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Self-similarity parameter estimation and reproduction property for non-Gaussian Hermite processes

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Abstract

We consider the class of all the Hermite processes $(Z_t^{(q,H)})_{t \in [0,1]}$ of order $q \in \mathbf{N}^*$ and with Hurst parameter $H \in (\frac{1}{2}, 1)$. The process $Z^{(q,H)}$ is H -selfsimilar, it has stationary increments and it exhibits long-range dependence identical to that of fractional Brownian motion (fBm). For $q = 1$, $Z^{(1,H)}$ is fBm, which is Gaussian; for $q = 2$, $Z^{(2,H)}$ is the Rosenblatt process, which lives in the second Wiener chaos; for any $q > 2$, $Z^{(q,H)}$ is a process in the q th Wiener chaos. We study the variations of $Z^{(q,H)}$ for any q , by using multiple Wiener -Itô stochastic integrals and Malliavin calculus.

We prove a reproduction property for this class of processes in the sense that the terms appearing in the chaotic decomposition of their variations give rise to other Hermite processes of different orders and with different Hurst parameters. We apply our results to construct a strongly consistent estimator for the self-similarity parameter H from discrete observations of $Z^{(q,H)}$; the asymptotics of this estimator, after appropriate normalization, are proved to be distributed like a Rosenblatt random variable (value at time 1 of a Rosenblatt process).with self-similarity parameter $1 + 2(H - 1)/q$.

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Key words: multiple Wiener integral, Hermite process, fractional Brownian motion, non-central limit theorem, quadratic variation, self-similarity, Malliavin calculus, parameter estimation.

1 Introduction

1.1 Background and motivation

The variations of a stochastic process play a crucial role in its probabilistic and statistical analysis. Best known is the quadratic variation of a semimartingale, which is crucial for its Itô formula; quadratic variation also has a direct utility in practice, in estimating unknown parameters, such as volatility in financial models, in the so-called “historical” context. For self-similar stochastic processes, the study of their variations constitutes a fundamental tool to construct good estimators of their self-similarity parameters. These processes are well suited to model various phenomena where long memory is an important factor (internet traffic, hydrology, econometrics, among others). The most important modeling task is then to determine or estimate the self-similarity parameter, because it is also typically responsible for the process’s long memory and its regularity properties. Estimating this parameter is thus an important research direction in theory

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and practice. Several approaches, such as wavelets, variations, maximum likelihood methods, have been proposed. We refer to the monograph [1] for a complete exposition.

The family of Gaussian processes known as fractional Brownian motion (fBm) is particularly interesting, and most popular among self-similar processes, because of fBm's stationary increments, its clear similarities and differences with standard Brownian motion, and the fact that its self-similarity parameter H , known as the Hurst parameter, is also immediately interpreted as both the memory length parameter (the correlation of unit length increments which are n time units apart decays slowly at the speed n^{2H-2}) and the regularity parameter (fBm is α -Hölder continuous on any bounded time interval for any $\alpha < H$).

Soon after fBm's inception, the study of its variations began in the 1970's and 1980's; of interest to us in the present article are several such studies of variations which uncovered a generalization of fBm to non-Gaussian processes known as the Rosenblatt process and other Hermite processes: [2], [5], [7], [20] or [21]. We briefly recall some relevant basic facts. We consider $(B_t^H)_{t \in [0,1]}$ a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. As such, B^H is the continuous centered Gaussian process with covariance function

$$R_H(t, s) = \mathbf{E}[B_t B_s] = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, 1].$$

Equivalently, $B_0 = 0$ and $\mathbf{E}[(B_s - B_t)^2] = |t - s|^{2H}$. It is the only continuous Gaussian process H which is self-similar with stationary increments. Consider the regular partition of the interval $[0, 1]$ as $0 = t_0 < t_1 < \dots < t_N = 1$ with $t_i = \frac{i}{N}$, and define the following q -variations

$$v_N^{(q)} = \sum_{i=0}^{N-1} H_q \left(\frac{B_{t_{i+1}}^H - B_{t_i}^H}{\left(\mathbf{E} \left[(B_{t_{i+1}}^H - B_{t_i}^H)^2 \right] \right)^{\frac{1}{2}}} \right) = \sum_{i=0}^{N-1} H_q \left((t_{i+1} - t_i)^{-H} (B_{t_{i+1}}^H - B_{t_i}^H) \right) \quad (1)$$

for any integer $q \geq 2$, where H_q is the Hermite polynomial of degree q . Then it follows from the above references that:

- for $0 \leq H \leq 1 - \frac{1}{2q}$ the limit in distribution of $N^{-\frac{1}{2}} v_N^{(q)}$ is a centered Gaussian random variable,
- for $1 - \frac{1}{2q} < H < 1$ the limit of $N^{q(1-H)-1} v_N^{(q)}$ is a Hermite random variable of order q with self-similarity parameter $q(H - 1) + 1$.

This latter random variable is non-Gaussian; it equals the value at time 1 of a Hermite process, which is a stochastic process in the q th Wiener chaos with the same covariance structure as fBm; as such, the Hermite process has stationary increments and shares the same self-similarity, regularity, and long memory properties as fBm; see Definition 1.

Because of a natural coupling, the last limit above also holds in $L^2(\Omega)$ (see [14]). In the critical case $H = 1 - \frac{1}{2q}$ the limit is still Gaussian but the normalization involves a logarithm. These results are widely applied to estimation problems; to avoid the barrier $H = \frac{3}{4}$ that occurs in the case $q = 2$, one can use "higher-order filters", which means that the increments of the fBm are replaced by higher-order increments, such as discrete versions of higher-order derivatives, in order to obtain a Gaussian limit for any H (see [9], [8], [4]).

Nonetheless, the appearance of Hermite random variables in the above non-central limit theorems begs the study of Hermite processes as such. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, self-similarity, and Hölder-regularity, allowing significant deviation from fBm and other Gaussian processes, without having to invoke non-linear stochastic differential equations based on fBm, thereby avoiding the notorious issues associated with such equations (see [15]). Just as in the case of fBm, the estimation of the Hermite process's parameter H is crucial for proper modeling; to our knowledge it has not been treated in the literature. We choose to tackle this issue by using variations methods, to find out how the above central and non-central limit theorems fit into a larger picture.

A note regarding the usage of the letter q : in the known results above, this letter was used to discuss the q -th variations (in the sense Hermite polynomials) for fBm data; in this article, in contrast to these classical works, we will use data from a Hermite process of order q , and will only consider its quadratic variation. The historical motivation in the above non-central limit theorem explains this coincidence of notation.

Lastly, we mention that various very recent and interesting results, which are beyond the scope of this article, have been proven for weighted power variations of stochastic processes such as fractional Brownian motion (see [11], [14], [13]), fractional Brownian sheets (see [18]), iterated Brownian motion (see [12]) or the solution of the stochastic heat equation (see [19] or [3]).

1.2 Main results: summary and discussion

In this article, we show results that are interesting from a theoretical viewpoint, such as the reproducing properties (the variations of Hermite processes give birth to other Hermite processes); and we provide an application to parameter estimation, in which care is taken to show that the estimators can be evaluated practically.

Let $Z^{(q,H)}$ be a Hermite process of order $q \in \mathbf{N}^*$ with self-similarity parameter $H \in (\frac{1}{2}, 1)$ (see Definition 1 further below). Define the *centered quadratic variation statistic*

$$V_N := \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2}{\mathbf{E} \left[\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2 \right]} - 1 \right].$$

Also define some constants which will recur throughout this article:

$$\begin{aligned} d(H) &: = \frac{(2(2H-1))^{1/2}}{(H+1)H^{1/2}}, \\ H' &: = 1 + \frac{(H-1)}{q}. \end{aligned} \tag{2}$$

1.2.1 Convergence of V_N

We prove that, under suitable normalization, the sequence $\{V_N\}_{N \in \mathbf{N}}$ converges in $L^2(\Omega)$ to a Rosenblatt random variable.

Theorem 1 *Let $H \in (1/2, 1)$ and $q \in \mathbf{N} \setminus \{0, 1\}$. Let $Z^{(q,H)}$ be a Hermite process of order q and self-similarity parameter H (see Definition 1). Define*

$$c = (q!)^{-1} \frac{(4H' - 3)^{1/2}(4H' - 2)^{1/2}}{2d(H)^2(H'(2H' - 1))^{q-1}}.$$

Then $cN^{(2-2H)/q}V_N$ converges in $L^2(\Omega)$ to a standard Rosenblatt random variable with parameter $H'' := \frac{2(H-1)}{q} + 1$, which is the value at time 1 of a Hermite process of order 2 and self-similarity parameter H'' .

The Rosenblatt random variable is a double integral with respect to the same Wiener process used to define the Hermite process; it is thus an element of the second Wiener chaos. In reference to (1) and the results described thereafter [$q = 2$ in (1) means normal convergence of normalized quadratic variation for $H \leq 3/4$ for fBm data], our Theorem 1 shows that fBm is the only Hermite process for which there exists a range for the parameter H allowing normal convergence of the quadratic variation, while for all other Hermite processes, convergence to a second chaos (Rosenblatt) random variable is universal for all H .

Our proofs are based on chaos expansion into multiple Wiener integrals and Malliavin calculus. The main line of the proof is as follows: since the variable $Z_t^{(q,H)}$ is an element of the q th Wiener chaos, the product formula for multiple integrals implies that the statistics V_N can be decomposed into a sum of

multiple integrals from the order 2 to the order $2q$. The dominant term in this decomposition, which gives the final renormalization order $N^{(2-2H)/q}$, is the term which is a double Wiener integral, and one proves it *always* converges to a Rosenblatt random variable; all other terms are of much lower orders, which is why the only remaining term, after renormalization, converges to a second chaos random variable. The difference with the fBm case comes from the limit of the term of order 2, which in that case is sometimes Gaussian and sometimes Rosenblatt-distributed, depending on the value of H (see [23] for instance).

1.2.2 Convergence of each term in the chaos decomposition of V_N : reproduction property

We also study the limits of the other terms in the decomposition of V_N , those of order higher than 2, which we can call $\{T_k : 2, 4, \dots, 2q\}$, and we obtain interesting facts: for every $k = 2, 4, \dots, 2(q-1)$, the normalization of the k th Wiener chaos term T_k of V_N has a limit which is a Hermite random variable of order k . We call this *the reproduction property* for Hermite processes, because from one Hermite process of order q , one can reconstruct other Hermite processes of all even orders from 2 to $2(q-1)$. As an exception to this rule, the normalized term of highest order $2q$ follows a dichotomy: it converges to a Hermite r.v. of order $2q$ if $H > 3/4$, but converges to Gaussian limit if $H \in (1/2, 3/4]$ (the lower half of this dichotomy has been noticed for $q = 2$ in [23]).

Let us compare this more detailed reproduction and dichotomy phenomenon with the basic result of Theorem 1 described in Section 1.2.1. If $q \geq 2$ (non-Gaussian data case), T_2 always converges to a Rosenblatt r.v., and is in fact responsible for the global behavior of the entire V_N , while T_{2q} converges instead to a Gaussian limit iff $H \leq 3/4$. But if $q = 1$ (case of fBm data), there is no distinction between T_2 and T_{2q} because they coincide with V_N itself; in order for the fBm case's classical dichotomy around $H = 3/4$ to be consistent with our results for $q \geq 2$, one sees that for fBm, the quadratic variation V_N , which is in the second Wiener chaos term, has nevertheless to be interpreted as “ T_{2q} ” rather than “ T_2 ”.

Specifically, we have the following detailed result.

Theorem 2 *Let $Z^{(q,H)}$ be a Hermite process with $q \geq 2$, as in the previous theorem. Consider the Wiener chaos expansion of V_N : we write*

$$V_N = \sum_{n=1}^q c_{2n} T_{2n}$$

where $c_{2q-2k} = k! \binom{q}{k}^2$ and T_{2n} is a multiple Wiener integral of order $2n$.

- For every $H \in (1/2, 1)$ and for every $k = 1, \dots, q-1$ the expression $(z_{k,H})^{-1} N^{(2-2H)k/q} T_{2k}$ converges in $L^2(\Omega)$ to $Z^{r,K}$, a Hermite random variable of order $r = 2k$ with self-similarity parameter $K = 2k(H' - 1) + 1$, where $z_{k,H}$ is the constant defined by

$$z_{q-k,H} = d(H)^2 [H'(2H' - 1)]^k ((H' - 1)k + 1)^{-1} (2(H' - 1) + 1)^{-1}.$$

- For every $H \in (1/2, 3/4)$, with $k = q$, we have convergence in law of $x_{1,H}^{-1/2} \sqrt{N} T_{2q}$ to a standard normal distribution, with $x_{1,H} = \left(\sum_{l=1}^{q-1} b_{2,H,l} + 1 + (1/2) \sum_{k=1}^{\infty} (2k^{2H} - (k+1)^{2H} - (k-1)^{2H})^2 \right)$ where $b_{2,H,k} = (q!)^2 (C_k^q)^2 2d(H)^4 a(H')^{2q} \left(1 + 1/2 \sum_{k=1}^{\infty} (2k^{2H} - (k+1)^{2H} - (k-1)^{2H})^2 \right)$.
- For every $H \in (3/4, 1)$, with $k = q$, we have convergence of $x_{2,H}^{-1/2} N^{2-2H} T_{2q}$ in $L^2(\Omega)$ to $Z^{2q, 2H-1}$, a Hermite r.v. of order $2q$ with parameter $2H - 1$; here $x_{2,H} = \left(\sum_{l=1}^{q-1} b_{1,H,l} + H^2(2H - 1)/(4H - 3) \right)$ and $b_{1,H,l} = (q!)^2 (C_l^q)^2 2d(H)^4 a(H')^{2q} \int_0^1 (1-x)x^{4H-4} dx$.
- For $H = 3/4$, with $k = q$, we have convergence in law of $\sqrt{N/\log N} x_{3,H}^{-1/2} T_{2q}$ to a standard normal distribution, with $x_{3,H} = \left(\sum_{l=1}^{q-1} b_{3,H,l} + 9/16 \right)$ and $b_{3,H,l} = (q!)^2 (C_l^q)^2 2d(H)^4 a(H')^{2q} (1/2)$.

Some of the aspects of this theorem had been discovered in the case of $q = 2$ (Rosenblatt process) in [23]. In that paper, the existence of a higher-order chaos term with normal convergence had been exhibited for the Rosenblatt process with $H < 3/4$, while the case of $H \geq 3/4$ had not been studied. The entire spectrum of convergences in Theorem 2 was not apparent in [23], however, because it was unclear whether the term T_2 's convergence to a Rosenblatt r.v. was due to the fact that we were dealing with input coming from a Rosenblatt process, or whether it was a more general function of the structure of the second Wiener chaos. Here we see that the second alternative is true. In addition one can say that the case of $q \in \{1, 2\}$ alone was insufficient to understand the origins of the dichotomies, or lack thereof, around $H = 3/4$.

The paper [23] had exhibited a remarkable structure of the Rosenblatt data when $H < 3/4$. In that case, as we see in Theorem 2, there are only two terms in the expansion of V_N , T_2 and T_4 ; moreover, and this is the remarkable feature, the proper normalization of the term T_2 converges in $L^2(\Omega)$ to none other than the Rosenblatt process at time 1. Since this value is part of the observed data, one can subtract it to take advantage of the Gaussian limit of the renormalized T_4 ; an application to parameter estimation with normal asymptotics was included in [23] for $H \leq 3/4$. In Theorem 2 above, if $q \geq 3$, even if $H \leq 3/4$, by which a Gaussian limit can be constructed from the renormalized T_{2q} , we still have at least two other terms in the sequence $T_2, T_4, \dots, T_{2q-2}$, and all but at most one of these will converge in $L^2(\Omega)$ to Hermite processes with different orders, all different from q , which implies that they are not directly observed. This means that our Theorem 2 proves that the operation performed with Rosenblatt data, subtracting an observed quantity to isolate T_{2q} and its Gaussian asymptotics, is *not* possible for any Hermite process with $q \geq 3$. If one resorts to the artificial assumption that we observe all the Hermite processes associated with a fixed Wiener process (coupling these processes via Definition 1), then one may perform a sequence of subtractions and compensations to isolate T_{2q} , generalizing the compensation operation done in one step in [23] for the Rosenblatt data. It seems unlikely to us that this assumption may be available in any realistic modeling situation; we do not comment further on such an extension.

1.2.3 Statistical estimation of H

The last aspect of this paper applies Theorem 1 to estimating the parameter H . Let S_N be the empirical mean of the individual squared increments

$$S_N = \frac{1}{N} \sum_{i=0}^{N-1} \left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2$$

and let

$$\hat{H}_N = -(\log S_N) / (2 \log N).$$

We show that \hat{H}_N is a strongly consistent estimator of H , and we show asymptotic Rosenblatt distribution for $N^{2(1-H)/q} (H - \hat{H}) \log N$. The fact that this estimator fails to be asymptotically normal is not a problem in itself. What is more problematic is the fact that if one tries to check one's assumptions on the data by comparing the asymptotics of \hat{H} with a Rosenblatt distribution, one has to know something about the speed of the normalization factor $N^{2(1-H)/q}$. Here, we prove in addition that one may replace H in this factor by \hat{H}_N , so that the asymptotic properties of \hat{H}_N can actually be checked.

Theorem 3 *The estimator \hat{H}_N is strongly consistent, i.e. $\lim_{N \rightarrow \infty} \hat{H}_N = H$ almost surely. Moreover there exists a standard Rosenblatt random variable R with self-similarity parameter $1 + 2(H - 1)/q$ such that*

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\left| 2N^{2(1-\hat{H}_N)/q} (H - \hat{H}_N) \log N - q! c_{1,H,q}^{1/2} R \right| \right] = 0,$$

where, with $a(H') := \left(1 + \frac{H-1}{q}\right) \left(1 + 2\frac{H-1}{q}\right)$, $c_{1,H,q} := 2 \left(\int_0^1 (1-x)x^{(2H'-2)(2q-2k)} dx\right) a(H')^{2q-2} (2q-2k)! d(H)^2$.

Replacing the constant $c_{1,H,q}$ by its value in terms of \hat{H}_N instead of H also leads to the above convergence. However, numerical results illustrating model validation based on the above theorem are beyond

the scope of this article; moreover such applications would be much more sensitive to the convergence speed than to the actual constants; therefore we omit the proof of this further improvement by which one may replace $c_{1,H,q}$ by $c_{1,\hat{H}_N,q}$ in Theorem 3.

1.3 Structure of the article

The structure of the remainder of this article is as follows. The above three theorems are proved in several steps; each intermediate result is labeled as its own proposition or theorem, with its own proof, the above three theorems being a summary of what these intermediate results achieve. Section 2 presents the results on multiplication in the Wiener chaos that will be needed for our proofs of the above three theorems. It also defines the class of Hermite processes. Section 3 is devoted to proving Theorem 1, by first finding the normalization function needed to make the mean square of V_N converge to 1, and then showing the asymptotic Rosenblatt distribution of the properly normalized V_N . Section 4 calculates the asymptotics of all the non-leading terms in the chaos expansion of V_N^2 ; for all terms of orders 4 to $2q-2$, calculations similar to Section 3 ensue. The case of the term of highest order $2q$ is similar when $H > 3/4$, but is more delicate when $H \leq 3/4$: the use of a new characterization of normal convergence in [16], based on a Malliavin derivative calculation, is required. Section 5 proves Theorem 3 by first showing that $2N^{2(1-H)/q}(H - \hat{H}_N) \log N$ converges in $L^2(\Omega)$ to a Rosenblatt random variable and then showing that one may replace H by \hat{H}_N in the power of N if one is content with $L^1(\Omega)$ -convergence. This restriction to L^1 -convergence is only illusory, since a final theorem in this section explains how to obtain these and all convergences in this paper in any $L^p(\Omega)$.

2 Preliminaries

2.1 Multiplication in Wiener Chaos

Let $(W_t)_{t \in [0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W . We refer to [15] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals.

Let $f \in \mathcal{S}$, i.e. f is an elementary function, meaning that

$$f = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} 1_{A_{i_1} \times \dots \times A_{i_m}}$$

where the coefficients satisfy $c_{i_1, \dots, i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoint. For a such step function f we define

$$I_n(f) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} W(A_{i_1}) \dots W(A_{i_m})$$

where we put $W([a,b]) = W_b - W_a$. It can be seen that the application I_n constructed above from \mathcal{S} to $L^2(\Omega)$ is an isometry on \mathcal{S} , i.e.

$$\mathbf{E}[I_n(f)I_m(g)] = n! \langle f, g \rangle_{L^2([0,1]^n)} \text{ if } m = n \quad (3)$$

and

$$\mathbf{E}[I_n(f)I_m(g)] = 0 \text{ if } m \neq n.$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where \tilde{f} denotes the symmetrization of f defined by $\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Since the set \mathcal{S} is dense in $L^2([0,1]^n)$ the mapping I_n can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that I_n can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_2} \dots \int_0^{t_n} f(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n};$$

here the integrals are of Itô type; this formula is easy to show for elementary f 's, and follows for general $f \in L^2([0, 1]^n)$ by a density argument.

The product for two multiple integrals can be expanded into a sum of multiple integrals (see [15]): if $f \in L^2([0, 1]^n)$ and $g \in L^2([0, 1]^m)$ then it holds that

$$I_n(f)I_m(g) = \sum_{l=0}^{m \wedge n} l! C_m^l C_n^l I_{m+n-2l}(f \otimes_l g) \quad (4)$$

where the contraction $f \otimes_l g$ belongs to $L^2([0, 1]^{m+n-2l})$ for $l = 0, 1, \dots, m \wedge n$ and it is given by

$$\begin{aligned} & (f \otimes_l g)(s_1, \dots, s_{n-l}, t_1, \dots, t_{m-l}) \\ &= \int_{[0,1]^l} f(s_1, \dots, s_{n-l}, u_1, \dots, u_l) g(t_1, \dots, t_{m-l}, u_1, \dots, u_l) du_1 \dots du_l. \end{aligned} \quad (5)$$

2.2 The Hermite process

The fractional Brownian process $(B_t^H)_{t \in [0,1]}$ with Hurst parameter $H \in (0, 1)$ can be written as

$$B_t^H = \int_0^t K^H(t, s) dW_s, \quad t \in [0, 1]$$

where $(W_t, t \in [0, T])$ is a standard Wiener process, the kernel $K^H(t, s)$ has the expression $c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du$ where $t > s$ and $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)} \right)^{1/2}$ and $\beta(\cdot, \cdot)$ is the Beta function. For $t > s$, the kernel's derivative is $\frac{\partial K^H}{\partial t}(t, s) = c_H \left(\frac{s}{t} \right)^{1/2-H} (t-s)^{H-3/2}$. Fortunately we will not need to use these expressions explicitly, since they will be involved below only in integrals whose expressions are known.

We will denote by $(Z_t^{(q,H)})_{t \in [0,1]}$ the Hermite process *with self-similarity parameter* $H \in (1/2, 1)$. Here $q \geq 1$ is an integer. The Hermite process can be defined in two ways: as a multiple integral with respect to the standard Wiener process $(W_t)_{t \in [0,1]}$; or as a multiple integral with respect to a fractional Brownian motion with suitable Hurst parameter (see Remark 1 below). We adopt the first approach throughout the paper. We refer to [14] or [22] for the following integral representation of Hermite processes.

Definition 1 *The Hermite process $(Z_t^{(q,H)})_{t \in [0,1]}$ of order $q \geq 1$ and with self-similarity parameter $H \in (\frac{1}{2}, 1)$ is given by*

$$Z_t^{(q,H)} = d(H) \int_0^t \dots \int_0^t dW_{y_1} \dots dW_{y_q} \left(\int_{y_1 \vee \dots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du \right), \quad t \in [0, 1] \quad (6)$$

where $K^{H'}$ is the usual kernel of the fractional Brownian motion and

$$H' = 1 + \frac{H-1}{q} \iff (2H'-2)q = 2H-2. \quad (7)$$

Of fundamental importance is the fact that the covariance of $Z^{(q,H)}$ is identical to that of fBm, namely

$$\mathbf{E} \left[Z_s^{(q,H)} Z_t^{(q,H)} \right] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

The constant $d(H)$, given in (2) on page 3, is chosen to avoid any additional multiplicative constants. We stress that $Z^{(q,H)}$ is far from Gaussian for $q > 1$, since it is formed of multiple Wiener integrals of order q .

Remark 1 Let B be a fractional Brownian motion with Hurst parameter H' given by (7) and denote by $I_q^{B^{H'}}$ the multiple integral of order q with respect to this process. The Hermite process can also be defined as

$$Z_t^{(q,H)} = c(H)I_q^{B^{H'}}(\mu_t), \quad t \in [0, 1] \quad (8)$$

where μ_t denotes the uniform measure on the diagonal D_t of $[0, t]^q$. More details can be found in [14].

The basic properties of the Hermite process are listed below:

- the Hermite process $Z^{(q)}$ is H -self-similar and it has stationary increments.
- the mean square of the increment is given by

$$\mathbf{E} \left[\left| Z_t^{(q,H)} - Z_s^{(q,H)} \right|^2 \right] = |t - s|^{2H}; \quad (9)$$

as a consequence, it follows with little extra effort from Kolmogorov's continuity criterion that $Z^{(q,H)}$ has Hölder-continuous paths of any exponent $\delta < H$.

- it exhibits long-range dependence in the sense that $\sum_{n \geq 1} \mathbf{E} \left[Z_1^{(q,H)} (Z_{n+1}^{(q,H)} - Z_n^{(q,H)}) \right] = \infty$. In fact, the summand in this series is of order n^{2H-2} . This property is identical to that of fBm since the processes share the same covariance structure, and the property is well-known for fBm with $H > 1/2$.
- for $q = 1$, $Z^{(1,H)}$ is standard fBm with Hurst parameter H , while for $q \geq 2$ the Hermite process is not Gaussian. In the case $q = 2$ this stochastic process is known as *the Rosenblatt process*.

3 Variations of the Hermite process

Recall the centered quadratic variation statistic V_N given in the introduction:

$$V_N = \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2}{\mathbf{E} \left[\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2 \right]} - 1 \right]. \quad (10)$$

Since $\mathbf{E} \left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2 = N^{-2H}$ by (9), we can write

$$V_N = N^{2H-1} \sum_{i=0}^{N-1} \left[\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2 - N^{-2H} \right].$$

Let $I_i := \left[\frac{i}{N}, \frac{i+1}{N} \right]$, $i = 0, \dots, N-1$. In preparation for calculating the variance of V_N we will find an explicit expansion of V_N in Wiener chaos. We have

$$Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} = I_q(f_{i,N})$$

where we used the notation

$$\begin{aligned} f_{i,N}(y_1, \dots, y_q) &= 1_{[0, \frac{i+1}{N}]}(y_1 \vee \dots \vee y_q) d(H) \int_{y_1 \vee \dots \vee y_q}^{\frac{i+1}{N}} \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du \\ &\quad - 1_{[0, \frac{i}{N}]}(y_1 \vee \dots \vee y_q) d(H) \int_{y_1 \vee \dots \vee y_q}^{\frac{i}{N}} \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du. \end{aligned}$$

Using the product formula for multiple integrals (4), we obtain

$$I_q(f_{i,N})I_q(f_{i,N}) = \sum_{l=0}^q l!(C_q^l)^2 I_{2q-2l}(f_{i,N} \otimes_l f_{i,N})$$

where the $f \otimes_l g$ denotes the l -contraction of the functions f and g given by (5). Let us compute these contractions; for $l = q$ we have

$$(f_{i,N} \otimes_q f_{i,N}) = q! \langle f_{i,N}, f_{i,N} \rangle_{L^2([0,1]^{\otimes q})} = \mathbf{E} \left[\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right)^2 \right] = N^{-2H}.$$

Throughout the paper the notation $\partial_1 K(t, s)$ will be used for $\partial_1 K^{H'}(t, s)$. For $l = 0$ we have

$$\begin{aligned} (f_{i,N} \otimes_0 f_{i,N})(y_1, \dots, y_q, z_1, \dots, z_q) &= (f_{i,N} \otimes f_{i,N})(y_1, \dots, y_q, z_1, \dots, z_q) \\ &= f_{i,N}(y_1, \dots, y_q) f_{i,N}(z_1, \dots, z_q) \end{aligned}$$

while for $1 \leq k \leq q-1$

$$\begin{aligned} &(f_{i,N} \otimes_k f_{i,N})(y_1, \dots, y_{q-k}, z_1, \dots, z_{q-k}) = d(H)^2 \int_{[0,1]^k} d\alpha_1 \dots d\alpha_k \\ &\times \left(1_{[0, \frac{i+1}{N}]^{q-k}}(y_i) 1_{[0, \frac{i+1}{N}]^k}(\alpha_i) \int_{y_1 \vee \dots \vee y_{q-k} \vee \alpha_1 \dots \vee \alpha_k}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \partial_1 K(u, \alpha_1) \dots \partial_1 K(u, \alpha_k) \right. \\ &- \left. 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^k}(\alpha_i) \int_{y_1 \vee \dots \vee y_{q-k} \vee \alpha_1 \dots \vee \alpha_k}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \partial_1 K(u, \alpha_1) \dots \partial_1 K(u, \alpha_k) \right) \\ &\times \left(1_{[0, \frac{i+1}{N}]^{q-k}}(z_i) 1_{[0, \frac{i+1}{N}]^k}(\alpha_i) \int_{z_1 \vee \dots \vee z_{q-k} \vee \alpha_1 \dots \vee \alpha_k}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) \partial_1 K(v, \alpha_1) \dots \partial_1 K(v, \alpha_k) \right. \\ &- \left. 1_{[0, \frac{i}{N}]^{q-k}}(z_i) 1_{[0, \frac{i}{N}]^k}(\alpha_i) \int_{z_1 \vee \dots \vee z_{q-k} \vee \alpha_1 \dots \vee \alpha_k}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) \partial_1 K(v, \alpha_1) \dots \partial_1 K(v, \alpha_k) \right) \end{aligned}$$

and by interchanging the order of the integration we get

$$\begin{aligned} &(f_{i,N} \otimes_k f_{i,N})(y_1, \dots, y_{q-k}, z_1, \dots, z_{q-k}) \\ &= d(H)^2 \left\{ 1_{[0, \frac{i+1}{N}]^{q-k}}(y_i) 1_{[0, \frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \right. \\ &\int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \\ &- 1_{[0, \frac{i+1}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\ &\int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \\ &- 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i+1}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\ &\int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \\ &+ 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^{q-k}}(z_i) \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \\ &\left. \int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) \left(\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha \right)^k \right\} \end{aligned}$$

and since

$$\int_0^{u \wedge v} \partial_1 K(u, \alpha) \partial_1 K(v, \alpha) d\alpha = a(H') |u - v|^{2H' - 2}$$

with $a(H') = H'(2H' - 1)$ we obtain

$$\begin{aligned} & (f_{i,N} \otimes_k f_{i,N})(y_1, \dots, y_{q-k}, z_1, \dots, z_{q-k}) \\ = & d(H)^2 a(H')^k \left\{ \right. \\ & 1_{[0, \frac{i+1}{N}]^{q-k}}(y_i) 1_{[0, \frac{i+1}{N}]^{q-k}}(z_i) \\ & \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u - v|^{(2H' - 2)k} \\ & - 1_{[0, \frac{i+1}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^{q-k}}(z_i) \\ & \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i+1}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u - v|^{(2H' - 2)k} \\ & - 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i+1}{N}]^{q-k}}(z_i) \\ & \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i+1}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u - v|^{(2H' - 2)k} \\ & + 1_{[0, \frac{i}{N}]^{q-k}}(y_i) 1_{[0, \frac{i}{N}]^{q-k}}(z_i) \\ & \left. \int_{y_1 \vee \dots \vee y_{q-k}}^{\frac{i}{N}} du \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \int_{z_1 \vee \dots \vee z_{q-k}}^{\frac{i}{N}} dv \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u - v|^{(2H' - 2)k} \right\} \end{aligned} \quad (11)$$

As a consequence, we can write

$$V_N = T_{2q} + c_{2q-2} T_{2q-2} + \dots + c_4 T_4 + c_2 T_2 \quad (12)$$

where

$$c_{2q-2k} := k! \binom{q}{k}^2 \quad (13)$$

are the combinatorial constants from the product formula for $0 \leq k \leq q - 1$, and

$$T_{2q-2k} := N^{2H-1} I_{2q-2k} \left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right). \quad (14)$$

where the integrands in the last formula above are given explicitly in (11). This Wiener chaos decomposition of V_N allows us to find V_N 's precise order of magnitude via its variance's asymptotics.

Proposition 1 *With*

$$c_{1,H} = \frac{4d(H)^4 (H'(2H' - 1))^{2q-2}}{(4H' - 3)(4H' - 2)}, \quad (15)$$

it holds that

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[c_{1,H}^{-1} N^{(2-2H')} c_2^{-2} V_N^2 \right] = 1.$$

Proof. We only need to estimate the $L^2(\Omega)$ norm of each term appearing in the chaos decomposition (12) of V_N , since these terms are orthogonal in $L^2(\Omega)$.

We can write, for $0 \leq k \leq q-1$,

$$\begin{aligned} \mathbf{E} [T_{2q-2k}^2] &= N^{4H-2} (2q-2k)! \left\| \left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right)^s \right\|_{L^2([0,1]^{2q-2k})}^2 \\ &= N^{4H-2} (2q-2k)! \sum_{i,j=0}^{N-1} \langle f_{i,N} \tilde{\otimes}_k f_{i,N}, f_{j,N} \tilde{\otimes}_k f_{j,N} \rangle_{L^2([0,1]^{2q-2k})} \end{aligned}$$

where $(g)^s = \tilde{g}$ and $f_{i,N} \tilde{\otimes}_k f_{i,N}$ denotes the symmetrization of the function $f_{i,N} \otimes_k f_{i,N}$. We will consider first the term T_2 obtained for $k = q-1$. In this case, the kernel $\sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N}$ is symmetric and we can avoid its symmetrization. Therefore

$$\begin{aligned} \mathbf{E} [T_2^2] &= 2! N^{4H-2} \left\| \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \right\|_{L^2([0,1]^2)}^2 \\ &= 2! N^{4H-2} \sum_{i,j=0}^{N-1} \langle f_{i,N} \otimes_{q-1} f_{i,N}, f_{j,N} \otimes_{q-1} f_{j,N} \rangle_{L^2([0,1]^2)}. \end{aligned}$$

We compute now the scalar product in the above expression. By using Fubini theorem, we end up with the following easier expression

$$\begin{aligned} &\langle f_{i,N} \otimes_{q-1} f_{i,N}, f_{j,N} \otimes_{q-1} f_{j,N} \rangle_{L^2([0,1]^2)} \\ &= a(H')^{2q} d(H)^4 \\ &\int_{I_i} \int_{I_i} \int_{I_j} \int_{I_j} |u-v|^{(2H'-2)(q-1)} |u'-v'|^{(2H'-2)(q-1)} |u-u'|^{2H'-2} |v-v'|^{2H'-2} dv' du' dv du \end{aligned}$$

Using the change of variables

$$y = \left(u - \frac{i}{N}\right)N$$

and similarly for the other variables, we now obtain

$$\begin{aligned} \mathbf{E} [T_2^2] &= 2d(H)^4 (H'(2H'-1))^{2q} N^{4H-2} N^{-4} N^{-(2H'-2)2q} \sum_{i,j=0}^{N-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ &\quad \times |y-z|^{(2H'-2)(q-1)} |y'-z'|^{(2H'-2)(q-1)} |y-y'+i-j|^{(2H'-2)} |z-z'+i-j|^{(2H'-2)} \end{aligned}$$

This can be viewed as the sum of a diagonal part ($i = j$) and a non-diagonal part ($i \neq j$), where the non-diagonal part is dominant, as the reader will readily check. Therefore, the behavior of $\mathbf{E} [T_2^2]$ will be given by

$$\begin{aligned} \mathbf{E} [T_2^2] &: = 2! d(H)^4 (H'(2H'-1))^{2q} N^{-2} 2 \sum_{i>j} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ &\quad \times |y-z|^{(2H'-2)(q-1)} |y'-z'|^{(2H'-2)(q-1)} |y-y'+i-j|^{(2H'-2)} |z-z'+i-j|^{(2H'-2)} \\ &= 2! d(H)^4 (H'(2H'-1))^{2q} N^{-2} 2 \sum_{i=0}^{N-1} \sum_{\ell=0}^{N-i+1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ &\quad \times |y-z|^{(2H'-2)(q-1)} |y'-z'|^{(2H'-2)(q-1)} |y-y'+\ell-1|^{(2H'-2)} |z-z'+\ell-1|^{(2H'-2)} \\ &= 2! d(H)^4 (H'(2H'-1))^{2q} N^{-2} 2 \sum_{\ell=0}^{N-1} (N-\ell+1) \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ &\quad \times |y-z|^{(2H'-2)(q-1)} |y'-z'|^{(2H'-2)(q-1)} |y-y'+\ell-1|^{(2H'-2)} |z-z'+\ell-1|^{(2H'-2)} \end{aligned}$$

As in [23] note that

$$\begin{aligned} & \frac{1}{N^2} \sum_{\ell=0}^{N-1} (N - \ell + 1) |y - y' + \ell - 1|^{(2H'-2)} |z - z' + \ell - 1|^{(2H'-2)} \\ = & N^{2(2H'-2)} \frac{1}{N} \sum_{\ell=0}^{N-1} \left(1 - \frac{\ell-1}{N}\right) \left|\frac{y-y'}{N} + \frac{\ell-1}{N}\right|^{2H'-2} \left|\frac{z-z'}{N} + \frac{\ell-1}{N}\right|^{2H'-2} \end{aligned}$$

Using a Riemann sum approximation argument we conclude that

$$\mathbf{E} \left[T_2'^2 \right] \sim \frac{4d(H)^4 (H'(2H'-1))^{2q-2}}{(4H'-3)(4H'-2)} N^{2(2H'-2)}$$

Therefore, it follows that

$$\mathbf{E} \left[c_{1,H}^{-1} N^{(2-2H')(2)} T_2^2 \right] \rightarrow_{N \rightarrow \infty} 1. \quad (16)$$

with $c_{1,H}$ as in (15).

Let us study now the terms T_4, \dots, T_{2q} given by (14). Here the function $\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N}$ is no longer symmetric but we will show that the behavior of its L^2 norm is dominated by $\mathbf{E} [T_2^2]$. Since for any square integrable function g one has $\|\tilde{g}\|_{L^2} \leq \|g\|_{L^2}$, we have for $k = 0, \dots, q-2$

$$\begin{aligned} \mathbf{E} [T_{2q-2k}^2] &= (2q-2k)! N^{4H-2} \left\| \sum_{i=0}^{N-1} f_{i,N} \tilde{\otimes}_k f_{i,N} \right\|_{L^2([0,1]^{2q-2k})}^2 \\ &\leq (2q-2k)! N^{4H-2} \left\| \sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right\|_{L^2([0,1]^{2q-2k})}^2 \\ &= (2q-2k)! N^{4H-2} \sum_{i,j=0}^{N-1} \langle f_{i,N} \otimes_k f_{i,N}, f_{j,N} \otimes_k f_{j,N} \rangle_{L^2([0,1]^{2q-2k})}; \end{aligned} \quad (17)$$

proceeding as above we can write

$$\begin{aligned} \mathbf{E} [T_{2q-2k}^2] &\leq (2q-2k)! (H'(2H'-1))^{2q} d(H)^4 N^{4H-2} \sum_{i,j=0}^{N-1} \int_{I_i} \int_{I_i} dy_1 dz_1 \int_{I_j} \int_{I_j} dy_1' dz_1' \\ &\quad \times |y_1 - z_1|^{(2H'-2)k} |y_1' - z_1'|^{(2H'-2)k} |y_1 - y_1'|^{(2H'-2)(q-k)} |z_1 - z_1'|^{(2H'-2)(q-k)} \end{aligned}$$

and using a change of variables as before,

$$\begin{aligned} & \mathbf{E} [T_{2q-2k}^2] \\ & \leq (2q-2k)! (H'(2H'-1))^{2q} d(H)^4 N^{4H-2} N^{-4} N^{-(2H'-2)2q} \sum_{i,j=0}^{N-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ & \quad \times |y - z|^{(2H'-2)k} |y' - z'|^{(2H'-2)k} |y - y' + i - j|^{(2H'-2)(q-k)} |z - z' + i - j|^{(2H'-2)(q-k)} \\ & = (2q-2k)! a(H')^{2q} d(H)^4 N^{(2H'-2)(2q-2k)} \frac{1}{N} 2 \sum_{\ell=0}^{N-1} \left(1 - \frac{\ell}{N}\right) \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ & \quad \times |y - z|^{(2H'-2)k} |y' - z'|^{(2H'-2)k} \left| \frac{y-y'}{N} + \frac{\ell-1}{N} \right|^{(2H'-2)(q-k)} \left| \frac{z-z'}{N} + \frac{\ell-1}{N} \right|^{(2H'-2)(q-k)}. \end{aligned} \quad (18)$$

Since off a diagonal term (again of lower order), the terms $\frac{z-z'}{N}$ are dominated by $\frac{\ell}{N}$ for large ℓ, N it follows that, for $1 \leq k \leq q-1$

$$\mathbf{E} \left[c_{1,H,q}^{-1} N^{(2-2H')(2q-2k)} T_{2q-2k}^2 \right] =_{N \rightarrow \infty} \mathcal{O}(1) \quad (19)$$

where

$$c_{1,H,q} = 2 \left(\int_0^1 (1-x)x^{(2H'-2)(2q-2k)} dx \right) (2q-2k)! d(H)^2 a(H')^{2q}. \quad (20)$$

The term T_{2q} (case $k=0$) has similar asymptotics for $H > \frac{3}{4}$, while for $H \in (\frac{1}{2}, \frac{3}{4})$ it has to be renormalized by N ; we omit the details here, which are contained in other proofs in this paper, especially those from Section 4; in any even, T_{2q} is dominated by T_2 in all cases.

It is now clear that the dominant term in the decomposition of V_N is the second chaos term. More specifically we have for any $k \leq q-2$,

$$\mathbf{E} \left[N^{2(2-2H')} T_{2q-2k}^2 \right] = O \left(N^{-2(2-2H')2(q-k-1)} \right). \quad (21)$$

Combining this with the orthogonality of chaos integrals, we immediately get that, up to terms that tend to 0, $N^{2-2H'} V_N$ and $N^{2-2H'} T_2$ have the same norm in $L^2(\Omega)$. This finishes the proof of the proposition. \blacksquare

Summarizing the spirit of the above proof, to understand the behavior of the renormalized sequence V_N it suffices to study the limit of the term

$$I_2 \left(N^{2H-1} N^{(2-2H')} \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \right) \quad (22)$$

with

$$\begin{aligned} & (f_{i,N} \otimes_{q-1} f_{i,N})(y, z) \\ = & d(H)^2 a(H')^{q-1} \\ & \left(1_{[0, \frac{i}{N}]}(y \vee z) \int_{I_i} \int_{I_i} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \right. \\ & + 1_{[0, \frac{i}{N}]}(y) 1_{I_i}(z) \int_{I_i} \int_z^{\frac{i+1}{N}} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \\ & + 1_{I_i}(y) 1_{[0, \frac{i}{N}]}(z) \int_y^{\frac{i+1}{N}} \int_{I_i} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \\ & \left. + 1_{I_i}(y) 1_{I_i}(z) \int_y^{\frac{i+1}{N}} \int_z^{\frac{i+1}{N}} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \right). \quad (23) \end{aligned}$$

We will see in the proof of the next theorem that, of the contribution of the four terms on the right-hand side of (23), only the first one does not tend to 0 in $L^2(\Omega)$. Hence the following notation will be useful: f_2^N will denote the integrand of the contribution to (22) corresponding to that first term, and r_2 will be the remainder of the integrand in (22). In other words,

$$f_2^N + r_2 = N^{2H-1} N^{(2-2H')} \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \quad (24)$$

and

$$\begin{aligned} f_2^N(y, z) & : = N^{2H-1} N^{(2-2H')} d(H)^2 a(H')^{q-1} \\ & \sum_{i=0}^{N-1} 1_{[0, \frac{i}{N}]}(y \vee z) \int_{I_i} \int_{I_i} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)}. \quad (25) \end{aligned}$$

Theorem 4 *The sequence given by (22) converges in $L^2(\Omega)$ as $N \rightarrow \infty$ to the constant $d(H)^2 a(H')^{q-1}$ times a standard Rosenblatt random variable $Z_1^{(2, 2H'-1)}$ with self-similarity parameter $2H'-1$ where $H' = 1 + \frac{H-1}{q}$. Consequently, we also have that $c_{1,H}^{-1/2} N^{(2-2H')} c_2^{-1} V_N$ converges in $L^2(\Omega)$ as $N \rightarrow \infty$ to the same Rosenblatt random variable.*

Proof: The first statement of the theorem is that $N^{2-2H'}T_2$ converges to

$$d(H)^2 a(H')^{q-1} Z_1^{(2,2H'-1)}$$

in $L^2(\Omega)$. From (22) it follows that T_2 is a second-chaos random variable, with kernel

$$N^{2H-1} \sum_{i=0}^{N-1} (f_{i,N} \otimes_{q-1} f_{i,N})$$

(see expression in (23)), so we only need to prove this kernel converges in $L^2([0,1]^2)$.

The first step is to prove that $r_2(y, z)$ defined in (24) converges to zero in $L^2([0,1]^2)$ as $N \rightarrow \infty$. The crucial fact is that the intervals I_i which are disjoint, appear in the expression of this term and this implies that the non-diagonal terms vanish when we take the square norm of the sum. Indeed,

$$\begin{aligned} & \|r_2\|_{L^2([0,1]^2)}^2 \\ = & N^{4H-2} N^{2(2-2H')} \sum_{i=0}^{N-1} \int_0^1 \int_0^1 dy dz \\ & \left(1_{[0, \frac{i}{N}]}(y) 1_{I_i}(z) \left(\int_{I_i} \int_z^{\frac{i+1}{N}} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \right)^2 \right. \\ & + 1_{I_i}(y) 1_{[0, \frac{i}{N}]}(z) \left(\int_y^{\frac{i+1}{N}} \int_{I_i} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \right)^2 \\ & \left. + 1_{I_i}(y) 1_{I_i}(z) \left(\int_y^{\frac{i+1}{N}} \int_z^{\frac{i+1}{N}} dv du \partial_1 K(u, y) \partial_1 K(v, z) |u-v|^{(2H'-2)(q-1)} \right)^2 \right) \\ \leq & N^{4H-2} N^{2(2H'-2)} \int_{I_i} \int_{I_i} \int_{I_i} \int_{I_i} du dv du' dv' \\ & |u-v|^{(2H'-2)(q-1)} |u'-v'|^{(2H'-2)(q-1)} |u-u'|^{2H'-2} |v-v'|^{2H'-2}. \end{aligned}$$

It follows from the calculations done to estimate $\mathbf{E}[T_2^2]$ in the proof of Proposition 1, that this term converges to zero as N goes to infinity; in fact it can easily be seen that the norm in L^2 of r_2 corresponds to the diagonal part in the evaluation in $\mathbf{E}[T_2^2]$ which is clearly dominated by the non-diagonal part, so this result comes as no surprise.

This shows $N^{(2-2H')}T_2$ is the sum of $I_2(f_2^N)$ and a term which goes to 0 in $L^2(\Omega)$. To conclude, we proceed in the following way: we show that for every $y, z \in [0, 1]$ the sequence $f_2^N(y, z)$ converges to the kernel of the Rosenblatt random variable with desired self-similarity parameter, immediately implying that the sequence $(f_{i,N} \otimes_{q-1} f_{i,N})(y, z)$ converges to the same kernel. Indeed, we have by definition

$$\begin{aligned} f_2^N(y, z) &= (H'(2H'-1))^{(q-1)} d(H)^2 N^{2H-1} N^{2-2H'} \\ &\quad \times 1_{[0, \frac{i}{N}]}(y \vee z) \sum_{i=0}^{N-1} \int_{I_i} \int_{I_i} |u-v|^{(2H'-2)(q-1)} \partial_1 K^{H'}(u, y) \partial_1 K^{H'}(v, z) \end{aligned}$$

and it is asymptotically equivalent to (in the sense that it has the same limit as)

$$\begin{aligned}
& N^{2H-1} N^{2-2H'} d(H)^2 (H'(2H'-1))^{(q-1)} \sum_{i=0}^{N-1} \int_{I_i} \int_{I_i} |u-v|^{(2H'-2)(q-1)} \\
& \quad \times \partial_1 K^{H'}(i/N, y) \partial_1 K^{H'}(i/N, z) dudv \\
= & (H'(2H'-1))^{(q-1)} d(H)^2 N^{2H-1} N^{2-2H'} \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N, y) \partial_1 K^{H'}(i/N, z) \mathbf{1}_{y \vee z < i/N} \\
& \quad \times \int_{I_i} \int_{I_i} |u-v|^{(2H'-2)(q-1)} dudv \\
= & (H'(2H'-1))^{(q-1)} [((2H'-2)(q-1)+1)((H'-1)(q-1)+1)]^{-1} \\
& \quad \times N^{2H-1} N^{(2-2H')} N^{(2-2H')(q-1)+2} \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N, y) \partial_1 K^{H'}(i/N, z) \mathbf{1}_{y \vee z < i/N} \\
= & (H'(2H'-1))^{(q-1)} [((2H'-2)(q-1)+1)((H'-1)(q-1)+1)]^{-1} \\
& \quad \times N^{-1} \sum_{i=0}^{N-1} \partial_1 K^{H'}(i/N, y) \partial_1 K^{H'}(i/N, z) \mathbf{1}_{y \vee z < i/N}
\end{aligned}$$

which is a Riemann sum that converges pointwise on $[0, 1]^2$ to the kernel of the Rosenblatt process at time 1. To obtain the convergence in $L^2([0, 1]^2)$ it is enough to show for example that the sequence $(f_{i,N} \otimes_{q-1} f_{i,N})_N$ is Cauchy in the space $L^2([0, 1]^2)$; this technical property follows from the proof of Theorem 8 in [23], to which the reader is referred.

In order to show that the full scaled variation $c_{1,H}^{-1/2} N^{(2-2H')} c_2^{-1} V_N$ converges in $L^2(\Omega)$ to the same Rosenblatt random variable as the normalized version of the quantity in (22), it is sufficient to show that, after normalization by $N^{2-2H'}$, each of the remaining terms in the chaos expansion (12) of V_N , converge to zero in $L^2(\Omega)$, i.e. that $N^{(2-2H')} T_{2q-2k}$ converge to zero in $L^2(\Omega)$, for all $1 \leq k < q-1$. From (21) we have

$$\mathbf{E} \left[N^{2(2-2H')} T_{2q-2k}^2 \right] = O \left(N^{-2(2-2H')2(q-k-1)} \right)$$

which is all that is needed, concluding the proof of the theorem. \blacksquare

4 Reproduction property for the Hermite process

We now study the limits of the other terms in the chaos expansion (12) of V_N . We will consider first the convergence of the term of greatest order T_{2q} in this expansion. The behavior of T_{2q} is interesting because it differs from the behavior of the all other terms: its suitable normalization possesses a Gaussian limit if $H \in (1/2, 3/4]$. Therefore it inherits, in some sense, the properties of the quadratic variations for the fractional Brownian motion (results in [23]).

We have

$$T_{2q} = N^{2H-1} I_{2q} \left(\sum_{i=0}^{N-1} f_{i,N} \otimes f_{i,N} \right)$$

and

$$\mathbf{E} [T_{2q}^2] = N^{4H-2} (2q)! \sum_{i,j=0}^{N-1} \langle f_{i,N} \otimes f_{i,N}, f_{j,N} \otimes f_{j,N} \rangle_{L^2([0,1]^{2q})}$$

We will use the following combinatorial formula: if f, g are two symmetric functions in $L^2([0, 1]^q)$, then

$$\begin{aligned}
& (2q)! \langle f \otimes f, g \otimes g \rangle_{L^2([0,1]^{2q})} \\
= & (q!)^2 \langle f \otimes f, g \otimes g \rangle_{L^2([0,1]^{2q})} + \sum_{k=1}^{q-1} \binom{q}{k}^2 (q!)^2 \langle f \otimes_k g, g \otimes_k f \rangle_{L^2([0,1]^{2q-2k})}
\end{aligned}$$

to obtain

$$\begin{aligned} \mathbf{E} [T_{2q}^2] &= N^{4H-2} (q!)^2 \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^q)}^2 \\ &\quad + N^{4H-2} (q!)^2 \sum_{i,j=0}^{N-1} \sum_{k=1}^{q-1} \binom{q}{k}^2 \langle f_{i,N} \otimes_k f_{j,N}, f_{j,N} \otimes_k f_{i,N} \rangle_{L^2([0,1]^{2q-2k})}. \end{aligned}$$

First note that

$$\mathbf{E} \left[\left(Z_{\frac{i+1}{N}}^{(q,H)} - Z_{\frac{i}{N}}^{(q,H)} \right) \left(Z_{\frac{j+1}{N}}^{(q,H)} - Z_{\frac{j}{N}}^{(q,H)} \right) \right] = \mathbf{E} [I_q(f_{i,N}) I_q(f_{j,N})] = q! \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^q)}$$

and so the covariance structure of $Z^{(q,H)}$ implies

$$(q!)^2 \sum_{i,j=0}^{N-1} \langle f_{i,N}, f_{j,N} \rangle_{L^2([0,1]^q)}^2 = \frac{1}{4} \sum_{i,j=0}^{N-1} \left[\left(\frac{i-j+1}{N} \right)^{2H} + \left(\frac{i-j-1}{N} \right)^{2H} - 2 \left(\frac{i-j}{N} \right)^{2H} \right]^2. \quad (26)$$

Secondly, the square norm of the contraction $f_{i,N} \otimes_k f_{j,N}$ has been computed before (actually its expression is obtained in the lines from formula (17) to formula (18)). By a simple polarization, we obtain

$$\begin{aligned} &N^{2H-1} \sum_{i,j=0}^{N-1} \langle f_{i,N} \otimes_k f_{j,N}, f_{j,N} \otimes_k f_{i,N} \rangle_{L^2([0,1]^{2q-2k})} \\ &= d(H)^4 a(H')^{2q} N^{4H-2} N^{(2H'-2)2q} \sum_{i,j=0}^{N-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy dz dy' dz' \\ &\quad |y-z+i-j|^{(2H'-2)k} |y'-z'+i-j|^{(2H'-2)k} |y-y'+i-j|^{(2H'-2)(q-k)} |z-z'+i-j|^{(2H'-2)(q-k)} \end{aligned}$$

and as in the proof of Proposition 1, we can find that if $H > \frac{3}{4}$ this term has to be renormalized by

$$b_{1,H,k} N^{(2-2H')2q} = b_{H,k} N^{4-4H}$$

where $b_{1,H,k} = (q!)^2 (C_k^q)^2 2d(H)^4 a(H')^{2q} \int_0^1 (1-x)x^{4H-4} dx$. If $H \in (\frac{1}{2}, \frac{3}{4})$, then, the same quantity will be renormalized by

$$b_{2,H,k} N$$

where $b_{2,H,k} = (q!)^2 (C_k^q)^2 2d(H)^4 a(H')^{2q} \left(1 + \frac{1}{2} \sum_{k=1}^{\infty} (2k^{2H} - (k+1)^{2H} - (k-1)^{2H})^2 \right)$ while for $H = \frac{3}{4}$ the renormalization is of order $b_{3,H,k} N (\log N)^{-1}$ with $b_{3,H,k} = (q!)^2 (C_k^q)^2 2d(H)^4 a(H')^{2q}$.

As a consequence, we find a sum whose behavior is well-known (it is the same as the mean square of the quadratic variations of the fractional Brownian motion, see e.g. [23]) and we get, for large N

$$\begin{aligned} \mathbf{E} [T_{2q}^2] &\sim \frac{1}{N} x_{1,H}, & \text{if } H \in \left(\frac{1}{2}, \frac{3}{4} \right) \\ \mathbf{E} [T_{2q}^2] &\sim N^{4H-4} x_{2,H}, & \text{if } H \in \left(\frac{3}{4}, 1 \right) \\ \mathbf{E} [T_{2q}^2] &\sim \frac{\log N}{N} x_{3,H}, & \text{if } H = \frac{3}{4} \end{aligned}$$

where $x_{1,H} = \left(\sum_{l=1}^{q-1} 1/b_{2,H,l} + 1 + (1/2) \sum_{k=1}^{\infty} (2k^{2H} - (k+1)^{2H} - (k-1)^{2H})^2 \right)$, $x_{2,H} = \left(\sum_{l=1}^{q-1} 1/b_{1,H,l} + H^2(2H-1)/(4H-3) \right)$ and $x_{3,H} = \left(\sum_{l=1}^{q-1} 1/b_{3,H,l} + 9/16 \right)$.

The fact that the normalizing factor for T_{2q} when $H < 3/4$ is $N^{-1/2}$ (in particular does not depend on H) is a tell-tale sign that normal convergence may occur. The next step is then indeed to show that $N^{-1/2} T_{2q}$ converges to a normal law if $H \leq 3/4$. We recall the following result (see Theorem 4 in [16], see also [17]).

Proposition 2 Let $F_N = I_n(f_N)$ be a sequence of square integrable random variables in the n -th Wiener chaos such that $\mathbf{E}[F_N^2] \rightarrow 1$ as $N \rightarrow \infty$. Then the following are equivalent:

- i) The sequence $(F_N)_{N \geq 0}$ converges to the normal law $N(0, 1)$.
- ii) $\lim_{N \rightarrow \infty} \|DF_N\|_{L^2[0,1]}^2 = n$ in $L^2(\Omega)$, where D is the Malliavin derivative w.r.t. the underlying Wiener process W .

Criterion (ii) is due to [16]; we refer to it as the Nualart–Ortiz-Latorre criterion.

Theorem 5

- a) Suppose that $H \in (\frac{1}{2}, \frac{3}{4})$ and let $F_N := x_{1,H}^{-1/2} \sqrt{N} T_{2q}$. Then, as $N \rightarrow \infty$, then sequence F_N convergence to the standard normal distribution $N(0, 1)$.
- b) Suppose $H = \frac{3}{4}$ and set $G_N := \sqrt{\frac{N}{\log N}} x_{3,H}^{-1/2} T_{2q}$. Then, as $N \rightarrow \infty$, then sequence G_N convergence to the standard normal distribution $N(0, 1)$.

Proof: From the computations preceding Proposition 2, Condition (i) in Proposition 2 has already been established: $\lim_{N \rightarrow \infty} \mathbf{E}[F_N^2] = 1$. It remains to check that Condition (ii) in Proposition 2 is satisfied. We calculate the Malliavin derivative of F_N . It holds that, for every $r \in [0, 1]$,

$$D_r F_N = x_{1,H}^{-1/2} \sqrt{N} N^{2H-1} (2q) I_{2q-1} \left(\sum_{i=0}^{N-1} (f_{i,N} \otimes f_{i,N})(\cdot, r) \right)$$

and

$$\begin{aligned} \|DF_N\|_{L^2[0,1]}^2 &= \int_0^1 |D_r F_N|^2 dr = x_{1,H}^{-1} N 4q^2 N^{4H-2} \\ &\cdot \int_0^1 dr \sum_{k=0}^{2q-1} k! (C_{2q-1}^k)^2 \sum_{i,j=0}^{N-1} I_{2(2q-1)-2k} ((f_{i,N} \otimes f_{i,N})(\cdot, r) \otimes_k (f_{j,N} \otimes f_{j,N})(\cdot, r)) \end{aligned} \quad (27)$$

First we note that:

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\|DF_N\|_{L^2[0,1]}^2 \right] = \lim_{N \rightarrow \infty} \mathbf{E} \left[\int_0^1 |D_r F_N|^2 dr \right] = 2q.$$

Indeed, this follows easily because for a random variable G in the n th Wiener chaos, we have

$$n \mathbf{E}[G^2] = \int_0^1 |D_r G|^2 dr.$$

To complete the proof of (ii), we only need to show that $\|DF_N\|_{L^2[0,1]}^2$ converges in $L^2(\Omega)$ to its mean. Since its mean is the term of order 0, corresponding to $k = 2q - 1$ in the expansion (27), our task is only to show that for all $k = 0, 1, \dots, 2q - 2$,

$$N^{4H-1} \int_0^1 dr \sum_{i,j=0}^{N-1} I_{2(2q-1)-2k} ((f_{i,N} \otimes f_{i,N})(\cdot, r) \otimes_k (f_{j,N} \otimes f_{j,N})(\cdot, r))$$

converges to 0 in $L^2(\Omega)$. Now fix k . To simplify the notation let $p = 2(2q - 1) - 2k$, and for each $i = 1, \dots, N$,

let $f_{i,N} \otimes f_{i,N} = f_{i,N}^{\otimes 2} = \phi_i$. We want the convergence to 0 as $N \rightarrow \infty$ of

$$\Delta_N \quad : \quad = N^{8H-2} \int_0^1 \int_0^1 dr dr' \sum_{i,j,i',j'=0}^{N-1} \mathbf{E} [I_p [\phi_i(\cdot, r) \otimes_k \phi_j(\cdot, r)] I_p [\phi_{i'}(\cdot, r') \otimes_k \phi_{j'}(\cdot, r')]] \quad (28)$$

$$\begin{aligned} &= p! N^{8H-2} \int_0^1 \int_0^1 dr dr' \sum_{i,j,i',j'=0}^{N-1} \langle \phi_i(\cdot, r) \otimes_k \phi_j(\cdot, r) ; \phi_{i'}(\cdot, r') \otimes_k \phi_{j'}(\cdot, r') \rangle_{L^2([0,1]^p)} \\ &= p! N^{8H-2} \sum_{i,j,i',j'=0}^{N-1} \left\langle \left(\int_0^1 dr \phi_i(\cdot, r) \otimes_k \phi_j(\cdot, r) \right) ; \left(\int_0^1 dr' \phi_{i'}(\cdot, r') \otimes_k \phi_{j'}(\cdot, r') \right) \right\rangle_{L^2([0,1]^p)} \\ &= p! N^{8H-2} \sum_{i,j,i',j'=0}^{N-1} \langle \phi_i \otimes_{k+1} \phi_j ; \phi_{i'} \otimes_{k+1} \phi_{j'} \rangle_{L^2([0,1]^p)} \end{aligned} \quad (29)$$

for all $k = 0, 1, \dots, 2q - 2$.

We are faced with the problem of calculating the quantity

$$A_{i,j,i',j'}(\ell, N) := \left\langle f_{i,N}^{\otimes 2} \otimes_\ell f_{j,N}^{\otimes 2} ; f_{i',N}^{\otimes 2} \otimes_\ell f_{j',N}^{\otimes 2} \right\rangle_{L^2([0,1]^p)}$$

for all $k+1 =: \ell = 1, \dots, 2q - 1$; we may not use the formulas, such as $\langle f_{i,N} \otimes_k f_{i,N} ; f_{j,N} \otimes_k f_{j,N} \rangle$, obtained in the previous section, directly for our purposes. Instead, we separate the cases $\ell \leq q$ and $\ell > q$, beginning with the first one, to write

$$\begin{aligned} &\left(f_{i,N}^{\otimes 2} \otimes_\ell f_{j,N}^{\otimes 2} \right) (x_1, \dots, x_{2q-\ell}, y_1, \dots, y_{2q-\ell}) \\ &= \int_{[0,1]^\ell} f_{i,N}(x_1, \dots, x_q) f_{i,N}(x_{q+1}, \dots, x_{2q-\ell}, r_1, \dots, r_\ell) \\ &\quad f_{j,N}(y_1, \dots, y_q) f_{j,N}(y_{q+1}, \dots, y_{2q-\ell}, r_1, \dots, r_\ell) dr_1 \dots dr_\ell \\ &= (f_{i,N} \otimes f_{j,N})(x_1, \dots, x_q, y_1, \dots, y_q) \cdot (f_{i,N} \otimes_\ell f_{j,N})(x_{q+1}, \dots, x_{2q-\ell}, y_{q+1}, \dots, y_{2q-\ell}). \end{aligned}$$

This immediately yields

$$A_{i,j,i',j'}(\ell, N) = \langle f_{i,N} ; f_{i',N} \rangle \langle f_{j,N} ; f_{j',N} \rangle \langle f_{i,N} \otimes_\ell f_{j,N} ; f_{i',N} \otimes_\ell f_{j',N} \rangle. \quad (30)$$

The first two terms in this product are given by the covariance structure of fractional Brownian motion (we already used this fact for instance in (26)):

$$\langle f_{i,N} ; f_{i',N} \rangle = \frac{1}{2q!} \left[\left(\frac{i-i'+1}{N} \right)^{2H} + \left(\frac{i-i'-1}{N} \right)^{2H} - 2 \left(\frac{i-i'}{N} \right)^{2H} \right].$$

If we ignore all the terms in the sum in (29) such that $|i-j| \leq 1$ (for this ‘‘tridiagonal’’, the reader will check that these terms sum up to a quantity of a lower order than the remainder of the sum), we then have that up to a constant $c = c(q, H)$,

$$|\langle f_{i,N} ; f_{i',N} \rangle| \leq c \frac{1}{N^2} \left| \frac{i-i'}{N} \right|^{2H-2}. \quad (31)$$

The calculation of the last term in the product (30) is not as easy; however, we may use calculations made in the previous section nonetheless. In particular, an expression for the quantity $\langle f_{i,N} \otimes_k f_{i,N} ; f_{j,N} \otimes_k f_{j,N} \rangle$ was derived from line (17) to line (18); this expression can be transformed via polarization into the following formula:

$$\begin{aligned} &\langle f_{i,N} \otimes_\ell f_{j,N} ; f_{i',N} \otimes_\ell f_{j',N} \rangle \\ &= N^{-4+2q(2-2H')} \int_{[0,1]^4} dy dy' dz dz' \left[|y-y'+i-i'|^{q-\ell} |z-z'+j-j'|^{q-\ell} \right. \\ &\quad \left. \cdot |y-z+i-j|^\ell |y'-z'+i'-j'|^\ell \right]^{-(2-2H')}. \end{aligned} \quad (32)$$

Again ignoring the indices such that

$$m := \min(|i - j|, |i - i'|, |j - j'|, |i' - j'|) \leq 1,$$

the expression in (32) is easily bounded above, as follows

$$|\langle f_{i,N} \otimes_\ell f_{j,N}; f_{i',N} \otimes_\ell f_{j',N} \rangle| \leq c \left[\left| \frac{i-j}{N} \right|^\ell \left| \frac{i'-j'}{N} \right|^\ell \left| \frac{i-i'}{N} \right|^{q-\ell} \left| \frac{j-j'}{N} \right|^{q-\ell} \right]^{-(2-2H')} N^{-4}. \quad (33)$$

Using (31) and (33), Δ_N , via the expression in (29), is now bounded above as

$$\begin{aligned} |\Delta_N| &\leq N^{8H-2} \sum_{i,j,i',j'=0}^{N-1} \left| \langle \phi_i \otimes_{k+1} \phi_j; \phi_{i'} \otimes_{k+1} \phi_{j'} \rangle_{L^2([0,1]^p)} \right| \\ &\leq (c + O_N(1)) N^{8H-10} \sum_{\substack{i,j,i',j'=0,\dots,N-1; \\ m \geq 2}} \left[\left| \frac{i-i'}{N} \right| \left| \frac{j-j'}{N} \right| \right]^{2H-2} \\ &\quad \cdot \left[\left| \frac{i-j}{N} \right|^\ell \left| \frac{i'-j'}{N} \right|^\ell \left| \frac{i-i'}{N} \right|^{q-\ell} \left| \frac{j-j'}{N} \right|^{q-\ell} \right]^{-(2-2H')} \\ &= (c + O_N(1)) N^{8H-6} \frac{1}{N^4} \sum_{\substack{i,j,i',j'=0,\dots,N-1; \\ m \geq 2}} \left[\left| \frac{i-i'}{N} \right| \left| \frac{j-j'}{N} \right| \right]^{-(2-2H)-(q-\ell)(2-2H')} \\ &\quad \cdot \left[\left| \frac{i-j}{N} \right| \left| \frac{i'-j'}{N} \right| \right]^{-\ell(2-2H')}. \end{aligned}$$

The quadruple sum in the last expression above cannot be compared directly to a Riemann sum, because the first negative power that appears therein is less than -1 . However, we can still easily evaluate the behavior of this sum. Introducing the variables $m = i - i'$, $\mu = i - j$, $\nu = i' - j'$ and $n = j - j'$ and the powers $\alpha = (2 - 2H) + (q - \ell)(2 - 2H')$ and $\beta = \ell(2 - 2H')$, note that $\alpha > 1$ and $\beta < 1$. The above sum, times the factor N^{-4} , is now bounded above by

$$\begin{aligned} &\frac{1}{N^2} \sum_{\mu, m \in \{-N, \dots, N\} - \{0\}} \left| \frac{\mu}{N} \right|^{-\alpha} \left| \frac{m}{N} \right|^{-\alpha} \frac{1}{N^2} \sum_{\nu \in \{-N, \dots, N\} - \{0\}} \left| \frac{\nu}{N} \right|^{-\beta} \left| \frac{n}{N} \right|^{-\beta} \\ &\leq c(\beta) N^{-2+2\alpha} \sum_{\mu, m \in \{-N, \dots, N\} - \{0\}} |\mu|^{-\alpha} |m|^{-\alpha} \\ &\leq c(\alpha, \beta) N^{2\alpha-2}. \end{aligned}$$

We have proved that Δ_N in (28), which we only need to prove converges to 0, is bounded as

$$\begin{aligned} |\Delta_N| &\leq N^{8H-6+2\alpha-2} \\ &= N^{-2\ell(2-2H')}. \end{aligned}$$

Recalling that we have $\ell = k + 1 \geq 1$, this shows the required convergence in the case $q \geq \ell$. The case $q < \ell \leq 2q - 1$ leads to identical calculations, and is omitted. This finishes the proof of the theorem. \blacksquare

Remark 2 *The above theorem shows that the case $q = 2$ is special; in this case the centered quadratic variations statistics can be decomposed into a sum of two terms T_4 and T_2 where the dominant term T_2 converges to a Rosenblatt random variable with self-similarity parameter H which is in fact the observed value of the process. As a consequence it is possible to define some adjusted variations that satisfy a central limit theorem by subtracting the "ill-behaved" term T_2 using only observed data (see [23], and the description of this phenomenon in the introduction).*

It is possible to give the limits of the remaining terms T_{2q-2} to T_2 appearing in the decomposition of the statistic V_N . All these renormalized terms converge to Hermite random variables of the same order as their indices (we have already proved this property in detail for T_2 in the previous section). This is a kind of “reproduction” of the Hermite processes through their variations.

Theorem 6

- For every $H \in (\frac{1}{2}, 1)$ and for every $k = 1, \dots, q-2$ we have

$$\lim_{N \rightarrow \infty} N^{(2-2H')(q-k)} T_{2q-2k} = z_{k,H} Z^{(2q-2k, (2q-2k)(H'-1)+1)}, \quad \text{in } L^2(\Omega) \quad (34)$$

where $Z^{(2q-2k, (2q-2k)(H'-1)+1)}$ denotes a Hermite random variable with self-similarity order $(2q-2k)(H'-1)+1$ and $z_{k,H} = d(H)^2 a(H')^k ((H'-1)k+1)^{-1} (2(H'-1)+1)^{-1}$.

- Moreover, if $H \in (\frac{3}{4}, 1)$ then

$$\lim_{N \rightarrow \infty} N^{2-2H} x_{2,H}^{-1/2} T_{2q} = Z^{(2q, 2H-1)}, \quad \text{in } L^2(\Omega). \quad (35)$$

Proof: Recall that, for $k = 1$ to $2q-2$ we have $T_{2k} = N^{2H-1} I_{2q-2k} \left(\sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right)$ with $f_{i,N} \otimes_k f_{i,N}$ given by (11). The first step of the proof is to observe that the limit of $N^{(2H'-2)(q-k)} T_{2q-2k}$ is given by

$$N^{(2H'-2)(q-k)} N^{2H-1} I_{2q-2k}(f_{2q-2k}^N)$$

with

$$\begin{aligned} f_{2q-2k}^N(y_1, \dots, y_{q-k}, z_1, \dots, z_{q-k}) &= d(H)^2 a(H')^k \sum_{i=0}^{N-1} 1_{[0, \frac{i}{N}]}(y_i) 1_{[0, \frac{i}{N}]}(z_i) \\ &\int_{I_i} \int_{I_i} \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \partial_1 K(v, z_1) \dots \partial_1 K(v, z_{q-k}) |u-v|^{(2H'-2)k} dv du. \end{aligned}$$

The argument leading to the above fact is the same as in Theorem 4: a remainder term r_{2q-2k} , which is defined by T_{2q-2k} minus $I_2(f_{2q-2k}^N)$ converges to zero similarly to the term r_2 in the proof of Theorem 4 because of the appearance of the indicator functions $1_{I_i}(y_i)$ or $1_{I_i}(z_i)$ in each of the terms that form this remainder.

The second step of the proof is to replace $\partial_1 K(u, y_i)$ by $\partial_1 K(\frac{i}{N}, y_i)$ and $\partial_1 K(v, z_i)$ by $\partial_1 K(\frac{i}{N}, z_i)$ on the interval I_i inside the integrals du and dv . The rest of the proof can be argued as in the proof of Theorem 4. The term $N^{(2-2H')(q-k)} f_{2q-2k}^N$ will have the same limit as

$$\begin{aligned} &N^{(2-2H')(q-k)} N^{2H-1} d(H)^2 a(H')^k \sum_{i=0}^{N-1} 1_{[0, \frac{i}{N}]}(y_i) 1_{[0, \frac{i}{N}]}(z_i) \\ &\partial_1 K\left(\frac{i}{N}, y_1\right) \dots \partial_1 K\left(\frac{i}{N}, y_{q-k}\right) \partial_1 K\left(\frac{i}{N}, z_1\right) \dots \partial_1 K\left(\frac{i}{N}, z_{q-k}\right) \int_{I_i} \int_{I_i} |u-v|^{(2-2H')k} dv du \\ &= d(H)^2 a(H')^k ((H'-1)k+1)^{-1} (2(H'-1)+1)^{-1} N^{(2-2H')(q-k)} N^{2H-1} N^{(2-2H')k+2} \\ &\sum_{i=0}^{N-1} 1_{[0, \frac{i}{N}]}(y_i) 1_{[0, \frac{i}{N}]}(z_i) \partial_1 K\left(\frac{i}{N}, y_1\right) \dots \partial_1 K\left(\frac{i}{N}, y_{q-k}\right) \partial_1 K\left(\frac{i}{N}, z_1\right) \dots \partial_1 K\left(\frac{i}{N}, z_{q-k}\right) \\ &= z_{k,H} \frac{1}{N} \sum_{i=0}^{N-1} 1_{[0, \frac{i}{N}]}(y_i) 1_{[0, \frac{i}{N}]}(z_i) \partial_1 K\left(\frac{i}{N}, y_1\right) \dots \partial_1 K\left(\frac{i}{N}, y_{q-k}\right) \partial_1 K\left(\frac{i}{N}, z_1\right) \dots \partial_1 K\left(\frac{i}{N}, z_{q-k}\right). \end{aligned}$$

Now, the last sum is a Riemann sum that converges pointwise and in $L^2([0, 1]^{2q-2k})$ (because the sequence f_{2q-2k}^N is a Cauchy sequence in this space) to $z_{k,H} L_1$ where

$$L_1(y_1, \dots, y_{q-k}, z_1, \dots, z_{q-k}) = \int_{y_1 \vee z_{q-k}}^1 \partial_1 K(u, y_1) \dots \partial_1 K(u, y_{q-k}) \partial_1 K(u, z_1) \dots \partial_1 K(u, z_{q-k}) du$$

which is the kernel of the Hermite random variable of order $2q - 2k$ with self-similarity parameter $(2q - 2k)(H' - 1) + 1$. The case $H \in (\frac{3}{4}, 1)$ and $k = 0$ follows analogously. \blacksquare

5 Consistent estimation of H and its asymptotics

In this section, we drop the notational dependence of $Z^{(H,q)}$ on H . Theorem 4 can be immediately applied to the statistical estimation of H . We note that with V_N in (10) and S_N defined by

$$S_N = \frac{1}{N} \sum_{i=0}^{N-1} \left(Z_{\frac{i+1}{N}}^{(q)} - Z_{\frac{i}{N}}^{(q)} \right)^2, \quad (36)$$

we have

$$1 + V_N = N^{2H} S_N \quad (37)$$

and

$$\mathbf{E}[S_N] = N^{-2H}.$$

so that

$$H = -\frac{\log \mathbf{E}[S_N]}{2 \log N}.$$

To form an estimator of H , since Theorem 4 implies that S_N evidently concentrates around its mean, we will use S_N in the role of an empirical mean, instead of its true mean; in other words we let

$$\hat{H} = \hat{H}_N = -\frac{\log S_N}{2 \log N}.$$

We immediately get from (37) that

$$\begin{aligned} \log(1 + V_N) &= 2H \log N + \log S_N \\ &= 2(H - \hat{H}) \log N. \end{aligned} \quad (38)$$

The first observation is that \hat{H}_N is strongly consistent for the Hurst parameter.

Proposition 3 *We have that \hat{H}_N converges to H almost surely as $N \rightarrow \infty$.*

Proof: Let us prove that V_N converges to zero almost surely as $N \rightarrow \infty$. We know that V_N converges to 0 in $L^2(\Omega)$ as $N \rightarrow \infty$ and an estimation for its variance is given by the formula (16). On the other hand this sequence is stationary because the increments of the Hermite process are stationary. We can therefore apply a standard argument to obtain the almost sure convergence for discrete stationary sequence under condition (16). This argument follows from Theorem 6.2, page 492 in [6] and it can be used exactly as in the proof of Proposition 1 in [4]. A direct elementary proof using the Borel-Cantelli lemma is almost as easy. The almost sure convergence of \hat{H} to H is then obtained immediately via (38). \blacksquare

Owing to the fact that V_N is of small magnitude (it converges to zero almost surely by the last proposition's proof), $\log(1 + V_N)$ can be confused with V_N . It then stands to reason that $(H - \hat{H}) N^{2-2H'} \log N$ is asymptotically Rosenblatt-distributed since the same holds for V_N by Theorem 4. Just as in that theorem, more is true: as we now show, the convergence to a Rosenblatt random variable occurs in $L^2(\Omega)$.

Proposition 4 *There is a standard Rosenblatt random variable R with self-similarity parameter $2H' - 1$ such that*

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\left| 2N^{2-2H'} (H - \hat{H}) \log N - c_2 c_{1,H,q}^{1/2} R \right|^2 \right] = 0$$

Proof: To simplify the notation, we denote $c_2 c_{1,H,q}^{1/2}$ by c in this proof. Theorem 4 signifies that a standard Rosenblatt r.v. R with parameter $2H' - 1$ exists such that

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\left| N^{2-2H'} V_N - cR \right|^2 \right] = 0.$$

From (38) we immediately get

$$\begin{aligned} & \mathbf{E} \left[\left| 2N^{2-2H'} (H - \hat{H}) \log N - cR \right|^2 \right] \\ &= \mathbf{E} \left[\left| N^{2-2H'} \log(1 + V_N) - cR \right|^2 \right] \\ &\leq 2\mathbf{E} \left[\left| N^{2-2H'} V_N - cR \right|^2 \right] + 2N^{4-4H'} \mathbf{E} \left[|V_N - \log(1 + V_N)|^2 \right]. \end{aligned}$$

Therefore, we only need to show that $\mathbf{E} \left[|V_N - \log(1 + V_N)|^2 \right] = o\left(N^{4H'-4}\right)$.

Consider the function $f(x) = |\log(1 + x)|$, defined for $x > -1$. We have for $x > -1/2$, $|f(x)| \leq 2|x|$. We can write

$$\begin{aligned} & \mathbf{E} \left[|V_N - \log(1 + V_N)|^2 \right] \\ &\leq 2\mathbf{E} \left[|V_N|^2 \right] + 2\mathbf{E} \left[|\log(1 + V_N)|^2 \right] \\ &= 2\mathbf{E} \left[|V_N|^2 \right] + 2\mathbf{E} \left[1_{V_N > \frac{-1}{2}} |\log(1 + V_N)|^2 \right] \\ &\quad + 2\mathbf{E} \left[1_{V_N < \frac{-1}{2}} |\log(1 + V_N)|^2 \right] \\ &\leq 2\mathbf{E} \left[|V_N|^2 \right] + 4\mathbf{E} \left[1_{V_N > \frac{-1}{2}} |V_N|^2 \right] + 2\mathbf{E} \left[1_{V_N < \frac{-1}{2}} |\log(1 + V_N)|^2 \right] \\ &= 6\mathbf{E} \left[|V_N|^2 \right] + 2\mathbf{E} \left[1_{V_N < \frac{-1}{2}} |\log(1 + V_N)|^2 \right] \\ &: = A + B \end{aligned}$$

The term A is bounded above and this is immediately dealt with using the following lemma.

Lemma 1 *For every $n \geq 2$, there is a constant c_n such that $\mathbf{E} \left[|V_N|^{2n} \right] \leq c_n N^{(4H'-4)n}$.*

Proof: In the proof of Proposition 1 we calculated the L^2 norm of the terms appearing in the decomposition of V_N , where V_N is a sum of multiple integrals. Therefore, from Proposition 5.1 of [10] we immediately get an estimate for any event moment of each term. Indeed, for $k = q - 1$

$$\mathbf{E} \left[T_2^{2n} \right] \leq 1 \cdot 3 \cdot 5 \cdots (4n - 1) (c_{1,H}^2 N^{4H'-4})^n = c_n N^{(4H'-4)n}$$

and for $1 \leq k < q - 1$

$$\begin{aligned} \mathbf{E} \left[T_{2q-2k}^{2n} \right] &\leq 1 \cdot 3 \cdot 5 \cdots (2(2q - 2k)n) (c_{1,q,H}^2 N^{(2q-2k)(2H'-2)})^n \\ &= c_{q,n} N^{(2q-2k)(2H'-2)n} \end{aligned}$$

The dominant term is again the T_2 term and the result follows immediately. ■

For the term B , we write $\mathbf{1}_{V_N < -1/2}$ as $\mathbf{1}_{V_N < -1/2} \mathbf{1}_{S_N < 2^{-1}N^{-2H}}$. Now using Hölder's inequality, we write

$$B \leq \mathbf{P}^{1/p} [V_N < -1/2] \mathbf{E}^{1/p'} \left[\mathbf{1}_{S_N < 2^{-1}N^{-2H}} |\log(1 + V_N)|^{2p'} \right]. \quad (39)$$

To deal with the expectation term, recall that $\log(1 + V_N) = 2H \log N + \log S_N$, and thus

$$\mathbf{E} \left[\mathbf{1}_{S_N < 2^{-1}N^{-2H}} |\log(1 + V_N)|^{2p'} \right] \leq c_p (2H \log N)^{2p'} + c_p \mathbf{E} \left[\mathbf{1}_{S_N < 2^{-1}N^{-2H}} \left(\log \frac{1}{S_N} \right)^{2p'} \right]. \quad (40)$$

We estimate the last expectation by returning to the definition of S_N in (36). First note that with $f(x) = \mathbf{1}_{0 < x < 2^{-1}N^{-2H}} \log^{2p'}(1/x)$, and $g(x) = \mathbf{1}_{0 < x < 1} \log^{2p'}(1/x)$, we have $f(x) \leq g(x)$ and $g(x)$ is convex. Hence using Jensen's inequality on the empirical average defining S_N ,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{S_N < 2^{-1}N^{-2H}} |\log S_N|^{2p'} \right] \\ & \leq \mathbf{E} [g(S_N)] = \mathbf{E} \left[g \left(\frac{1}{N} \sum_{i=0}^{N-1} \left(Z_{\frac{i+1}{N}}^{(q)} - Z_{\frac{i}{N}}^{(q)} \right)^2 \right) \right] \\ & \leq \frac{1}{N} \sum_{i=0}^{N-1} \mathbf{E} \left[g \left(\left(Z_{\frac{i+1}{N}}^{(q)} - Z_{\frac{i}{N}}^{(q)} \right)^2 \right) \right]. \end{aligned} \quad (41)$$

Next we can bound $g(x)$ by $c_{\alpha,p} x^{-\alpha}$ for any $\alpha \in (0, 1)$ and a constant $c_{\alpha,p}$ large enough. Moreover by scaling, $Z_{\frac{i+1}{N}}^{(q)} - Z_{\frac{i}{N}}^{(q)}$ has the same law as $N^{-H} Z$ where Z has the law of the Hermite process $Z^{(q)}$ at time 1. Hence

$$\begin{aligned} \mathbf{E} \left[g \left(\left(Z_{\frac{i+1}{N}}^{(q)} - Z_{\frac{i}{N}}^{(q)} \right)^2 \right) \right] & \leq c_{\alpha,p} \mathbf{E} \left[\left(Z_{\frac{i+1}{N}}^{(q)} - Z_{\frac{i}{N}}^{(q)} \right)^{-2\alpha} \right] \\ & = c_{\alpha,p} N^{2\alpha H} \mathbf{E} \left[|Z|^{-2\alpha} \right] \\ & = c'_{\alpha,p} N^{2\alpha H} \end{aligned} \quad (42)$$

where $c'_{\alpha,p}$ is another constant depending only on α and p , which is finite as long as $\alpha < 1/2$, because Z has a bounded density at the origin. Combining (40), (41), and (42), we get

$$\mathbf{E} \left[\mathbf{1}_{S_N < 2^{-1}N^{-2H}} |\log(1 + V_N)|^{2p'} \right] \leq c_p (2H \log N)^{2p'} + c_p c'_{\alpha,p} N^{2\alpha H}. \quad (43)$$

To finish controlling B , we only need to evaluate the tail behavior of V_N , as in

$$\begin{aligned} \mathbf{P} [V_N < -1/2] & = \mathbf{P} \left[cN^{2-2H'} V_N < -2^{-1}cN^{2-2H'} \right] \\ & = \mathbf{P} \left[cN^{2-2H'} V_N - R + R < -2^{-1}cN^{2-2H'} \right]. \end{aligned}$$

To lighten the notation we let $U_N = cN^{2-2H'} V_N - R$ and $x_N = 2^{-1}cN^{2-2H'}$. Thus

$$\begin{aligned} \mathbf{P} [V_N < -1/2] & \leq \mathbf{P} [U_N < -x_N/2] + \mathbf{P} [R < -x_N/2] \\ & \leq \mathbf{P} [|U_N| > x_N/2] + \mathbf{P} [|R| > x_N/2] \\ & \leq \mathbf{E} [|U_N|^4] 2c^{-1}N^{-(8-8H')} + \exp \left(-c'N^{2-2H'} \right). \end{aligned}$$

Asymptotically the last term is negligible compared to the second-to-last one. Moreover by Lemma 1, $\mathbf{E} [|U_N|^4]$ is bounded. Therefore, for some constant c'' and N large,

$$\mathbf{P} [V_N < -1/2] \leq c'' N^{-(8-8H')}.$$

Combining this with (39) and (43), we get for large N ,

$$B \leq c''' N^{-(8-8H')\frac{1}{p}+2\alpha H\frac{1}{p}}.$$

By choosing α close enough to 0, we guarantee that $B = o(N^{4H'-4})$, which finishes the proof of the proposition. \blacksquare

A difficulty arises when applying the above proposition for model validation when checking the asymptotic distribution of the estimator \hat{H} : the normalization constant $N^{2-2\hat{H}'} \log N$ depends on H . While it is not always obvious that one may replace this instance of H' by its estimator, in our situation, because of the $L^2(\Omega)$ convergences, this is legitimate, as the following theorem shows.

Theorem 7 *Let $\hat{H}' = 1 + (\hat{H} - 1)/q$. There is a standard Rosenblatt random variable R with self-similarity parameter $2H' - 1$ such that*

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\left| 2N^{2-2\hat{H}'} (H - \hat{H}) \log N - c_2 c_{1,H,q}^{1/2} R \right| \right] = 0.$$

Proof: By the previous proposition, it is sufficient to prove that

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\left| (N^{2-2\hat{H}'} - N^{2-2H'}) (H - \hat{H}) \log N \right| \right] = 0.$$

We decompose the probability space depending on whether \hat{H} is far or not from its mean. For a fixed value $\varepsilon > 0$ which will be chosen later, it is most convenient to define the event

$$D = \left\{ \hat{H} > q\varepsilon/2 + 2H - 1 \right\}.$$

We have

$$\begin{aligned} & \mathbf{E} \left[\left| (N^{2-2\hat{H}'} - N^{2-2H'}) (H - \hat{H}) \log N \right| \right] \\ &= \mathbf{E} \left[\mathbf{1}_D \left| (N^{2-2\hat{H}'} - N^{2-2H'}) (H - \hat{H}) \log N \right| \right] \\ & \quad + \mathbf{E} \left[\mathbf{1}_{D^c} \left| (N^{2-2\hat{H}'} - N^{2-2H'}) (H - \hat{H}) \log N \right| \right] \\ &= : A + B. \end{aligned}$$

We study A first. Let us introduce the shorthand notation $x = \max(2 - 2H', 2 - 2\hat{H}')$ and $y = \min(2 - 2H', 2 - 2\hat{H}')$. We may write

$$\begin{aligned} \left| N^{2-2\hat{H}'} - N^{2-2H'} \right| &= e^{x \log N} - e^{y \log N} \\ &= e^{y \log N} (e^{(x-y) \log N} - 1) \\ &\leq N^y (\log N) (x - y) N^{x-y} \\ &= 2 (\log N) N^x \left| H' - \hat{H}' \right| \\ &= 2q^{-1} (\log N) N^x \left| H - \hat{H} \right|. \end{aligned}$$

Thus

$$\begin{aligned} A &\leq 2q^{-1} \mathbf{E} \left[\mathbf{1}_D N^x \left| H - \hat{H} \right|^2 \log^2 N \right] \\ &= 2q^{-1} \mathbf{E} \left[N^{x-(4-4H')} \mathbf{1}_D N^{4-4H'} \left| H - \hat{H} \right|^2 \log^2 N \right]. \end{aligned}$$

Now choose $\varepsilon \in (0, 2 - 2H')$. In this case, if $\omega \in D$, and $x = 2 - 2H'$, we get $x - (4 - 4H') = -x < -\varepsilon$. On the other hand, for $\omega \in D$ and $x = 2 - 2\hat{H}'$, we get $x - (4 - 4H') = 2 - 2\hat{H}' - (4 - 4H') < -\varepsilon$ as well. In conclusion, on D ,

$$x - (4 - 4H') < -\varepsilon,$$

which implies immediately

$$A \leq N^{-\varepsilon} 2q^{-1} \mathbf{E} \left[N^{4-4H'} \left| H - \hat{H} \right|^2 \log^2 N \right];$$

this prove that A tends to 0 as $N \rightarrow \infty$, since the last expectation above is bounded (converges to a constant) by Proposition 4.

Now we may study B . We are now operating with $\omega \in D^c$. In other words,

$$H - \hat{H} > 1 - H - q\varepsilon/2.$$

Still using $\varepsilon < 2 - 2H'$, this implies that $H > \hat{H}$. Consequently, it is not inefficient to bound $\left| N^{2-2H'} - N^{2-2\hat{H}'} \right|$ above by $N^{2-2\hat{H}'}$. In the same fashion, we bound $\left| H - \hat{H} \right|$ above by H . Hence we have, using Hölder's inequality with the powers $2q$ and $p^{-1} + (2q)^{-1} = 1$.

$$\begin{aligned} B &\leq H \log N \mathbf{E} \left[\mathbf{1}_{D^c} N^{2-2\hat{H}'} \right] \\ &\leq H \log N \mathbf{P}^{1/p} [D^c] \mathbf{E}^{1/(2q)} \left[N^{(2-2\hat{H}')2q} \right]. \end{aligned} \quad (44)$$

From Proposition 4, by Chebyshev's inequality, we have

$$\mathbf{P}^{1/p} [D^c] \leq \frac{\mathbf{E}^{1/p} \left[\left| H - \hat{H} \right|^2 \right]}{(1 - H - q\varepsilon/2)^{2/p}} \leq c_{q,H} N^{-(4-4H')/p} \quad (45)$$

for some constant $c_{q,H}$ depending only on q and H . Dealing with the other term in the upper bound for B is a little less obvious. We must return to the definition of \hat{H} . By (38) we have

$$\begin{aligned} 1 + V_N &= N^{2(H-\hat{H})} = N^{2q(H'-\hat{H}')} \\ &= N^{2q(2-2\hat{H}')} N^{-2q(2-2H')}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{E}^{1/(2q)} \left[N^{(2-2\hat{H}')2q} \right] &\leq N^{2-2H'} \mathbf{E}^{1/(2q)} [1 + V_N] \\ &\leq 2N^{2-2H'}. \end{aligned} \quad (46)$$

Plugging (45) and (46) back into (44), we get

$$B \leq 2H c_{q,H} (\log N) N^{-(4-4H')(2/p-1)}.$$

Given that $2q \geq 4$ and p is conjugate to $2q$, we have $p \leq 4/3$ so that $2/p - 1 \geq 1/2 > 0$, which proves that B goes to 0 as $N \rightarrow \infty$. This finishes the proof of the theorem. \blacksquare

Finally we state the extension of our results to $L^p(\Omega)$ -convergence.

Theorem 8 *The convergence in Theorem 7 holds in any $L^p(\Omega)$. In fact, the $L^2(\Omega)$ -convergences in all other results in this paper can be replaced by $L^p(\Omega)$ -convergences.*

Sketch of proof. We only give the outline of the proof. Lemma 1 works because, in analogy to the Gaussian case, random variables in a fixed Wiener chaos of order p have moments of all orders, and the relations between the various moments are given by equalities using a set of constants which depend only on p . This Lemma can be used immediately to prove the extension of Proposition 1 that

$$\mathbf{E} \left[V_N^{2p} \right] \simeq c_p N^{p(4H'-4)}.$$

Proving a new version of Theorem 4 with $L^p(\Omega)$ -convergence can base itself on the above result, and requires a careful reevaluation of the various terms involved. Proposition 4 can then be extended to $L^p(\Omega)$ -convergence thanks to the new $L^p(\Omega)$ versions of Theorem 4 and Proposition 1, and Theorem 7 follows easily from this new version of Proposition 1. ■

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