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The regularity and exponential decay of solution for a linear wave equation associated with two-point boundary conditions

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Abstract

This paper is concerned with the existence and the regularity of global solutions to the linear wave equation associated the boundary conditions of two-point type. We also investigate the decay properties of the global solutions to this problem by the construction of a suitable Lyapunov functional.

Keyword: Faedo-Galerkin method; Global existence; nonlinear wave equation; two-point boundary conditions.

AMS subject classification: 35L05, 35L15, 35L70, 37B25.

1 Introduction

The wave equation

$$u_{tt} - \Delta u = f(x, t, u, u_t),$$

associated with the different boundary conditions, has been extensively studied by many authors, see [1 -8] and references therein. In the above mentioned papers, the existence and regularity of solutions, the asymptotic behavior and asymptotic expansion of solutions have received much attention.

In [8], Santos also studied the asymptotic behavior of the solutions to a coupled system of wave equations having integral convolutions as memory terms. Their main result showed that the solution of that system decays uniformly in time, with rates depending on the rate of decay of the kernel of the convolutions.

In this paper we consider the following initial-boundary value problem for the linear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x, t) \text{ in } (0, 1) \times (0, \infty), \quad (1.1)$$

$$u_x(0, t) = h_0 u(0, t) + \lambda_0 u_t(0, t) + \tilde{h}_1 u(1, t) + \tilde{\lambda}_1 u_t(1, t) + g_0(t), \quad (1.2)$$

$$-u_x(1, t) = h_1 u(1, t) + \lambda_1 u_t(1, t) + \tilde{h}_0 u(0, t) + \tilde{\lambda}_0 u_t(0, t) + g_1(t), \quad (1.3)$$

$$u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \quad (1.4)$$

where $h_0, h_1, \lambda_0, \lambda_1, \tilde{h}_0, \tilde{h}_1, \tilde{\lambda}_0, \tilde{\lambda}_1, K, \lambda$ are constants and $\tilde{u}_0, \tilde{u}_1, f, g_0, g_1$ are given functions.

The rest of this paper consists of four sections. In section 2, we present some notations and lemmas that will be used to establish our results. In section 3, we investigate the existence and uniqueness of a weak solution and so a strong solutions of problem (1.1) – (1.4) with the convenable conditions. Section 4 is devoted to the study of the regularity of solutions. Finally, in fifth section, we prove that the exponential decay properties of the global solutions are similar to that of the functionals f, g_0, g_1 .

2 Preliminaries

Let $\Omega = (0, 1)$ and $Q_T = \Omega \times (0, T)$, for $T > 0$. In what follows we will denote

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx, \quad \|v\| = \sqrt{\langle v, v \rangle},$$

and $\|\cdot\|_1$ is a equivalent norm in $H^1(\Omega)$, defined by

$$\|v\|_1 = (v^2(0) + \|v_x\|^2)^{1/2}.$$

We also denote $u(x, t), u_t(x, t), u_{tt}(x, t), u_x(x, t)$ and $u_{xx}(x, t)$ by $u(t), u'(t), u''(t), u_x(t), u_{xx}(t)$, respectively, when no confusion arises.

Consider a symmetric bilinear form $a(u, v)$ on $H^1(\Omega) \times H^1(\Omega)$ by setting

$$a(u, v) = \langle u_x, v_x \rangle + h_0 u(0)v(0) + h_1 u(1)v(1).$$

We state here some preliminary results that will be used in the sequel.

Lemma 2.1. *The imbedding $H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \|v\|_1, \quad \text{for all } v \in H^1(\Omega). \quad (2.1)$$

Lemma 2.2. *Let $h_0 > 0$ and $h_1 \geq 0$. Then, the symmetric bilinear form $a(\cdot, \cdot)$ is continuous on $H^1(\Omega) \times H^1(\Omega)$ and coercive on $H^1(\Omega)$, i.e.,*

$$(i) \quad |a(u, v)| \leq C_1 \|u\|_1 \|v\|_1, \quad \forall u, v \in H^1(\Omega),$$

$$(ii) \quad a(v, v) \geq C_0 \|v\|_1^2, \quad \forall v \in H^1(\Omega),$$

where $C_0 = \min\{1, h_0\}$ and $C_1 = \max\{1, h_0, 2h_1\}$.

Lemma 2.3. *Let $\lambda_0, \lambda_1 > 0$ and $\tilde{\lambda}_0, \tilde{\lambda}_1 \in \mathbb{R}$ such that $(\tilde{\lambda}_0 + \tilde{\lambda}_1)^2 - 4\lambda_0\lambda_1 < 0$. Then we have*

$$\lambda_0 x^2 + \lambda_1 y^2 + (\tilde{\lambda}_0 + \tilde{\lambda}_1)xy \geq \frac{1}{2}\mu_{\min}(x^2 + y^2), \quad \forall x, y \in \mathbb{R},$$

where

$$\mu_{\min} = \frac{1}{4} \left[-(\tilde{\lambda}_0 + \tilde{\lambda}_1)^2 + 4\lambda_0\lambda_1 \right] \min \left\{ \frac{1}{\lambda_0}, \frac{1}{\lambda_1} \right\} > 0.$$

The proof of these lemmas are straightforward. We shall omit the details.

Remark 2.4. *From the Lemma 2.2 we deduce that*

$$C_0 \|v\|_1^2 \leq \|v\|_a^2 \leq C_1 \|v\|_1^2, \quad \forall v \in H^1(\Omega), \quad (2.2)$$

where $\|\cdot\|_a$ is the norm on $H^1(\Omega)$ generated by the symmetric bilinear form $a(\cdot, \cdot)$, i.e.,

$$\|v\|_a = \sqrt{a(v, v)}, \quad \forall v \in H^1(\Omega).$$

3 Existence and uniqueness of solutions

In this section, we assume that $h_0, \lambda_0, \lambda_1$ are positive constants, h_1 is nonnegative constant and $K, \lambda, \tilde{h}_0, \tilde{h}_1, \tilde{\lambda}_0, \tilde{\lambda}_1$ are constants verifying the condition

$$|\tilde{\lambda}_0 + \tilde{\lambda}_1| < 2\sqrt{\lambda_0\lambda_1}. \quad (3.1)$$

Theorem 3.1. *Let $T > 0$ and assume that $g_0, g_1 \in L^2(0, T)$, $f \in L^1(0, T; L^2(\Omega))$. Then, for each $(\tilde{u}_0, \tilde{u}_1) \in H^1(\Omega) \times L^2(\Omega)$, the problem (1.1) – (1.4) has a unique weak solution u satisfying*

$$u \in L^\infty(0, T; H^1(\Omega)), \quad u_t \in L^\infty(0, T; L^2(\Omega)),$$

and

$$u(0, \cdot), u(1, \cdot) \in H^1(0, T).$$

Proof. The proof consists of step 1 - 4.

Step 1. The Faedo-Galerkin approximation. Let $\{w_j\}$ be a denumerable base of $H^1(\Omega)$. We find the approximate solution of problem (1.1) – (1.4) in the form

$$u_m(t) = \sum_{j=1}^m c_{mj}(t)w_j,$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations

$$\begin{aligned} & \langle u_m''(t), w_j \rangle + a(u_m(t), w_j) + \left(\lambda_0 u_m'(0, t) + \tilde{h}_1 u_m(1, t) + \tilde{\lambda}_1 u_m'(1, t) \right) w_j(0) \\ & + \left(\lambda_1 u_m'(1, t) + \tilde{h}_0 u_m(0, t) + \tilde{\lambda}_0 u_m'(0, t) \right) w_j(1) + \langle K u_m(t) + \lambda u_m'(t), w_j \rangle \\ & = -g_0(t)w_j(0) - g_1(t)w_j(1) + \langle f(t), w_j \rangle, \quad 1 \leq j \leq m, \end{aligned} \quad (3.2)$$

with the initial conditions

$$u_m(0) = u_{0m} = \sum_{j=1}^m \alpha_{mj} w_j \rightarrow \tilde{u}_0 \text{ strongly in } H^1(\Omega), \quad (3.3)$$

and

$$u'_m(0) = u_{1m} = \sum_{j=1}^m \beta_{mj} w_j \rightarrow \tilde{u}_1 \text{ strongly in } L^2(\Omega). \quad (3.4)$$

From the assumptions of Theorem 3.1, system (3.2)–(3.4) has solution $u_m(t)$ on some interval $[0, T_m]$. The following estimates allow one to take $T_m = T$, for all m .

Step 2. A priori estimates. Multiplying the j^{th} equation of (3.2) by $c'_{mj}(t)$ and summing up with respect to j , afterwards, integrating by parts with respect to the time variable from 0 to t , we get after some rearrangements

$$\begin{aligned} S_m(t) &= S_m(0) - 2\tilde{h}_0 \int_0^t u_m(0, s) u'_m(1, s) ds - 2\tilde{h}_1 \int_0^t u_m(1, s) u'_m(0, s) ds \\ &\quad - 2K \int_0^t \langle u_m(s), u'_m(s) \rangle ds - 2\lambda \int_0^t \|u'_m(s)\|^2 ds \\ &\quad - 2 \int_0^t g_0(s) u'_m(0, s) ds - 2 \int_0^t g_1(s) u'_m(1, s) ds + 2 \int_0^t \langle f(s), u'_m(s) \rangle ds \\ &= S_m(0) + \sum_{i=1}^7 I_i, \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} S_m(t) &= \|u'_m(t)\|^2 + \|u_m(t)\|_a^2 \\ &\quad + 2 \int_0^t \left[\lambda_0 |u'_m(0, s)|^2 + \lambda_1 |u'_m(1, s)|^2 + (\tilde{\lambda}_0 + \tilde{\lambda}_1) u'_m(0, t) u'_m(1, t) \right] ds. \end{aligned} \quad (3.6)$$

By Lemma 2.3, it follows from (3.6), that

$$S_m(t) \geq \mu_0 X_m(t), \quad (3.7)$$

where

$$X_m(t) = \|u'_m(t)\|^2 + \|u_m(t)\|_1^2 + \int_0^t \left(|u'_m(0, s)|^2 + |u'_m(1, s)|^2 \right) ds, \quad (3.8)$$

and $\mu_0 = \min \{C_0, \mu_{min}\}$.

Now, using the inequalities (2.1) – (2.2) and the following inequalities

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \forall a, b \in \mathbb{R}, \forall \varepsilon > 0, \quad (3.9)$$

$$|u_m(0, t)| \leq \|u_m(t)\|_{C^0(\bar{\Omega})} \leq \sqrt{2} \|u_m(t)\|_1 \leq \sqrt{2X_m(t)}, \quad (3.10)$$

$$\|u_m(t)\|^2 \leq 2 \|u_{0m}\|^2 + 2 \int_0^t \|u'_m(s)\|^2 ds \leq 2 \|u_{0m}\|^2 + 2 \int_0^t X_m(s) ds, \quad (3.11)$$

we shall estimate respectively the terms on the right-hand side of (3.5) as follows

$$\begin{aligned} I_1 &= -2\tilde{h}_0 \int_0^t u_m(0, s) u'_m(1, s) ds \\ &\leq \frac{1}{\varepsilon} |\tilde{h}_0|^2 \int_0^t |u_m(0, s)|^2 ds + \varepsilon \int_0^t |u'_m(1, s)|^2 ds \\ &\leq \frac{2}{\varepsilon} |\tilde{h}_0|^2 \int_0^t X_m(s) ds + \varepsilon X_m(t), \end{aligned} \quad (3.12)$$

$$I_2 = -2\tilde{h}_1 \int_0^t u_m(1, s) u'_m(0, s) ds \leq \frac{2}{\varepsilon} |\tilde{h}_1|^2 \int_0^t X_m(s) ds + \varepsilon X_m(t), \quad (3.13)$$

$$I_3 = -2K \int_0^t \langle u_m(s), u'_m(s) \rangle ds \leq 2\sqrt{2}|K| \int_0^t X_m(s) ds, \quad (3.14)$$

$$I_4 = -2\lambda \int_0^t \|u'_m(s)\|^2 ds \leq 2|\lambda| \int_0^t X_m(s) ds, \quad (3.15)$$

$$\begin{aligned} I_5 &= -2 \int_0^t g_0(s) u'_m(0, s) ds \leq \frac{1}{\varepsilon} \|g_0\|_{L^2(0, T)}^2 + \varepsilon \int_0^t |u'_m(0, s)|^2 ds \\ &\leq \frac{1}{\varepsilon} \|g_0\|_{L^2(0, T)}^2 + \varepsilon X_m(t), \end{aligned} \quad (3.16)$$

$$I_6 = -2 \int_0^t g_1(s) u'_m(1, s) ds \leq \frac{1}{\varepsilon} \|g_1\|_{L^2(0, T)}^2 + \varepsilon X_m(t), \quad (3.17)$$

$$I_7 = 2 \int_0^t \langle f(s), u'_m(s) \rangle ds \leq \int_0^T \|f(s)\| ds + \int_0^t \|f(s)\| X_m(s) ds. \quad (3.18)$$

On the other hand, using (3.3) – (3.4), (3.6) and the assumption $(\tilde{u}_0, \tilde{u}_1) \in H^1(\Omega) \times L^2(\Omega)$, we have

$$S_m(0) = \|u_{1m}\|^2 + \|u_{0m}\|_a^2 \leq \tilde{C}_1 \text{ for all } m, \quad (3.19)$$

where \tilde{C}_1 is a constant depending only on \tilde{u}_0 , \tilde{u}_1 , h_0 and h_1 .

Combining (3.5), (3.7), (3.12)-(3.19), we obtain

$$\begin{aligned} (\mu_0 - 4\varepsilon) X_m(t) &\leq \tilde{C}_1 + \frac{1}{\varepsilon} \|g_0\|_{L^2(0, T)}^2 + \frac{1}{\varepsilon} \|g_1\|_{L^2(0, T)}^2 + \int_0^T \|f(s)\| ds \\ &\quad + \int_0^t \left[\frac{2}{\varepsilon} \left(|\tilde{h}_0|^2 + |\tilde{h}_1|^2 \right) + 2\sqrt{2}|K| + 2|\lambda| + \|f(s)\| \right] X_m(s) ds, \end{aligned} \quad (3.20)$$

for all $\varepsilon > 0$. By choosing $\varepsilon > 0$ such that $\mu_0 - 4\varepsilon > 0$, it follows from (3.20) that

$$X_m(t) \leq M_T^{(1)} + \int_0^t N_T^{(1)}(s) X_m(s) ds, \quad (3.21)$$

where

$$M_T^{(1)} = (\mu_0 - 4\varepsilon)^{-1} \left(\tilde{C}_1 + \frac{1}{\varepsilon} \|g_0\|_{L^2(0, T)}^2 + \frac{1}{\varepsilon} \|g_1\|_{L^2(0, T)}^2 + \int_0^T \|f(s)\| ds \right),$$

and

$$N_T^{(1)}(s) = (\mu_0 - 4\varepsilon)^{-1} \left[\frac{2}{\varepsilon} \left(|\tilde{h}_0|^2 + |\tilde{h}_1|^2 \right) + 2\sqrt{2}|K| + 2|\lambda| + \|f(s)\| \right], \quad N_T^{(1)} \in L^1(0, T).$$

By Gronwall's lemma, we deduce from (3.21), that

$$X_m(t) \leq M_T^{(1)} \exp \left(\int_0^t N_T^{(1)}(s) ds \right) \leq C_T, \text{ for all } t \in [0, T], \quad (3.22)$$

where C_T is a positive constant depending only on T .

Step 3. Limiting process. From (3.8) and (3.22), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \rightarrow u & \text{in } L^\infty(0, T; H^1(\Omega)) & \text{weak}^*, \\ u'_m \rightarrow u' & \text{in } L^\infty(0, T; L^2(\Omega)) & \text{weak}^*, \\ u_m(0, \cdot) \rightarrow u(0, \cdot) & \text{in } H^1(0, T) & \text{weakly}, \\ u_m(1, \cdot) \rightarrow u(1, \cdot) & \text{in } H^1(0, T) & \text{weakly}. \end{cases} \quad (3.23)$$

By the compactness lemma of Lions [5: p.57] and the imbedding $H^1(0, T) \hookrightarrow C^0([0, T])$, we can deduce from (3.23) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \rightarrow u & \text{strongly in } L^2(Q_T), \\ u_m(0, \cdot) \rightarrow u(0, \cdot) & \text{strongly in } C^0([0, T]), \\ u_m(1, \cdot) \rightarrow u(1, \cdot) & \text{strongly in } C^0([0, T]). \end{cases} \quad (3.24)$$

Passing to the limit in (3.2)-(3.4) by (3.23) and (3.24) we have u satisfying the equation

$$\begin{aligned} \frac{d}{dt} \langle u'(t), v \rangle + a(u(t), v) + \left(\lambda_0 u'(0, t) + \tilde{h}_1 u(1, t) + \tilde{\lambda}_1 u'(1, t) \right) v(0) \\ + \left(\lambda_1 u'(1, t) + \tilde{h}_0 u(0, t) + \tilde{\lambda}_0 u'(0, t) \right) v(1) + \langle Ku(t) + \lambda u'(t), v \rangle \\ = -g_0(t)v(0) - g_1(t)v(1) + \langle f(t), v \rangle, \quad \forall v \in H^1(\Omega), \end{aligned} \quad (3.25)$$

in $L^2(0, T)$ weakly, and

$$u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1. \quad (3.26)$$

The existence of the theorem is proved completely.

Step 4. Uniqueness of the solution. Let u_1, u_2 be two weak solutions of problem (1.1) – (1.4), such that

$$\begin{cases} u_i \in L^\infty(0, T; H^1(\Omega)), \quad u'_i \in L^\infty(0, T; L^2(\Omega)), \\ u_i(0, \cdot), u_i(1, \cdot) \in H^1(0, T), \quad i = 1, 2. \end{cases} \quad (3.27)$$

Then $u = u_1 - u_2$ is the weak solution of the following problem

$$\begin{cases} u_{tt} - u_{xx} + Ku + \lambda u_t = 0, \quad (x, t) \in Q_T, \\ u_x(0, t) = h_0 u(0, t) + \lambda_0 u_t(0, t) + \tilde{h}_1 u(1, t) + \tilde{\lambda}_1 u_t(1, t), \\ -u_x(1, t) = h_1 u(1, t) + \lambda_1 u_t(1, t) + \tilde{h}_0 u(0, t) + \tilde{\lambda}_0 u_t(0, t), \\ u(x, 0) = 0, \quad u_t(x, 0) = 0. \end{cases} \quad (3.28)$$

By using the lemma in [8, Lemma 2.4, p. 1799], we deduce that

$$\begin{aligned} \|u'(t)\|^2 + \|u(t)\|_a^2 + 2 \int_0^t \langle Ku(s) + \lambda u'(s), u'(s) \rangle ds \\ + 2 \int_0^t \left[\lambda_0 |u'(0, s)|^2 ds + \lambda_1 |u'(1, s)|^2 ds + \left(\tilde{\lambda}_0 + \tilde{\lambda}_1 \right) u'(0, s) u'(1, s) \right] ds \\ + 2\tilde{h}_1 \int_0^t u(1, s) u'(0, s) ds + 2\tilde{h}_0 \int_0^t u(0, s) u'(1, s) ds. \end{aligned} \quad (3.29)$$

Putting

$$\sigma(t) = \|u'(t)\|^2 + \|u(t)\|_1^2 + \mu_{\min} \int_0^t \left[|u'(0, s)|^2 + |u'(1, s)|^2 \right] ds. \quad (3.30)$$

From (3.29), (3.30) and Lemma 2.3, we prove, in a similar manner to that in the above part, that

$$\left(1 - \frac{2\varepsilon}{\mu_{\min}}\right) \sigma(t) \leq 2 \left[\frac{1}{\varepsilon} \left(|\tilde{h}_0|^2 + |\tilde{h}_1|^2 \right) + \sqrt{2}|K| + |\lambda| \right] \int_0^t \sigma(s) ds. \quad (3.31)$$

Choosing $\varepsilon > 0$, with $1 - 2\varepsilon\mu_{\min}^{-1} > 0$. Using Gronwall's lemma, it follows from (3.30)-(3.31), that $\sigma(t) \equiv 0$, i.e., $u_1 \equiv u_2$. The theorem 3.1 is proved completely. \square

Theorem 3.2. *Let $T > 0$ and assume that $g_0, g_1 \in H^1(0, T)$, $f, f_t \in L^2(Q_T)$. Then, for each $(\tilde{u}_0, \tilde{u}_1) \in H^2(\Omega) \times H^1(\Omega)$, the problem (1.1) – (1.4) has a unique weak solution u satisfying*

$$u \in L^\infty(0, T; H^2(\Omega)), \quad u_t \in L^\infty(0, T; H^1(\Omega)), \quad u_{tt} \in L^\infty(0, T; L^2(\Omega)), \quad (3.32)$$

and

$$u(0, \cdot), u(1, \cdot) \in H^2(0, T). \quad (3.33)$$

Proof. The proof consists of Steps 1-4.

Step 1. The Faedo-Galerkin approximation. Let $\{w_j\}$ be a denumerable base of $H^2(\Omega)$. We find the approximate solution of problem (1.1) – (1.4) in the form

$$u_m(t) = \sum_{j=1}^m c_{mj}(t) w_j,$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations (3.2), with the initial conditions

$$\begin{cases} u_m(0) = u_{0m} = \sum_{j=1}^m \alpha_{mj} w_j \rightarrow \tilde{u}_0 \text{ strongly in } H^2, \\ u'_m(0) = u_{1m} = \sum_{j=1}^m \beta_{mj} w_j \rightarrow \tilde{u}_1 \text{ strongly in } H^1. \end{cases} \quad (3.34)$$

From the assumptions of Theorem 3.2, system (3.2) and (3.34) has solution $u_m(t)$ on some interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all m .

Step 2. A priori estimates. By same arguments as in proof of Theorem 3.1, we obtain

$$X_m(t) \leq C_T, \text{ for all } t \in [0, T], \quad m \in \mathbb{Z}^+, \quad (3.35)$$

where $X_m(t)$ defined by (3.8) and C_T always indicating a bound depending on T .

Now, differentiating (3.2) with respect to t , we have

$$\begin{aligned} & \langle u'''_m(t), w_j \rangle + a(u'_m(t), w_j) + \left(\lambda_0 u''_m(0, t) + \tilde{h}_1 u'_m(1, t) + \tilde{\lambda}_1 u''_m(1, t) \right) w_j(0) + \\ & \left(\lambda_1 u''_m(1, t) + \tilde{h}_0 u'_m(0, t) + \tilde{\lambda}_0 u''_m(0, t) \right) w_j(1) + \langle K u'_m(t) + \lambda u''_m(t), w_j \rangle \\ & = -g'_0(t) w_j(0) - g'_1(t) w_j(1) + \langle f'(t), w_j \rangle, \end{aligned} \quad (3.36)$$

for all $j = 1, 2, \dots, m$.

Multiplying the j^{th} equation of (3.36) by $c'_{mj}(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t , we have after some rearrangements

$$\begin{aligned}
\tilde{S}_m(t) &= \tilde{S}_m(0) - 2\tilde{h}_0 \int_0^t u'_m(0, s) u''_m(1, s) ds - 2\tilde{h}_1 \int_0^t u'_m(1, s) u''_m(0, s) ds \\
&\quad - 2K \int_0^t \langle u'_m(s), u''_m(s) \rangle ds - 2\lambda \int_0^t \|u''_m(s)\|^2 ds \\
&\quad - 2 \int_0^t g'_0(s) u''_m(0, s) ds - 2 \int_0^t g'_1(s) u''_m(1, s) ds + 2 \int_0^t \langle f'(s), u''_m(s) \rangle ds \\
&= \tilde{S}_m(0) + \sum_{i=1}^7 J_i,
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
\tilde{S}_m(t) &= \|u''_m(t)\|^2 + \|u'_m(t)\|_a^2 \\
&\quad + 2 \int_0^t \left[\lambda_0 |u''_m(0, s)|^2 + \lambda_1 |u''_m(1, s)|^2 + (\tilde{\lambda}_0 + \tilde{\lambda}_1) u''_m(0, t) u''_m(1, t) \right] ds.
\end{aligned} \tag{3.38}$$

Using (3.34), (3.38) and Lemma 2.1, we have

$$\tilde{S}_m(0) = \|u''_m(0)\|^2 + \|u_{1m}\|_a^2 \leq \tilde{C}_2, \text{ for all } m, \tag{3.39}$$

where \tilde{C}_2 is a constant depending only on $\tilde{u}_0, \tilde{u}_1, f(\cdot, 0), K$ and λ . On the other hand, by Lemma 2.3, it follows from (3.39) that

$$\tilde{S}_m(t) \geq \mu_0 \tilde{X}_m(t), \tag{3.40}$$

where

$$\tilde{X}_m(t) = \|u''_m(t)\|^2 + \|u'_m(t)\|_1^2 + \int_0^t \left(|u''_m(0, s)|^2 + |u''_m(1, s)|^2 \right) ds, \tag{3.41}$$

and $\mu_0 = \min\{C_0, \mu_{\min}\}$.

By estimating the terms J_i , ($i = 1, 2, \dots, 7$) on the right-hand side of (3.37) as in the proof of Theorem 3.1, we get

$$\tilde{X}_m(t) \leq M_T^{(2)} + \int_0^t N_T^{(2)}(s) \tilde{X}_m(s) ds, \tag{3.42}$$

where

$$M_T^{(2)} = \frac{2}{\mu_0} \left[\tilde{C}_2 + \frac{8}{\mu_0} \|g'_0\|_{L^2(0, T)}^2 + \frac{8}{\mu_0} \|g'_1\|_{L^2(0, T)}^2 + \int_0^T \|f'(s)\| ds \right],$$

and

$$N_T^{(2)}(s) = \frac{2}{\mu_0} \left[\frac{16}{\mu_0} (\tilde{h}_0^2 + \tilde{h}_1^2) + 2\sqrt{2}|K| + 2|\lambda| + \|f'(s)\| \right], \quad N_T^{(2)} \in L^1(0, T).$$

From (3.42) and applying Gronwall's inequality, we obtain that

$$\tilde{X}_m(t) \leq M_T^{(2)} \exp \left(\int_0^t N_T^{(2)}(s) ds \right) \leq C_T, \text{ for all } t \in [0, T]. \tag{3.43}$$

Step 3. Limiting process. From (3.8), (3.35), (3.41) and (3.43), we deduce the existence of a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \rightarrow u & \text{in } L^\infty(0, T; H^1(\Omega)) & \text{weak}^*, \\ u'_m \rightarrow u' & \text{in } L^\infty(0, T; H^1(\Omega)) & \text{weak}^*, \\ u''_m \rightarrow u'' & \text{in } L^\infty(0, T; L^2(\Omega)) & \text{weak}^*, \\ u_m(0, \cdot) \rightarrow u(0, \cdot) & \text{in } H^2(0, T) & \text{weakly}, \\ u_m(1, \cdot) \rightarrow u(1, \cdot) & \text{in } H^2(0, T) & \text{weakly}. \end{cases} \quad (3.44)$$

By the compactness lemma of Lions [5, p.57] and the imbeddings $H^1(0, T) \hookrightarrow C^0([0, T])$, $H^2(0, T) \hookrightarrow C^1([0, T])$, we can deduce from (3.44) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \rightarrow u & \text{strongly in } L^2(Q_T), \text{ and a.e. } (x, t) \in Q_T, \\ u'_m \rightarrow u' & \text{strongly in } L^2(Q_T), \text{ and a.e. } (x, t) \in Q_T, \\ u_m(0, \cdot) \rightarrow u(0, \cdot) & \text{strongly in } C^1([0, T]), \\ u_m(1, \cdot) \rightarrow u(1, \cdot) & \text{strongly in } C^1([0, T]). \end{cases} \quad (3.45)$$

Passing to the limit in (3.2) and (3.34) by (3.44)-(3.45) we have u satisfying the problem

$$\begin{aligned} & \langle u''(t), v \rangle + a(u(t), v) + \left(\lambda_0 u'(0, t) + \tilde{h}_1 u(1, t) + \tilde{\lambda}_1 u'(1, t) \right) v(0) \\ & + \left(\lambda_1 u'(1, t) + \tilde{h}_0 u(0, t) + \tilde{\lambda}_0 u'(0, t) \right) v(1) + \langle Ku(t) + \lambda u'(t), v \rangle \\ & = -g_0(t)v(0) - g_1(t)v(1) + \langle f(t), v \rangle, \quad \forall v \in H^1(\Omega), \end{aligned} \quad (3.46)$$

$$u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1. \quad (3.47)$$

On the other hand, it follows from (3.44)_{1,2,3} and (3.46), that

$$u_{xx} = u'' + Ku + \lambda u' - f \in L^\infty(0, T; L^2(\Omega)). \quad (3.48)$$

Thus, $u \in L^\infty(0, T; H^2(\Omega))$ and the existence of solution is proved completely.

Step 4. Uniqueness of the solution of problem (1.1) – (1.4) is similarly proved as in Theorem 3.1 and we will omit here. \square

Remark 3.3. Noting that with the regularity obtained by (3.32)-(3.33), it follows that the problem (1.1) – (1.4) has a unique strong solution u satisfying

$$\begin{cases} u \in C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \\ u_t \in L^\infty(0, T; H^1(\Omega)), u_{tt} \in L^\infty(0, T; L^2(\Omega)), \\ u(0, \cdot), u(1, \cdot) \in H^2(0, T). \end{cases}$$

4 The regularity of solutions

In this section, we study the regularity of solution of problem (1.1) – (1.4). For this purpose, we also assume that the constants $h_0, h_1, \lambda_0, \lambda_1, K, \lambda, \tilde{h}_0, \tilde{h}_1, \tilde{\lambda}_0, \tilde{\lambda}_1$ satisfy the conditions as in section 3. Furthermore, we will impose the following stronger assumptions, with $r \in \mathbb{N}$.

(A1) $\tilde{u}_0 \in H^{r+2}(\Omega)$ and $\tilde{u}_1 \in H^{r+1}(\Omega)$.

(A2) The function $f(x, t)$ satisfies

$$\frac{\partial^r f}{\partial x^j \partial t^{r-j}} \in L^\infty(0, T; L^2(\Omega)), \quad 0 \leq j \leq r,$$

$$\frac{\partial^\nu f}{\partial t^\nu} \in L^2(0, T; L^2(\Omega)), \quad 0 \leq \nu \leq r+1,$$

and

$$\frac{\partial^\mu f}{\partial t^\mu}(\cdot, 0) \in H^1(\Omega), \quad 0 \leq \mu \leq r-1.$$

(A3) $g_0, g_1 \in H^{r+1}(0, T)$, $r \geq 1$.

Formally differentiating problem (1.1)–(1.4) with respect to time up to order r and letting $u^{[r]} = \frac{\partial^r u}{\partial t^r}$ we are led to consider the solution $u^{[r]}$ of problem $(Q^{[r]})$:

$$\begin{cases} Lu^{[r]} = f^{[r]}(x, t), & (x, t) \in Q_T, \\ B_0 u^{[r]} = g_0^{[r]}(t), \quad B_1 u^{[r]} = g_1^{[r]}(t), \\ u^{[r]}(x, 0) = u_0^{[r]}(x), \quad u_t^{[r]}(x, 0) = u_1^{[r]}(x), \end{cases} \quad (Q^{[r]})$$

where

$$\begin{aligned} Lu^{[r]} &= u_{tt}^{[r]} - u_{xx}^{[r]} + Ku^{[r]} + \lambda u_t^{[r]}, \\ B_0 u^{[r]} &= u_x^{[r]}(0, t) - h_0 u^{[r]}(0, t) - \lambda_0 u_t^{[r]}(0, t) - \tilde{h}_1 u^{[r]}(1, t) - \tilde{\lambda}_1 u_t^{[r]}(1, t), \\ B_1 u^{[r]} &= -u_x^{[r]}(1, t) - h_1 u^{[r]}(1, t) - \lambda_1 u_t^{[r]}(1, t) - \tilde{h}_0 u^{[r]}(0, t) - \tilde{\lambda}_0 u_t^{[r]}(0, t), \end{aligned}$$

the functions $u_0^{[r]}$ and $u_1^{[r]}$ are defined by the recurrence formulas

$$\begin{aligned} u_0^{[0]} &= \tilde{u}_0, \quad u_0^{[r]} = u_1^{[r-1]}, \quad r \geq 1, \\ u_1^{[0]} &= \tilde{u}_1, \quad u_1^{[r]} = u_{0xx}^{[r-1]} - Ku_0^{[r-1]} - \lambda u_1^{[r-1]} + \frac{\partial^{r-1} f}{\partial t^{r-1}}(x, 0), \quad r \geq 1, \end{aligned}$$

and

$$f^{[r]} = \frac{\partial^r f}{\partial t^r}, \quad g_i^{[0]} = g_i, \quad g_i^{[r]} = \frac{d^r g_i}{dt^r}, \quad r \geq 1, \quad i = 0, 1.$$

From the assumptions (A1)–(A3) we deduce that $u_0^{[r]}, u_1^{[r]}, f^{[r]}, g_0^{[r]}$ and $g_1^{[r]}$ satisfy the conditions of Theorem 3.2. Hence, the problem $(Q^{[r]})$ has a unique weak solution $u^{[r]}$ such that

$$\begin{cases} u^{[r]} \in L^\infty(0, T; H^2(\Omega)) \cap C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), \\ u_t^{[r]} \in L^\infty(0, T; H^1(\Omega)), \quad u_{tt}^{[r]} \in L^\infty(0, T; L^2(\Omega)), \\ u^{[r]}(0, \cdot), \quad u^{[r]}(1, \cdot) \in H^2(0, T). \end{cases} \quad (4.1)$$

Moreover, from the uniqueness of weak solution we have $u^{[r]} = \frac{\partial^r u}{\partial t^r}$. Hence we deduce from (4.1) that the solution u of problem (1.1)–(1.4) satisfy

$$\begin{aligned} u &\in C^{r-1}(0, T; H^2(\Omega)) \cap C^r(0, T; H^1(\Omega)) \cap C^{r+1}(0, T; L^2(\Omega)), \\ \frac{\partial^r u}{\partial t^r} &\in L^\infty(0, T; H^2(\Omega)) \cap C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), \\ \frac{\partial^{r+1} u}{\partial t^{r+1}} &\in L^\infty(0, T; H^1(\Omega)), \\ \frac{\partial^{r+2} u}{\partial t^{r+2}} &\in L^\infty(0, T; L^2(\Omega)), \\ u(0, \cdot), \quad u(1, \cdot) &\in H^{r+2}(0, T). \end{aligned} \quad (4.2)$$

Next we shall prove by induction on r that

$$\frac{\partial^{r+2-j}u}{\partial t^{r+2-j}} \in L^\infty(0, T; H^j(\Omega)), \quad 0 \leq j \leq r+2. \quad (4.3)$$

With $r = 1$, it follows from (4.2) that

$$u' \in L^\infty(0, T; H^2(\Omega)), \quad u'' \in L^\infty(0, T; H^1(\Omega)), \quad u''' \in L^\infty(0, T; L^2(\Omega)). \quad (4.4)$$

On the other hand, from (1.1), (4.4) and the assumption (A2) we deduce that

$$u_{xxx} = u''_x + Ku_x + \lambda u'_x - f_x \in L^\infty(0, T; L^2(\Omega)).$$

Thus, $u \in L^\infty(0, T; H^3(\Omega))$ and (4.3) hold for $r = 1$. Suppose by induction that (4.3) holds for $r - 1$, i.e.,

$$\frac{\partial^{r+1-j}u}{\partial t^{r+1-j}} \in L^\infty(0, T; H^j(\Omega)), \quad 0 \leq j \leq r+1. \quad (4.5)$$

We shall prove that (4.3) holds. It follows from (4.2) that

$$\frac{\partial^{r+2-j}u}{\partial t^{r+2-j}} \in L^\infty(0, T; H^j(\Omega)), \quad \text{for } j = 0, 1, 2. \quad (4.6)$$

Let $j \in \{3, 4, \dots, r+2\}$ and put $\theta = r+2-j$. We have from (1.1)

$$\frac{\partial^{r+2}u}{\partial x^j \partial t^\theta} = \frac{\partial^{r+2}u}{\partial x^{j-2} \partial t^{\theta+2}} + K \frac{\partial^r u}{\partial x^{j-2} \partial t^\theta} + \lambda \frac{\partial^{r+1}u}{\partial x^{j-2} \partial t^{\theta+1}} - \frac{\partial^r f}{\partial x^{j-2} \partial t^\theta}. \quad (4.7)$$

On the other hand, it follows from (4.5) and the assumption (A2), that

$$\frac{\partial^\theta u}{\partial t^\theta} \in L^\infty(0, T; H^{j-1}(\Omega)), \quad \frac{\partial^{\theta+1}u}{\partial t^{\theta+1}}, \quad \frac{\partial^\theta f}{\partial t^\theta} \in L^\infty(0, T; H^{j-2}(\Omega)). \quad (4.8)$$

Combining (4.6), (4.7) and (4.8), by induction arguments on j , we conclude that (4.3) holds.

Hence we have the following theorem

Theorem 4.1. *Let (A1)-(A3) hold. Then the unique solution $u(x, t)$ of problem (1.1)-(1.4) satisfies (4.2) and (4.3). Furthermore*

$$u \in H^{r+2}(Q_T) \cap \left(\bigcap_{j=0}^{r+1} C^{r+1-j}(0, T; H^j(\Omega)) \right). \quad (4.9)$$

5 Exponential decay of solutions

In this section we assume that $K > 0$ and $\lambda > 0$. Let $u(x, t)$ be a strong solution of problem (1.1)-(1.4). In order to obtain the decay result, we use the functional

$$\Gamma(t) = E(t) + \delta \psi(t), \quad (5.1)$$

where δ is a positive constant and

$$E(t) = \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|u(t)\|_a^2 + \frac{K}{2} \|u(t)\|^2, \quad (5.2)$$

$$\psi(t) = \langle u(t), u'(t) \rangle + \frac{\lambda}{2} \|u(t)\|^2 + \frac{\lambda_0}{2} u^2(0, t) + \frac{\lambda_1}{2} u^2(1, t). \quad (5.3)$$

Lemma 5.1. *There exist the constants β_1, β_2 such that*

$$\beta_1 E(t) \leq \Gamma(t) \leq \beta_2 E(t), \quad (5.4)$$

when $\delta < \frac{C_0}{2}$.

Proof. By using Lemma 2.1, it's easy to obtain the following estimate

$$\Gamma(t) \leq \frac{1+\delta}{2} \|u'(t)\|^2 + \frac{1}{2} \left[1 + \frac{2\delta}{C_0} (1 + \lambda + \lambda_0 + \lambda_1) \right] \|u(t)\|_a^2 + \frac{K}{2} \|u(t)\|^2,$$

which implies that

$$\Gamma(t) \leq \beta_2 E(t),$$

where

$$\beta_2 = 1 + \frac{2\delta}{C_0} (1 + \lambda + \lambda_0 + \lambda_1).$$

Similar, we have

$$\Gamma(t) \geq \frac{1-\delta}{2} \|u'(t)\|^2 + \left(\frac{1}{2} - \frac{\delta}{C_0} \right) \|u(t)\|_a^2 + \frac{K}{2} \|u(t)\|^2.$$

Thus, if $\delta < \frac{C_0}{2}$ then $\Gamma(t) \geq \beta_1 E(t)$, where $\beta_1 = 1 - \frac{2\delta}{C_0} > 0$. Lemma 5.1 is proved. \square

Lemma 5.2. *The functional $E(t)$ defined by (5.2), satisfies*

$$\begin{aligned} E'(t) &\leq \left(\frac{\varepsilon_1}{2} - \lambda \right) \|u'(t)\|^2 + \left(\varepsilon_1 - \frac{\mu_{\min}}{2} \right) \left[|u'(0,t)|^2 + |u'(1,t)|^2 \right] \\ &\quad + \frac{1}{\varepsilon_1 C_0} \left(\tilde{h}_0^2 + \tilde{h}_1^2 \right) \|u(t)\|_a^2 + \frac{1}{2\varepsilon_1} \left[g_0^2(t) + g_1^2(t) + \|f(t)\|^2 \right], \end{aligned} \quad (5.5)$$

for all $\varepsilon_1 > 0$.

Proof. Multiplying (1.1) by $u'(x, t)$ and integrating over $[0, 1]$, we get

$$\begin{aligned} E'(t) &= -\lambda \|u'(t)\|^2 - \left\{ \lambda_0 |u'(0,t)|^2 + \lambda_1 |u'(1,t)|^2 \right. \\ &\quad \left. + \left(\tilde{\lambda}_0 + \tilde{\lambda}_1 \right) u'(0,t) u'(1,t) \right\} - \tilde{h}_0 u(0,t) u'(1,t) \\ &\quad - \tilde{h}_1 u(1,t) u'(0,t) - g_0(t) u'(0,t) - g_1(t) u'(1,t) + \langle f(t), u'(t) \rangle. \end{aligned} \quad (5.6)$$

By Lemma 1.3 we have

$$\begin{aligned} &\lambda_0 |u'(0,t)|^2 + \lambda_1 |u'(1,t)|^2 + \left(\tilde{\lambda}_0 + \tilde{\lambda}_1 \right) u'(0,t) u'(1,t) \\ &\geq \frac{\mu_{\min}}{2} \left[|u'(0,t)|^2 + |u'(1,t)|^2 \right]. \end{aligned} \quad (5.7)$$

It follows from (5.6) and (5.7) that

$$\begin{aligned} E'(t) &\leq -\lambda \|u'(t)\|^2 - \frac{\mu_{\min}}{2} \left[|u'(0,t)|^2 + |u'(1,t)|^2 \right] \\ &\quad - \tilde{h}_0 u(0,t) u'(1,t) - \tilde{h}_1 u(1,t) u'(0,t) \\ &\quad - g_0(t) u'(0,t) - g_1(t) u'(1,t) + \langle f(t), u'(t) \rangle. \end{aligned} \quad (5.8)$$

On the other hand, for $\varepsilon_1 > 0$,

$$\begin{aligned} -\tilde{h}_0 u(0, t) u'(1, t) &\leq \frac{\varepsilon_1}{2} |u'(1, t)|^2 + \frac{1}{2\varepsilon_1} \tilde{h}_0^2 u^2(0, t) \\ &\leq \frac{\varepsilon_1}{2} |u'(1, t)|^2 + \frac{1}{\varepsilon_1 C_0} \tilde{h}_0^2 \|u(t)\|_a^2, \end{aligned} \quad (5.9)$$

$$-\tilde{h}_1 u(1, t) u'(0, t) \leq \frac{\varepsilon_1}{2} |u'(0, t)|^2 + \frac{1}{\varepsilon_1 C_0} \tilde{h}_1^2 \|u(t)\|_a^2, \quad (5.10)$$

$$\begin{aligned} -g_0(t) u'(0, t) - g_1(t) u'(1, t) &\leq \frac{\varepsilon_1}{2} \left[|u'(0, t)|^2 + |u'(1, t)|^2 \right] \\ &\quad + \frac{1}{2\varepsilon_1} [g_0^2(t) + g_1^2(t)], \end{aligned} \quad (5.11)$$

$$\langle f(t), u'(t) \rangle \leq \|f(t)\| \|u'(t)\| \leq \frac{\varepsilon_1}{2} \|u'(t)\|^2 + \frac{1}{2\varepsilon_1} \|f(t)\|^2. \quad (5.12)$$

Combining (5.8) - (5.12), it is easy to see that (5.5) holds. The proof is complete. \square

Lemma 5.3. *The functional $\psi(t)$ defined by (5.3) satisfies*

$$\begin{aligned} \psi'(t) &\leq \|u'(t)\|^2 + \left(\frac{2}{C_0} |\tilde{h}_0 + \tilde{h}_1| + \frac{5\varepsilon_2}{C_0} - 1 \right) \|u(t)\|_a^2 \\ &\quad + \frac{1}{2\varepsilon_2} (\tilde{\lambda}_0^2 + \tilde{\lambda}_1^2) \left[|u'(0, t)|^2 + |u'(1, t)|^2 \right] \\ &\quad + \frac{1}{2\varepsilon_2} \left[\|f(t)\|^2 + g_0^2(t) + g_1^2(t) \right], \end{aligned} \quad (5.13)$$

for all $\varepsilon_2 > 0$.

Proof. Multiplying the equation (1.1) by $u(x, t)$ and integrating over $[0, 1]$, we have

$$\begin{aligned} \psi'(t) &= \|u'(t)\|^2 - \|u(t)\|_a^2 - K \|u(t)\|^2 - (\tilde{h}_0 + \tilde{h}_1) u(0, t) u(1, t) \\ &\quad - \tilde{\lambda}_0 u'(0, t) u(1, t) - \tilde{\lambda}_1 u(0, t) u'(1, t) \\ &\quad - g_0(t) u(0, t) - g_1(t) u(1, t) + \langle f(t), u(t) \rangle. \end{aligned} \quad (5.14)$$

By some estimations as in proof of Lemma 5.2, we deduce the conclusion of Lemma. \square

Theorem 5.4. *Assume that*

$$\sigma(t) \leq \sigma_1 \exp(-\sigma_2 t), \text{ for all } t \geq 0, \quad (5.15)$$

where σ_1, σ_2 are two positive constants and

$$\sigma(t) = \|f(t)\|^2 + g_0^2(t) + g_1^2(t).$$

Then, there exist positive constants γ_1, γ_2 such that

$$E(t) \leq \gamma_1 \exp(-\gamma_2 t), \text{ for all } t \geq 0, \quad (5.16)$$

for any strong solution of the problem (1.1)-(1.4), where \tilde{h}_0 and \tilde{h}_1 are chosen small enough.

Proof. It follows from (5.1), (5.5) and (5.13), that

$$\begin{aligned}
\Gamma'(t) &\leq \left(\delta + \frac{\varepsilon_1}{C_0} - \lambda \right) \|u'(t)\|^2 \\
&\quad + \left[\frac{1}{\varepsilon_1 C_0} (\tilde{h}_0^2 + \tilde{h}_1^2) + \delta \left(\frac{2}{C_0} |\tilde{h}_0 + \tilde{h}_1| + \frac{5\varepsilon_2}{C_0} - 1 \right) \right] \|u(t)\|_a^2 \\
&\quad + \left[\varepsilon_1 - \frac{\mu_{\min}}{2} + \frac{\delta}{2\varepsilon_2} (\tilde{\lambda}_0^2 + \tilde{\lambda}_1^2) \right] \left[|u'(0, t)|^2 + |u'(1, t)|^2 \right] \\
&\quad + \frac{1}{2} \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2} \right) \left[\|f(t)\|^2 + g_0^2(t) + g_1^2(t) \right],
\end{aligned} \tag{5.17}$$

for all $\varepsilon_1, \varepsilon_2 > 0$.

Let $\varepsilon_1 < \min\{C_0\lambda, \frac{\mu_{\min}}{2}\}$, $\varepsilon_2 < \frac{1}{5}C_0$ and

$$\delta < \min \left\{ \frac{C_0}{2}, \lambda - \frac{\varepsilon_1}{C_0}, \frac{2\varepsilon_2}{\tilde{\lambda}_0^2 + \tilde{\lambda}_1^2} \left(\frac{\mu_{\min}}{2} - \varepsilon_1 \right) \right\}.$$

Then, by choosing \tilde{h}_0, \tilde{h}_1 satisfy

$$\frac{1}{\varepsilon_1 C_0} (\tilde{h}_0^2 + \tilde{h}_1^2) + \frac{2\delta}{C_0} |\tilde{h}_0 + \tilde{h}_1| < \delta \left(1 - \frac{5\varepsilon_2}{C_0} \right),$$

we deduce from (5.4) and (5.17) that there exists a constant $\gamma < \sigma_2$ such that

$$\Gamma'(t) \leq -\delta\Gamma(t) + \frac{1}{2} \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2} \right) \sigma(t), \text{ for all } t \geq 0. \tag{5.18}$$

Combining (5.4), (5.15) and (5.18), we get (5.16). Theorem 5.4 is completely proved. \square

We can extend the above theorem to weak solutions by using density arguments.

6 Numerical results

Consider the following problem:

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x, t) \text{ in } (0, 1) \times (0, \infty), \tag{6.1}$$

$0 < x < 1, 0 < t < T$, with boundary conditions

$$\begin{aligned}
u_x(0, t) &= h_0 u(0, t) + \lambda_0 u_t(0, t) + \tilde{h}_1 u(1, t) + \tilde{\lambda}_1 u_t(1, t) + g_0(t), \\
-u_x(1, t) &= h_1 u(1, t) + \lambda_1 u_t(1, t) + \tilde{h}_0 u(0, t) + \tilde{\lambda}_0 u_t(0, t) + g_1(t),
\end{aligned} \tag{6.2}$$

and initial conditions

$$u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \tag{6.3}$$

where $K = \lambda = \lambda_0 = \lambda_1 = 1$, $\tilde{\lambda}_0 = \tilde{\lambda}_1 = \frac{-1}{2}$, $h_0 = h_1 = 1$, $\tilde{h}_0 = \frac{1}{2}$, $\tilde{h}_1 = \frac{-3}{2}$ are constants and the functions $\tilde{u}_0, \tilde{u}_1, g_0, g_1$ and f are defined by

$$u_0(x) = 1 + (x^2 - x)^2, \quad \tilde{u}_1(x) = -1 - (x^2 - x)^2, \tag{6.4}$$

$$g_0(t) = e^{-t}, \quad g_1(t) = -e^{-t}, \quad (6.5)$$

$$f(x, t) = (x^4 - 2x^3 - 11x^2 - 12x - 1) e^{-t}. \quad (6.6)$$

The exact solution of the problem (6.1) – (6.3) with $\tilde{u}_0, \tilde{u}_1, g_0, g_1$ and f defined in (6.4) – (6.6) respectively, is the function U_{ex} given by

$$U_{ex}(x, t) = (x^4 - 2x^3 + x^2 + 1) e^{-t}. \quad (6.7)$$

To solve problem (6.1) – (6.3) numerically, we consider the differential system for the unknowns $U_j(t) \equiv u(x_j, t)$, $V_j(t) = \frac{dU_j}{dt}(t)$, with $x_j = j\Delta x$, $\Delta x = \frac{1}{N}$, $j = 0, 1, \dots, N$:

$$\left\{ \begin{array}{l} \frac{dU_j}{dt}(t) = V_j(t), \quad j = \overline{0, N}, \\ \frac{dV_0}{dt}(t) = -\left(\frac{1+h_0\Delta x}{(\Delta x)^2} + K\right) U_0(t) + \frac{1}{(\Delta x)^2} U_1(t) - \frac{\tilde{h}_1}{\Delta x} U_N(t) \\ \quad -\left(\frac{\lambda_0}{\Delta x} + \lambda\right) V_0(t) - \frac{\tilde{\lambda}_1}{\Delta x} V_N(t) - \frac{1}{\Delta x} g_0(t) + f_0(t), \\ \frac{dV_j}{dt}(t) = \frac{U_{j-1}(t) - 2U_j(t) + U_{j+1}(t)}{(\Delta x)^2} - KU_j(t) - \lambda V_j(t) + f(x_j, t), \quad j = \overline{1, N-1}, \\ \frac{dV_N}{dt}(t) = -\frac{\tilde{h}_0}{\Delta x} U_0(t) + \frac{1}{(\Delta x)^2} U_{N-1}(t) - \left(\frac{1+h_1\Delta x}{(\Delta x)^2} + K\right) U_N(t) \\ \quad -\frac{\tilde{\lambda}_0}{\Delta x} V_0(t) - \left(\frac{\lambda_1}{\Delta x} + \lambda\right) V_N(t) - \frac{1}{\Delta x} g_1(t) + f_N(t), \\ U_j(0) = \tilde{u}_0(x_j), \quad V_j(0) = \tilde{u}_1(x_j), \quad j = \overline{0, N}. \end{array} \right. \quad (6.8)$$

Then system (6.8) is equivalent to:

$$\frac{d}{dt} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_N \\ V_0 \\ V_1 \\ \vdots \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} 0 & & & & 1 & & & & \\ 0 & 0 & & & & 1 & & & \\ & \ddots & \ddots & & & & \ddots & & \\ & & \ddots & \ddots & & & & \ddots & \\ & & & \ddots & \ddots & & & & \ddots \\ & & & & 0 & 0 & & & 1 \\ \hline \tilde{\gamma}_0 & \alpha & & & \tilde{\gamma}_N & & & & \\ \alpha & \gamma & \alpha & & & -\lambda & & & \\ & \ddots & \ddots & \ddots & & & \ddots & & \\ & & \alpha & \gamma & \alpha & & & -\lambda & \\ \hat{\gamma}_0 & & & \alpha & \hat{\gamma}_N & \hat{\delta}_0 & & & \hat{\delta}_N \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_N \\ V_0 \\ V_1 \\ \vdots \\ \vdots \\ V_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ F_0 \\ F_1 \\ \vdots \\ \vdots \\ F_N \end{bmatrix} \quad (6.9)$$

where

$$\left\{ \begin{array}{l} \alpha = \frac{1}{(\Delta x)^2}, \quad \gamma = -K - \frac{2}{(\Delta x)^2} = -K - 2\alpha, \\ \tilde{\gamma}_0 = -\left(K + \frac{1+h_0\Delta x}{(\Delta x)^2}\right), \quad \tilde{\gamma}_N = -\frac{\tilde{h}_1}{\Delta x}, \quad \tilde{\delta}_0 = -(\frac{\lambda_0}{\Delta x} + \lambda), \quad \tilde{\delta}_N = -\frac{\tilde{\lambda}_1}{\Delta x}, \\ \hat{\gamma}_0 = -\frac{\tilde{h}_0}{\Delta x}, \quad \hat{\gamma}_N = -\left(K + \frac{1+h_1\Delta x}{(\Delta x)^2}\right), \quad \hat{\delta}_0 = -\frac{\tilde{\lambda}_0}{\Delta x}, \quad \hat{\delta}_N = -(\frac{\lambda_1}{\Delta x} + \lambda). \\ F_j = F_j(t) = f_j(t) = f(x_j, t), \quad j = \overline{1, N-1}, \\ F_0 = F_0(t) = -\frac{1}{\Delta x}g_0(t) + f_0(t), \\ F_N = F_N(t) = -\frac{1}{\Delta x}g_1(t) + f_N(t), \end{array} \right. \quad (6.10)$$

Rewritten (6.9)

$$\frac{d}{dt}X(t) = AX(t) + F(t), \quad (6.11)$$

$$\left\{ \begin{array}{l} X(t) = (U_0(t), U_1(t), \dots, U_N(t), V_0(t), V_1(t), \dots, V_N(t))^T \in \mathbb{R}^{2N+2}, \\ F(t) = (0, 0, \dots, 0, F_0, F_1, \dots, F_N)^T \in \mathbb{R}^{2N+2}, \\ A = \begin{bmatrix} O & E \\ \tilde{A} & \tilde{B} \end{bmatrix}, \end{array} \right. \quad (6.12)$$

$$E = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{\gamma}_0 & \alpha & & & \tilde{\gamma}_N \\ \alpha & \gamma & \alpha & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & \gamma & \alpha \\ \hat{\gamma}_0 & & & \alpha & \hat{\gamma}_N \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{\delta}_0 & & & & \tilde{\delta}_N \\ & -\lambda & & & \\ & & \ddots & & \\ & & & -\lambda & \\ \hat{\delta}_0 & & & & \hat{\delta}_N \end{bmatrix} \quad (6.13)$$

To solve the linear differential system (6.11), we use a spectral method with a time step $\Delta t = 0.08$ and a spacial step $\Delta x = 0.1$

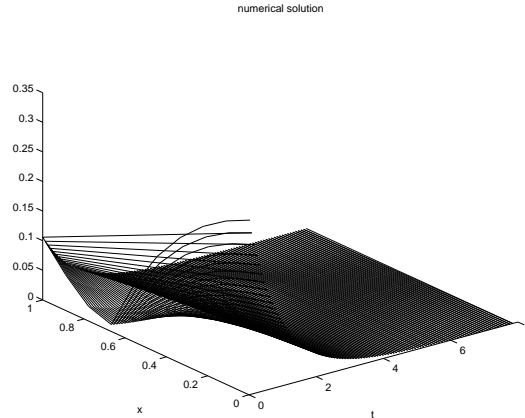


Figure 1: Numerical solution

In fig.1 we have drawn the approximated solution of the problem 6.1-6.3 while fig.2 represents his corresponding exact solution 6.7.

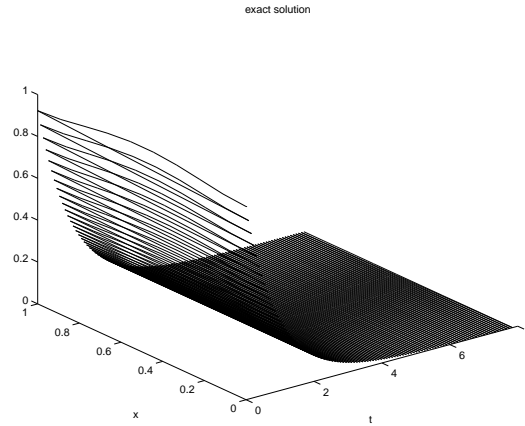


Figure 2: Exact solution

The fig.3 corresponds to the surface $(x, t) \mapsto u(x, t)$ approximated solution in the case where $f(x, t) = 0$. So in both cases we notice the very good decay of these surfaces from $T = 0$ to $T = 8$.

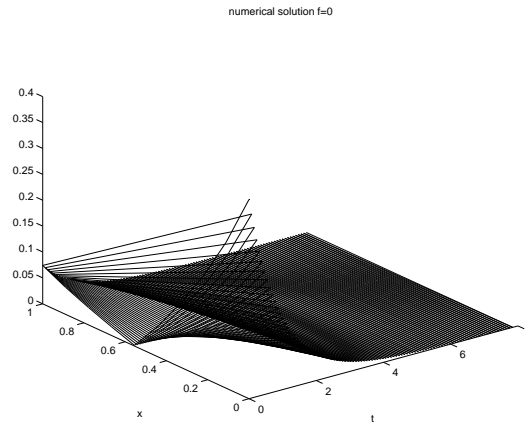


Figure 3: case f=0

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