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Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods *

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Abstract

We derive in this paper a unified framework for a priori and a posteriori error analysis of mixed finite element discretizations of second-order elliptic problems. It is based on the classical primal weak formulation, the postprocessing of the potential proposed in [T. Arbogast and Z. Chen, On the implementation of mixed methods as nonconforming methods for second-order elliptic problems, *Math. Comp.* **64** (1995), 943–972], and the discrete Friedrichs inequality. Our analysis in particular avoids any notion of the (discrete) inf–sup condition and in a straightforward manner and under minimal necessary assumptions, all known convergence and superconvergence results are recovered. The same framework then turns out to lead to optimal a posteriori error bounds. In particular, estimators for all families and orders of mixed finite element methods on grids consisting of simplices or rectangular parallelepipeds are derived. They give a guaranteed and fully computable upper bound on the energy error, represent error local lower bounds, and are robust under some conditions on the diffusion–dispersion tensors. They are thus suitable for both overall error control and adaptive mesh refinement. Moreover, the developed abstract framework and a posteriori error estimates are very general and apply to any locally conservative method. We finally prove that in parallel and simultaneously in converse to Galerkin finite element methods, under some circumstances, the weak solution is the orthogonal projection of the postprocessed mixed finite element approximation onto the $H_0^1(\Omega)$ space and also establish several links between mixed finite element approximations and some generalized weak solutions.

Key words: mixed finite element methods, a priori error estimates, inf–sup condition, postprocessing, discrete Friedrichs inequality, locally conservative methods, a posteriori error estimates, guaranteed upper bound, orthogonal projection, generalized weak solution

1 Introduction

We consider in this paper the model problem

$$-\nabla \cdot (\mathbf{S}\nabla p) = f \quad \text{in } \Omega, \quad (1.1a)$$

$$p = 0 \quad \text{on } \partial\Omega, \quad (1.1b)$$

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where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal (polyhedral) domain (open, bounded, and connected set), \mathbf{S} is a symmetric, bounded, and uniformly positive definite tensor, and $f \in L^2(\Omega)$. The classical primal weak formulation consists in finding $p \in H_0^1(\Omega)$ such that

$$(\mathbf{S}\nabla p, \nabla\varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \quad (1.2)$$

(see Section 2.1 below for the details on the notation). The problem (1.1a)–(1.1b) can be equivalently written as the first-order system

$$\mathbf{u} = -\mathbf{S}\nabla p \quad \text{in } \Omega, \quad (1.3a)$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \quad (1.3b)$$

$$p = 0 \quad \text{on } \partial\Omega, \quad (1.3c)$$

which leads to the weak mixed formulation, consisting in finding $\mathbf{u} \in \mathbf{H}(\text{div}, \Omega)$ and $p \in L^2(\Omega)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\text{div}, \Omega), \quad (1.4a)$$

$$(\nabla \cdot \mathbf{u}, \phi) = (f, \phi) \quad \forall \phi \in L^2(\Omega). \quad (1.4b)$$

Note that this formulation is equivalent to (1.2) in the sense that $p = p$ and $\mathbf{u} = -\mathbf{S}\nabla p$, which is straightforward to show, cf. Quarteroni and Valli [49, Section 7.1]. In particular, this equivalence is sufficient to conclude the well-posedness of (1.4a)–(1.4b), it is not necessary to resort to the saddle-point mixed theory, presented, e.g., in Brezzi and Fortin [20].

We are interested in mixed finite element approximations to (1.4a)–(1.4b), which consist in finding $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in \Phi_h$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (1.5a)$$

$$(\nabla \cdot \mathbf{u}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi_h. \quad (1.5b)$$

Here $\Phi_h \subset L^2(\Omega)$ and $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$ are some of the usual finite-dimensional spaces defined on a mesh \mathcal{T}_h of simplices or rectangular parallelepipeds, see Section 4.1 below and Brezzi and Fortin [20] or Roberts and Thomas [53]. The main purposes of this paper are the following: i) present a unified framework for both a priori and a posteriori error analysis of mixed finite element methods; ii) base this framework entirely on the primal weak formulation (1.2) (and its above-cited direct equivalence with (1.4a)–(1.4b)) on the continuous level and on postprocessing and the discrete Friedrichs inequality on the discrete level; in particular, the notion of the inf–sup and the discrete inf–sup condition is completely avoided; iii) arrive at optimal a priori estimates (under minimal necessary assumptions); iv) present new (and optimal) a posteriori error estimates; v) obtain these results with as simple as possible proofs; vi) present some new (to the best of the author’s knowledge) properties of the mixed finite element methods.

A priori error estimates for mixed finite element methods are usually obtained by means of the saddle-point theory of Brezzi [17] and Babuška [10]. Traditionally, the natural norms of the spaces $\mathbf{H}(\text{div}, \Omega)$ and $L^2(\Omega)$ are used, but mesh-dependent norms can be employed instead, cf. Babuška et al. [11]. Postprocessing of p_h into \tilde{p}_h is then usually used for the double purpose of giving an improved approximation to p and facilitating the implementation of mixed methods, cf. Arnold and Brezzi [9], Bramble and Xu [16], Stenberg [54], Chen [25], and Arbogast and Chen [8]. In combination with mesh-dependent norms, it has also previously been used in order to obtain error estimates in, e.g., Lovadina and Stenberg [42], see also the references therein. Some other results are presented by Marini and Pietra [44] and in [25] and [8]. Nice links between the mixed

finite element and nonconforming finite element methods are then in particular given in [9, 25, 8], Marini [43], Chen [26], or [35, 62]. Recently, Cockburn and Gopalakrishnan [28, 29] showed that analysis of mixed methods can entirely be based on the hybridization (cf. Section 4.3 below) and lifting operators and demonstrated interesting relations between the different mixed methods. Let us also mention that very tight links between mixed finite element and finite volume methods exists, see Younès et al. [65] and [57] and the references therein.

A posteriori error estimates for mixed finite element methods were started in the works of Alonso [7], Braess and Verfürth [15], Carstensen [23], Hoppe and Wohlmuth [37], Achchab et al. [2], Wohlmuth and Hoppe [64], Carstensen and Bartels [24], Kirby [40], El Alaoui and Ern [32], Wheeler and Yotov [63], and Lovadina and Stenberg [42]. For some discussion of these results, we refer to [59]. Recently, new works appeared. Repin and Smolianski [52] are able to give a guaranteed upper bound, which may however not be sufficiently precise for inhomogeneous \mathbf{S} and general domains and boundary conditions. Moreover, no local efficiency is shown. Nicaise and Creusé [46] improve the results of [23] and extend them to the anisotropic case. Kim [39] presents estimates applicable to any locally conservative method, as is the case of the estimates presented here. Bounds up to an undetermined constant are given in a mesh-dependent norm, which contains a weighted jump term for the potential. The enrichment of the natural energy norm by these jump terms on the one hand facilitates the analysis, but on the other one, is questionable in particular for inhomogeneous and anisotropic tensors \mathbf{S} and not necessary. The results of Repin et al. [51] are only valid under the hypothesis that $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ and $p_h \in H_0^1(\Omega)$, which is not the case of (1.5a)–(1.5b) (see also Section 6.4.2 below for further remarks on this point). Larson and Målqvist [41] give energy norm error estimates for the flux. The upper bound again features an unknown constant and, moreover, no local efficiency is proved. Finally, optimal a posteriori error estimates in the lowest-order Raviart–Thomas–Nédélec case were first presented in [59, 58] and then also by Ainsworth [6].

We first in Section 3 of this paper, after collecting some preliminaries in Section 2, give an abstract estimate on the energy norm of the difference between two arbitrary vector fields. This estimate will then be used in order to obtain both a priori and a posteriori estimates on the error in the approximation of \mathbf{u} in a straightforward way. In section 4 we then recall and some basic facts about mixed finite element methods and in particular the postprocessing of [8] and, for the lowest-order Raviart–Thomas–Nédélec case, that of [59]. This postprocessing is then the basis for optimal a priori and a posteriori error estimates on the error in the approximation of p .

We carry out the a priori error analysis in Section 5. We highlight here its main ideas for the case $\mathbf{S} = \mathbb{I}$ (\mathbb{I} denotes the identity matrix). Typically, one has $\mathbf{V}_h \cdot \mathbf{n}|_{\mathcal{E}_h} = \mathbb{P}_k(\mathcal{E}_h)$ in mixed finite element methods, where \mathcal{E}_h is the set of sides (edges if $d = 2$ and faces if $d = 3$). Our main assumption is that there exists a space M_h such that i) M_h is continuous enough in the sense that it is contained in the space of functions such that the jumps of their traces are orthogonal to the polynomials from $\mathbb{P}_k(\mathcal{E}_h)$; ii) M_h is large enough in the sense that it contains the space of piecewise polynomials of order $(k + 1)$; iii) for all $K \in \mathcal{T}_h$ and all $\xi_h \in M_h(K)$, $(\nabla \xi_h, \mathbf{v}_h)_K = 0$ for all $\mathbf{v}_h \in \mathbf{V}_h(K)$ implies $\nabla \xi_h = 0$. Finally, we suppose that one can construct a postprocessed potential $\tilde{p}_h \in M_h$ such that the $\mathbf{L}^2(\Omega)$ -orthogonal projection of $-\nabla \tilde{p}_h$ onto $\prod_{K \in \mathcal{T}_h} \mathbf{V}_h(K)$ is \mathbf{u}_h . This is the situation of the postprocessing of [8]. Recalling that moreover the $L^2(\Omega)$ -orthogonal projection of $\nabla \cdot \mathbf{u}_h$ onto Φ_h equals that of f by (1.5b), we note that this fully mimics the continuous setting where $\mathbf{u} \in \mathbf{H}(\text{div}, \Omega)$, $p \in H_0^1(\Omega)$, and (1.3a)–(1.3b) holds true. Now proving the equivalence between the energy seminorms on $M_h(K)$ and the $\mathbf{L}^2(K)$ -orthogonal projection of $-\nabla M_h(K)$ onto $\mathbf{V}_h(K)$ for each element K enables us to relate the energy error in $p - \tilde{p}_h$ to the one in $\mathbf{u} - \mathbf{u}_h$, easily obtained itself from the above-mentioned abstract estimate for vector functions. $L^2(\Omega)$ estimates then follow by the discrete Friedrichs inequality. We also show that using the postprocessing

of [59] in the lowest-order Raviart–Thomas–Nédélec case, much of the above can be avoided and one obtains the estimates for $p - \tilde{p}_h$ in an extremely simple way. Finally, by construction, p_h is the $L^2(\Omega)$ -orthogonal projection of \tilde{p}_h onto Φ_h , so that the estimates for the error in $p - p_h$ are easily recovered. The analysis still relies on the appropriate vector interpolation operator of each mixed finite element method, satisfying the commuting diagram property, see [20, Section III.3]. On the other hand, the inf–sup condition is completely avoided by the postprocessing and the discrete Friedrichs inequality.

In Section 6, we extend the a posteriori error estimates for the lowest-order Raviart–Thomas–Nédélec case of [59] to other families of mixed finite elements, all orders, and grids consisting of rectangular parallelepipeds. Using the techniques that go back to the Prager–Synge equality [48], we present here estimates which give a guaranteed upper bound (contain no undetermined constant), are locally efficient (represent a local lower bound for the actual error), and are easily computable. Moreover, numerical experiments show that they give effectivity indices (overestimation factors) very close to the optimal value of one. Finally, they are robust with respect to the weak solution regularity, and inhomogeneities in \mathbf{S} under the “monotonicity” assumption (cf. Bernardi and Verfürth [14, Hypothesis 2.7]). A fully optimal estimate can be easily obtained in our setting for an approximation couple $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ and $s_h \in H_0^1(\Omega)$, as we show in Section 6.4.2. Using the abstract framework for the error between two arbitrary vector fields of Section 3, we first give estimates for the energy error in the approximation of \mathbf{u} . It consists of two parts. The first one is generally given by $\inf_{s \in H_0^1(\Omega)} \|\mathbf{u}_h + \mathbf{S}\nabla s\|_*$, expressing the measure of how close \mathbf{u}_h is to a gradient of a $H_0^1(\Omega)$ -potential in the vector energy norm $\|\cdot\|_*$. In practice, the indicator of an element K is given by $\|\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\text{Os}}(\tilde{p}_h))\|_{*,K}$, where \mathcal{I}_{Os} is the Oswald averaging operator. The second one is the residual term (sometimes considered separately and call “data oscillation term”), given by $C_P^{1/2} h_K c_{\mathbf{S},K}^{-1/2} \|f - P_{\Phi_h}(f)\|_K$, where h_K is the diameter of K , $c_{\mathbf{S},K}$ is the smallest eigenvalue of \mathbf{S} on K , $C_P = 1/\pi^2$ is the constant from the Poincaré inequality, P_{Φ_h} is the $L^2(\Omega)$ -orthogonal projection onto Φ_h , and $\|\cdot\|$ is the L^2 norm. Such an estimator in particular improves on estimators of the type $h_K \|\mathbf{u}_h + \mathbf{S}\nabla p_h\|_K$, found in many of the above-cited works. Remark that the latter one in particular reduces to $h_K \|\mathbf{u}_h\|_K$ in low order mixed finite element methods, i.e., the weighted $L^2(\Omega)$ -norm of the approximate flux, which reflects no approximation. Next, using the framework introduced in [59] and [39], we give estimates for the energy error in the approximation of p .

The a posteriori error estimates developed in this paper are very general and apply directly to any locally conservative method, such as the finite volume one, cf. Eymard et al. [34], Aavatsmark et al. [1], or Droniou and Eymard [31], mimetic finite difference, cf. Brezzi et al. [21], covolume, cf. Chou et al. [27], and other. For related results, we refer to [61]. They are given for a general diffusion tensor, require no additional regularity of the weak solution, no saturation assumption, and no use of the Helmholtz decomposition. They allow for grids consisting of rectangular parallelepipeds, which can be very useful in practice, where such grids are extensively used. Combinations of simplices and rectangular parallelepipeds in one grid and extensions to nonmatching grids could also be considered along the lines of the analysis in [61] and [33]. Homogeneous Dirichlet boundary conditions are only considered for the simplicity of the exposition; for inhomogeneous Dirichlet/Neumann boundary conditions, we refer, e.g., to [39, 61].

Finally, in Section 7, we give some complements on mixed finite element methods. In particular, we show that under certain conditions, the weak solution p is the orthogonal projection of the postprocessed mixed finite element approximation \tilde{p}_h onto the $H_0^1(\Omega)$ space. This stands in parallel and simultaneously in converse to Galerkin finite element methods, where the approximate solution is the orthogonal projection of the weak solution onto the discrete space. We also show that mixed finite element approximations have close relations to some generalized weak solutions, independently of the smoothness of the tensor \mathbf{S} .

2 Preliminaries

We set up in this section the notation for meshes and functional spaces used throughout the paper, define scalar- and vector-valued bilinear forms and energy (semi-)norms, and finally recall the Oswald interpolation operator.

2.1 Notation

We shall work in this paper with triangulations \mathcal{T}_h which for all $h > 0$ consist either of closed simplices or of closed rectangular parallelepipeds K such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. We suppose that \mathcal{T}_h are conforming (matching), i.e., such that if $K, L \in \mathcal{T}_h$, $K \neq L$, then $K \cap L$ is either an empty set or a common face, edge, or vertex of K and L . Let h_K denote the diameter of K and let $h := \max_{K \in \mathcal{T}_h} h_K$. We denote by \mathcal{E}_h the set of all sides of \mathcal{T}_h , by $\mathcal{E}_h^{\text{int}}$ the set of interior, by $\mathcal{E}_h^{\text{ext}}$ the set of exterior, and by \mathcal{E}_K the set of all the sides of an element $K \in \mathcal{T}_h$; h_σ then stands for the diameter of $\sigma \in \mathcal{E}_h$. We will also use the notation \mathcal{T}_K ($\tilde{\mathcal{E}}_K$, respectively) for such $L \in \mathcal{T}_h$ ($\sigma \in \mathcal{E}_h$) which share at least a vertex with a $K \in \mathcal{T}_h$. Similarly, \mathcal{T}_V is the set of such $K \in \mathcal{T}_h$ that contain a node V . Later on, we will sometimes need the assumption that \mathcal{T}_h are shape-regular in the sense that there exists a constant $\kappa_{\mathcal{T}} > 0$ such that $\min_{K \in \mathcal{T}_h} \kappa_K \geq \kappa_{\mathcal{T}}$ for all $h > 0$, where $\kappa_K := |K|/h_K^d$.

Next, for $K \in \mathcal{T}_h$, \mathbf{n} will always denote its exterior normal vector; we shall also employ the notation \mathbf{n}_σ for a normal vector of a side $\sigma \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for interior sides and coinciding with the exterior normal of Ω for exterior sides. For $\sigma \in \mathcal{E}_h^{\text{int}}$ shared by $K, L \in \mathcal{T}_h$ such that \mathbf{n}_σ points from K to L and a function $\varphi \in H^1(\mathcal{T}_h)$ (see below for the notation), we shall define the jump operator $[[\cdot]]$ by

$$[[\varphi]] := (\varphi|_K)|_\sigma - (\varphi|_L)|_\sigma.$$

We put $[[\varphi]]_\sigma := \varphi|_\sigma$ for any $\sigma \in \mathcal{E}_h^{\text{ext}}$.

For a given domain $S \subset \mathbb{R}^d$, we shall hereafter employ the standard functional notations $L^2(S)$, $H^q(S)$, $H_0^1(S)$, cf. [4]. In particular, we note by $(\cdot, \cdot)_S$ the $L^2(S)$ inner product, by $\|\cdot\|_S$ the associated norm (we omit the index S when $S = \Omega$), and by $|S|$ the Lebesgue measure of S . Let next $\mathbf{H}(\text{div}, S) = \{\mathbf{v} \in \mathbf{L}^2(S); \nabla \cdot \mathbf{v} \in L^2(S)\}$ and let $\langle \cdot, \cdot \rangle_{\partial S}$ stand for the $(d-1)$ -dimensional $L^2(\partial S)$ -inner product on ∂S or the appropriate duality pairing on ∂S . We will also need the space $\tilde{\mathbf{H}}(\text{div}, S) = \{\mathbf{v} \in \mathbf{L}^q(S); \nabla \cdot \mathbf{v} \in L^2(S)\}$, $q \geq 2$, cf. [20, Section III.3.3]. For a given partition \mathcal{T}_h of Ω , let $H^1(\mathcal{T}_h) := \{\varphi \in L^2(\Omega); \varphi|_K \in H^1(K) \forall K \in \mathcal{T}_h\}$ be the broken Sobolev space. Also, we let $W_0(\mathcal{T}_h)$ and $W_h(\mathcal{T}_h)$ be the spaces of functions with jumps of traces across the sides orthogonal to, respectively, constants and polynomials of $\mathbf{V}_h \cdot \mathbf{n}|_\sigma$ for each $\sigma \in \mathcal{E}_h$,

$$W_0(\mathcal{T}_h) := \{\varphi \in H^1(\mathcal{T}_h); \langle [[\varphi]], 1 \rangle_\sigma = 0 \quad \forall \sigma \in \mathcal{E}_h\}, \quad (2.1a)$$

$$W_h(\mathcal{T}_h) := \{\varphi \in H^1(\mathcal{T}_h); \langle [[\varphi]], \psi_h \rangle_\sigma = 0 \quad \forall \psi_h \in \mathbf{V}_h \cdot \mathbf{n}|_\sigma \quad \forall \sigma \in \mathcal{E}_h\}. \quad (2.1b)$$

Clearly, $W_0(\mathcal{T}_h), W_h(\mathcal{T}_h) \not\subset H_0^1(\Omega)$ but there is “less and less nonconformity” in $W_h(\mathcal{T}_h)$ with increasing order of the method. Finally, the weak gradient on $H^1(\Omega)$ and the piecewise weak gradient on $H^1(\mathcal{T}_h)$ are both denoted by the ∇ sign and similarly for the weak divergence $\nabla \cdot$. To simplify the notation, we systematically use the convention $0/0 = 0$ throughout the text.

Finally, we denote by $c_{\mathbf{S}, \Omega}, C_{\mathbf{S}, \Omega}$ the best constants such that $c_{\mathbf{S}, \Omega} \mathbf{v} \cdot \mathbf{v} \leq \mathbf{S} \mathbf{v} \cdot \mathbf{v} \leq C_{\mathbf{S}, \Omega} \mathbf{v} \cdot \mathbf{v}$, $c_{\mathbf{S}, \Omega} > 0$, $C_{\mathbf{S}, \Omega} > 0$, for all $\mathbf{v} \in \mathbb{R}^d$ and a.e. in Ω . Similar notations $c_{\mathbf{S}, K}$, $C_{\mathbf{S}, K}$, and $c_{\mathbf{S}, \mathcal{T}_K}$ for $K \in \mathcal{T}_h$ will also be employed.

2.2 Bilinear forms and energy (semi-)norms

Let the symmetric bilinear form \mathcal{B} acting on scalars be defined by

$$\mathcal{B}(p, \varphi) := (\mathbf{S}\nabla p, \nabla \varphi), \quad p, \varphi \in H^1(\mathcal{T}_h), \quad (2.2)$$

whereas its vector counterpart \mathcal{A} acting on vectors by

$$\mathcal{A}(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{S}^{-1}\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega). \quad (2.3)$$

Note that the primal weak formulation (1.2) can be rewritten equivalently using the above forms \mathcal{B} and \mathcal{A} as: find $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \quad (2.4)$$

or

$$\mathcal{A}(\mathbf{S}\nabla p, \mathbf{S}\nabla \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega), \quad (2.5)$$

as

$$\mathcal{B}(p, \varphi) = \mathcal{A}(\mathbf{S}\nabla p, \mathbf{S}\nabla \varphi) \quad \forall p, \varphi \in H^1(\mathcal{T}_h), \quad (2.6)$$

which will turn out to be useful later. Let us also define the energy seminorm on the space $H^1(\mathcal{T}_h)$

$$\|\varphi\|^2 := \mathcal{B}(\varphi, \varphi) = \|\mathbf{S}^{\frac{1}{2}}\nabla \varphi\|^2, \quad \varphi \in H^1(\mathcal{T}_h), \quad (2.7)$$

which becomes a norm on $W_0(\mathcal{T}_h)$ thanks to the discrete Friedrichs inequality

$$\|\varphi\|_{\Omega} \leq C_{\text{DF}}^{\frac{1}{2}} \|\nabla \varphi\| \quad \forall \varphi \in W_0(\mathcal{T}_h), \quad \forall h > 0, \quad (2.8)$$

where C_{DF} only depends on $\kappa_{\mathcal{T}}$ and $\inf_{\mathbf{b} \in \mathbb{R}^d} \{\text{thick}_{\mathbf{b}}(\Omega)\}$, cf. [56, Theorem 5.4]. Similarly, let the energy norm for vectors be given by

$$\|\mathbf{v}\|_*^2 := \mathcal{A}(\mathbf{v}, \mathbf{v}) = \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|^2, \quad \mathbf{v} \in \mathbf{L}^2(\Omega). \quad (2.9)$$

Note in particular that by (2.6),

$$\|\varphi\| = \|\mathbf{S}\nabla \varphi\|_* \quad \forall \varphi \in H^1(\mathcal{T}_h). \quad (2.10)$$

By the Cauchy–Schwarz inequality, one also immediately has

$$\mathcal{B}(p, \varphi) \leq \|p\| \|\varphi\| \quad \forall p, \varphi \in H^1(\mathcal{T}_h), \quad (2.11a)$$

$$\mathcal{A}(\mathbf{u}, \mathbf{v}) \leq \|\mathbf{u}\|_* \|\mathbf{v}\|_* \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega). \quad (2.11b)$$

We will also use the “div–energy” norm for vectors, defined as

$$\|\mathbf{v}\|_{*, \text{div}}^2 := \|\mathbf{v}\|_*^2 + \|\nabla \cdot \mathbf{v}\|^2, \quad \mathbf{v} \in \mathbf{H}(\text{div}, \Omega). \quad (2.12)$$

Let us finally recall that, for $K \in \mathcal{T}_h$, the Poincaré inequality states that

$$\|\varphi - \pi_0(\varphi)\|_K^2 \leq C_{\text{P}} h_K^2 \|\nabla \varphi\|_K^2 \quad \forall \varphi \in H^1(K), \quad (2.13)$$

where π_l denotes the $L^2(\Omega)$ -orthogonal projection onto piecewise polynomials of degree l . Thanks to the convexity of simplices and rectangular parallelepipeds, $C_{\text{P}} = 1/\pi^2$, cf. [47, 13].

2.3 Oswald interpolation operator

We shall work later with piecewise polynomial approximations \tilde{p}_h to p , nonconforming in the sense that $\tilde{p}_h \notin H_0^1(\Omega)$ but satisfying $\tilde{p}_h \in W_h(\mathcal{T}_h)$ ($\tilde{p}_h \in H^1(\mathcal{T}_h)$ in general). It will also turn out that we will need their conforming (continuous, contained in $H_0^1(\Omega)$) interpolant. We will use for this purpose the Oswald one, previously considered, e.g., in [3, 38, 33] and analyzed in detail in [38, 22].

If \mathcal{T}_h consist of simplices, let $\mathbb{R}_n(\mathcal{T}_h) := \mathbb{P}_n(\mathcal{T}_h)$ denote the space of piecewise polynomials of total degree at most n on each simplex (without any continuity requirement on the sides). Similarly, if \mathcal{T}_h consist of rectangular parallelepipeds, let $\mathbb{R}_n(\mathcal{T}_h) := \mathbb{Q}_n(\mathcal{T}_h)$ denote the space of piecewise polynomials of degree at most n in each variable. The Oswald interpolation operator $\mathcal{I}_{\text{Os}} : \mathbb{R}_n(\mathcal{T}_h) \rightarrow \mathbb{R}_n(\mathcal{T}_h) \cap H_0^1(\Omega)$ is defined as follows: given a function $\varphi_h \in \mathbb{R}_n(\mathcal{T}_h)$, the value of $\mathcal{I}_{\text{Os}}(\varphi_h)$ is prescribed at the Gauss–Lobatto nodes on rectangular parallelepipeds and Lagrangian nodes on simplices of $\mathbb{R}_n(\mathcal{T}_h) \cap H_0^1(\Omega)$ by the average of the values of φ_h at this node,

$$\mathcal{I}_{\text{Os}}(\varphi_h)(V) = \frac{1}{|\mathcal{T}_V|} \sum_{K \in \mathcal{T}_V} \varphi_h|_K(V).$$

Note that the interpolant is in particular equal to $\varphi_h|_K(V)$ at a node V lying in the interior of some $K \in \mathcal{T}_h$, while at boundary nodes, the value of $\mathcal{I}_{\text{Os}}(\varphi_h)$ is set to zero. The following results have been proved in [22, Lemmas 3.2 and 5.3 and Remark 3.2] and [38, Theorem 2.2]:

Lemma 2.1 (Oswald interpolation operator). *Let \mathcal{T}_h be shape-regular, let $\varphi_h \in \mathbb{R}_n(\mathcal{T}_h)$, and let $\mathcal{I}_{\text{Os}}(\varphi_h)$ be constructed as described above. Then*

$$\|\nabla(\varphi_h - \mathcal{I}_{\text{Os}}(\varphi_h))\|_K^2 \leq C \sum_{\sigma \in \tilde{\mathcal{E}}_K} h_\sigma^{-1} \|[\varphi_h]\|_\sigma^2,$$

where the constant C depends only on the space dimension d , on the maximal polynomial degree n , and on the shape regularity parameter $\kappa_{\mathcal{T}}$.

3 Abstract framework

We develop in the first part of this section an abstract estimate on the energy norm of the difference between two arbitrary vector fields which will enable us to easily carry out both the a priori and a posteriori error analysis of mixed finite element methods in a unified way. In the second part of this section, we give a slightly improved version of the estimate, suitable for a posteriori error estimation.

3.1 A general abstract estimate

Following the approach introduced in [59, Lemma 7.1], we have the following abstract result:

Theorem 3.1 (General abstract estimate). *Let $\mathbf{v}, \mathbf{w}, \mathbf{t} \in \mathbf{L}^2(\Omega)$ be arbitrary. Then*

$$\|\|\mathbf{v} - \mathbf{w}\|\|_* \leq \|\|\mathbf{w} - \mathbf{t}\|\|_* + \left| \mathcal{A} \left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{\|\|\mathbf{v} - \mathbf{t}\|\|_*} \right) \right|.$$

Proof. Let us first suppose that $\|\|\mathbf{v} - \mathbf{w}\|\|_* \leq \|\|\mathbf{v} - \mathbf{t}\|\|_*$. We then have

$$\begin{aligned} \|\|\mathbf{v} - \mathbf{t}\|\|_*^2 &= \mathcal{A}(\mathbf{v} - \mathbf{t}, \mathbf{v} - \mathbf{t}) = \mathcal{A}(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{t}) + \mathcal{A}(\mathbf{w} - \mathbf{t}, \mathbf{v} - \mathbf{t}) \\ &\leq \|\|\mathbf{v} - \mathbf{t}\|\|_* \mathcal{A} \left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{\|\|\mathbf{v} - \mathbf{t}\|\|_*} \right) + \|\|\mathbf{w} - \mathbf{t}\|\|_* \|\|\mathbf{v} - \mathbf{t}\|\|_*, \end{aligned}$$

using the bilinearity of $\mathcal{A}(\cdot, \cdot)$ and (2.11b). In view of the assumption, this finishes the proof in the first case.

If $\|\mathbf{v} - \mathbf{t}\|_* \leq \|\mathbf{v} - \mathbf{w}\|_*$ holds, then

$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|_*^2 &= \mathcal{A}(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}) = \mathcal{A}(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{t}) + \mathcal{A}(\mathbf{v} - \mathbf{w}, \mathbf{t} - \mathbf{w}) \\ &\leq \|\mathbf{v} - \mathbf{t}\|_* \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{\|\mathbf{v} - \mathbf{t}\|_*}\right) + \|\mathbf{v} - \mathbf{w}\|_* \|\mathbf{w} - \mathbf{t}\|_* \\ &\leq \|\mathbf{v} - \mathbf{w}\|_* \left| \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{\|\mathbf{v} - \mathbf{t}\|_*}\right) \right| + \|\mathbf{v} - \mathbf{w}\|_* \|\mathbf{w} - \mathbf{t}\|_*, \end{aligned}$$

whence again the assertion follows. Thus the proof is complete. \square

Remark 3.2 (General abstract estimate). *Using the triangle inequality, the bilinearity of $\mathcal{A}(\cdot, \cdot)$, and (2.11b), we immediately have*

$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|_* &\leq \|\mathbf{w} - \mathbf{t}\|_* + \|\mathbf{v} - \mathbf{t}\|_* = \|\mathbf{w} - \mathbf{t}\|_* + \mathcal{A}\left(\mathbf{v} - \mathbf{t}, \frac{\mathbf{v} - \mathbf{t}}{\|\mathbf{v} - \mathbf{t}\|_*}\right) \\ &\leq \|\mathbf{w} - \mathbf{t}\|_* + \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{\|\mathbf{v} - \mathbf{t}\|_*}\right) + \mathcal{A}\left(\mathbf{w} - \mathbf{t}, \frac{\mathbf{v} - \mathbf{t}}{\|\mathbf{v} - \mathbf{t}\|_*}\right) \\ &\leq 2\|\mathbf{w} - \mathbf{t}\|_* + \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{\|\mathbf{v} - \mathbf{t}\|_*}\right). \end{aligned}$$

The estimate of Theorem 3.1 is superior to this simple bound by removing the factor 2 at the term $\|\mathbf{w} - \mathbf{t}\|_*$. In comparison to Theorem 3.3 below, the advantage of Theorem 3.1 is that any triple of functions from $\mathbf{L}^2(\Omega)$ can be chosen. Moreover, it turns out that it is extensible to the convection–diffusion–reaction framework, where it in addition shows advantageous that $\mathbf{t} \in \mathbf{L}^2(\Omega)$ in the second argument of $\mathcal{A}(\cdot, \cdot)$ can be chosen, cf. [59].

3.2 A Pythagorean estimate

Following the approach introduced in Kim [39, Lemma 4.4], we have the following estimate:

Theorem 3.3 (Pythagorean abstract estimate). *Let \mathbf{v} be such that $\mathbf{v} = -\mathbf{S}\nabla\vartheta$ for some $\vartheta \in H_0^1(\Omega)$ and let $\mathbf{w} \in \mathbf{L}^2(\Omega)$ be arbitrary. Let next $\psi \in H_0^1(\Omega)$ be the solution of the problem*

$$\mathcal{B}(\psi, \varphi) = \mathcal{A}(-\mathbf{w}, \mathbf{S}\nabla\varphi) \quad \forall \varphi \in H_0^1(\Omega). \quad (3.1)$$

Then

$$\|\mathbf{v} - \mathbf{w}\|_*^2 = \|\mathbf{w} + \mathbf{S}\nabla\psi\|_*^2 + \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} + \mathbf{S}\nabla\psi}{\|\mathbf{v} + \mathbf{S}\nabla\psi\|_*}\right)^2. \quad (3.2)$$

Moreover,

$$\|\mathbf{w} + \mathbf{S}\nabla\psi\|_* = \inf_{s \in H_0^1(\Omega)} \|\mathbf{w} + \mathbf{S}\nabla s\|_*. \quad (3.3)$$

Proof. Let us first note that there exists a unique solution to the problem (3.1) by the Riesz representation theorem, as $\mathcal{A}(\mathbf{w}, -\mathbf{S}\nabla(\cdot))$ is a continuous linear form. Note as well that (3.1) can be equivalently written, using (2.6), as

$$\mathcal{A}(\mathbf{S}\nabla\psi + \mathbf{w}, \mathbf{S}\nabla\varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega). \quad (3.4)$$

Using this characterization for $\varphi = \psi - \vartheta$, we thus have

$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|_*^2 &= \mathcal{A}(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}) = \mathcal{A}(\mathbf{v} - \mathbf{w}, \mathbf{v} + \mathbf{S}\nabla\psi) + \mathcal{A}(\mathbf{v} - \mathbf{w}, -\mathbf{S}\nabla\psi - \mathbf{w}) \\ &= \mathcal{A}(\mathbf{v} + \mathbf{S}\nabla\psi, \mathbf{v} + \mathbf{S}\nabla\psi) - 2\mathcal{A}(\mathbf{S}\nabla\psi + \mathbf{w}, \mathbf{v} + \mathbf{S}\nabla\psi) + \mathcal{A}(\mathbf{w} + \mathbf{S}\nabla\psi, \mathbf{w} + \mathbf{S}\nabla\psi) \\ &= \|\mathbf{v} + \mathbf{S}\nabla\psi\|_*^2 + \|\mathbf{w} + \mathbf{S}\nabla\psi\|_*^2, \end{aligned}$$

employing also the definition and the symmetry of $\mathcal{A}(\cdot, \cdot)$. The proof is finished by noticing that

$$\|\mathbf{v} + \mathbf{S}\nabla\psi\|_* = \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} + \mathbf{S}\nabla\psi}{\|\mathbf{v} + \mathbf{S}\nabla\psi\|_*}\right)$$

and that

$$\|\mathbf{w} + \mathbf{S}\nabla\psi\|_*^2 = \mathcal{A}(\mathbf{w} + \mathbf{S}\nabla s, \mathbf{w} + \mathbf{S}\nabla\psi) \leq \|\mathbf{w} + \mathbf{S}\nabla s\|_* \|\mathbf{w} + \mathbf{S}\nabla\psi\|_*$$

for an arbitrary $s \in H_0^1(\Omega)$, whence (3.3) follows. \square

This Pythagorean estimate, as we will see later, gives a slightly more precise upper bound in a posteriori error estimates.

4 The mixed finite element method

We recall here some known basic facts about the mixed finite element method, namely the existence and uniqueness of discrete solutions, hybridization, and, most importantly, the postprocessing of [59] and [8] in the lowest-order Raviart–Thomas–Nédélec case and that of [8] in general. First, however, we start by giving the examples of the most common mixed finite element spaces.

4.1 Examples of local mixed finite element spaces

Table 1 lists the most common mixed finite element spaces $\mathbf{V}_h(K) \times \Phi_h(K)$ on an element $K \in \mathcal{T}_h$. The notation **RTN** stands for the Raviart–Thomas [50] space on triangles and rectangles and the Nédélec [45] space on tetrahedra and rectangular parallelepipeds if $d = 3$ and **BDM** for the Brezzi–Douglas–Marini [19] space on triangles and rectangles and the Brezzi–Douglas–Durán–Fortin [18] space on tetrahedra and rectangular parallelepipeds if $d = 3$. In the notation, “s” stands for simplices, “r” for rectangular parallelepipeds, $\mathbf{P}_{2,k}^* := r\nabla \times (x^{k+1}y) + s\nabla \times (xy^{k+1})$, $r, s \in \mathbb{R}$, and $\mathbf{P}_{3,k}^* := \sum_{i=0}^k \{r_i \nabla \times (0, 0, xy^{i+1}z^{k-i})^t + s_i \nabla \times (x^{k-i}yz^{i+1}, 0, 0)^t + t_i \nabla \times (0, x^{i+1}y^{k-i}z, 0)^t\}$, $r_i, s_i \in \mathbb{R}$, with $\nabla \times$ the curl operator. We have here denoted by k the biggest polynomial space contained in $\mathbf{V}_h(K)$ and by l that in $\Phi_h(K)$. Then $\mathbf{V}_h := \Pi_{K \in \mathcal{T}_h} \mathbf{V}_h(K) \cap \mathbf{H}(\text{div}, \Omega)$ and $\Phi_h := \Pi_{K \in \mathcal{T}_h} \Phi_h(K)$. Note in particular that whereas $\mathbf{V}_h(K)$ are local unconstrained spaces, the fact that $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$ imposes the normal trace continuity of all $\mathbf{v}_h \in \mathbf{V}_h$, i.e., $\mathbf{v}_h|_K \cdot \mathbf{n}_{\sigma_{K,L}} = \mathbf{v}_h|_L \cdot \mathbf{n}_{\sigma_{K,L}}$ for all $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$ shared by elements K and L . For a general reference to mixed finite element methods, we refer to Brezzi and Fortin [20] or Roberts and Thomas [53].

In the rest of the paper, we shall sometimes considered apart the following particular case:

Assumption (A) (Lowest-order Raviart–Thomas–Nédélec case)

The spaces \mathbf{V}_h and Φ_h are formed by $\mathbf{RTN}_0^s(K)$ or $\mathbf{RTN}_0^r(K)$ from Table 1 and the tensor \mathbf{S} is piecewise constant on simplices and piecewise constant and diagonal on rectangular parallelepipeds.

Space	d	element	$\mathbf{V}_h(K)$	$\Phi_h(K)$	$\mathbf{V}_h(K) \cdot \mathbf{n} _\sigma$	k	l
$\mathbf{RTN}_k^s(K)$	2, 3	simplex	$\mathbb{P}_k^d + \mathbb{P}_k \mathbf{x}$	\mathbb{P}_k	\mathbb{P}_k	≥ 0	$= k$
$\mathbf{RTN}_k^r(K)$	2, 3	rect. par.	$\mathbb{Q}_k^d + \mathbb{Q}_k \mathbf{x}$	\mathbb{Q}_k	\mathbb{P}_k ($d = 2$), \mathbb{Q}_k ($d = 3$)	≥ 0	$= k$
$\mathbf{BDM}_k^s(K)$	2, 3	simplex	\mathbb{P}_k^d	\mathbb{P}_{k-1}	\mathbb{P}_k	≥ 1	$= k - 1$
$\mathbf{BDM}_k^r(K)$	2, 3	rect. par.	$\mathbb{P}_k^d \oplus \mathbf{P}_{d,k}^*$	\mathbb{P}_{k-1}	\mathbb{P}_k	≥ 1	$= k - 1$

Table 1: Examples of local mixed finite element spaces

4.2 Existence and uniqueness of the discrete solutions

For the sake of completeness and also to stress its simplicity, we recall here the proof of existence and uniqueness of the discrete mixed finite element solution.

Corollary 4.1 (Existence and uniqueness of the discrete mixed finite element solution). *Let $\nabla \cdot \mathbf{V}_h = \Phi_h$. Then there exists a unique solution to the problem (1.5a)–(1.5b).*

Proof. Problem (1.5a)–(1.5b) is a square linear finite-dimensional system. It thus suffices to prove that $f = 0$ implies $\mathbf{u}_h = 0$ and $p_h = 0$. Put $\phi_h = p_h$ in (1.5b) and $\mathbf{v}_h = \mathbf{u}_h$ in (1.5a) and sum the equations. This gives $(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{u}_h) = 0$, whence $\mathbf{u}_h = 0$ follows. Consequently, $(p_h, \nabla \cdot \mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in \mathbf{V}_h$, whence $p_h = 0$ follows by the assumption $\nabla \cdot \mathbf{V}_h = \Phi_h$. \square

4.3 Hybridization

The hybridization technique allows to relax the normal trace continuity constraint $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$ while imposing it instead with the aid of Lagrange multipliers. The unconstrained flux space is given by $\tilde{\mathbf{V}}_h := \prod_{K \in \mathcal{T}_h} \mathbf{V}_h(K)$, where $\mathbf{V}_h(K)$ are the local spaces on each element, and the Lagrange multipliers space Λ_h is the space of (discontinuous) piecewise polynomials on $\mathcal{E}_h^{\text{int}}$ such that for all $\sigma \in \mathcal{E}_h^{\text{int}}$, $\mu_h|_\sigma \in \mathbf{V}_h \cdot \mathbf{n}|_\sigma$. With these notations, the hybridized version of (1.5a)–(1.5b) consists in finding $\mathbf{u}_h \in \tilde{\mathbf{V}}_h$, $p_h \in \Phi_h$, and $\lambda_h \in \Lambda_h$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) + \sum_{K \in \mathcal{T}_h} \langle \mathbf{v}_h \cdot \mathbf{n}, \lambda_h \rangle_{\partial K \setminus \partial \Omega} = 0 \quad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h, \quad (4.1a)$$

$$(\nabla \cdot \mathbf{u}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi_h, \quad (4.1b)$$

$$\sum_{K \in \mathcal{T}_h} \langle \mathbf{u}_h \cdot \mathbf{n}, \mu_h \rangle_{\partial K \setminus \partial \Omega} = 0 \quad \forall \mu_h \in \Lambda_h. \quad (4.1c)$$

It is well known and easy to show that p_h, \mathbf{u}_h from (1.5a)–(1.5b) and (4.1a)–(4.1c) coincide; λ_h then provides an additional approximation to p . Let us also recall that λ_h can be postprocessed locally from (1.5a)–(1.5b); on each $\sigma \in \mathcal{E}_h^{\text{int}}$, $\sigma \in \mathcal{E}_K$ for some $K \in \mathcal{T}_h$, it is given by

$$\langle \mathbf{v}_h \cdot \mathbf{n}, \lambda_h \rangle_\sigma = -(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K + (p_h, \nabla \cdot \mathbf{v}_h)_K \\ \forall \mathbf{v}_h \in \mathbf{V}_h(K) \text{ such that } (\mathbf{v}_h \cdot \mathbf{n})|_\gamma = 0 \quad \forall \gamma \in \mathcal{E}_K, \gamma \neq \sigma,$$

so that it is not necessary to implement (4.1a)–(4.1c) in order to obtain it.

4.4 Postprocessing

Seemingly, there is no direct analogy of the link $\mathbf{u} = -\mathbf{S}\nabla p$ at the discrete level in the mixed finite element method. It is sometimes even said that the distinctive feature of the mixed finite

element method is that the discrete flux \mathbf{u}_h has “more regularity” than the discrete potential p_h , in a sense that it is a polynomial of a higher degree. We shall see in this section that this is only an impression and that the link $\mathbf{u}_h \approx -\mathbf{S}\nabla p_h$ can easily be recovered by postprocessing.

Different postprocessing techniques for mixed finite elements have been introduced in the past. Let us cite the works of Arnold and Brezzi [9], Bramble and Xu [16], Stenberg [54], Chen [25], Arbogast and Chen [8], and, for the lowest-order Raviart–Thomas–Nédélec case, the author [59]. It will turn out that for our purposes, the postprocessing of [59] and [8] under Assumption (A) and that of [8] in general will be optimal. We now recall it here.

4.4.1 Postprocessing in the lowest-order Raviart–Thomas–Nédélec case

Under Assumption (A), the following postprocessing has been proposed in [59, Section 4.1] on simplicial meshes and in [8, Sections 6 and 9] (cf. also [61, Section 3.2]) on meshes consisting of rectangular parallelepipeds: construct $\tilde{p}_h \in \mathbb{P}_2(\mathcal{T}_h)$ such that

$$-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K \quad \forall K \in \mathcal{T}_h, \quad (4.2a)$$

$$\pi_0(\tilde{p}_h|_K) = p_h|_K \quad \forall K \in \mathcal{T}_h. \quad (4.2b)$$

Note that \tilde{p}_h is in general not a full second-order polynomial and that it is only built on each $K \in \mathcal{T}_h$ from the given degrees of freedom, so that its construction cost is negligible.

In general, \tilde{p}_h is nonconforming in the sense that $\tilde{p}_h \notin H_0^1(\Omega)$ but it is shown in [59, Lemma 6.1] that $\tilde{p}_h \in W_0(\mathcal{T}_h)$ on simplicial meshes; for meshes of rectangular parallelepipeds, see [8]. Hence, at least the mean values of \tilde{p}_h on the sides of \mathcal{T}_h are continuous (and equal to zero on $\partial\Omega$). Moreover, these means of traces coincide with the Lagrange multipliers λ_h of the hybridized version (4.1a)–(4.1c) of (1.5a)–(1.5b), see [59, Lemma 6.4] and [8].

4.4.2 Postprocessing in the general case

It turns out that in the general case, there does not exist \tilde{p}_h such that (4.2a) is true. Then the postprocessing by Arbogast and Chen [8] proposes a weak form of this relation. This postprocessing is a generalization of the postprocessing proposed originally by Arnold and Brezzi [9] and Chen [25] and it is defined as follows. Let P_{Φ_h} be the $L^2(\Omega)$ -orthogonal projection onto Φ_h , $P_{\tilde{\mathbf{V}}_h}$ the $\mathbf{L}^2(\Omega)$ -orthogonal projection onto $\tilde{\mathbf{V}}_h$ with respect to the scalar product $(\mathbf{S}^{-1}\cdot, \cdot)$, and P_{Λ_h} the $L^2(\mathcal{E}_h^{\text{int}})$ -orthogonal projection onto Λ_h , i.e.,

$$P_{\Phi_h} : L^2(\Omega) \rightarrow \Phi_h \quad \text{for } \phi \in L^2(\Omega), \quad (\phi - P_{\Phi_h}(\phi), \phi_h) = 0 \quad \forall \phi_h \in \Phi_h, \quad (4.3a)$$

$$P_{\tilde{\mathbf{V}}_h} : \mathbf{L}^2(\Omega) \rightarrow \tilde{\mathbf{V}}_h \quad \text{for } \mathbf{v} \in \mathbf{L}^2(\Omega), \quad (\mathbf{S}^{-1}(\mathbf{v} - P_{\tilde{\mathbf{V}}_h}(\mathbf{v})), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h, \quad (4.3b)$$

$$P_{\Lambda_h} : L^2(\mathcal{E}_h^{\text{int}}) \rightarrow \Lambda_h \quad \text{for } \mu \in L^2(\mathcal{E}_h^{\text{int}}), \quad (\mu - P_{\Lambda_h}(\mu), \mu_h)_{\mathcal{E}_h^{\text{int}}} = 0 \quad \forall \mu_h \in \Lambda_h. \quad (4.3c)$$

The postprocessed potential $\tilde{p}_h \in M_h$ is then defined by

$$P_{\Phi_h}(\tilde{p}_h) = p_h, \quad (4.4a)$$

$$P_{\Lambda_h}(\tilde{p}_h) = \lambda_h. \quad (4.4b)$$

Note that employing (4.4a)–(4.4b) in (4.1a) and using $\nabla \cdot \mathbf{V}_h(K) = \Phi_h(K)$ and $\mathbf{V}_h(K) \cdot \mathbf{n}|_{\partial K \setminus \partial\Omega} = \Lambda_h(K)$ gives, for all $K \in \mathcal{T}_h$,

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K - (\tilde{p}_h, \nabla \cdot \mathbf{v}_h)_K + \langle \mathbf{v}_h \cdot \mathbf{n}, \tilde{p}_h \rangle_{\partial K \setminus \partial\Omega} = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K).$$

Employing the Green theorem for the two last terms of the above expression then leads to

$$(\mathbf{S}^{-1}(\mathbf{u}_h + \mathbf{S}\nabla\tilde{p}_h), \mathbf{v}_h)_K = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K) \quad \forall K \in \mathcal{T}_h,$$

which is nothing but

$$P_{\tilde{\mathbf{V}}_h}(-\mathbf{S}\nabla\tilde{p}_h) = \mathbf{u}_h. \quad (4.5)$$

The finite-dimensional spaces M_h for the individual families and types of elements are detailed in [8] (cf. also [25]); principally, they consist of piecewise polynomial spaces augmented with bubble functions. They are usually nonconforming in the sense that $M_h \not\subset H_0^1(\Omega)$. We also remark that whereas for a given space M_h , $\tilde{p}_h \in M_h$ satisfying (4.4a)–(4.4b) is prescribed uniquely, the space M_h itself for a given method is not defined in a unique way; there in particular exist several different spaces for the lowest-order Raviart–Thomas elements on triangles.

For the analysis of this paper, along with (4.4a)–(4.4b), we will only need the three following characterizing properties of the spaces M_h :

$$M_h \subset W_h(\mathcal{T}_h), \quad (4.6a)$$

$$\mathbb{P}_{k+1}(\mathcal{T}_h) \subset M_h, \quad (4.6b)$$

$$(\nabla\xi_h, \mathbf{v}_h)_K = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K) \quad \Rightarrow \quad \nabla\xi_h = 0 \quad \forall \xi_h \in M_h(K), \forall K \in \mathcal{T}_h. \quad (4.6c)$$

The first property simply ensures that there is “enough continuity” in M_h , the second one guarantees that M_h is “large enough”, and the last one ensures the “compatibility” of ∇M_h with \mathbf{V}_h . Note that (4.6c) in particular implies $\dim(M_h(K)) \leq \dim(\mathbf{V}_h(K)) + 1$ (and consequently $\dim(M_h) \leq \dim(\tilde{\mathbf{V}}_h) + 1$).

5 A priori error analysis

We show in this section that with the abstract result of Theorem 3.1, it is immediate to get the a priori error estimates for the flux in the form $\|\mathbf{u} - \mathbf{u}_h\|_* \leq \|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|_*$, where $I_{\mathbf{V}_h}$ is the vector interpolation operator of each mixed finite element method (satisfying the commuting diagram property), see [20, Section III.3]. Consequently, we easily recover the known a priori error estimates for the flux. We next focus on the postprocessing of Section 4.4.1 under Assumption (A). We show that in this case, thanks to (4.2a), the a priori error estimates for the potential are again completely straightforward. In general, the postprocessing of Section 4.4.2 has to be used and the properties of (4.5) and (4.6c) have to be exploited but still quite easily, all known a priori error estimates for the approximate potential p_h are reproduced. We in particular recover them from the estimates for \tilde{p}_h , which themselves are derived first and seem to be new. Moreover, all the estimates are optimal and given under minimal necessary assumptions, improving thus some of the previously known results.

Throughout this section, we shall suppose that \mathcal{T}_h is shape-regular with a constant $\kappa_{\mathcal{T}}$. We always give a detailed form of the estimates up to the form with the error between the exact solution and its interpolate. Obtaining the final error estimates is then a question of application of interpolation estimates, presented, e.g., in [20, 49, 53]. For the sake of completeness, we include these final results, supposing the full necessary regularity.

5.1 Estimates for the flux

A straightforward application of Theorem 3.1 gives the following result:

Theorem 5.1 (Abstract a priori estimate for the flux). *Let \mathbf{u} given by (1.4a)–(1.4b) belong to the space $\tilde{\mathbf{H}}(\text{div}, S)$ and let \mathbf{u}_h be given by (1.5a)–(1.5b). Let next $I_{\mathbf{V}_h}$ be the mixed interpolation operator, see [20, Section III.3]. Then*

$$\|\|\mathbf{u} - \mathbf{u}_h\|\|_* \leq \|\|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|\|_*. \quad (5.1)$$

Proof. Put $\mathbf{v} = \mathbf{u}_h$, $\mathbf{w} = \mathbf{u}$, and $\mathbf{t} = I_{\mathbf{V}_h}(\mathbf{u})$ in Theorem 3.1. This gives

$$\|\|\mathbf{u}_h - \mathbf{u}\|\|_* \leq \|\|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|\|_* + \left| \mathcal{A} \left(\mathbf{u}_h - \mathbf{u}, \frac{\mathbf{u}_h - I_{\mathbf{V}_h}(\mathbf{u})}{\|\|\mathbf{u}_h - I_{\mathbf{V}_h}(\mathbf{u})\|\|_*} \right) \right|.$$

Notice that the properties of the interpolation operator $I_{\mathbf{V}_h}$ imply

$$\mathcal{A}(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - I_{\mathbf{V}_h}(\mathbf{u})) = 0. \quad (5.2)$$

Indeed, it follows by subtracting (1.4a) from (1.5a) and using (2.3) that

$$\mathcal{A}(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) = (p_h - p, \nabla \cdot \mathbf{v}_h)$$

for all $\mathbf{v}_h \in \mathbf{V}_h$. It suffices to put $\mathbf{v}_h = \mathbf{u}_h - I_{\mathbf{V}_h}(\mathbf{u})$ and to notice that $\nabla \cdot (\mathbf{u}_h - I_{\mathbf{V}_h}(\mathbf{u})) = 0$, which follows from (1.5b) and from the commuting diagram property [20, Proposition III.3.7], to see (5.2). Hence the result follows. \square

Using the interpolation estimates, see, e.g., [20, 49, 53] we infer from the previous results and (1.5b) the following corollary:

Corollary 5.2 (A priori estimates for the flux). *Let \mathbf{u} be given by (1.4a)–(1.4b) and \mathbf{u}_h by (1.5a)–(1.5b). Then*

$$\begin{aligned} \|\|\mathbf{u} - \mathbf{u}_h\|\|_* &\leq Ch^{k+1}, \\ \|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,\text{div}} &\leq Ch^{l+1}. \end{aligned}$$

Remark 5.3 (Discrete inf–sup condition/discrete Friedrichs inequality in flux estimates). *Note that by the above estimate, no link other than $\nabla \cdot \mathbf{V}_h = \Phi_h$ between the spaces Φ_h and \mathbf{V}_h is needed for the convergence and a priori error estimates for the flux. In particular, no discrete inf–sup condition, or, equivalently, as we shall see later, no discrete Friedrichs inequality needs to hold at this stage.*

5.2 Estimates for the postprocessed potential in the lowest-order Raviart–Thomas–Nédélec case

As the proof of the following theorem shows, a priori error estimates for the postprocessed potential \tilde{p}_h under Assumption (A) are straightforward.

Theorem 5.4 (A priori estimates for the postprocessed potential \tilde{p}_h in the lowest-order Raviart–Thomas–Nédélec case). *Let Assumption (A) hold, let \mathbf{u} , p be given by (1.4a)–(1.4b), \mathbf{u}_h , p_h by (1.5a)–(1.5b), and \tilde{p}_h by (4.2a)–(4.2b). Then*

$$\begin{aligned} \|p - \tilde{p}_h\| &= \|\|\mathbf{u} - \mathbf{u}_h\|\|_* \leq Ch, \\ \|p - \tilde{p}_h\|_1 &\leq C\|p - \tilde{p}_h\|. \end{aligned}$$

Proof. For the first estimate, it is sufficient to note that (2.10) in combination with (4.2a) gives $\|p - \tilde{p}_h\| = \|\|\mathbf{u} - \mathbf{u}_h\|\|_*$ and use the result of Corollary 5.2. The second estimate is then directly implied by the fact that $\tilde{p}_h \in W_0(\mathcal{T}_h)$ and the discrete Friedrichs inequality (2.8). \square

5.3 Estimates for the postprocessed potential in the general case

In the general case, one no more has (4.2a), whence $\|p - \tilde{p}_h\| = \|\mathbf{u} - \mathbf{u}_h\|_*$ and $\|\tilde{p}_h\| = \|\mathbf{u}_h\|_*$ no more holds true. As however the following lemma shows, there is still a strong particular connection between $\|\tilde{p}_h\|$ and $\|\mathbf{u}_h\|_*$.

Lemma 5.5 (Equivalence between the energy seminorms on $M_h(K)$ and $P_{\tilde{\mathbf{V}}_h}(-\mathbf{S}\nabla M_h(K))$). *There holds*

$$\|P_{\tilde{\mathbf{V}}_h}(-\mathbf{S}\nabla\xi_h)\|_{*,K} \leq \|\xi_h\|_K \leq C_{\text{eq}}\|P_{\tilde{\mathbf{V}}_h}(-\mathbf{S}\nabla\xi_h)\|_{*,K} \quad \forall K \in \mathcal{T}_h, \forall \xi_h \in M_h(K) \quad (5.3)$$

and thus, in particular,

$$\|\mathbf{u}_h\|_* \leq \|\tilde{p}_h\| \leq C_{\text{eq}}\|\mathbf{u}_h\|_*.$$

More generally,

$$\|\nabla\xi_h\|_K \leq C_K \sup_{\mathbf{v}_h \in \mathbf{V}_h(K)} \frac{(\nabla\xi_h, \mathbf{v}_h)_K}{\|\mathbf{v}_h\|_K} \quad \forall K \in \mathcal{T}_h, \forall \xi_h \in M_h(K). \quad (5.4)$$

Proof. We have

$$\|P_{\tilde{\mathbf{V}}_h}(-\mathbf{S}\nabla\xi_h)\|_{*,K} \leq \|-\mathbf{S}\nabla\xi_h\|_{*,K} = \|\xi_h\|_K$$

by the fact that $P_{\tilde{\mathbf{V}}_h}$ is the $\mathbf{L}^2(K)$ -orthogonal projection onto $\mathbf{V}_h(K)$ with respect to the scalar product $(\mathbf{S}^{-1}\cdot, \cdot)_K$, whose norm is $\|\cdot\|_{*,K}$, and by (2.10). Supposing for the moment the validity of (5.4), we now prove that also the other inequality in the first assertion of the lemma holds true. Let $K \in \mathcal{T}_h$ and $\xi_h \in M_h(K)$ be given. First note that by (5.4), the definition (4.3b) of $P_{\tilde{\mathbf{V}}_h}$, the Cauchy–Schwarz inequality, the assumption on \mathbf{S} , and (2.9),

$$\begin{aligned} \|\nabla\xi_h\|_K &\leq C_K \sup_{\mathbf{v}_h \in \mathbf{V}_h(K)} \frac{(\mathbf{S}^{-1}\mathbf{S}\nabla\xi_h, \mathbf{v}_h)_K}{\|\mathbf{v}_h\|_K} = C_K \sup_{\mathbf{v}_h \in \mathbf{V}_h(K)} \frac{(\mathbf{S}^{-1}P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla\xi_h), \mathbf{v}_h)_K}{\|\mathbf{v}_h\|_K} \\ &\leq C_K \|\mathbf{S}^{-1}P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla\xi_h)\|_K \leq \frac{C_K}{c_{\mathbf{S},K}^{1/2}} \|P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla\xi_h)\|_{*,K}. \end{aligned}$$

Hence

$$\|\xi_h\|_K \leq C_{\mathbf{S},K}^{1/2} \|\nabla\xi_h\|_K \leq C_K \frac{C_{\mathbf{S},K}^{1/2}}{c_{\mathbf{S},K}^{1/2}} \|P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla\xi_h)\|_{*,K}$$

by (2.7), the assumption on \mathbf{S} , and the previous estimate, which gives the right inequality in (5.3) with $C_{\text{eq}} := \max_{K \in \mathcal{T}_h} \{C_K C_{\mathbf{S},K}^{1/2} / c_{\mathbf{S},K}^{1/2}\}$. Finally, the validity of (5.4) on a reference element \hat{K} with a constant only dependent on the maximal polynomial degree of $M_h(\hat{K})$ follows from (4.6c). Thus (5.4), with C_K only dependent on the maximal polynomial degree of $M_h(K)$ and on κ_K follows by the Piola transformation and scaling arguments. \square

Theorem 5.6 (A priori estimates for the postprocessed potential \tilde{p}_h in the general case). *Let \mathbf{u} , p be given by (1.4a)–(1.4b), \mathbf{u}_h , p_h by (1.5a)–(1.5b), and \tilde{p}_h by (4.4a)–(4.4b). Then*

$$\|p - \tilde{p}_h\| \leq C \left(\inf_{s_h \in M_h} \|p - s_h\| + \|\mathbf{u} - \mathbf{u}_h\|_* + \|\mathbf{u} - P_{\tilde{\mathbf{V}}_h}(\mathbf{u})\|_* \right) \leq Ch^{k+1}, \quad (5.5)$$

$$\|p - \tilde{p}_h\|_1 \leq C \|p - \tilde{p}_h\|. \quad (5.6)$$

Proof. Let $s_h \in M_h$ be arbitrary. Using (5.3), (4.5), adding and subtracting \mathbf{u} and $P_{\tilde{\mathbf{V}}_h}(\mathbf{u})$, using that $\mathbf{u} = -\mathbf{S}\nabla p$, and finally employing the triangle inequality, the fact that $P_{\tilde{\mathbf{V}}_h}$ is the $\mathbf{L}^2(\Omega)$ -orthogonal projection onto $\tilde{\mathbf{V}}_h$ with respect to the scalar product $(\mathbf{S}^{-1}\cdot, \cdot)$, and (2.10), we have

$$\begin{aligned} \|\tilde{p}_h - s_h\| &\leq C_{\text{eq}} \|P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla(\tilde{p}_h - s_h))\|_* = C_{\text{eq}} \|\mathbf{u}_h - P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla s_h)\|_* \\ &= C_{\text{eq}} \|\mathbf{u}_h + \mathbf{u} - \mathbf{u} + P_{\tilde{\mathbf{V}}_h}(\mathbf{u}) + P_{\tilde{\mathbf{V}}_h}(\mathbf{S}\nabla(p - s_h))\|_* \\ &\leq C_{\text{eq}} (\|\mathbf{u} - \mathbf{u}_h\|_* + \|\mathbf{u} - P_{\tilde{\mathbf{V}}_h}(\mathbf{u})\|_* + \|p - s_h\|). \end{aligned}$$

Thus the first line of (5.5) follows by the triangle inequality $\|p - \tilde{p}_h\| \leq \|p - s_h\| + \|\tilde{p}_h - s_h\|$, (4.6b), Corollary 5.2, and the approximation properties of $P_{\tilde{\mathbf{V}}_h}$. Estimate (5.6) then again follows immediately by the discrete Friedrichs inequality (2.8). \square

5.4 Estimates for the original potential

In this short section, we easily recover the estimates for the original potential p_h from the previous results.

Theorem 5.7 (A priori estimates for the original potential p_h). *Let \mathbf{u}, p be given by (1.4a)–(1.4b), \mathbf{u}_h, p_h by (1.5a)–(1.5b), and \tilde{p}_h by (4.2a)–(4.2b) or (4.4a)–(4.4b). Then*

$$\|p - p_h\| \leq \|p - P_{\Phi_h}(p)\| + \|p - \tilde{p}_h\| \leq Ch^{l+1}.$$

Proof. Using (4.4a), adding and subtracting $P_{\Phi_h}(p)$, employing the triangle inequality, and finally the fact that P_{Φ_h} is the $L^2(\Omega)$ -orthogonal projection onto Φ_h , we have

$$\begin{aligned} \|p - p_h\| &= \|p - P_{\Phi_h}(\tilde{p}_h)\| = \|p - P_{\Phi_h}(p) + P_{\Phi_h}(p - \tilde{p}_h)\| \\ &\leq \|p - P_{\Phi_h}(p)\| + \|P_{\Phi_h}(p - \tilde{p}_h)\| \leq \|p - P_{\Phi_h}(p)\| + \|p - \tilde{p}_h\|. \end{aligned}$$

The final estimate then follows by Theorem 5.6 and the approximation properties of P_{Φ_h} . \square

5.5 Superconvergence estimates for the original potential

For the sake of completeness, we show in this section the superconvergence estimates for the original potential p_h , following essentially [30], [20, Section V.3], and [29]. Note that we reduce the assumptions necessary in [29].

Assumption (B) (Elliptic regularity)

For each $g_h \in \Phi_h$, the weak solution of the problem

$$\mathbf{r} = -\mathbf{S}\nabla q \quad \text{in } \Omega, \quad (5.7a)$$

$$\nabla \cdot \mathbf{r} = g_h \quad \text{in } \Omega, \quad (5.7b)$$

$$q = 0 \quad \text{on } \partial\Omega \quad (5.7c)$$

satisfies

$$\|q\|_2 + |\mathbf{r}|_1 \leq C_{\text{ER}} \|g_h\|. \quad (5.8)$$

Theorem 5.8 (Superconvergence estimates for the original potential p_h). *Let \mathbf{u} , p be given by (1.4a)–(1.4b) and \mathbf{u}_h , p_h by (1.5a)–(1.5b). Let next Assumption (B) hold. Then if $l = k$,*

$$\|P_{\Phi_h}(p) - p_h\| \leq Ch(\|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|_* + \|\nabla \cdot (\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u}))\|) \leq Ch^{k+2},$$

and if $k \geq 1$ and $(\mathbf{u} - I_{\mathbf{V}_h} \mathbf{u}, \mathbf{1})_K = 0$ for each $K \in \mathcal{T}_h$,

$$\|P_{\Phi_h}(p) - p_h\| \leq Ch(\|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|_*) \leq Ch^{k+2}.$$

Proof. We use the characterization

$$\|P_{\Phi_h}(p) - p_h\| = \sup_{g_h \in \Phi_h} \frac{(P_{\Phi_h}(p) - p_h, g_h)}{\|g_h\|}.$$

We next develop, using the definition (4.3a) of the P_{Φ_h} orthogonal projection, the fact that $\nabla \cdot I_{\mathbf{V}_h}(\mathbf{r}) = g_h$, and subtracting (1.5a) from (1.4a)

$$\begin{aligned} (P_{\Phi_h}(p) - p_h, g_h) &= (p - p_h, g_h) = (p - p_h, \nabla \cdot I_{\mathbf{V}_h}(\mathbf{r})) = (\mathbf{S}^{-1}(\mathbf{u} - \mathbf{u}_h), I_{\mathbf{V}_h}(\mathbf{r})) \\ &= (\mathbf{S}^{-1}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), I_{\mathbf{V}_h}(\mathbf{r})) + (\mathbf{S}^{-1}(I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h), I_{\mathbf{V}_h}(\mathbf{r})) \\ &= (\mathbf{S}^{-1}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), I_{\mathbf{V}_h}(\mathbf{r}) - \mathbf{r}) + (\mathbf{S}^{-1}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), \mathbf{r}) \\ &\quad + (\mathbf{S}^{-1}(I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h), I_{\mathbf{V}_h}(\mathbf{r}) - \mathbf{r}) + (\mathbf{S}^{-1}(I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h), \mathbf{r}). \end{aligned}$$

We now first note that for the last term, we have

$$(\mathbf{S}^{-1}(I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h), \mathbf{r}) = -(I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h, \nabla q) = (\nabla \cdot (I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h), q) = 0,$$

employing (5.7a), the Green theorem, and the fact that $\nabla \cdot (I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h) = 0$. Next, the first term can be estimated by, employing (5.8),

$$\begin{aligned} (\mathbf{S}^{-1}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), I_{\mathbf{V}_h}(\mathbf{r}) - \mathbf{r}) &\leq \|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|_* \|I_{\mathbf{V}_h}(\mathbf{r}) - \mathbf{r}\|_* \leq Ch \|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|_* |\mathbf{r}|_1 \\ &\leq CC_{\text{ER}} h \|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\|_* \|g_h\|. \end{aligned}$$

The third term can be estimated similarly, using in addition the triangle inequality and (5.1). Finally, there are two ways to estimate the second term. Firstly,

$$\begin{aligned} (\mathbf{S}^{-1}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), \mathbf{r}) &= -(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u}), \nabla q) = (\nabla \cdot (\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), q) \\ &= (\nabla \cdot (\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), q - \pi_0(q)) \\ &\leq C_{\text{P}}^{\frac{1}{2}} h \|\nabla \cdot (\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u}))\| \|q\|_1 \leq C_{\text{P}}^{\frac{1}{2}} C_{\text{ER}} h \|\nabla \cdot (\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u}))\| \|g_h\| \end{aligned}$$

employing (5.7a), the Green theorem, the fact that $(\nabla \cdot (\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), 1)_K = 0$ for all $K \in \mathcal{T}_h$, the Poincaré inequality (2.13), and (5.8). Alternatively, if $k \geq 1$ and $(\mathbf{u} - I_{\mathbf{V}_h} \mathbf{u}, \mathbf{1})_K = 0$ for each $K \in \mathcal{T}_h$, then

$$\begin{aligned} (\mathbf{S}^{-1}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})), \mathbf{r}) &= (I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}, \nabla q) = (I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}, \nabla q - \pi_0(\nabla q)) \\ &\leq C_{\text{P}}^{\frac{1}{2}} h \|I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}\| \|q\|_2 \leq C_{\text{P}}^{\frac{1}{2}} C_{\text{ER}} C_{\mathbf{S}, \Omega}^{\frac{1}{2}} h \|\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})\| \|g_h\|, \end{aligned}$$

employing also the Poincaré inequality (2.13), the assumption on \mathbf{S} , and the definition of the energy norm (2.9). Combining the above estimates proves the assertions of the theorem. \square

5.6 Superconvergence estimates for the postprocessed potential

Using the results of the previous section, we establish here in a straightforward way superconvergence estimates for the postprocessed potential \tilde{p}_h .

Theorem 5.9 (Superconvergence estimates for the postprocessed potential \tilde{p}_h). *Let \mathbf{u} , p be given by (1.4a)–(1.4b), \mathbf{u}_h , p_h by (1.5a)–(1.5b), and \tilde{p}_h by (4.2a)–(4.2b) or (4.4a)–(4.4b). Then*

$$\|p - \tilde{p}_h\| \leq Ch\|p - \tilde{p}_h\| + \|P_{\Phi_h}(p) - p_h\|.$$

If in particular Assumption (B) holds and if either $l = k$ or $k \geq 1$ and $(\mathbf{u} - I_{\mathbf{V}_h}\mathbf{u}, \mathbf{1})_K = 0$ for each $K \in \mathcal{T}_h$, then

$$\|p - \tilde{p}_h\| \leq Ch^{k+2}.$$

Proof. We have, using the triangle inequality, the fact that P_{Φ_h} is the $L^2(\Omega)$ -orthogonal projection onto Φ_h , (4.4a), and the Poincaré inequality (2.13),

$$\begin{aligned} \|p - \tilde{p}_h\| &= \|p - \tilde{p}_h - P_{\Phi_h}(p - \tilde{p}_h) + P_{\Phi_h}(p - \tilde{p}_h)\| \leq \|p - \tilde{p}_h - \pi_0(p - \tilde{p}_h)\| + \|P_{\Phi_h}(p) - p_h\| \\ &\leq C_{\mathbf{P}}^{\frac{1}{2}}h|p - \tilde{p}_h|_1 + \|P_{\Phi_h}(p) - p_h\| \leq \frac{C_{\mathbf{P}}^{1/2}h}{c_{\mathbf{S},\Omega}^{1/2}}\|p - \tilde{p}_h\| + \|P_{\Phi_h}(p) - p_h\|. \end{aligned}$$

□

6 A posteriori error analysis

We show in this section that with the abstract result of Theorem 3.1 (or with the slight improvement of Theorem 3.3), it is also immediate to get an optimal framework for a posteriori error estimates for the flux in mixed finite element methods. For the potential, a similar framework developed in [59, 39, 33] is adopted. We finally give fully computable versions of all the estimates, prove their local efficiency, discuss their robustness, and present some extensions.

6.1 Estimates for the flux

We state and prove here our a posteriori error estimates for the flux, first in an abstract and then in a fully computable form.

6.1.1 Abstract estimates

An application of Theorem 3.1 gives the following result, which we state as generally as possible; in practice, \mathbf{u}_h is given by (1.5a)–(1.5b).

Theorem 6.1 (Abstract a posteriori estimate for the flux and its efficiency). *Let \mathbf{u} be given by (1.4a)–(1.4b) and let $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ be arbitrary. Then*

$$\|\|\mathbf{u} - \mathbf{u}_h\|\|_* \leq \inf_{s \in H_0^1(\Omega)} \|\|\mathbf{u}_h + \mathbf{S}\nabla s\|\|_* + \eta_{\mathbf{R}} \leq \|\|\mathbf{u} - \mathbf{u}_h\|\|_* + \eta_{\mathbf{R}}, \quad (6.1)$$

where

$$\eta_{\mathbf{R}} := \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{\mathbf{P}}h_K^2}{c_{\mathbf{S},K}} \|f - P_{\Phi_h}(f)\|_K^2 \right\}^{\frac{1}{2}}.$$

Proof. The right inequality in (6.1) is straightforward by putting $s = p$ and noticing that $\mathbf{u} = -\mathbf{S}\nabla p$ by the equivalence of (1.2) and (1.4a)–(1.4b). For the left one, put $\mathbf{v} = \mathbf{u}$, $\mathbf{w} = \mathbf{u}_h$, and $\mathbf{t} = -\mathbf{S}\nabla s$, with $s \in H_0^1(\Omega)$ arbitrary, in Theorem 3.1. This gives

$$\|\|\mathbf{u} - \mathbf{u}_h\|\|_* \leq \|\|\mathbf{u}_h + \mathbf{S}\nabla s\|\|_* + \left| \mathcal{A} \left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} + \mathbf{S}\nabla s}{\|\|\mathbf{u} + \mathbf{S}\nabla s\|\|_*} \right) \right|.$$

Next put $\varphi := (p - s)/\|p - s\| \in H_0^1(\Omega)$ and rewrite the second term of the above expression as $|\mathcal{A}(\mathbf{u} - \mathbf{u}_h, -\mathbf{S}\nabla\varphi)|$, employing $\mathbf{u} = -\mathbf{S}\nabla p$ and (2.10). Next, by the equivalent definition of the weak solution (2.5),

$$\mathcal{A}(\mathbf{u}, -\mathbf{S}\nabla\varphi) = (f, \varphi),$$

whereas

$$\mathcal{A}(\mathbf{u}_h, -\mathbf{S}\nabla\varphi) = -(\mathbf{u}_h, \nabla\varphi) = (P_{\Phi_h}(f), \varphi)$$

by (2.3), the Green theorem, and the assumption on \mathbf{u}_h . Hence

$$\mathcal{A}(\mathbf{u} - \mathbf{u}_h, -\mathbf{S}\nabla\varphi) = (f - P_{\Phi_h}(f), \varphi).$$

This last expression can easily be estimated by

$$\begin{aligned} (f - P_{\Phi_h}(f), \varphi) &= \sum_{K \in \mathcal{T}_h} (f - P_{\Phi_h}(f), \varphi)_K = \sum_{K \in \mathcal{T}_h} (f - P_{\Phi_h}(f), \varphi - \pi_0(\varphi))_K \\ &\leq \sum_{K \in \mathcal{T}_h} \|f - P_{\Phi_h}(f)\|_K \|\varphi - \pi_0(\varphi)\|_K \leq \sum_{K \in \mathcal{T}_h} \|f - P_{\Phi_h}(f)\|_K C_P^{\frac{1}{2}} h_K \|\nabla\varphi\|_K \\ &\leq \sum_{K \in \mathcal{T}_h} \|f - P_{\Phi_h}(f)\|_K \frac{C_P^{1/2}}{c_{\mathbf{S},K}^{1/2}} h_K \|\varphi\|_K \leq \eta_{\mathbf{R}} \|\varphi\|, \end{aligned}$$

employing the fact that zero-order polynomial are always in Φ_h , which implies $(f - P_{\Phi_h}(f), \varphi)_K = (f - P_{\Phi_h}(f), \varphi - \pi_0(\varphi))_K$, the Cauchy–Schwarz inequality, the Poincaré inequality (2.13), (2.7), and once again the Cauchy–Schwarz inequality. The assertion of the theorem follows by the fact that $\|\varphi\| = 1$. \square

Remark 6.2 (Nature of the estimate of Theorem 6.1). *Theorem 6.1 shows that the error in a vector field $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ is measured by how close \mathbf{u}_h is to a gradient of a $H_0^1(\Omega)$ -potential, up to the term $\eta_{\mathbf{R}}$.*

Remark 6.3 (Residual term). *The term $\eta_{\mathbf{R}}$ is sometimes referred to as the “data oscillation term”, because it only depends on the variation of the source function f , and considered separately from the actual a posteriori error estimate. If $f \in H^{l+1}(\mathcal{T}_h)$, this term is clearly of order $\mathcal{O}(h^{l+2})$. Thus it is superconvergent for those mixed finite elements methods where $\|\|\mathbf{u} - \mathbf{u}_h\|\|_*$ is of order $\mathcal{O}(h^{l+1})$, namely the Raviart–Thomas–Nédélec ones. This is, however, not always the case, namely for the Brezzi–Douglas–Marini family, where $\|\|\mathbf{u} - \mathbf{u}_h\|\|_*$ is of order $\mathcal{O}(h^{l+2})$. In this second case in particular, it is thus important not to separate $\eta_{\mathbf{R}}$ from the estimate and use $h_K \|f - P_{\Phi_h}(f)\|_K$ with the correct weighting given by the Poincaré constant C_P and the material constant $c_{\mathbf{S},K}$.*

Remark 6.4 (Efficiency of the abstract estimate of Theorem 6.1). *When the term $\eta_{\mathbf{R}}$ is superconvergent (see Remark 6.3), the estimate of Theorem 6.1 is optimal, i.e., it also represents a lower bound for the error, up to $\eta_{\mathbf{R}}$. We will in Theorem 6.7 below see that (local) efficiency also holds for $\eta_{\mathbf{R}}$ in any case. Another possibility to work with the term $\eta_{\mathbf{R}}$ is to derive estimates in the $\|\|\cdot\|\|_{*,\text{div}}$ -norm, as we do it below.*

Employing Theorem 3.3 instead of Theorem 3.1, we can easily get the following slightly improved version of Theorem 6.1:

Corollary 6.5 (Improved abstract a posteriori estimate for the flux and its efficiency). *Let \mathbf{u} be given by (1.4a)–(1.4b) and let $\mathbf{u}_h \in \mathbf{H}(\operatorname{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ be arbitrary. Then*

$$\|\|\mathbf{u} - \mathbf{u}_h\|\|_*^2 \leq \inf_{s \in H_0^1(\Omega)} \|\|\mathbf{u}_h + \mathbf{S}\nabla s\|\|_*^2 + \eta_{\mathbb{R}}^2 \leq \|\|\mathbf{u} - \mathbf{u}_h\|\|_*^2 + \eta_{\mathbb{R}}^2.$$

This version is particularly suitable to derive in a straightforward way an estimate in the $\|\|\cdot\|\|_{*,\operatorname{div}}$ -norm:

Theorem 6.6 (Abstract $\|\|\cdot\|\|_{*,\operatorname{div}}$ -norm a posteriori estimate for the flux and its efficiency). *Let \mathbf{u} be given by (1.4a)–(1.4b) and let $\mathbf{u}_h \in \mathbf{H}(\operatorname{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ be arbitrary. Then*

$$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,\operatorname{div}}^2 \leq \inf_{s \in H_0^1(\Omega)} \|\|\mathbf{u}_h + \mathbf{S}\nabla s\|\|_*^2 + \|f - P_{\Phi_h}(f)\|^2 + \eta_{\mathbb{R}}^2 \leq \|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,\operatorname{div}}^2 + \eta_{\mathbb{R}}^2.$$

Note that now the term $\eta_{\mathbb{R}}$, by its definition, converges by one order faster than $\|f - P_{\Phi_h}(f)\|$. Hence, in contrast to Theorem 6.1 (see also Remark 6.4), the $\|\|\cdot\|\|_{*,\operatorname{div}}$ -norm setting gives an optimal global abstract efficiency, up to the now always (also in the Brezzi–Douglas–Marini-like cases) superconvergent term $\eta_{\mathbb{R}}$. On the other hand, however, the term $\|f - P_{\Phi_h}(f)\|$ is generally of order $\mathcal{O}(h^{l+1})$, which may dominate the error in the Brezzi–Douglas–Marini-like cases, where $\|\|\mathbf{u} - \mathbf{u}_h\|\|_*$ is of order $\mathcal{O}(h^{l+2})$. As this term is entirely data-dependent, we believe that, although Theorem 6.6 gives an optimal abstract estimate and efficiency, $\|\|\cdot\|\|_{*,\operatorname{div}}$ -norm estimate is not suitable for a posteriori error estimation, as previously noted in, e.g., [42, Remark 3.4].

6.1.2 Fully computable estimates

Employing Corollary 6.5 and Theorem 6.6, we see that in order to give a fully computable a posteriori error estimate, we only need to specify a function $s \in H_0^1(\Omega)$. This choice is of course particularly important for the precision of the estimate and it is also crucial in order to prove a local efficiency. Clearly, $-\mathbf{S}\nabla s$ has to be as close as possible to \mathbf{u}_h . In view of this fact, we are led to first consider \tilde{p}_h given by (4.2a)–(4.2b) in the lowest-order Raviart–Thomas–Nédélec case and by (4.4a)–(4.4b) otherwise, for \mathbf{u}_h the mixed finite element solution given by (1.5a)–(1.5b). Recall that \mathbf{u}_h directly equals $-\mathbf{S}\nabla\tilde{p}_h$ under Assumption (A) and that \mathbf{u}_h is very close to $-\mathbf{S}\nabla\tilde{p}_h$ in general by (4.5). The last step is then to “smooth” \tilde{p}_h into a conforming function and for exactly this reason, we have in Section 2.3 introduced the Oswald interpolation operator. Hence (a general version of) our fully computable a posteriori error estimate is as follows:

Theorem 6.7 (Fully computable a posteriori estimates for the flux). *Let \mathbf{u} be given by (1.4a)–(1.4b) and let $\mathbf{u}_h \in \mathbf{H}(\operatorname{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ and $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let the potential estimator be given by*

$$\eta_{\mathbb{P},K} := \|\|\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\text{Os}}(\tilde{p}_h))\|\|_{*,K}, \quad (6.2)$$

the residual estimator by

$$\eta_{\mathbb{R},K} := \frac{C_{\mathbb{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - P_{\Phi_h}(f)\|_K, \quad (6.3)$$

and the divergence estimator by

$$\eta_{\mathbb{D},K} := \|f - P_{\Phi_h}(f)\|_K. \quad (6.4)$$

Then

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_*^2 &\leq \sum_{K \in \mathcal{T}_h} (\eta_{\mathbb{P},K}^2 + \eta_{\mathbb{R},K}^2), \\ \|\mathbf{u} - \mathbf{u}_h\|_{*,\text{div}}^2 &\leq \sum_{K \in \mathcal{T}_h} (\eta_{\mathbb{P},K}^2 + \eta_{\mathbb{R},K}^2 + \eta_{\mathbb{D},K}^2). \end{aligned}$$

Remark 6.8 (Constants in Theorem 6.7). *Remark that there are no undetermined constants in the estimates of Theorem 6.7. Moreover, the leading estimators $\eta_{\mathbb{P},K}$ and $\eta_{\mathbb{D},K}$ are completely constant-free and the only constants figure in the residual estimator $\eta_{\mathbb{R},K}$, which is likely to be superconvergent, see Remark 6.3.*

6.2 Estimates for the potential

We state and prove here our a posteriori error estimates for the potential, first in an abstract and then in a fully computable form.

6.2.1 Abstract estimates

Building on the approaches of [59, Lemma 7.1] and [39, Lemma 4.4], the following has been shown in [33, Theorem 4.1]:

Theorem 6.9 (Abstract a posteriori estimate for the potential and its efficiency). *Let p be the weak potential given by (1.2) and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then*

$$\begin{aligned} \|p - \tilde{p}_h\|^2 &\leq \inf_{s \in H_0^1(\Omega)} \|\tilde{p}_h - s\|^2 \\ &\quad + \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} ((f - \nabla \cdot \mathbf{t}, \varphi) - (\mathbf{S}\nabla\tilde{p}_h + \mathbf{t}, \nabla\varphi))^2 \\ &\leq 2\|p - \tilde{p}_h\|^2. \end{aligned} \tag{6.5}$$

Remark 6.10 (Nature of the estimate of Theorem 6.9). *Theorem 6.9 shows that the error in a scalar field $\tilde{p}_h \in H^1(\mathcal{T}_h)$ is measured by how close \tilde{p}_h is to the space $H_0^1(\Omega)$, how close the approximate diffusive flux $-\mathbf{S}\nabla\tilde{p}_h$ is to the space $\mathbf{H}(\text{div}, \Omega)$, and how small the residual $f - \nabla \cdot \mathbf{t}$ can be.*

6.2.2 Fully computable estimates in the energy norm

Analogously to the proof of Theorem 6.1 (cf. [33, Section 4] where the proof is given in full details in the discontinuous Galerkin setting), we have the following result. We again state it generally; in practice, it will be used for the postprocessed approximation \tilde{p}_h of Section 4.4 and the mixed finite element approximate flux \mathbf{u}_h given by (1.5a)–(1.5b). Recall in this respect that the postprocessed potential \tilde{p}_h belongs to $W_0(\mathcal{T}_h)$ and that $\|\cdot\|$ is a norm on $W_0(\mathcal{T}_h)$ thanks to the discrete Friedrichs inequality (2.8), whence the justification of the “energy norm” (and not just seminorm) in the title of this section.

Theorem 6.11 (Fully computable energy a posteriori estimate for the potential). *Let p be given by (1.2) and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ be arbitrary. Let the nonconformity estimator be given by*

$$\eta_{\text{NC},K} := \|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|_K, \tag{6.6}$$

the diffusive flux estimator by

$$\eta_{\text{DF},K} := \|\mathbf{u}_h + \mathbf{S}\nabla\tilde{p}_h\|_{*,K}, \quad (6.7)$$

and the residual estimator by (6.3). Then

$$\|p - \tilde{p}_h\|^2 \leq \sum_{K \in \mathcal{T}_h} \{\eta_{\text{NC},K}^2 + (\eta_{\text{DF},K} + \eta_{\text{R},K})^2\}.$$

Remark 6.12 (Constants in Theorem 6.11). *We note that similar observation to that of Remark 6.8 holds here true as well.*

6.2.3 Fully computable estimates in the $L^2(\Omega)$ -norm

The energy norm estimate of the previous section is aimed to be used for the postprocessed approximation \tilde{p}_h of Section 4.4. Using this result, we now derive $L^2(\Omega)$ -norm estimates, first for \tilde{p}_h and then for the original approximate potential p_h . As it will however appear, these estimates are somewhat “less nice” than those of the previous section, as they in particular feature several, albeit known, constants in the leading terms. In our opinion, $L^2(\Omega)$ -norm is not optimal for a posteriori error estimates in mixed finite elements and we believe that trying to directly and only derive estimates for p_h in the $L^2(\Omega)$ -norm was the bottleneck of a lot of previous works on a posteriori error estimates in mixed finite element methods.

We first give an $L^2(\Omega)$ -norm estimate for \tilde{p}_h , again in the most general setting possible:

Corollary 6.13 (A posteriori estimate for \tilde{p}_h in the $L^2(\Omega)$ -norm). *Let p be given by (1.2) and let $\tilde{p}_h \in W_0(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ be arbitrary. Then*

$$\|p - \tilde{p}_h\|^2 \leq \frac{C_{\text{DF}}}{c_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \{\eta_{\text{NC},K}^2 + (\eta_{\text{DF},K} + \eta_{\text{R},K})^2\},$$

where $\eta_{\text{NC},K}$, $\eta_{\text{DF},K}$, and $\eta_{\text{R},K}$ are given respectively by (6.6), (6.7), and (6.3).

Proof. Immediate from Theorem 6.11, using the fact that $(p - \tilde{p}_h) \in W_0(\mathcal{T}_h)$, the discrete Friedrichs inequality (2.8), and (2.7). \square

We conclude this section by an $L^2(\Omega)$ -norm estimate for p_h , following trivially from Corollary 6.13 by the triangle inequality; in practice, again p_h and \mathbf{u}_h are given by (1.5a)–(1.5b) and \tilde{p}_h by (4.2a)–(4.2b) or (4.4a)–(4.4b):

Corollary 6.14 (A posteriori estimate for p_h in the $L^2(\Omega)$ -norm). *Let p be given by (1.2) and let $p_h \in \Phi_h$, $\tilde{p}_h \in W_0(\mathcal{T}_h)$, and $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$ be arbitrary. Then*

$$\|p - p_h\| \leq \left\{ \frac{C_{\text{DF}}}{c_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \{\eta_{\text{NC},K}^2 + (\eta_{\text{DF},K} + \eta_{\text{R},K})^2\} \right\}^{\frac{1}{2}} + \|\tilde{p}_h - p_h\|,$$

where $\eta_{\text{NC},K}$, $\eta_{\text{DF},K}$, and $\eta_{\text{R},K}$ are given respectively by (6.6), (6.7), and (6.3).

6.3 Local efficiency

To conclude this section, we prove here local efficiency of the a posteriori error estimators of Theorems 6.7 and 6.11.

Theorem 6.15 (Local efficiency of estimators of Theorems 6.7 and 6.11). *Let f be piecewise polynomial of order m and let \mathbf{u}, p be given by (1.4a)–(1.4b). Let next \mathcal{T}_h be shape-regular and let \mathbf{u}_h and p_h be given by (1.5a)–(1.5b) and \tilde{p}_h by (4.2a)–(4.2b) or (4.4a)–(4.4b). Let finally the a posteriori error estimators $\eta_{P,K}$, $\eta_{R,K}$, $\eta_{NC,K}$, and $\eta_{DF,K}$ be given respectively by (6.2), (6.3), (6.6), and (6.7). Then*

$$\begin{aligned}\eta_{P,K} &\leq \eta_{DF,K} + \eta_{NC,K}, \\ \eta_{DF,K} &\leq \|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,K} + \|p - \tilde{p}_h\|_K, \\ \eta_{NC,K} &\leq C \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},\mathcal{T}_K}}} \|p - \tilde{p}_h\|_{\mathcal{T}_K}, \\ \eta_{R,K} &\leq \tilde{C} \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}} \|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,K},\end{aligned}$$

where the constant C depends only on the space dimension d , the maximal polynomial degree n of \tilde{p}_h , and the shape regularity parameter $\kappa_{\mathcal{T}}$ and \tilde{C} depends only on d , the polynomial degree m of f , and $\kappa_{\mathcal{T}}$.

Proof. We have for $\eta_{DF,K}$

$$\eta_{DF,K} \leq \|\|\mathbf{u}_h + \mathbf{S}\nabla p\|\|_{*,K} + \|\|\mathbf{S}\nabla p - \mathbf{S}\nabla \tilde{p}_h\|\|_{*,K} = \|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,K} + \|p - \tilde{p}_h\|_K$$

by the triangle inequality and (2.10). The estimate for $\eta_{P,K}$ is obtained analogously. Next, the inequality

$$h_\sigma^{-\frac{1}{2}} \|\llbracket \tilde{p}_h \rrbracket\|_\sigma \leq C \sum_{L;\sigma \in \mathcal{E}_L} \|\nabla(\tilde{p}_h - \varphi)\|_L$$

was established in [3, Theorem 10] for $\tilde{p}_h \in W_0(\mathcal{T}_h)$, simplicial meshes, $\sigma \in \mathcal{E}_h^{\text{int}}$, and an arbitrary $\varphi \in H^1(\Omega)$. It generalizes easily to rectangular parallelepipeds and to the case $\sigma \in \mathcal{E}_h^{\text{ext}}$ and $\varphi \in H_0^1(\Omega)$; here C depends only on d and $\kappa_{\mathcal{T}}$. Thus we have for the nonconformity estimator

$$\begin{aligned}\eta_{NC,K}^2 &= \|\|\tilde{p}_h - \mathcal{I}_{Os}(\tilde{p}_h)\|\|_K^2 \leq CC_{\mathbf{S},K} \sum_{\sigma \in \tilde{\mathcal{E}}_K} h_\sigma^{-1} \|\llbracket \tilde{p}_h \rrbracket\|_\sigma^2 \\ &\leq CC_{\mathbf{S},K} \sum_{L \in \mathcal{T}_K} \|\nabla(p - \tilde{p}_h)\|_L^2 \leq C \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},\mathcal{T}_K}} \sum_{L \in \mathcal{T}_K} \|p - \tilde{p}_h\|_L^2,\end{aligned}$$

using Lemma 2.1 and the above estimate, with C depending only on d , n , and $\kappa_{\mathcal{T}}$. Finally,

$$\|f - P_{\Phi_h}(f)\|_K = \|f - \nabla \cdot \mathbf{u}_h\|_K \leq CC_{\mathbf{S},K}^{1/2} h_K^{-1} \|\|\mathbf{u} - \mathbf{u}_h\|\|_{*,K}$$

with C depending only on d , $\kappa_{\mathcal{T}}$, and m follows standardly by using the element bubble function, the equivalence of norms on finite-dimensional spaces, the definition (1.2) of the weak solution, the Green theorem, the Cauchy–Schwarz inequality, the definition (2.9) of the energy norm, and the inverse inequality, cf. [55] or [59, Lemma 7.6]. Hence the estimate for $\eta_{R,K}$ follows. \square

Remark 6.16 (Maximal polynomial degree n of \tilde{p}_h). *Note that by the postprocessing of Section 4.4, $\tilde{p}_h \in M_h$, usually a nonconforming polynomial space enriched with bubbles. The “maximal polynomial degree n of \tilde{p}_h ”, with the notation of Section 2.3, will then simply correspond to the highest polynomial/bubble degree used.*

6.4 Extensions

We present here two extensions of the previous results. First of all, following Bernardi and Verfürth [14] and Ainsworth [5] and using the Oswald interpolation operator with diffusion tensor-dependent weights as in [33], one can obtain estimates robust with respect to inhomogeneities under the “monotonicity” assumption. Secondly, improving on the situation studied in [33] in the discontinuous Galerkin setting, we show that for mixed finite elements, our estimates are robust with respect to all inhomogeneities, anisotropies, polynomial degree, and mesh regularity for the error in $\|\mathbf{u} - \mathbf{u}_h\|_*$ and $\|p - s_h\|$, where $s_h \in H_0^1(\Omega)$ is arbitrary. The same holds true for the triple error in \mathbf{u} , s_h , and \tilde{p}_h .

6.4.1 Estimates robust with respect to inhomogeneities under the “monotonicity” assumption

With the notation of Section 2.3, let

$$\mathcal{I}_{\text{Os},\mathbf{S}}(\varphi_h)(V) = \frac{1}{\sum_{K \in \mathcal{T}_V} C_{\mathbf{S},K}^{1/2}} \sum_{K \in \mathcal{T}_V} C_{\mathbf{S},K}^{1/2} \varphi_h|_K(V).$$

Then all the estimates of Sections 6.1 and 6.2 hold true with \mathcal{I}_{Os} replaced by $\mathcal{I}_{\text{Os},\mathbf{S}}$. Clearly, the difference between \mathcal{I}_{Os} and $\mathcal{I}_{\text{Os},\mathbf{S}}$ is the use of the diffusion tensor-dependent weights in the latter. We first make the following assumption (cf. [14, Hypothesis 2.7]):

Assumption (C) (Monotonicity of the distribution of $C_{\mathbf{S},K}$)

For any two elements $L, M \in \mathcal{T}_h$ which share at least one point, there exists a connected path passing from L to M through element sides such that the function $C_{\mathbf{S},K}$ is monotone along this path.

We then have the following result:

Theorem 6.17 (Local efficiency robust with respect to inhomogeneities under Assumption (C)). *Let all the assumptions of Theorem 6.15 hold, with \mathcal{I}_{Os} replaced by $\mathcal{I}_{\text{Os},\mathbf{S}}$. Let next Assumption (C) hold. Then*

$$\begin{aligned} \eta_{\text{P},K} &\leq \eta_{\text{DF},K} + \eta_{\text{NC},K}, \\ \eta_{\text{DF},K} &\leq \|\mathbf{u} - \mathbf{u}_h\|_{*,K} + \|p - \tilde{p}_h\|_K, \\ \eta_{\text{NC},K} &\leq C \max_{K \in \mathcal{T}_K} \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}} \|p - \tilde{p}_h\|_{\mathcal{T}_K}, \\ \eta_{\text{R},K} &\leq \tilde{C} \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}} \|\mathbf{u} - \mathbf{u}_h\|_{*,K}, \end{aligned}$$

where the constant C depends only on the space dimension d , the maximal polynomial degree n of \tilde{p}_h , and the shape regularity parameter $\kappa_{\mathcal{T}}$ and \tilde{C} depends only on d , the polynomial degree m of f , and $\kappa_{\mathcal{T}}$.

Unfortunately, for the above robustness result, the “monotonicity” assumption is crucial. This excludes the most interesting cases where the weak solution is singular. For conforming discretizations, estimates robust in all cases are presented in [60]. The generalization to the nonconforming case seems not to be straightforward and represents an ongoing work.

6.4.2 Estimates robust with respect to inhomogeneities, anisotropies, polynomial degree, and mesh regularity for flux- and potential-conforming approximations

Combining Theorems 6.7 and 6.11 for the upper bound and the triangle inequality and the estimate for $\eta_{R,K}$ from Theorem 6.15 for the local efficiency, we can state the following result:

Theorem 6.18 (Optimal a posteriori error estimate for flux- and potential-conforming approximations). *Let \mathbf{u}, p be given by (1.4a)–(1.4b) and let $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = P_{\Phi_h}(f)$, $\tilde{p}_h \in H^1(\mathcal{T}_h)$, and $s_h \in H_0^1(\Omega)$ be arbitrary. Let next the a posteriori error estimators $\eta_{P,K}$, $\eta_{R,K}$, $\eta_{\text{NC},K}$, and $\eta_{\text{DF},K}$ be given respectively by (6.2), (6.3), (6.6), and (6.7), with $\mathcal{I}_{\text{Os}}(\tilde{p}_h)$ replaced by s_h . Then*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_*^2 + \|p - s_h\|^2 &\leq \sum_{K \in \mathcal{T}_h} \{\eta_{P,K}^2 + \eta_{R,K}^2 + (\eta_{P,K} + \eta_{R,K})^2\}, \\ \|\mathbf{u} - \mathbf{u}_h\|_*^2 + \|p - \tilde{p}_h\|^2 + \|p - s_h\|^2 &\leq \sum_{K \in \mathcal{T}_h} \{\eta_{P,K}^2 + \eta_{R,K}^2 + (\eta_{P,K} + \eta_{R,K})^2 \\ &\quad + \eta_{\text{NC},K}^2 + (\eta_{\text{DF},K} + \eta_{R,K})^2\}. \end{aligned}$$

Moreover,

$$\begin{aligned} \eta_{P,K} &\leq \|\mathbf{u} - \mathbf{u}_h\|_{*,K} + \|p - s_h\|_K, \\ \eta_{\text{DF},K} &\leq \|\mathbf{u} - \mathbf{u}_h\|_{*,K} + \|p - \tilde{p}_h\|_K, \\ \eta_{\text{NC},K} &\leq \|p - \tilde{p}_h\|_K + \|p - s_h\|_K. \end{aligned}$$

Finally, the residual estimators $\eta_{R,K}$ represent a higher-order term in the Raviart–Thomas–Nédéc case whenever $f \in H^{l+1}(\mathcal{T}_h)$, see Remark 6.3. In any case, when f is piecewise polynomial of order m and \mathcal{T}_h shape-regular, then

$$\eta_{R,K} \leq \tilde{C} \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}} \|\mathbf{u} - \mathbf{u}_h\|_{*,K},$$

where \tilde{C} depends only on d , the polynomial degree m of f , and $\kappa_{\mathcal{T}}$.

Remark 6.19 (Theorem 6.18). *Theorem 6.18 shows that the a posteriori error estimates presented in this paper also give an upper bound for the error including $\|p - s_h\|$ ($\|p - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|$ in the mixed finite element setting). In this case, possibly up to the residual term, they become robust with respect to all the diffusion tensor \mathbf{S} , the space dimension d , the maximal polynomial degree of \mathbf{u}_h , s_h , and \tilde{p}_h , and the mesh shape regularity. Moreover, a maximal overestimation factor (effectivity index) is guaranteed. Note also that in order to solve the possible issue of the residual term, the estimates can be given for $\|\mathbf{u} - \mathbf{u}_h\|_{*,\text{div}}$ as in Theorem 6.7.*

Remark 6.20 (Optimal efficiency). *It is to the estimates of Theorem 6.18 that the results of Repin et al. [51] should be compared. Basically, giving optimal a posteriori error estimates for approximations which are both flux- and potential-conforming is trivial.*

7 Complements on mixed finite element methods

We give here some complements on mixed finite element methods that are to our knowledge not presented in any of the standard textbooks or other references mentioned in the introduction. We start by showing that under the assumption that the source function f belongs to the space Φ_h , some orthogonal projection relations are valid in the mixed finite element method, parallel and complementary to the conforming finite element method. We next show that mixed finite element approximate solutions are directly equal to or very close to some generalized weak solutions.

7.1 Orthogonal projection properties

We first give the following characterization, valid for any mixed finite element scheme.

Theorem 7.1 (Vector orthogonal projection property). *Let $f \in \Phi_h$, let p be given by (1.4a)–(1.4b), and let $\mathbf{u}_h \in \mathbf{H}(\operatorname{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = f$ be arbitrary. Then*

$$\| \mathbf{S}\nabla p + \mathbf{u}_h \|_* = \inf_{s \in H_0^1(\Omega)} \| \mathbf{u}_h + \mathbf{S}\nabla s \|_*, \quad (7.1)$$

or, equivalently,

$$\mathcal{A}(\mathbf{S}\nabla p + \mathbf{u}_h, \mathbf{S}\nabla \varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega). \quad (7.2)$$

Proof. Property (7.1) follows immediately from (6.1) under the assumption $f \in \Phi_h$. To see (7.2) is then standard; alternatively, putting $\mathbf{w} = \mathbf{u}_h$ in (3.1) and using $f \in \Phi_h$ implies

$$\mathcal{A}(-\mathbf{u}_h, \mathbf{S}\nabla \varphi) = (-\mathbf{u}_h, \nabla \varphi) = (f, \varphi)$$

by the Green theorem and (1.5b). Hence $\psi = p$ and (7.2) coincides with (3.4). \square

Remark 7.2 (Vector orthogonal projection property). *In the conforming finite element method for (1.1a)–(1.1b), the approximate solution $p_h \in X_h$ with $X_h := \mathbb{R}_k(\mathcal{T}_h) \cap H_0^1(\Omega)$ is characterized by*

$$\mathcal{B}(p_h, \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in X_h$$

and satisfies

$$\begin{aligned} \| p - p_h \| &= \inf_{s_h \in X_h} \| p - s_h \|, \\ \mathcal{B}(p - p_h, \varphi_h) &= 0 \quad \forall \varphi_h \in X_h. \end{aligned}$$

This means that it is the $H_0^1(\Omega)$ -orthogonal projection of the exact potential p onto X_h with respect to the scalar product $\mathcal{B}(\cdot, \cdot)$ (and the associated scalar energy norm (2.7)). We denote this projection by P_{X_h} . Theorem 7.1 says that in the mixed finite element method, under the condition that $f \in \Phi_h$, the exact flux $\mathbf{u} = -\mathbf{S}\nabla p$ is the $\mathbf{L}^2(\Omega)$ -orthogonal projection of the approximate flux \mathbf{u}_h onto $\mathbf{S}\nabla H_0^1(\Omega)$ with respect to the scalar product $\mathcal{A}(\cdot, \cdot)$ (and the associated vector energy norm (2.9)). Note the parallel but also the exchange of the roles between the exact and approximate solutions: in the conforming finite element method, the approximate solution is the orthogonal projection of the exact one, whereas in the mixed finite element case, the exact solution is the orthogonal projection of the approximate one.

The following characterization is only valid in the lowest-order Raviart–Thomas–Nédélec cases:

Theorem 7.3 (Scalar orthogonal projection property). *Let Assumption (A) hold, let $f \in \Phi_h$, and let p be given by (1.4a)–(1.4b), \mathbf{u}_h, p_h by (1.5a)–(1.5b), and \tilde{p}_h by (4.2a)–(4.2b). Then*

$$\| p - \tilde{p}_h \| = \inf_{s \in H_0^1(\Omega)} \| \tilde{p}_h - s \|,$$

or, equivalently,

$$\mathcal{B}(p - \tilde{p}_h, \varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega).$$

Proof. Immediate from (7.1) and (7.2) using (1.3a), (4.2a), and (2.10). \square

Remark 7.4 (Scalar orthogonal projection property). *Under assumptions of Theorem 7.3, the exact potential p is the $W_0(\mathcal{T}_h)$ -orthogonal projection of the approximate postprocessed potential \tilde{p}_h onto $H_0^1(\Omega)$ with respect to the scalar product $\mathcal{B}(\cdot, \cdot)$ (and the associated scalar energy norm (2.7)). We denote this projection by $P_{H_0^1}$. Here, the parallel to the conforming finite element method is even stronger, compare it with Remark 7.2. The situation is graphically illustrated in Figure 1.*

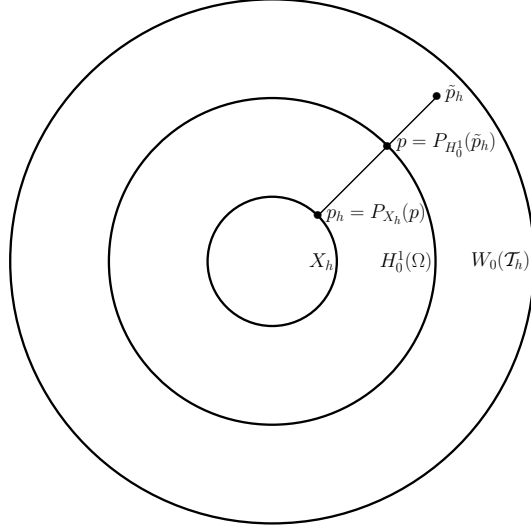


Figure 1: Graphical visualization of the relations between the postprocessed lowest-order mixed finite solution \tilde{p}_h , weak solution p , and conforming finite element solution p_h when $f \in \Phi_h$.

7.2 Generalized weak solutions and mixed finite elements

We develop here the ideas of [59, Section 5.4] on the relation between mixed finite element approximate solutions and certain generalized weak solutions. For some results comparing the mixed and (generalized) finite element approximate solutions, we refer to Babuška and Osborn [12] and Falk and Osborn [36].

By a generalized weak solution, we understand a function $\tilde{p} \in W_h(\mathcal{T}_h)$ such that

$$(\mathbf{S}\nabla\tilde{p}, \nabla\varphi) = (f, \varphi) \quad \forall \varphi \in W_h(\mathcal{T}_h). \quad (7.3)$$

Note that (2.7), (2.11a), and the discrete Friedrichs inequality (2.8) ensure the existence and uniqueness of the solution of (7.3). This generalized weak solution is dependent on the given mesh \mathcal{T}_h and also on the normal components of the space \mathbf{V}_h by the definition (2.1b) of the space $W_h(\mathcal{T}_h)$. Note also that $H_0^1(\Omega) \subset W_h(\mathcal{T}_h)$, whence the term “generalized”.

Theorem 7.5 (A posteriori estimates for the generalized weak solutions). *Let \tilde{p} be given by (7.3), $\tilde{\mathbf{u}}$ by $\tilde{\mathbf{u}} := -\mathbf{S}\nabla\tilde{p}$, \mathbf{u}_h , p_h by (1.5a)–(1.5b), and \tilde{p}_h by (4.2a)–(4.2b) or (4.4a)–(4.4b). Then*

$$\begin{aligned} \|\tilde{\mathbf{u}} - \mathbf{u}_h\|^2 &\leq \sum_{K \in \mathcal{T}_h} (\eta_{\text{DF},K}^2 + \eta_{\text{R},K}^2), \\ \|\tilde{p} - \tilde{p}_h\|^2 &\leq \sum_{K \in \mathcal{T}_h} (\eta_{\text{DF},K} + \eta_{\text{R},K})^2. \end{aligned}$$

where the diffusive flux estimator $\eta_{\text{DF},K}$ is given by (6.7) and the residual estimator $\eta_{\text{R},K}$ by (6.3).

Proof. By replacing $H_0^1(\Omega)$ by $W_h(\mathcal{T}_h)$ in Theorem 3.3, putting $\mathbf{v} = \tilde{\mathbf{u}}$, $\mathbf{w} = \mathbf{u}_h$, and using (3.3), one comes to the equivalent of (3.2)–(3.3) in the form

$$\|\tilde{\mathbf{u}} - \mathbf{u}_h\|_*^2 = \inf_{s \in W_h(\mathcal{T}_h)} \|\mathbf{u}_h + \mathbf{S}\nabla s\|_*^2 + \mathcal{A} \left(\tilde{\mathbf{u}} - \mathbf{u}_h, \frac{\tilde{\mathbf{u}} + \mathbf{S}\nabla\psi}{\|\tilde{\mathbf{u}} + \mathbf{S}\nabla\psi\|_*} \right)^2.$$

We next put $\varphi := (\tilde{p} - \psi) / \|\tilde{p} - \psi\| \in W_h(\mathcal{T}_h)$ and rewrite the second term of the above expression as $\mathcal{A}(\tilde{\mathbf{u}} - \mathbf{u}_h, -\mathbf{S}\nabla\varphi)$, employing $\tilde{\mathbf{u}} = -\mathbf{S}\nabla\tilde{p}$ and (2.10). Next, by (2.3) and the definition of the generalized weak solution (7.3),

$$\mathcal{A}(\tilde{\mathbf{u}}, -\mathbf{S}\nabla\varphi) = (f, \varphi),$$

whereas

$$\mathcal{A}(\mathbf{u}_h, -\mathbf{S}\nabla\varphi) = -(\mathbf{u}_h, \nabla\varphi) = \sum_{K \in \mathcal{T}_h} \{(\nabla \cdot \mathbf{u}_h, \varphi)_K - \langle \mathbf{u}_h \cdot \mathbf{n}, \varphi \rangle_{\partial K}\} = (P_{\Phi_h}(f), \varphi) \quad (7.4)$$

by (2.3), the Green theorem, the fact that $\mathbf{u}_h \in \mathbf{V}_h$ and $\varphi \in W_h(\mathcal{T}_h)$, and (1.5b). Note the importance of the definition (2.1b) of the space $W_h(\mathcal{T}_h)$, by which the term $\sum_{K \in \mathcal{T}_h} \langle \mathbf{u}_h \cdot \mathbf{n}, \varphi \rangle_{\partial K} = \sum_{\sigma \in \mathcal{E}_h} \langle \mathbf{u}_h \cdot \mathbf{n}, \llbracket \varphi \rrbracket \rangle_\sigma$ disappears. Hence

$$\mathcal{A}(\tilde{\mathbf{u}} - \mathbf{u}_h, -\mathbf{S}\nabla\varphi) = (f - P_{\Phi_h}(f), \varphi).$$

Estimating this term exactly as in the proof of Theorem 6.1 and putting $s = \tilde{p}_h$, the estimate for $\tilde{\mathbf{u}} - \mathbf{u}_h$ follows.

Similarly as in the vector case, instead of (6.5), one in the present setting gets

$$\|\tilde{p} - \tilde{p}_h\|^2 \leq \inf_{s \in W_h(\mathcal{T}_h)} \|\tilde{p}_h - s\|^2 + \sup_{\varphi \in W_h(\mathcal{T}_h), \|\varphi\|=1} \mathcal{B}(\tilde{p} - \tilde{p}_h, \varphi)^2.$$

Whereas the first term disappears by putting $s = \tilde{p}_h$, we have for the second one, adding and subtracting $(\mathbf{u}_h, \nabla\varphi)$,

$$\mathcal{B}(\tilde{p} - \tilde{p}_h, \varphi) = (f, \varphi) - (\mathbf{S}\nabla\tilde{p}_h + \mathbf{u}_h, \nabla\varphi) + (\mathbf{u}_h, \nabla\varphi).$$

The estimate for $\tilde{p} - \tilde{p}_h$ now follows by (7.4) and the Cauchy–Schwarz inequality. \square

Remark 7.6 (A posteriori estimates for the generalized weak solutions). *Note that the essential difference of the estimates of Theorem 7.5 and of those of Theorems 6.7 and 6.11 are that the nonconformity estimator $\eta_{\text{NC},K}$ given by (6.6) and the potential estimator $\eta_{\text{P},K}$ given by (6.2), the two estimators penalizing the nonconformity in \tilde{p}_h through the introduction of the Oswald interpolate $\mathcal{I}_{\text{Os}}(\tilde{p}_h)$, are not present, since the generalized solution \tilde{p} itself is in the space $W_h(\mathcal{T}_h)$ as \tilde{p}_h . Note also that under Assumption (A), the diffusive flux estimators $\eta_{\text{DF},K}$ vanish, whereas for $f \in \Phi_h$, the residual estimators $\eta_{\text{R},K}$ vanish. Thus in the lowest-order Raviart–Thomas–Nédéc case and for elementwise constant f , $\tilde{p} = \tilde{p}_h$ (and $\tilde{\mathbf{u}} = \mathbf{u}_h$). We refer to [59, Sections 5.4 and 5.6] for a more detailed discussion of this special case.*

The proof of the following theorem is straightforward, using the same techniques as those in the proof of Theorem 6.15.

Theorem 7.7 (Local efficiency of estimators of Theorem 7.5). *Let the assumptions of Theorem 7.5 be verified. Then*

$$\eta_{\text{DF},K} \leq \|\tilde{\mathbf{u}} - \mathbf{u}_h\|_{*,K} + \|\tilde{p} - \tilde{p}_h\|_K.$$

Moreover, the residual estimators $\eta_{\text{R},K}$ represent a higher-order term in the Raviart–Thomas–Nédéc case whenever $f \in H^{l+1}(\mathcal{T}_h)$, see Remark 6.3. In any case, when f is piecewise polynomial of order m and \mathcal{T}_h shape-regular, then

$$\eta_{\text{R},K} \leq \tilde{C} \sqrt{\frac{C_{\text{S},K}}{c_{\text{S},K}}} \|\tilde{\mathbf{u}} - \mathbf{u}_h\|_{*,K},$$

where \tilde{C} depends only on d , the polynomial degree m of f , and $\kappa_{\mathcal{T}}$.

Remark 7.8 (Local efficiency of estimators of Theorem 7.5). *Note that, possibly up to the residual term, the a posteriori error estimate of Theorem 7.5 is according to Theorem 7.7 robust with respect to all the diffusion tensor \mathbf{S} , the space dimension d , the maximal polynomial degree n of \tilde{p}_h , and the mesh shape regularity.*

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