Monotone Concave Operators: An application to the existence and uniqueness of solutions to the Bellman equation

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Monotone concave operators: An application to the existence and uniqueness of solutions to the Bellman equation*

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Abstract

We propose a new approach to the issue of existence and uniqueness of solutions to the Bellman equation, exploiting an emerging class of methods, called *monotone map methods*, pioneered in the work of Krasnosel'skii (1964) and Krasnosel'skii-Zabreiko (1984). The approach is technically simple and intuitive. It is derived from geometric ideas related to the study of fixed points for monotone concave operators defined on partially order spaces.

Keywords: Dynamic Programming; Bellman Equation; Unbounded Returns
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1 Introduction

Many economic models deal explicitly with situations in which economic agents operate through time in deterministic or stochastic environments. The introduction of dynamics inevitably directed the study of formal economic models in the context of recursive dynamic programs. Recursive methods have proven to be important tools for analyzing theoretical models with equilibrium outcomes described by complex dynamics.

A key ingredient of the recursive approach involves the analysis of the associated Bellman equation. If the value function solves the Bellman equation, the associated policy function characterizes the paths that are optimal with respect to the original program. An important step amounts to show that Bellman's maximizing operator has a unique solution in a certain class of functions.

A systematic treatment of the recursive approach can be found in Stokey, Lucas and Prescott 1989. However, with few exceptions their analysis is based on contraction techniques and therefore it is restricted to apply in models with bounded return functions. The approach proposed by Boyd (1990), Becker and Boyd (1997) and further developed by Durán (2000) resolves partially the problem of unbounded returns, since it covers models with bounded from below but unbounded from above return functions. The argument relies on relaxing boundedness by considering functions that obey a growth condition. It is in this new space of functions that they obtain the contraction property (weighted contraction property) for the maximizing operator. Another very interesting approach has been proposed by Streufert (1990). His idea is based on the notion of biconvergence, a limiting condition ensuring that returns of any feasible path are sufficiently discounted from above (upper convergence) and sufficiently discounted from below (lower convergence).

The common critique to the contraction technique in the unbounded case is based on the following two points: (i) first it is not always obvious to find a weighted norm to obtain the contraction property of the maximizing operator, and (ii) the introduction of a weighted norm implies existence and uniqueness of the solution in a certain space of continuous functions.

The shortcomings of the weighted approach become more serious when we consider programs with unbounded from below returns, since in those cases $-\infty$ is a solution to the Bellman equation. This is also a problem in Streufert (1990), since in cases where a one-stage return of $-\infty$ is admissible, the return function fails to be lower convergent.

In a first attempt to address the problem of unbounded returns, Alvarez and Stokey (1998) study a wide class of homogeneous problems. They show that the principle of optimality applies to problems of this sort. Their method of proof is based on finding restrictions that bound the growth rates of state

variables from above along any feasible path, and, for the case where the utility may attain $-\infty$, from below along at least one feasible path. In their analysis both decreasing and increasing returns technologies are excluded.

A recent approach to deal with unbounded returns has been proposed by Le Van and Morhaim (2002). Their argument does not depend on a contraction mapping technique but rather builds on important insights derived from the assumptions imposed on the return functions. The method proposed exploits the well known fact that the value function is a solution to the Bellman equation (Stokey et al. 1989). Under general conditions on technology, they are able to show that the value function is upper semicontinuous and satisfies a kind of *transversality* condition. Restricting subsequently themselves to the space of functions satisfying these two properties, they prove that the value function is the unique solution to Bellman functional equation. The results in Le Van and Morhaim have been further generalized by Le Van and Vailakis (2005) to deal with problems with recursive but not necessarily additively separable preferences.

In a recent paper, Rincón-Zapatero and Rodríguez-Palmero (2003) reconsidered the contraction approach. But instead of considering the usual normed space of functions in which the Bellman operator fails to be a contraction, they focus on metric spaces which are different depending on the characteristics of the problem. The underlying idea is based on the fact that one can choose an appropriate metric in the space of continuous functions to make use of the right properties of the operators and then apply a contraction argument[1]. The advantage of the metric approach is that in some cases it implies existence and uniqueness of the solution to the Bellman equation in the whole class of continuous functions. This is the case when returns are bounded from below and when the discounting factor satisfies suitable bounds. However, in models with returns unbounded from below, the approach limits its scope, since in those cases uniqueness can only be guaranteed in the appropriate space of functions that satisfy a kind of *transversality* condition as was first argued by Le Van and Morhaim (2002).

Our paper builds on the recent contributions of Le Van-Morhaim and Rincón-Zapatero and Rodríguez-Palmero. Borrowing elements from these two papers we propose a synthetic frame to the study of dynamic programming problems with time-additively separable objectives and bounded or unbounded (below/above) returns. The method proposed exploits an emerging class of methods, called *monotone map methods*, pioneered in the work of Krasnosel'skii (1964) and Krasnosel'skii-Zabreiko (1984). The approach is technically simple

[1]The proposed method in Rincón-Zapatero and Rodríguez-Palmero (2003) is based on the ingenious idea of constructing a special metric space. Unfortunately, a crucial step in their analysis is not correct. We clarify this point later on.

and intuitive. It is derived from geometric ideas related to the study of fixed points for monotone concave operators defined on partially ordered spaces.

Qualitative geometric methods for the investigation of solutions to nonlinear equations are not new in economics. Previous applications can be found among others in Coleman (1991, 2000), Kennan (2001), Datta, Mirman and Reffett (2002), Datta, Mirman, Morand and Reffett (2002) and Morand and Reffett (2003). The advantage of these methods is that they are constructive and can therefore be used to translate conditions in existence theorems into conditions which entail the applicability of various approximation methods.

In order to explain the underlying idea of our approach we follow Krasnosel'skii and Zabreiko (1984, section 6.46). Consider the scalar equation $x = \phi(x)$, where $\phi(x)$ is a continuous and non-negative function on $[0, \infty)$. Uniqueness of a positive solution is guaranteed if the function ϕ is strictly concave. In this case ϕ satisfies $\phi(tx) > t\phi(x)$ for $x > 0$ and $0 < t < 1$. The description in terms of those inequalities is very useful since it generalizes in a natural way to monotone concave operators defined on partially ordered spaces. It turns out that equations with concave operators defined on partially ordered spaces show a behavior which is rather similar to scalar equations.

The paper is organized as follows. Section 2 presents a fixed point theorem that is central in our analysis. We state the theorem in an abstract way, independently of the problem we study, since we believe that it may have a broader scope and be applied to different situations. Concerning the uniqueness of a fixed point our results are related to the ones found in Krasnosel'skii and Zabreiko (1984, Theorems 46.1, 46.2 and 49.2). But their results (applied to abstract concave operators defined on a cone of a Banach space) are more difficult and hold under restrictions that we do not need for our analysis. The existence result is closely related to Knaster-Tarski Theorem. Section 3 contains our main results. Taking as a starting point the usual form of the maximizing operator we first define an alternative maximizing operator, defined on the positive cone of continuous functions, that has a multiplicative form. By construction, the fixed points of the new multiplicative Bellman operator coincide with the fixed points of the original Bellman operator. We subsequently show how the aforementioned fixed point theorem can be applied to address the issue of existence and uniqueness of solutions to the Bellman operator. This is the subject of subsections 3.1 and 3.2. In terms of the assumptions we impose, we follow closely Le Van and Morhaim (2002) and Rincón-Zapatero and Rodríguez-Palmero (2003). Our purpose is to show how the theorem can be used to provide a unified framework that encompasses all the results established by these two papers. An additional advantage of our approach is that all proofs are worked out with simple and standard mathematical arguments. Although we do not provide explicit examples, it should be clear that all applications found in Le

Van and Morhaim (2002) and Rincón-Zapatero and Rodríguez-Palmero (2003) are particular cases of our setting.

2 A Fixed Point Theorem

Let Y be a topological space. Denote by $P^+(Y)$ the space of functions over Y with images on $\mathbb{R}_+$. Consider the following subset of $P^+(Y)$:

$$SB^+(Y) = \left\{ f \in P^+(Y) : \lambda_f := \inf_{x \in Y} f(x) > 0 \text{ and } \mu_f := \sup_{x \in Y} f(x) < \infty \right\}.$$

Theorem 2.1. *[Uniqueness] Let T be an increasing operator from $P^+(Y)$ into $P^+(Y)$ that satisfies the following property:*

$$\textit{(P.1)} \quad \forall \alpha \in (0,1],\ \exists \gamma \in (\alpha, 1] \text{ such that } T(\alpha f) \geq \gamma T f,\ \forall f \in P^+(Y).$$

We have the following results:

(i) Property (P.1) is equivalent to property:

$$\textit{(P.2)} \quad \forall \alpha > 1,\ \exists \gamma \in [1, \alpha) \text{ such that } T(\alpha f) \leq \gamma T f,\ \forall f \in P^+(Y).$$

(ii) Given any $\alpha \in (0,1)$ (respectively $\alpha > 1$) and any function $f \in P^+(Y)$ the operator T has at most one fixed point $\widehat{f}$ in the interval $[\alpha f, f]$ (respectively in $[f, \alpha f]$).

(iii) Let $\widehat{f}$ be a fixed point of the operator T. Assume that there exist a function $f \in P^+(Y)$ and two constants $\alpha_f \in (0,1)$, $\alpha'_f > 1$ such that $\alpha_f f \leq \widehat{f} \leq \alpha'_f f$. Then, $T^n f \to \widehat{f}$ uniformly on any set $K \subset Y$ such that $\sup_{x \in K} \widehat{f}(x) < \infty$. In particular, if $\sup_{x \in Y} \widehat{f}(x) < \infty$, then $T^n f \to \widehat{f}$ uniformly on Y.

(iv) Let $A(Y)$ be a subset of $SB^+(Y)$. The operator T has at most one fixed point, denoted $\widehat{f}$, on the set $A(Y)$. In addition, for any function $f \in A(Y)$, $T^n f \to \widehat{f}$ uniformly as $n \to +\infty$.

Proof. (i) Let $f \in P^+(Y)$ and $\alpha > 1$. Define the function $g : x \in Y \to \alpha f(x)$. Since $f = \frac{1}{\alpha} g$, property (P.1) implies that there exists $\gamma \in [1, \alpha)$ such that

$$T\left(\frac{1}{\alpha} g\right) \geq \frac{1}{\gamma} T(g).$$

It follows that $T(\alpha f) \leq \gamma T(f)$. By the same way we prove easily the converse.

(ii) Let $\alpha \in (0,1)$, $f \in P^+(Y)$ and $\widehat{f}_1, \widehat{f}_2 \in P^+(Y)$ be two fixed points of T in the interval $[\alpha f, f]$. In this case, for any $x \in Y$ we have:

$$\alpha f(x) \leq \widehat{f}_1(x) \leq f(x)$$

$$\alpha f(x) \leq \widehat{f}_2(x) \leq f(x).$$

It follows that $\alpha\widehat{f}_1 \leq \widehat{f}_2$ and $\alpha\widehat{f}_2 \leq \widehat{f}_1$. Let $\alpha^* = \sup\{\kappa > 0 : \kappa\widehat{f}_1(x) \leq \widehat{f}_2(x), \forall x \in Y\}$ and $\alpha^{**} = \sup\{\kappa > 0 : \kappa\widehat{f}_2(x) \leq \widehat{f}_1(x), \forall x \in Y\}$. If $\alpha^* = \alpha^{**} = 1$, it follows directly that $\widehat{f}_1 = \widehat{f}_2$. If one of them, say $\alpha^* \in (0,1)$, then $\widehat{f}_2 = T\widehat{f}_2 \geq T(\alpha^*\widehat{f}_1) \geq \gamma\widehat{f}_1$ with $\gamma \in (\alpha^*, 1]$: a contradiction to the definition of α^*. The uniqueness result in $[f, \alpha f]$ with $\alpha > 1$ follows in a similar way.

(iii) If $\widehat{f}(x) = 0$ for any $x \in Y$ the claim is true. Assume that there exists $x \in Y$ such that $\widehat{f}(x) > 0$. By hypothesis we have:

$$\frac{1}{\alpha'_f}\widehat{f} \leq f \leq \frac{1}{\alpha_f}\widehat{f}.$$

Define the real-valued functions:

$$\phi(t) = \max\{\tau : T(t\widehat{f}) \geq \tau\widehat{f}\} \text{ and } \psi(t) = \min\{\tau : T(t\widehat{f}) \leq \tau\widehat{f}\}.$$

Observe that both ϕ and ψ are non-decreasing. We show that both functions are continuous. Observe that when $\widehat{f}(x) > 0$ the monotonicity of T implies that for any $t \geq 1$:

$$\frac{T(t\widehat{f})(x)}{\widehat{f}(x)} \geq 1.$$

Moreover, from property (P.1), it follows that for any $t < 1$ there exists $\gamma(t) \in (t, 1]$ such that:

$$\frac{T(t\widehat{f})(x)}{\widehat{f}(x)} \geq \gamma(t) > t.$$

Observe that:

$$\phi(t) = \inf_{\{x \in Y : \widehat{f}(x) > 0\}} \frac{T(t\widehat{f})(x)}{\widehat{f}(x)}.$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing sequence such that $t_n \to t$. Since ϕ is non-decreasing we have:

$$\lim_{n\to\infty} \phi(t_n) \geq \phi(t).$$

For any $x \in Y$ such that $\widehat{f}(x) > 0$, property (P.2) implies that:

$$\frac{T(t_n\widehat{f})(x)}{\widehat{f}(x)} = \frac{T(\frac{t_n}{t}t\widehat{f})(x)}{\widehat{f}(x)} \leq \frac{t_n}{t}\frac{T\left(t\widehat{f}\right)(x)}{\widehat{f}(x)}.$$

But in this case we have:

$$\phi(t_n) \leq \frac{t_n}{t}\phi(t).$$

Taking the limits as $n \to \infty$ we obtain:

$$\lim_{n\to\infty} \phi(t_n) \leq \phi(t).$$

A similar argument can apply in case where $\{t_n\}_n$ is an increasing convergent sequence. It follows that for any arbitrary sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to t$ we have:

$$\lim_{n\to\infty} \phi(t_n) = \phi(t).$$

The continuity of function ψ can be established in a similar way.

It is obvious that $1 > \phi(t) > t$, for $t < 1$ and $1 < \psi(t) < t$, for $t > 1$. Furthermore, $\phi(1) = \psi(1) = 1$.

Let $t_0 = \frac{1}{\alpha'_f}$ and $s_0 = \frac{1}{\alpha_f}$. Consider the sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{s_n\}_{n\in\mathbb{N}}$ defined in the following way:

$$t_n = \phi(t_{n-1}) \text{ and } s_n = \psi(s_{n-1}).$$

Observe that $t_n \to 1$ and $s_n \to 1$. Moreover, since $t_0\widehat{f} \leq f \leq s_0\widehat{f}$, for any $n > 1$ we have:

$$t_n\widehat{f} \leq T^n f \leq s_n\widehat{f}.$$

Let $K \subset Y$. For any $x \in K$, the last inequalities imply that for any $n > 1$:

$$\begin{aligned} \left|T^n f(x) - \widehat{f}(x)\right| &\leq \max\{(s_n - 1), (1 - t_n)\}\widehat{f}(x) \\ &\leq \max\{(s_n - 1), (1 - t_n)\} \sup_{x\in K} \widehat{f}(x). \end{aligned}$$

This proves the claim.

(iv) Let $f \in A(Y) \subseteq SB^+(Y)$. We first show that T has at most one fixed point $\widehat{f}$ in the interval $[0, f]$. Assume that $\widehat{f}_1, \widehat{f}_2 \in A(Y)$ are two fixed points of T in the interval $[0, f]$. For any $x \in Y$ we have:

$$\begin{aligned} \frac{\widehat{f}_1(x)}{f(x)} &\geq \frac{\lambda_{\widehat{f}_1}}{\mu_f} = \alpha_{\widehat{f}_1} \\ \frac{\widehat{f}_2(x)}{f(x)} &\geq \frac{\lambda_{\widehat{f}_2}}{\mu_f} = \alpha_{\widehat{f}_2}. \end{aligned}$$

Let $\alpha = \min\{\alpha_{\widehat{f}_1}, \alpha_{\widehat{f}_2}\}$. Observe that $\alpha \in (0, 1)$ and that the following inequalities are true:

$$\begin{aligned} \alpha f &\leq \widehat{f}_1 \leq f \\ \alpha f &\leq \widehat{f}_2 \leq f. \end{aligned}$$

A direct application of claim (ii) implies that $\widehat{f}_1 = \widehat{f}_2$.

Assume now that $\widehat{f}_1, \widehat{f}_2 \in A(Y)$ are two fixed points of T on $A(Y)$. Define the function $f : x \in Y \to f(x) = \max\{\widehat{f}_1(x), \widehat{f}_2(x)\}$. Observe that $f \in A(Y)$ and that:

$$\begin{aligned} 0 &\leq \widehat{f}_1 \leq f \\ 0 &\leq \widehat{f}_2 \leq f. \end{aligned}$$

Given that T has at most one fixed point $\widehat{f}$ in the interval $[0, f]$, it follows that $\widehat{f_1} = \widehat{f_2}$.. Let $f \in A(Y)$ and $\widehat{f}$ be the fixed point of T on $A(Y)$. There always exist $\alpha_f \in (0,1)$ and $\alpha'_f > 1$ such that $\alpha_f f \leq \widehat{f} \leq \alpha'_f f$. A direct application of claim (iii) implies that $T^n f \to \widehat{f}$ uniformly as $n \to +\infty$. ■

The results in Theorem 2.1 do not imply that a fixed point of the operator T exists. We show below that a fixed point exists provided that the operator T satisfies some additional conditions. Working in this direction, we assume that the operator T satisfies the following property:

(P.3) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $P^+(Y)$ converging to $f \in P^+(Y)$, then for all $x \in Y$, $\liminf_n Tf_n(x) \geq Tf(x)$.

Consider the following subset of $P^+(Y)$:

$$B^+(Y) = \{f \in P^+(Y) : f \text{ is bounded from above on any compact set of } Y\}.$$

We have the following result.

Theorem 2.2. *[Existence] Let T be a monotone operator mapping $P^+(Y)$ into $P^+(Y)$ that satisfies properties (P.1) and (P.3).*

(i) Assume there exists $f_1 \in P^+(Y)$ such that $f_1 \leq Tf_1$. If the sequence $\{T^n f_1\}_{n\in\mathbb{N}}$ has a pointwise limit $\widehat{f} \in P^+(Y)$, then $\widehat{f}$ is a fixed point of T on $P^+(Y)$.

(ii) Assume there exist two functions $f_1, f_2 \in B^+(Y)$ such that:

$$f_1(x) \leq Tf_1(x) \leq Tf_2(x) \leq f_2(x), \ \forall x \in Y.$$

Then, the pointwise limit of $\{T^n f_1\}_{n\in\mathbb{N}}$ exists, belongs to $B^+(Y)$ and is a fixed point of T on $B^+(Y)$.

Proof. (i) Since $T^n f_1 \leq \widehat{f}$, it follows that $\widehat{f} \leq T\widehat{f}$. The operator satisfies property (P.3), in which case, for any $x \in Y$ we have:

$$\widehat{f}(x) = \liminf_n T(T^n f_1)(x) \geq T\widehat{f}(x).$$

Therefore, we conclude that $\widehat{f} = T\widehat{f}$.

(ii) For any $n \in \mathbb{N}$, for any $x \in Y$, we have:

$$f_1(x) \leq T^n f_1(x) \leq T^n f_2(x) \leq f_2(x).$$

The sequence $\{T^n f_1(x)\}_{n\in\mathbb{N}}$ is non-decreasing and bounded, in which case $\widehat{f}(x) = \lim_n T^n f_1(x)$ exists. Moreover, $\widehat{f}(x) \leq f_2(x)$ for any $x \in Y$, so $\widehat{f} \in B^+(Y)$. Apply statement (i) to conclude that $\widehat{f} = T\widehat{f}$. ■

3 Applications to Dynamic Programming

Our purpose is to study a general class of dynamic programming problems stated in the following reduced form:

$$\begin{cases} V(x_0) = \sup \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \\ \text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \ \forall t \\ \qquad\quad x_0 \in X \text{ is given,} \end{cases} \tag{1}$$

where $F : graph\Gamma \to \mathbb{R} \cup \{-\infty\}$ is the return function, $\beta \in (0,1)$ is the discounting factor, X is a topological space, $\Gamma : X \to 2^X$ is a technological correspondence and V is the value function. The following assumptions are typically made in this context.

Assumption (H1). Γ is a continuous, nonempty, compact-valued correspondence.

Assumption (H2). The function F is continuous at any $(x,y) \in graph\Gamma$ such that $F(x,y) > -\infty$. Moreover, if $F(x,y) = -\infty$ and $\lim_n (x_n, y_n) = (x,y)$, then $\lim_n F(x_n, y_n) = -\infty$.

We are interested in the relation between the value function V and the solutions to the Bellman operator:

$$Tf(x) = \sup\{F(x,y) + \beta f(y) : y \in \Gamma(x)\}. \tag{2}$$

We would like the value function V to be a fixed point of the Bellman operator, and conversely, the Bellman operator T to have a unique fixed point that coincides with the value function V. We proceed by introducing some notation required to study this relation.

Let

$$\Pi(x_0) = \{\widetilde{x} = (x_1, ..., x_t, ...) \in X^\infty : x_{t+1} \in \Gamma(x_t), \forall t \geq 0\}$$

denote the set of feasible sequences from x_0.

Lemma 3.1. *Assume (**H1**). Then:*

(i) For any $x_0 \in X$, $\Pi(x_0)$ is compact for the product topology.

(ii) The correspondence Π is continuous for the product topology.

Proof. See Appendix. ■

Remark 3.1. Le Van and Morhaim (2002) provide a proof of statement (ii) in Lemma 3.1 imposing additional restrictions on the technological correspondence Γ. It appears that Assumption (**H1**) is sufficient to prove the claim[2].

Let

$$u : \widetilde{x} \in \Pi(x_0) \to u(\widetilde{x}) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})$$

denote the total discounted returns. We subsequently define the set:

$$\Pi^0(x_0) = \left\{ \widetilde{x} \in \Pi(x_0) : u(\widetilde{x}) = \sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1}) \text{ exists and } u(\widetilde{x}) > -\infty \right\}.$$

of feasible programs from x_0 consistent with bounded from below total discounted utility.

Consider the following maximization problem:

$$\begin{cases} \widehat{V}(x_0) = \sup e^{\sum\limits_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})} \\ \text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \ \forall t \\ \qquad\quad x_0 \in X. \end{cases} \tag{3}$$

Observe that $\widehat{V} = e^V$. The Bellman operator associated with problem (3) has now the following multiplicative form:

If $f \in P^+(X)$:

$$\widehat{T}f(x) = \sup\{e^{F(x,y)}(f(y))^\beta : y \in \Gamma(x), \ x \in X\}. \tag{4}$$

Let $C^+(X)$ denote the set of continuous and non-negative functions, i.e.

$$C^+(X) = \{f(x) \geq 0 : f \text{ is continuous on } X\}.$$

If $f \in C^+(X)$ then:

$$\widehat{T}f(x) = \max\{e^{F(x,y)}(f(y))^\beta : y \in \Gamma(x), \ x \in X\}. \tag{5}$$

The operator $\widehat{T}$ is well defined on $P^+(X)$ (respectively on $C^+(X)$), and given assumption (**H1**)-(**H2**) we have that $\widehat{T}$ maps $P^+(X)$ (respectively $C^+(X)$) into $P^+(X)$ (respectively $C^+(X)$). In what follows we study the connection between the solutions to the multiplicative Bellman equation, $\widehat{T}f = f$ and the value function $\widehat{V}$. In particular, we apply Theorems 2.1 and 2.2 to show that under suitable conditions the operator $\widehat{T}$ has a unique fixed point that coincides with the value function $\widehat{V}$. By construction, the solutions of the multiplicative Bellman equation are related to the solutions of the original Bellman equation. Our first step amounts to show that the operator $\widehat{T}$ satisfies properties (P.1) and (P.3).

[2]We are grateful to V. Filipe Martins-da-Rocha for bringing this point to our attention.

Lemma 3.2. *Under Assumptions (**H1**)-(**H2**), the operator $\widehat{T}$ satisfies properties (P.1) and (P.3).*

Proof. See Appendix. ■

The following proposition is crucial for our analysis. It is stated here for further reference.

Proposition 3.1. *Let Assumptions (**H1**)-(**H2**) be satisfied. Assume further that there exist two functions f_1, f_2 in $B^+(X)$ that satisfy:*

(a) $f_1 \leq \widehat{T} f_1 \leq \widehat{T} f_2 \leq f_2$.

(b) $f_2(x_0) = 0$ for any $x_0 \in X$ such that $\Pi^0(x_0) = \emptyset$.

(c) For any $x_0 \in X$, for any $\widetilde{x} \in \Pi^0(x_0)$,

$$\lim_{T \to +\infty} (f_1(x_T))^{\beta^T} = \lim_{T \to +\infty} (f_2(x_T))^{\beta^T} = 1.$$

Then:

(i) The operator $\widehat{T}$ has a unique fixed point $\widehat{f}$ on the set $[f_1, f_2]$. Moreover, for any $f \in [f_1, f_2]$ we have $\widehat{T}^n f \to \widehat{f}$ pointwise.

(ii) Suppose that condition (c) is strengthened in the following way:

(c′) For any compact set $K \subseteq X$, for any $\varepsilon > 0$, there exists $N(K, \varepsilon)$, such that, for any $x_0 \in K$, for any $\widetilde{x} \in \Pi^0(x_0)$, for any $T \geq N(K, \varepsilon)$,

$$(1 - \varepsilon) \leq (f_i(x_T))^{\beta^T} \leq (1 + \varepsilon), \ i = 1, 2.$$

The operator $\widehat{T}$ has a unique fixed point $\widehat{f}$ on the set $[f_1, f_2]$. In addition, $\widehat{f}$ is the uniform limit on any compact set of any sequence $\{\widehat{T}^n f\}_{n \in \mathbb{N}}$ with $f \in [f_1, f_2]$. Moreover, if $[f_1, f_2]$ contains a continuous function then $\widehat{f}$ is continuous.

Proof. See Appendix. ■

3.1 Returns bounded from below

In this section we deal with problems where both the total discounted utility u and the period return function F are bounded from below. In what follows Assumption (**H2**) is replaced by:

Assumption (H2′). The return function $F : graph\Gamma \to \mathbb{R}$ is continuous.

Given a subset K of X denote

$$\Gamma(K) = \cup_{x \in K} \Gamma(x).$$

We impose the following restrictions on the topological space X and the technological correspondence Γ.

Assumption (H3). $X = \cup_j K_j$ where $\{K_j\}_{j\in\mathbb{N}}$ is a countable increasing sequence of nonempty and compact subsets of X such that $\Gamma(K_j) \subseteq K_j$, $\forall j \in \mathbb{N}$.

For any $j \in \mathbb{N}$ define:

$$\lambda_j = \min_{x\in K_j}\{\min_{y\in\Gamma(x)} F(x,y)\} \text{ and } \mu_j = \max_{x\in K_j}\{\max_{y\in\Gamma(x)} F(x,y)\}.$$

Observe that the sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ is non-increasing while the sequence $\{\mu_j\}_{j\in\mathbb{N}}$ is non-decreasing. Denote

$$C^{++}(X) = \{f(x) > 0 : f \text{ is continuous on } X\}.$$

Theorem 3.1. *Assume **(H1)-(H2′)-(H3)**. Let*

$$A(X) = \{f \in B^+(X) : \forall j \in \mathbb{N},\ \forall x \in K_j,\ e^{\frac{\lambda_j}{(1-\beta)}} \le f(x) \le e^{\frac{\mu_j}{(1-\beta)}}\}.$$

Then, $\widehat{T}$ has a unique fixed point $\widehat{f}$ on $A(X)$ which is continuous and coincides with the value function $\widehat{V}$. In particular, $\widehat{f}$ is the unique fixed of $\widehat{T}$ on the set $C^{++}(X)$. In addition, for any $f \in A(X)$, $\widehat{T}^n f \to \widehat{V}$ uniformly on any compact set of X.

Proof. Define the functions f_1 and f_2 as follows:

$$\begin{aligned} x \in K_1,\ f_1(x) &= e^{\frac{\lambda_1}{(1-\beta)}},\ f_2(x) = e^{\frac{\mu_1}{(1-\beta)}} \\ x \in K_2\backslash K_1,\ f_1(x) &= e^{\frac{\lambda_2}{(1-\beta)}},\ f_2(x) = e^{\frac{\mu_2}{(1-\beta)}}. \\ x \in K_j\backslash K_{j-1},\ f_1(x) &= e^{\frac{\lambda_j}{(1-\beta)}},\ f_2(x) = e^{\frac{\mu_j}{(1-\beta)}} \text{ and so on.} \end{aligned}$$

The functions f_1 and f_2 belong to $A(X)$. Moreover, we have:

$$f_1 \le \widehat{T}(f_1) \le \widehat{T}(f_2) \le f_2.$$

It is easy to check that $\widehat{T}$ maps $A(X)$ into $A(X)$. We claim that condition (c′) in Proposition 3.1 is satisfied. Fix some $j \in \mathbb{N}$. For any $x_0 \in K_j$ and any $\widetilde{x} \in \Pi(x_0)$ we have that $x_t \in K_j$, $\forall t$. It follows that for any $\varepsilon > 0$, there exists N, such that, for any $T \ge N$, for any $x_0 \in K_j$, for any $\widetilde{x} \in \Pi(x_0)$ we have:

$$1 - \varepsilon \le (f_1(x_T))^{\beta^T} \le 1 + \varepsilon.$$

Similarly, we have:

$$1 - \varepsilon \le (f_2(x_T))^{\beta^T} \le 1 + \varepsilon.$$

From Proposition 3.1, $\widehat{T}$ has a unique fixed point $\widehat{f}$ in $[f_1, f_2]$. Moreover, $\widehat{f}$ is the uniform limit on any compact set K of $\{\widehat{T}^n f\}_{n\in\mathbb{N}}$ with $f \in A(X)$. Let α be any constant that belongs to the open interval (λ_1, μ_1). Observe that the

function g, defined by $g(x) = e^{\alpha}$ for any $x \in X$, is continuous and belongs to $[f_1, f_2]$. This implies that $\widehat{f}$ is continuous.

The value function $\widehat{V}$ is a fixed point of the Bellman operator $\widehat{T}$ (see Theorem 4.2 in Stokey et al.). It is easy to verify that $\widehat{V}$ belongs to $A(X)$ and therefore it coincides with $\widehat{f}$. In general, if $\widehat{h}$ is a fixed point of the operator $\widehat{T}$ on the set $C^{++}(X)$, then $\widehat{h}$ belongs to $A(X)$ and therefore it coincides with $\widehat{V}$. This shows that $\widehat{V}$ is the unique fixed point of $\widehat{T}$ on the set $C^{++}(X)$. ■

Remark 3.2. (i) Rincón-Zapatero and Rodríguez-Palmero (2003) obtain the same result (Theorem 3 in the corresponding paper). However, following their approach may be problematic since the proof of their theorem relies on a result which is not correct[3]. Moreover, following our approach, the continuity of the value function $\widehat{V}$ comes as a consequence of being the unique solution to the multiplicative Bellman equation.

(ii) Observe that uniqueness is implied for the whole space $C^{++}(X)$. The implication for the Bellman operator T is that uniqueness is implied for the space of functions:

$$A(X) = \{f \in C(X) : f(x) > -\infty \text{ for any } x \in X\}.$$

(iii) The critical step in the proof relies on the fact that the operator $\widehat{T}$ maps $[f_1, f_2]$ into $[f_1, f_2]$. In that process, assuming $\Gamma(K_j) \subseteq K_j$ for any $j \in \mathbb{N}$ plays a crucial role and simultaneously allows us to establish uniqueness in the class of continuous functions. However, there are situations where the assumption $\Gamma(K_j) \subseteq K_j$ does not hold. The next step is to identify alternative restrictions that replace this rather restrictive assumption. An important observation is that under Assumption (**H1**) it is always possible to find an increasing sequence $\{K_j\}_{j\in\mathbb{N}}$ of compact sets covering X such that $\Gamma(K_j) \subseteq K_{j+1}$, $\forall j \in \mathbb{N}$. Along this line we replace (**H3**) by the following assumption.

Denote

$$||\psi||_{K_j} = \max_{x\in K_j} |\psi(x)|$$

where $\psi(x) = \max_{y\in\Gamma(x)} F(x, y)$.

Assumption (H3′). $X = \cup_j K_j$ where $\{K_j\}_{j\in\mathbb{N}}$ is a countable increasing sequence of nonempty, compact subsets of X such that $\Gamma(K_j) \subseteq K_{j+1}$, $\forall j \in \mathbb{N}$ and

$$\sum_{j=1}^{+\infty} \beta^j ||\psi||_{K_j} = M < +\infty.$$

[3]The proof of Theorem 3 in Rincón-Zapatero and Rodríguez-Palmero (2003) makes use of Theorem 1 which in turn relies on Proposition (1b). However, Proposition (1b) is not correct. Refer to Matkowski and Nowak (2008) for a counterexample.

Remark 3.3. (i) Assume that $X = \mathbb{R}^n_+$ and that $y \in \Gamma(x) \Rightarrow \|y\| \leq \gamma \|x\|$ with $\gamma > 1$ (this is assumed in Le Van and Morhaim (2002)). In this case, one can construct a sequence $\{K_j\}_{j\in\mathbb{N}}$ satisfying the restrictions imposed in Assumption (**H3′**) by simply letting $K_j = \overline{B(0, \gamma^j j)} \cap X$.

Given any $j \in \mathbb{N}$ define:

$$m_j = \sum_{l=j}^{+\infty} \beta^{l-j} ||\psi||_{K_l}.$$

Observe that $\{m_j\}_{j=1}^{\infty}$ is a non-decreasing sequence. The following result is a natural application of Proposition 3.1 to the multiplicative Bellman operator in cases where Assumption (**H3′**) is satisfied.

Theorem 3.2. *Assume (**H1**)-(**H2′**)-(**H3′**). Let*

$$A(X) = \{f \in B^+(X) : \forall K_j,\ \forall x \in K_j,\ e^{-m_j} \leq f(x) \leq e^{m_j}\}.$$

Then, $\widehat{T}$ has a unique fixed point $\widehat{f}$ on $A(X)$ which is continuous and coincides with the value function $\widehat{V}$. In addition, for any $f \in A(X)$, $\widehat{T}^n f \to \widehat{V}$ uniformly on any compact set of X.

Proof. Define the functions f_1 and f_2 as follows:

$$\begin{aligned} x \in K_1,\ f_1(x) &= e^{-m_1},\ f_2(x) = e^{m_1} \\ x \in K_2 \backslash K_1,\ f_1(x) &= e^{-m_2},\ f_2(x) = e^{m_2}. \\ x \in K_j \backslash K_{j-1},\ f_1(x) &= e^{-m_j},\ f_2(x) = e^{m_j} \text{ and so on.} \end{aligned}$$

The functions f_1 and f_2 belong to $A(X)$. Moreover, we have:

$$f_1 \leq \widehat{T}(f_1) \leq \widehat{T}(f_2) \leq f_2.$$

It is easy to check that $\widehat{T}$ maps $A(X)$ into $A(X)$. We claim that condition (c′) in Proposition 3.1 is satisfied. Fix some $j \in \mathbb{N}$. For any $x_0 \in K_j$, for any $\widetilde{x} \in \Pi(x_0)$ we have:

$$e^{-\beta^T m_{j+T}} \leq (f_1(x_T))^{\beta^T} \leq e^{\beta^T m_{j+T}}.$$

Recall that $\beta^T m_{j+T} \to 0$ when $T \to +\infty$. It follows that for any $\varepsilon > 0$, there exists N, such that, for any $T \geq N$, for any $x_0 \in K_j$, for any $\widetilde{x} \in \Pi(x_0)$ we have:

$$1 - \varepsilon \leq (f_1(x_T))^{\beta^T} \leq 1 + \varepsilon.$$

Similarly, we have:

$$1 - \varepsilon \leq (f_2(x_T))^{\beta^T} \leq 1 + \varepsilon.$$

From Proposition 3.1, $\widehat{T}$ has a unique fixed point $\widehat{f}$ in $[f_1, f_2]$. Moreover, $\widehat{f}$ is the uniform limit on any compact set K of $\{\widehat{T}^n f\}_{n\in\mathbb{N}}$ with $f \in A(X)$. Observe that the function g, defined by $g(x) = 1$ for any $x \in X$, is continuous and belongs to $[f_1, f_2]$. This implies that $\widehat{f}$ is continuous.

It is easy to verify that for any $j \in \mathbb{N}$, for any $x \in K_j$, we have $\widehat{V}(x) \leq e^{m_j}$. Now, given any $x_0 \in K_j$, consider the feasible sequence $\{x_t\}_{t\geq 0}$ that satisfies $F(x_t, x_{t+1}) = \psi(x_t)$ for any t. It follows that:

$$\widehat{V}(x_0) \geq e^{\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})} \geq e^{-\sum_{t=0}^{+\infty} \beta^t |\psi(x_t)|} = e^{-m_j}.$$

In other words, $\widehat{V} \in A(X)$. Since $\widehat{V}$ satisfies the Bellman equation (see Theorem 4.2 in Stokey et al.), it is a fixed point of $\widehat{T}$ in $A(X)$ and hence it coincides with $\widehat{f}$. ■

Remark 3.4. (i) The last theorem is equivalent to Theorem 4 in Rincón-Zapatero and Rodríguez-Palmero (2003). They assume that there exists $c > 1$ with $c\beta < 1$ such that:

$$\sum_{j=1}^{+\infty} c^{-j} ||\psi||_{K_j} = M < +\infty$$

while we impose a weaker assumption:

$$\sum_{j=1}^{+\infty} \beta^j ||\psi||_{K_j} = M < +\infty.$$

(ii) The continuity of the value function $\widehat{V}$ comes as a consequence of being the unique solution to the multiplicative Bellman equation.

(iii) The implication for the Bellman operator T is that uniqueness is implied for the space of functions:

$$A(X) = \{f \in C^+(X) : \forall K_j,\ \forall x \in K_j,\ -m_j \leq f(x) \leq m_j\}.$$

(iv) iii) One may wonder whether the value function may be the limit of any sequence $\{\widehat{T}^n f\}_{n\in\mathbb{N}}$ where f is any function in $C^{++}(X)$. We give below an example showing that this is not possible. Let $X = \mathbb{R}_+$, $\Gamma(x) = [0,\ \frac{x}{\beta}]$ and $F(x, y) = \ln\left[\frac{x}{\beta} - y + 1\right]$. Define:

$$K_1 = \left[0,\ \frac{1}{\beta}\right],\ K_2 = \left[0,\ \frac{1}{\beta^2}\right],\ K_n = \left[0,\ \frac{1}{\beta^n}\right],\ \text{and so on.}$$

Obviously, the sequence $\{K_j\}_{j\in\mathbb{N}}$ is increasing, $X = \cup_j K_j$ and $\Gamma(K_j) = K_{j+1}$. We have $\widehat{V}(x) = \lim_n \widehat{T}^n(1) = e^{\frac{1}{1-\beta}\ln\left[\frac{(1-\beta)}{\beta}x+1\right]}$. Take $f(x) = e^x$, $\forall x$ and compute $\widehat{T}^n f(x)$. We obtain:

$$\widehat{T}^n f(x) = e^{(1+\beta+\ldots+\beta^{n-1})[\beta-\ln\beta-1]} e^x.$$

Hence, $T^n f(x) \to e^{\frac{1}{1-\beta}[\beta - \ln \beta - 1]} e^x \neq \widehat{V}(x)$ when $x > 0$.

The function $f(x) = e^x$ does not belongs to the set $A(X)$ in Theorem 3.2. Indeed, the sequence $\{\frac{1}{\beta}, \frac{1}{\beta^2}, ... \frac{1}{\beta^t}, ...\}$ belongs to $\Pi(1)$. For j large enough we have:

$$e^{m_j} = e^{\beta^{-j} \sum_{l=j}^{+\infty} \beta^l ||\psi||_{K_l}} < e^{\beta^{-j}}.$$

3.2 Returns unbounded from below

3.2.1 Unbounded total discounted returns

Following Rincón-Zapatero and Rodríguez-Palmero (2003) we first consider the case where the return function F is bounded from below but there are feasible programs $\widetilde{x} \in \Pi(x_0)$ such that the total discounted utility $u(\widetilde{x})$ may take the value $-\infty$. We impose the following assumption.

Assumption (H3″). (a) Let X be a Banach space. Moreover, $X = \cup_j K_j$ where $\{K_j\}_{j \in \mathbb{N}}$ is a countable increasing sequence of nonempty, convex and compact subsets of X.

(b) Fix some $j \in \mathbb{N}$. Define the truncated correspondence Γ_j as follows:

$$\Gamma_j(x) = \begin{cases} \Gamma(x) & if\ x \in K_j \\ \Gamma(P_j(x)) & if\ x \notin K_j. \end{cases}$$

where $P_j(x)$ denotes the projection of $x \in X$ on the set K_j. The technological correspondence Γ_j satisfies the following property:

$$\forall x \in X,\ \Gamma_j(x) \subseteq \Gamma(x).$$

(c) Let F^- denote the negative part of the return function F, i.e. $F^- = \min(0, F)$. For each $\widetilde{x} \in \Pi^0(x_0)$, all $x_0 \in X$, there exists $a \in X$ with $a \in \Gamma(x_t) \cap \Gamma(a)$ for all t large enough, and such that, $\lim_t \beta^t F^-(x_t, a) = 0$.

Remark 3.5. Note that the requirements (a) and (b) in Assumption (**H3″**) are satisfied, for instance, when $\Gamma(x) = [0, f(x)]$ and f is a non-decreasing function from $\mathbb{R}_+$ into $\mathbb{R}_+$. In general, (a) and (b) in Assumption (**H3″**) are satisfied (see Appendix B (claim (i)) in Rincón-Zapatero and Rodríguez-Palmero (2003)) provided that

(a) X is a nonempty, closed, convex and comprehensive subset of $\mathbb{R}^n_+$
(b) graphΓ satisfies: $\widehat{x} \geq x \Rightarrow (\widehat{x}, y) \in \text{graph}\Gamma,$ for all $(x, y) \in \text{graph}\Gamma$.

Fix some $j \in \mathbb{N}$ and consider the operator $\widehat{T}_j$ defined as follows:

$$\forall f \in B^+(X),\ \forall x \in X,\ \widehat{T}_j f(x) = \sup_{y \in \Gamma_j(x)} \{e^{F(x,y)} (f(y))^\beta\}.$$

Observe that $\widehat{T}_j$ maps $B^+(X)$ into $B^+(X)$. We have the following result.

Lemma 3.3. *Under Assumptions (**H1**)-(**H2′**)-(**H3″**), the operator $\widehat{T}_j$ has a unique fixed point $\widehat{f}_j$ that belongs to $C^{++}(X)$.*

Proof. It easy to check that the operator $\widehat{T}_j$ satisfies properties (P.1) and (P.3). Under Assumption (**H3″**) we have $\Gamma_j(x) \subseteq \Gamma(K_j)$, all $x \in X$. It follows that there exists a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact sets such that $X = \cup_i K_i$ and $\Gamma_j(K_i) \subset K_i$ for any i. With this in mind the proof of the claim parallels the one in Theorem 3.1. ■

Let us consider the sequence $\{\widehat{f}_j\}_{j\in\mathbb{N}}$ where each $\widehat{f}_j$ is the unique continuous fixed point of the operator $\widehat{T}_j$ on $C^{++}(X)$. We impose the following assumption.

Assumption (H4). There exists an upper-semicontinuous function g such that $f_j \leq g$ for all j and

$$\limsup_t [g(x_t)]^{\beta^t} \leq 1, \ \forall \widetilde{x} \in \Pi(x_0).$$

Remark 3.6. (i) It is important to provide conditions under which a function g with the aforementioned properties exists. This is the case (see Rincón-Zapatero and Rodríguez-Palmero (2003) Remark 5(ii)) when there exists an upper-semicontinuous function $\omega : X \to \mathbb{R}_+$ such that:

$$\psi \leq \omega \text{ and } \max_{y\in\Gamma(x)} \omega(y) \leq \gamma\omega(x) \text{ for all } x \in X,$$

where $\gamma > 0$, $\beta\gamma < 1$ and $\psi(x) = \max_{y\in\Gamma(x)} F(x, y)$. We simply let:

$$g(x) = e^{\frac{\omega(x)}{1-\beta\gamma}}.$$

In Le Van and Morhaim (2002) it is assumed (Assumptions (**H2**), (**H7**) in their paper) that $y \in \Gamma(x) \Rightarrow \|y\| \leq \gamma \|x\|$ with $\gamma > 1$ and that $\forall (x, y) \in graph\Gamma$

$$F(x, y) \leq A + B(\|x\| + \|y\|) \text{ with } A \geq 0, B \geq 0.$$

In this case, $\omega(x) = A + B(1 + \gamma) \|x\|$.

We have the following results.

Lemma 3.4. *Under Assumptions (**H1**)-(**H2**)-(**H3″**)-(**H4**), the sequence $\{\widehat{f}_j\}_{j\in\mathbb{N}}$, where $\widehat{f}_j$ is the unique continuous fixed point of the operator $\widehat{T}_j$, is increasing and bounded. In addition, $\widehat{f} := \sup_j \widehat{f}_j$ coincides with the value function $\widehat{V}$.*

Proof. It follows from Appendix B (claim (ii)) and Theorem 5 in Rincón-Zapatero and Rodríguez-Palmero (2003). ■

Lemma 3.5. *Assume **(H1)**-**(H2)**-**(H3′)**-**(H4)**. Then, the operator $\widehat{T}$ has at most one fixed point on the set:*

$$A(X) = \left\{ f \in B^+(X) : \limsup_t (f(x_t))^{\beta^t} \leq 1,\ \forall \widetilde{x} \in \Pi(x_0) \right\}.$$

Proof. See Appendix. ■

Theorem 3.3. *Assume **(H1)**-**(H2)**-**(H3′)**-**(H4)**. Then:*

(i) The value function $\widehat{V}$ is the unique continuous fixed-point of $\widehat{T}$ on the set:

$$A(X) = \left\{ f \in B^+(X) : \begin{array}{c} \forall x_0 \in X,\ \forall \widetilde{x} \in \Pi(x_0),\ \lim_t \sup (f(x_t))^{\beta^t} \leq 1 \\ \forall x_0 \in X,\ \forall \widetilde{x} \in \Pi^0(x_0),\ \lim_t (f(x_t))^{\beta^t} = 1. \end{array} \right\}$$

(ii) For any $x \in \mathbb{R}_+$, $\widehat{V}(x) = \lim_n \widehat{T}^n f(x)$ where $f \in A(X)$ and verifies $Tf \leq f$. If in addition $f \in C^{++}(X)$, then the convergence is uniform on any compact subset of X. In particular, for any $\widehat{f}_j$ the sequence $(\widehat{T}^n \widehat{f}_j)_{n \in \mathbb{N}}$ converges to $\widehat{V}$ uniformly on any compact subset of X.

Proof. (i) It is easy to check that $\widehat{T}$ maps $A(X)$ into $A(X)$. The value function $\widehat{V}$ is a fixed point of the operator $\widehat{T}$ (see Theorem 4.2 in Stokey et al.). Since $\widehat{V}$ coincides with the function $\widehat{f} := \sup_j \widehat{f}_j$ we have:

$$\forall x_0 \in X,\ \forall \widetilde{x} \in \Pi(x_0), \lim_{t \to +\infty} \sup (\widehat{V}(x_t))^{\beta^t} \leq 1.$$

Moreover, for any $\widetilde{x} \in \Pi^0(x_0)$, one has:

$$0 < e^{u(\widetilde{x})} \leq \left[e^{\sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})} (\widehat{V}(x_{T+1}))^{\beta^{T+1}} \right].$$

Then:

$$1 = e^{\left[u(\widetilde{x}) - \lim_T \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \right]} \leq \lim_T \inf (\widehat{V}(x_{T+1}))^{\beta^{T+1}}.$$

It follows that $\widehat{V}$ belongs to $A(X)$ and by Lemma 3.5 is the unique fixed point of $\widehat{T}$ on the set $A(X)$.

The function $\widehat{f} := \sup_j \widehat{f}_j$ is lower semi-continuous as the supremum of continuous functions. Since the value function $\widehat{V}$ coincides with the function $\widehat{f}$ is is lower semi-continuous. We next show that $\widehat{V}$ is also upper-semicontinuous at any $x_0 \in X$. Let $(x_0^n)_{n \in \mathbb{N}}$ be a sequence that converges to x_0. The continuity of Π implies that there exists a sequence $(\widetilde{x}^n)_{n \in \mathbb{N}}$ with $\widetilde{x}^n \in \Pi^0(x_0^n)$ such that $\widetilde{x}^n \to \widetilde{x} \in \Pi^0(x_0)$ and

$$\begin{aligned}\widehat{V}(x_0^n) &= e^{\sum_{t=0}^{\infty}\beta^t F(x_t^n,x_{t+1}^n)} \\ &= e^{F(x_0^n,x_1^n)+\beta F(x_1^n,x_2^n)+...+\beta^{n-1}F^-(x_{T-1}^n,x_T^n)}[\widehat{V}(x_T^n)]^{\beta^T} \\ &\leq e^{F(x_0^n,x_1^n)+\beta F(x_1^n,x_2^n)+...+\beta^{n-1}F^-(x_{T-1}^n,x_T^n)}[g(x_T^n)]^{\beta^T}.\end{aligned}$$

Since the function g is upper semi-continuous it follows that

$$\limsup_n \widehat{V}(x_0^n) \leq e^{F(x_0,x_1)+\beta F(x_1,x_2)+...+\beta^{n-1}F^-(x_{T-1},x_T)}[g(x_T)]^{\beta^T}$$

Taking the limits as $T \to \infty$ we get:

$$\limsup_n \widehat{V}(x_0^n) \leq u(\widetilde{x}) \leq \widehat{V}(x_0).$$

(ii) It follows from Le Van and Morhaim (2002, Theorem 4.2(ii)). ■

Remark 3.7. Theorem 3.3 is similar to Theorem 5 in Rincón-Zapatero and Rodríguez-Palmero (2003). We rely on their method based on truncating the technological correspondence, and approaching the fixed point by means of a sequence of fixed points of the truncated problems. Working in this direction the crucial step is Lemma 3.3[4].

3.2.2 Unbounded period returns

We next look to problems where the return function F can take the value $-\infty$ at some points of the technological set. The following assumption is satisfied in a large class of economic applications.

Assumption (H3‴). (a) Let $X = \mathbb{R}^n_+$ and assume that

$$\begin{array}{ll}\text{(a.1)} & \Gamma(0) = \{0\} \text{ and } \forall x \neq 0,\ \exists y \in \Gamma(x)\backslash\{0\} \\ \text{(a.2)} & F(0,0) = -\infty \text{ and } \forall x \neq 0,\ \exists y \in \Gamma(x),\ F(x,y) > -\infty \\ \text{(a.3)} & \Pi^0(x_0) \neq \varnothing \text{ if } x_0 \neq 0.\end{array}$$

(b) There exist two functions $f_1, f_2 \in C^+(X\backslash\{0\})$ such that

$$\begin{array}{ll}\text{(b.1)} & f_1 \leq \widehat{T}f_1 \leq \widehat{T}f_2 \leq f_2 \\ \text{(b.2)} & f_2(0) = 0. \\ \text{(b.3)} & \forall x_0 \neq 0,\ \forall \widetilde{x} \in \Pi^0(x_0),\ \lim_T (f_1(x_T))^{\beta^T} = \lim_T (f_2(x_T))^{\beta^T} = 1.\end{array}$$

[4]This step is also crucial in Rincón-Zapatero and Rodríguez-Palmero (2003). However, to prove the existence of a sequence $\{\widehat{f}_j\}_{j\in\mathbb{N}}$ with each $\widehat{f}_j$ be the unique continuous fixed point of the operator $\widehat{T}_j$ on $C^{++}(X)$ they make use of Theorem 3 which relies on the incorrect Proposition (1b).

Consider the following set of functions:

$$A(X) = \{f \in C^+(X\backslash\{0\}) : f(0) = 0 \text{ and } f_1 \leq f \leq f_2\}.$$

We have the following result.

Theorem 3.4. *Under Assumptions (**H1**)-(**H2**)-(**H3'''**), the operator $\widehat{T}$ has a unique fixed point $\widehat{f}$ on the set $A(X)$ which coincides with the value function $\widehat{V}$. In addition, for any $f \in [f_1, f_2]$, $\widehat{T}^n f \to \widehat{V}$ pointwise.*

Proof. The operator $\widehat{T}$ maps $A(X)$ into $A(X)$. From Proposition 3.1 it follows that $\widehat{T}$ has a unique fixed point $\widehat{f}$ on the set $A(X)$. We have to show that $\widehat{V}$ belongs to $A(X)$. Let $x_0 \neq 0$ and take $\widetilde{x} \in \Pi^0(x_0)$ such that

$$\widehat{V}(x_0) = e^{\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})}.$$

$$\begin{aligned} f_2(x_0) &\geq \widehat{T}^N f_2(x_0) \\ &\geq e^{F(x_0,x_1)+\beta F(x_1,x_2)+\ldots+\beta^{N-1}F^-(x_{N-1},x_N)}[f_2(x_N)]^{\beta^N}. \end{aligned}$$

Taking the limit as $T \to \infty$ we get:

$$f_2(x_0) \geq \widehat{V}(x_0).$$

A similar argument shows that $f_1(x_0) \leq \widehat{V}(x_0)$. Moreover, $\widehat{V}(0) = 0$.

To prove the claim we have to show that $\widehat{V} \in C^+(X\backslash\{0\})$[5]. Let $x_0 \neq 0$ and take $\widetilde{x} \in \Pi^0(x_0)$ such that

$$\widehat{V}(x_0) = e^{\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})}.$$

We claim that $\widehat{T}^n f_2(x_0) \geq \widehat{V}(x_0)$ for any $n > 0$. Indeed,

$$\begin{aligned} f_2(x_0) &\geq \widehat{T}^n f_2(x_0) \\ &\geq e^{F(x_0,x_1)+\beta F(x_1,x_2)+\ldots+\beta^{n-1}F^-(x_{n-1},x_n)}[f_2(x_n)]^{\beta^n} \\ &\geq e^{F(x_0,x_1)+\beta F(x_1,x_2)+\ldots+\beta^{n-1}F^-(x_{n-1},x_n)}[\widehat{V}(x_n)]^{\beta^n} \\ &= \widehat{V}(x_0). \end{aligned}$$

A similar type argument shows that $\widehat{T}^n f_1(x_0) \leq \widehat{V}(x_0)$ for any $n > 0$. Define $f_1^n := \widehat{T}^n f_1$ and $f_2^n := \widehat{T}^n f_2$. Let $x_0 > 0$ and consider a convergent sequence $x_0^m \to x_0$. Since f_1^n, f_2^n are continuous we have $\lim_m f_1^n(x_0^m) = f_1^n(x_0)$ and $\lim_m f_2^n(x_0^m) = f_2^n(x_0)$. It follows that

$$\begin{aligned} -f_2^n(x_0) &\leq -\widehat{V}(x_0) \leq -f_1^n(x_0) \\ f_1^n(x_0) = \lim_m f_1^n(x_0^m) &\leq \lim_m \widehat{V}(x_0^m) \leq \lim_m f_2^n(x_0^m) = f_2^n(x_0). \end{aligned}$$

[5]We are grateful to Jorge Durán for suggesting this method of proof.

Therefore,

$$\left|\lim_m \widehat{V}(x_0^m) - \widehat{V}(x_0)\right| \leq f_2^n(x_0) - f_1^n(x_0).$$

From Proposition 3.1, for any $x_0 > 0$, we have:

$$\lim_n [f_2^n(x_0) - f_1^n(x_0)] = 0.$$

We have proved that $\widehat{V} \in C^+(X\backslash\{0\})$. ■

Remark 3.8. (i) Theorem 3.3 is similar to Theorem 6 in Rincón-Zapatero and Rodríguez-Palmero (2003).

(ii) A function f_1 with the specified properties can always be found if one assumes that for all $x \neq 0$ there is a continuous selection q of Γ with $U(x, q(x)) > -\infty$ (this is the case in Rincón-Zapatero and Rodríguez-Palmero, Assumption DP3′).

Concerning the existence of function f_2 observe the following. If the set X is compact one may take:

$$f_2(x) = e^{\frac{M}{1-\beta}}$$

where $M = \left\{0, \max_{\substack{y\in\Gamma(x)\\ x\in X}} F(x,y)\right\}$. In cases where there exists an upper-semicontinuous function $\omega : X \to \mathbb{R}_+$ such that:

$$\text{(a) } \psi \leq \omega \text{ and } \max_{y\in\Gamma(x)} \omega(y) \leq \gamma\omega(x) \text{ for all } x \in X$$

$$\text{(b) } \lim_T (\omega(x_T))^{\beta^T} = 1,$$

where $\gamma > 0$, $\beta\gamma < 1$ and $\psi(x) = \max_{y\in\Gamma(x)} F(x,y)$, one may choose $f_2(x) = e^{\frac{\omega(x)}{1-\beta\gamma}}$.

When the technological correspondence satisfies $\Gamma(K_j) \subseteq K_j$ for any $j \in \mathbb{N}$ (as in Rincón-Zapatero and Rodríguez-Palmero, Theorem 6), then the functions f_1 and f_2 can be chosen as in Theorem 3.1 for any $x \neq 0$.

(iii) For our multiplicative operator, uniqueness of solutions is implied on the set $A(X)$. The implication of this result for the standard Bellman operator T, is that uniqueness is established for the space of functions:

$$A(X) = \{f \in C(X\backslash\{0\}) : f(0) = -\infty \text{ and } f_1 \leq f \leq f_2\}.$$

(iv) One may wonder whether the value function may be the limit of any sequence $\{\widehat{T}^n f\}_{n\in\mathbb{N}}$ where f is any function in $C^{++}(X)$. We give below an example showing that this is not possible. Indeed, let $X = \mathbb{R}_+$, $\Gamma(x) = [0,\ \frac{x}{\beta}]$ and $F(x,y) = \ln\left[\frac{x}{\beta} - y\right]$. Define:

$$K_1 = \left[0, \frac{1}{\beta}\right],\ K_2 = \left[0, \frac{1}{\beta^2}\right], \ldots, K_n = \left[0, \frac{1}{\beta^n}\right], \text{ and so on.}$$

Obviously, the sequence $\{K_j\}_{j\in\mathbb{N}}$ is increasing, $X = \cup_j K_j$ and $\Gamma(K_j) = K_{j+1}$. We have $\widehat{V}(x) = \lim_n \widehat{T}^n(1) = e^{\frac{1}{1-\beta}\ln\left[\frac{(1-\beta)}{\beta}x\right]}$. Take $f(x) = e^x$, $\forall x$ and compute $\widehat{T}^n f(x)$. We obtain:

$$\widehat{T}^n f(x) = e^{-(1+\beta+...+\beta^{n-1})(\ln\beta+1)} e^x.$$

Hence $T^n f(x) \to e^{\frac{-1}{1-\beta}(\ln\beta+1)} e^x \neq \widehat{V}(x)$ when $x > 0$.

The function $f(x) = e^x$ does not belongs to the set $A(X)$ in Theorem 3.4. Indeed, the sequence $\{\frac{1}{\beta}, \frac{1}{\beta^2}, ...\frac{1}{\beta^t}, ...\}$ belongs to $\Pi(1)$. For any $t \geq 0$, we have $(f(\frac{1}{\beta^t}))^{\beta^t} = e$, and therefore $\lim_{t\to+\infty} (f(\frac{1}{\beta^t}))^{\beta^t} > 1$.

4 Appendix-Proofs

4.1 Lemma 3.1

Proof. (i) The proof is standard. (ii) We only prove that Π is lower hemi-continuous. Let $x_0 \in X$, $\widetilde{x} = (x_1, x_2, ..., x_t, ...) \in \Pi(x_0)$ and $(x_0^n)_n$ be a sequence converging to x_0. Consider a compact neighborhood of x_0 denoted by B. For $n \in \mathbb{N}$ large enough, $x_0^n \in B$. Since Γ is continuous, for n large enough, $\Pi(x_0^n)$ is included in $\prod_{i=0}^{\infty} \Gamma^i(B)$ which is a compact set of the product topology.

Since Γ is lower hemi-continuous, there exists a sequence $(x_0^{n_1}, x_1^{n_1})_{n_1}$ where $x_1^{n_1} \in \Gamma(x_0^{n_1}), \forall n_1$ and $(x_0^{n_1})_{n_1}$ is a subsequence of $(x_0^n)_n$ such that $x_1^{n_1} \to x_1$. Using again the lower hemi-continuity of Γ, there exists a sequence $(x_0^{n_2}, x_1^{n_2}, x_2^{n_2})_{n_2}$ where $x_2^{n_2} \in \Gamma(x_1^{n_2}), \forall n_2$ and $(x_1^{n_2})_{n_2}$ is a subsequence of $(x_1^{n_2})$ such that $x_2^{n_2} \to x_2$. By induction, there exists a sequence $(x_0^{n_k}, x_1^{n_k}, x_2^{n_k}, ..., x_k^{n_k})_{n_k}$ where $(x_0^{n_k}, x_1^{n_k}, x_2^{n_k}, ..., x_k^{n_k}) \to (x_0, x_1, x_2, ..., x_k)$ when $n_k \to \infty$. Now define:

$$\begin{array}{ll}
\widetilde{z}^{n_1} = (x_1^{n_1}, \widetilde{y}^{n_1}) \in \Pi(x_0^{n_1}) & \text{with } \widetilde{y}^{n_1} \in \Pi(x_1^{n_1}) \\
\widetilde{z}^{n_2} = (x_1^{n_2}, x_2^{n_2}, \widetilde{y}^{n_2}) \in \Pi(x_0^{n_2}) & \text{with } \widetilde{y}^{n_2} \in \Pi(x_2^{n_2}) \\
\qquad\qquad ... & \\
\widetilde{z}^{n_k} = (x_1^{n_k}, x_2^{n_k}, ..., , x_k^{n_k}, \widetilde{y}^{n_k}) \in \Pi(x_0^{n_k}) & \text{with } \widetilde{y}^{n_k} \in \Pi(x_k^{n_k})
\end{array}$$

Since $x_0^{n_k} \to x_0$, and since for n_k large enough, $\Pi(x_0^{n_k})$ is included in a compact set of the product topology, one can assume that $(\widetilde{z}^{n_k})_k$ converges, and let us denote $\widetilde{z}$ its limit. Fix t. By construction, for k large enough, $z_t^{n_k} = x_t^{n_k}$. Hence $\widetilde{z} = \widetilde{x}$. ■

4.2 Lemma 3.2

Proof. The proof (P.1) is easy. Let us show that (P.3) is satisfied. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $P^+(X)$ that converges to $f \in P^+(X)$. Let $x \in X$. For any

$y \in \Gamma(x)$ we have:

$$\widehat{T} f_n(x) \geq e^{F(x,y)} (f_n(y))^{\beta}.$$

This in turn implies that:

$$\liminf_n \widehat{T} f_n(x) \geq e^{F(x,y)} (f(y))^{\beta},$$

and therefore:

$$\liminf_n \widehat{T} f_n(x) \geq \widehat{T} f(x).$$

■

4.3 Proposition 3.1

Proof. (i) Let $\widehat{f} := \lim_{n \to +\infty} \widehat{T}^n f_1$ denote the pointwise limit of $\{\widehat{T}^n f_1\}_{n \in \mathbb{N}.}$. From Theorem 2.2(ii) this limit exists, belongs to $B^+(X)$ and is a fixed point of $\widehat{T}$ on $B^+(X)$. It is therefore a fixed point of $\widehat{T}$ on the set $[f_1, f_2]$.

Let $x_0 \in X$ and $\widetilde{x} \in \Pi^0(x_0)$. For any T we have:

$$\widehat{f}(x_0) = \widehat{T}\widehat{f}(x_0) \geq e^{\sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1})} (\widehat{f}_1(x_T))^{\beta^T}.$$

Since $\lim_{T \to +\infty} (\widehat{f}_1(x_T))^{\beta^T} = 1$

$$\widehat{f}(x_0) \geq e^{\sum_{t=0}^{+\infty} \beta^t F(x_t, x_{t+1})}. \tag{1}$$

Assume that $\widehat{T}$ has another fixed point $\widehat{g} \in [f_1,\ f_2]$. It must be the case that

$$\widehat{f} \leq \widehat{g}. \tag{2}$$

Let $\alpha > 1$. For any $x_0 \in X$, there exists $x_1 \in \Gamma(x_0)$ such that:

$$\widehat{g}(x_0) = \widehat{T}\widehat{g}(x_0) \leq \alpha [e^{F(x_0, x_1)} (\widehat{g}(x_1))^{\beta}].$$

But there is also $x_2 \in \Gamma(x_1)$ such that:

$$\widehat{g}(x_1) \leq \alpha [e^{F(x_1, x_2)} (\widehat{g}(x_2))^{\beta}]$$

and

$$\widehat{g}(x_0) \leq \alpha^{1+\beta} [e^{F(x_0, x_1) + \beta F(x_1, x_2)} (\widehat{g}(x_2))^{\beta^2}].$$

By induction, it follows that for any $T > 0$

$$\widehat{g}(x_0) \leq \alpha^{\frac{1-\beta^T}{1-\beta}} \left(e^{\sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1})} (\widehat{g}(x_T))^{\beta^T} \right).$$

Since $\lim_{T\to+\infty}(\widehat{g}(x_T))^{\beta^T}=1$, by letting $T\to+\infty$ we get:

$$\widehat{g}(x_0)\leq \alpha^{\frac{1}{1-\beta}}e^{\sum\limits_{t=0}^{+\infty}\beta^t F(x_t,x_{t+1})}. \tag{3}$$

Combining inequalities (1), (2) and (3) we have:

$$\widehat{f}(x_0)\leq \widehat{g}(x_0)\leq \alpha^{\frac{1}{1-\beta}}\widehat{f}(x_0).$$

From Theorem 2.1(ii), it follows that $\widehat{f}(x_0)=\widehat{g}(x_0)$ for any $x_0\in X$ such that $\Pi^0(x_0)\neq\emptyset$. For any $x_0\in X$ such that $\Pi^0(x_0)=\emptyset$, it follows that $\widehat{f}(x_0)=\widehat{g}(x_0)=0$.

We next show that $\{\widehat{T}^n f_2\}_{n\in\mathbb{N}}$ converges to $\widehat{f}$ pointwise. For any $x_0\in X$ such that $\Pi^0(x_0)=\emptyset$, we have $\widehat{T}^n f_2(x_0)\to\widehat{f}(x_0)=0$. Let $x_0\in X$ be such that $\Pi^0(x_0)\neq\emptyset$. Fix some $N\in\mathbb{N}$. For any $n>N$, there exists $x_1^n\in\Gamma(x_0)$ such that:

$$\widehat{T}^n f_2(x_0)=\widehat{T}(\widehat{T}^{n-1}f_2)(x_0)=e^{F(x_0,x_1^n)}(\widehat{T}^{n-1}f_2(x_1^n))^{\beta}.$$

Similarly, there exists $x_2^n\in\Gamma(x_1^n)$ such that:

$$\widehat{T}^{n-1}f_2(x_1^n)=e^{F(x_1^n,x_2^n)}(\widehat{T}^{n-2}f_2(x_2^n))^{\beta}$$

and hence:

$$\widehat{T}^n f_2(x_0)=e^{F(x_0,x_1^n)+\beta F(x_1^n,x_2^n)}(\widehat{T}^{n-2}f_2(x_2^n))^{\beta^2}.$$

By induction,

$$\begin{aligned}\widehat{T}^n f_2(x_0) &= e^{F(x_0,x_1^n)+\ldots+\beta^{N-1}F(x_{N-1}^n,x_N^n)}(\widehat{T}^{n-N}f_2(x_N^n))^{\beta^N}\\ &\leq e^{F(x_0,x_1^n)+\ldots+\beta^{N-1}F(x_{N-1}^n,x_N^n)}(f_2(x_N^n))^{\beta^N}.\end{aligned}$$

Observe also that

$$\begin{aligned}\widehat{f}(x_0)=\widehat{T}^N\widehat{f}(x_0) &\geq e^{F(x_0,x_1^n)+\ldots+\beta^{N-1}F(x_{N-1}^n,x_N^n)}(\widehat{f}(x_N^n))^{\beta^N}\\ &\geq e^{F(x_0,x_1^n)+\ldots+\beta^{N-1}F(x_{N-1}^n,x_N^n)}(f_1(x_N^n))^{\beta^N}.\end{aligned}$$

Combining the last inequalities one gets:

$$1\leq\frac{\widehat{T}^n\widehat{f}_2(x_0)}{\widehat{f}(x_0)}\leq\frac{(f_2(x_N^n))^{\beta^N}}{(f_1(x_N^n))^{\beta^N}}.$$

Let $\varepsilon>0$ be given. There exists $N(\varepsilon,x_0)$, such that, for any $n>N(\varepsilon,x_0)$

$$1\leq\frac{\widehat{T}^n\widehat{f}_2(x_0)}{\widehat{f}(x_0)}\leq 1+\varepsilon.$$

The claim has been proved.

(ii) The proof of uniform convergence on compact subsets of X is standard. If $f\in[f_1,f_2]$ is continuous, uniform convergence of $\{\widehat{T}^n f\}_{n\in\mathbb{N}}$ implies that $\widehat{f}$ is continuous. ■

4.4 Lemma 3.5

Proof. (i) Let $\widehat{f}$ be a fixed point of $\widehat{T}$ on $B^+(X)$ that satisfies:

$$\limsup_t \left[\widehat{f}(x_t)\right]^{\beta^t} \leq 1,\ \forall \widetilde{x} \in \Pi(x_0).$$

Let $x_0 \in X$. Assumption **H3″** implies that for any $\widetilde{x} \in \Pi^0(x_0)$

$$\begin{aligned} \widehat{f}(x_0) &= \widehat{T}^n \widehat{f}(x_0) \\ &\geq e^{F(x_0,x_1)+\beta F(x_1,x_2)+...+\beta^{n-1}F^-(x_{n-1},a)}[\widehat{f}(a)]^{\beta^n}. \end{aligned}$$

Taking the limits as $n \to \infty$ we get:

$$e^{\sum_{t=0}^{\infty} \beta^t F(x_t,x_{t+1})} \leq \widehat{f}(x_0),\ \forall \widetilde{x} \in \Pi(x_0). \tag{1}$$

Let $\widehat{h}$ be another fixed point of $\widehat{T}$ on set $A(X)$. Let $\alpha > 1$. There exists $x_1 \in \Gamma(x_0)$ such that

$$\widehat{h}(x_0) \leq \alpha e^{F(x_0,x_1)}(\widehat{h}(x_1))^{\beta}.$$

There also exists $x_2 \in \Gamma(x_1)$ such that

$$\widehat{h}(x_1) \leq \alpha e^{F(x_1,x_2)}(\widehat{h}(x_2))^{\beta}.$$

and hence

$$\widehat{h}(x_0) \leq \alpha^{1+\beta} e^{F(x_0,x_1)+\beta F(x_1,x_2)}(\widehat{h}(x_2))^{\beta^2}.$$

By induction

$$\widehat{h}(x_0) \leq \alpha^{\frac{1}{1-\beta}} e^{F(x_0,x_1)+\beta F(x_1,x_2)+...+\beta^t F(x_{t-1},x_t)}[\widehat{h}(x_t)]^{\beta^t}.$$

Let $t \to +\infty$. We have:

$$\widehat{h}(x_0) \leq \alpha^{\frac{1}{1-\beta}} e^{\sum_{t=0}^{+\infty} F(x_t,x_{t+1})}.$$

It follows that:

$$\widehat{h}(x_0) \leq \alpha^{\frac{1}{1-\beta}} \widehat{f}(x_0). \tag{2}$$

Inequality (1) actually holds for any fixed point. Let $\widetilde{x}' \in \Pi^0(x_0)$ be such that $\widehat{f}(x_0) = e^{\sum_{t=0}^{\infty} \beta^t F(x'_t,x'_{t+1})}$. It follows that:

$$\widehat{f}(x_0) \leq \widehat{h}(x_0). \tag{3}$$

Combining (2) and (3) we get

$$\widehat{f}(x_0) \leq \widehat{h}(x_0) \leq \alpha^{\frac{1}{1-\beta}} \widehat{f}(x_0).$$

From Theorem 2.1(ii), we must have $\widehat{h} = \widehat{f}$. ■

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