqueness of the local minimizers. Indeed, we obtain

$$\sigma_{loc} = \begin{cases} \bar{k}\dfrac{[u]}{L}, & 0 \leq [u] < L\left(\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right), \\ \bar{\Sigma}, & L\left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right) \leq [u] \\ & \quad < L\left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right), \\ \bar{k}\dfrac{[u]}{L} - \bar{\alpha}, & [u] \geq L\left(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \dfrac{\bar{\alpha}\bar{\chi}}{2k}\right), \end{cases} \tag{50}$$

where $\bar{\Sigma}$ can take any value in $[\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \bar{\alpha}\bar{\chi}, \bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \bar{\alpha}\bar{\chi}]$. Therefore, local minimizers give rise to multiple contact laws, all of which are included in the dashed parallelogram depicted in Figure 5.

4.3. The Role of Nucleation and the Kinetics

One way of overcoming the stress indeterminacy associated with metastable equilibrium solutions consists in adopting a criterion for selecting a path between local minimizers. In line with Aberayatne and Knowles [9, 12, 13, 20], we introduce nucleation and kinetic conditions into the equilibrium problem for the bar. In this section we briefly outline these conditions. In the context of the dynamic problem involving elastic bars. Aberayatne and

Then, according to Aberayatne and Knowles, a phase boundary nucleates provided that the driving force exerted on it, shortly after it nucleates, exceeds some critical value f_{nuc}(>0):

$$f_\varepsilon \geq f_{nuc}. \tag{54}$$

This condition determines the relative displacement δ at which the point $(\sigma_\varepsilon, u'(\varepsilon L))$ leaves the first ascending branch of the stress-strain diagram. Once a phase boundary has nucleated. it propagates according to the kinetic (or evolution) condition, which relates the driving, force to the velocity of the phase boundary

$$f_e = \phi(\dot{s}). \tag{55}$$

The restriction $z\phi(z) \geq 0$ is imposed by the dissipation inequality (51). Condition (55) rules out most of the local minimizers and uniquely determines a path in the stress-strain plane between them. Different paths arise, depending on the type of kinetic function ϕ involved.

Here we adopt the classical linear kinetics $\phi(z) = mz$ and the pinning kinetics $\phi(z) = m\sqrt{(ax^2 - b)_{+z}}$. The latter models the presence of defects slowing down the phase boundary motion [10]. This gives the following evolution of parameter s:

$$s(\delta) = (s(0) + K_2)e^{-K_3\delta} + K_1\delta - K_2 \text{ (linear)}, \tag{56}$$

$$m\sqrt{(a\dot{s}^2 - b)} = K_4 + K_5\delta L^{-1} + K_6 s \text{ (pinning)}. \tag{57}$$

The constants K_i are given in terms of the material parameters. To study the asymptotic behavior of these equations, we assume $m = \bar{m}\varepsilon^{-1}$. Taking $\bar{s}$ to denote the limit of s as ε tends to zero, we obtain the contact laws given by Eq. (69) with

$$\bar{\Sigma} = \bar{k}[u]/L - \bar{s}\bar{\alpha} \tag{58}$$

$$\bar{s}(\delta) = (\bar{s}(0) + L_2)e^{-L_3\delta} + L_1\delta - L_2 \text{ (linear)}, \tag{59}$$

$$\bar{m}\sqrt{(a\dot{\bar{s}}^2 - b)} = L_4 + L_5\bar{c}(\bar{\alpha}\bar{\chi})^{-1}\delta L^{-1} + L_6\bar{s} \text{ (pinning)}. \tag{60}$$

The new constants L_i are again completely given in terms of $\bar{\alpha}, \bar{\chi}$, etc. These contact laws are outlined in Figure 5.

4.4. Convex Decomposition

Another approach consists of selecting a particular solution numerically between the set of local energy minimizers. In this section, we present the convex decomposition procedure (DC) [18]. First of all, we note that the energy in the thin layer can be decomposed as follows:

$$w_a(e) = \phi_1(e) - \phi_2(e) \tag{61}$$

$$\phi_1(e) = \frac{k}{2}e^2 + c \tag{62}$$

$$\phi_2(\varepsilon) = \max(\alpha|\varepsilon| - c, 0) \tag{63}$$

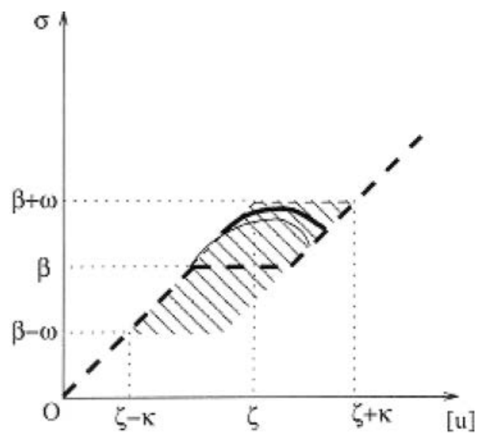

FIG. 5. Contact law obtained by energy minimization. To make the figure clearer, we have defined $\zeta = c(\bar{\alpha}\bar{\chi})^{-1}$, $\kappa = \bar{\alpha}\bar{\chi}\bar{k}^{-1}$, $\beta = \bar{k}(\bar{\alpha}\bar{\chi})^{-1} - \bar{\alpha}\bar{\chi}/2$, $\omega = \bar{\alpha}\bar{\chi}/2$. The shaded region is the contact law domain corresponding to local minimizers. Contact law corresponding to global minimizers (Maxwell line): - -. Contact laws corresponding to local minimizers: — linear kinetic, — pinning kinetic, - - convex decomposition.

Knowles assume that the nucleation and propagation of a phase boundary are governed by a relation between the normal speed on the phase boundary and the driving force f_e acting on it. As discussed in [9]. one could adopt a quasi-static context in which the inertial effects are neglected and the boundary datum δ is taken as a time parameter. All possible energy minimizing solutions are again described by (i), (ii) and (iii). We now write the dissipation inequality, which states that the dissipation, given by the rate of change in the total energy (the sum of the potential energy and the kinetic energy) minus the power of the work done at the boundary must be non negative at each equilibrium displacement. In the present context, since the total energy coincides with the potential energy, the dissipation inequality takes the form

$$D = \frac{dE(\bar{u})}{dt} - \sigma_\varepsilon \frac{d\delta}{dt} \geq 0. \tag{51}$$

Here $\bar{u}$ is given by Eq. (40). Substituting Eq. (40) into the above dissipation inequality and differentiating, we get

$$D = \dot{s}\left(\omega_\alpha\left(\frac{\sigma_\varepsilon}{k}\right) - \omega_\alpha\left(\frac{\sigma_\varepsilon + \alpha}{k}\right) + \frac{\sigma_\varepsilon \alpha}{k}\right) \tag{52}$$

where $\dot{s} = \frac{d\lambda}{d\delta}\varepsilon L$ plays the role of the velocity of the phase boundary at $x = \lambda\varepsilon L$. The driving force f_ε exerted on the phase boundary is defined as the quantity which multiplies $\hat{s}$:

$$f_\varepsilon = \left(w_\alpha\left(\frac{\sigma_\varepsilon}{k}\right) - w_\alpha\left(\frac{\sigma_\varepsilon + \alpha}{k}\right) + \frac{\sigma_\varepsilon \alpha}{k}\right)$$
$$= \frac{\alpha}{k}(\sigma_\varepsilon - \sigma_M). \tag{53}$$

The minimization of the energy $w_a(e)$ leads to the minimization of the following Lagrangian of the second kind:

$$L_2(e,\tau) = \int_{\varepsilon L}^{L} w_b(e)dl + \int_0^{\varepsilon L} \phi_1(e)dl + \phi_2^*(\tau) - \int_0^{\varepsilon L} e\tau dl \quad (64)$$

$$\phi_2^*(\tau) = \sup_e \left(\int_0^{\varepsilon L} e\tau dl - \int_0^{\varepsilon L} \phi_2(e)dl \right) \quad (65)$$

This problem can be solved by performing a sequence of minimizations on each variable e and τ (this method is described in detail in [18].) In our case, the solution of this minimization problem is obtained in a single iteration except for the two limit values of σ_ε, $kc(\alpha\chi)^{-1}$ and $kc(\alpha\chi)^{-1} + \alpha\chi$, which require two iterations. The solution of the minimization problem has three branches, the first two of which correspond to bars made entirely of austenite and martensite, respectively,

$$\delta/L \le kc(\alpha\chi\hat{k}_\varepsilon)^{-1} + \alpha\varepsilon k^{-1}, \;\; \sigma_\varepsilon = \hat{k}_\varepsilon(\delta L^{-1} - \alpha\varepsilon k^{-1}). \quad (66)$$

$$\delta/L \ge kc(\alpha\chi\hat{k}_\varepsilon)^{-1}, \;\; \sigma_\varepsilon = \hat{k}_\varepsilon \delta L^{-1}. \quad (67)$$

The third branch corresponds to a mixture.

$$\begin{aligned} &kc(\alpha\chi\hat{k}_\varepsilon)^{-1} \le \delta L^{-1} \le kc(\alpha\chi\hat{k}_\varepsilon)^{-1} + \alpha\varepsilon k^{-1}, \\ &\sigma_\varepsilon = kc(\alpha\chi)^{-1}. \end{aligned} \quad (68)$$

Assuming that ε tends to zero, we obtain the following contact law, which is plotted in Figure 5:

$$\sigma_{DC} = \begin{cases} \bar{k}\dfrac{[u]}{L}, & 0 \le [u] < L\bar{c}(\bar{\alpha}\bar{\chi})^{-1}, \\ \bar{k}\dfrac{\bar{c}}{\bar{\alpha}\bar{\chi}}, & L\bar{c}(\bar{\alpha}\bar{\chi})^{-1} \le [u] < L\left(\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \dfrac{\bar{\alpha}\bar{\chi}}{k}\right), \\ \bar{k}\dfrac{[u]}{L} - \bar{\alpha}\bar{\chi}, & [u] \ge L\left(\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \dfrac{\bar{\alpha}\bar{\chi}}{k}\right). \end{cases} \quad (69)$$

5. CONCLUSIONS

In this study, the asymptotic behavior of a thin layer characterized by a nonconvex, piecewise quadratic energy and bounded by two linearly elastic bodies is analyzed. To obtain the contact law which describes the behavior of a layer with a vanishing thickness, two different approaches were tested. First, we have dealt with a two-dimensional equilibrium problem and performed matched asymptotic expansions of both stress and displacement fields. After a suitable rescaling of the material parameters (see Eq. (15)), we obtained the contact laws described by Eq. (30), (31), and which are shown in Figures (3) (4). We then considered the possibility of obtaining the contact law by performing energy minimization. A detailed study of the computation of the $\Gamma-$ limit of the energy when the thickness and the rigidity of the adhesive tend to zero has been previously published in [21]. In the one-dimensional setting, it is possible to completely solve the minimization problem and to calculate the corresponding contact laws. Although this approach consists of simply solving an example, it is the first attempt ever made to model one-dimensional contact laws, as shown in Figure 5, taking nucleation and the kinetics of the martensite transformation process into account.

REFERENCES

1. Suquet, P., “Discontinuities and plasticity,” *Nonsmooth mechanics and applications* 315–323, Springer-Verlag (1988).
2. Licht, C., “Comportement asymptotique d’une bande dissipative mince de faible rigidité,” *Compte Rendu Académie des Sciences Série I.* **317**, 429–433 (1992).
3. Licht, C., and Michaille, G., “Une modélisation du comportement d’un joint collé élastique,” *Compte Rendu Académie des Sciences Série I.* **322**, 295–300 (1996).
4. Licht, C., and Michaille, G., “A modelling of elastic adhesive bonded joints,” *Adv. Math. Sci. Appl.*, **7**, 711–740 (1997).
5. Ould-Khaoua, A., Lebon, F., Licht, C., and G., Michaille, “Thin layers in elasticity: a theoretical and numerical study,” In ASME, editor, *Proceedings of the 1996 ESDA Conference*, volume 4, 171–178, 1996.
6. Lebon, F., Ould Khaoua, A., and Licht C., “Numerical study of soft adhesively bonded joints in finite elasticity,” *Comp. Mech.* **21**, 134–140 (1997).
7. Frémond M., “Matériaux á mémoire de forme,” *Compte Rendu Académie des Sciences* **304**, 239–214 (1987).
8. Ericksen, J. L., “Equilibrium of bars,” *J. Elast.* **5**, 191–201, (1991).
9. Aberavatne, R., Bhattacharya K., and Knowles, J. K., “Strain-energy functions with multiple local minima: modeling phase transformations using finite thermoelasticity,” *Nonlinear elasticity: Theory and application* (ed. Y. Fu and R. W. Ogden), Cambridge University Press, 2000.
10. Aberayatne, R., Chu, C., and James, R. D., “Kinetics of materials with wiggly energies: theory and application to the evolution of twinning microstructures in a Cu-Al-Ni shape memory alloy,” *Phil. Mag.* **A73**, 457–497, (1996).
11. Del Piero, G., and Rizzoni, R. “Weak local energy minimizers in finite elasticity,” in preparation.
12. Aberayatne, R., and Knowles, J. K., “On the driving traction acting on a surface of strain discontinuity in a continuum,” *J. Mech. Phys. Solids*, **38**, 345–360 (1990).
13. Aberayatne, R., Knowles, J. K., “Kinetic relations and the propagation of phase boundaries in solids,” *Arch. Rational. Mech. Anal.* **114**, 119–154 (1991).
14. Sanchez-Hubert, J., and Sanchez-Palencia, E., *Introduction aux méthodes asymptotiques et á l’homogénisation*. Masson. Paris, 1992.
15. Panagiotopoulos, P. D., *Inequality problems in mechanics and applications Convex and nonconver energy functions*, Birkhauser, 1985.
16. Morean, J. J., Panagiotopoulos, P. D., and Strang, G., *Topics in nonsmooth mechanics*. Birkhauser, 1988.
17. Mistakidis E. S., and Panagiotopoulos, P. D., “Numerical treatment of problems involving nonmonotone boundary or stress-strain laws,” *Comp. and Struct.*, **64**, 553–565 (1997).
18. Pagano, S., Alart, P., and Maisonneuve, O., “Solid-solid phase transition modelling. Local and global minimizations of non-convex and relaxed potetials. Isothermal case for shape memory alloys,” *International Journal of Engineering Sciences* **36**, 1143–1172 (1998).
19. Sagan, H., *An introduction to the calculus of variations*, Dover Publications, New York, 1992.
20. Lebon, F., and Rizzoni, R., “Aymptotic analysis of soft thin layers with nonconvex energy,” 16th AIMETA Congress of Theoretical and Applied Mechanics, Ferrara, 2003.
21. Dacorogna, B., *Direct methods in the calculus of variations*, Springer-Verlag, Berlin, 1989.
22. Klarbring A., “Derivation of the adhesively bonded joints by the asymptotic expansion method,” *International Journal of Engineering Science* **29**, 493–512 (1991).
23. Lebon, F., Ronel-Idrissi, S., “Asymptotic analysis of Mohr-Coulomb and Drucker Prager soft thin layers,” *International Journal of Steel and Composite Structures* **4**, 133–148 (2004).