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**BOUNDARY OF THE RAUZY FRACTAL SETS IN $\mathbb{R} \times \mathbb{C}$
GENERATED BY $P(x) = x^4 - x^3 - x^2 - x - 1$**

F. DURAND AND A. MESSAOUDI

ABSTRACT. We study the boundary of the 3-dimensional Rauzy fractal $\mathcal{E} \subset \mathbb{R} \times \mathbb{C}$ generated by the polynomial $P(x) = x^4 - x^3 - x^2 - x - 1$. The finite automaton characterizing the boundary of \mathcal{E} is given explicitly. As a consequence we prove that the set \mathcal{E} has 18 neighborhoods where 6 of them intersect the central tile \mathcal{E} in a point. Our construction shows that the boundary is generated by an iterated function system starting with 2 compact sets.

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1. INTRODUCTION

Consider $A = \{1, 2, 3\}$ as an alphabet. Let A^* be the set of finite words on A and $\sigma : A \rightarrow A^*$ be the map (called Tribonacci substitution) defined by

$$\sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1.$$

We extend σ to $A^{\mathbb{N}}$ by concatenation : $\sigma(a_0 \cdots a_n \dots) = \sigma(a_0) \cdots \sigma(a_n) \dots$. It is clear that σ has a unique fixed point $u : \sigma(u) = u \in A^{\mathbb{N}}$. The dynamical system associated to σ is the couple (Ω, S) where $S : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ is the shift map $(S((x_n)_{n \in \mathbb{N}})) = (x_{n+1})_{n \in \mathbb{N}}$ and Ω is the S -orbit closure of $u : \Omega = \overline{\{S^n u | n \in \mathbb{N}\}}$. It is well known that (Ω, S) is minimal, uniquely ergodic and of zero entropy (see [Q87, F02] for more details).

In 1982, G. Rauzy [R82] studied the Tribonacci substitution σ . He proved that the dynamical system generated by σ is measure theoretically conjugate to an exchange of domains X_1, X_2, X_3 in a compact tile $X = X_1 \cup X_2 \cup X_3$. The set X is the classical two-dimensional Rauzy fractal. It has been extensively studied and is related to many topics : numeration systems [M00, M06, M05], geometrical representation of symbolic dynamical systems [AI01, AIS01, CS01, HZ98, M98, T06, S96], multidimensional continued fractions and simultaneous approximations [ABI02, CHM01, C02, HM06], self-similar tilings [A99, A00, AI01, P99] and Markov partitions of Hyperbolic automorphisms of the torus [KV98, M98, P99].

Among the main properties of the set X , let us recall it is compact, connected, its interior is simply connected, its boundary is fractal and it induces a periodic tiling of \mathbb{R}^2 ([R82]).

It is possible to associate such a fractal set to a large class of substitutions over an alphabet with d letters (called unimodular Pisot substitutions). Let us call them *Rauzy fractals*. P. Arnoux and S. Ito [AI01] (see also [CS01]) proved that the dynamical system associated to such a substitution σ is measure theoretically conjugate to an exchange of domains X_1, \dots, X_d in the Rauzy fractal $X_\sigma = X_1 \cup \dots \cup X_d \subset \mathbb{R}^{d-1}$ provided that a certain combinatorial condition is true. All these sets X_σ are compact and generate a periodic tiling of \mathbb{R}^{d-1} .

There are different ways to define the Rauzy fractal associated to a given substitution σ over an alphabet of $d + 1$ letters. One is through numeration systems. Let $d \geq 2$ and a_1, a_2, \dots, a_d be integers such that $a_1 \geq a_2 \geq \dots \geq a_d \geq 1$. Consider $A = \{1, 2, \dots, d + 1\}$ as an alphabet. Let σ_d be the substitution defined by

$$\sigma_d(i) = \underbrace{11 \dots 1}_{a_i} (i + 1) \text{ if } i \leq d \text{ and } \sigma_d(d + 1) = 1.$$

We define the Rauzy fractal associated to σ_d as follows. Consider the sequence $(F_n)_{n \geq 0}$ defined by

$$F_{n+d+1} = a_1 F_{n+d} + a_2 F_{n+d-1} + \dots + a_d F_{n+1} + F_n, \forall n \geq 0,$$

with initial conditions (called *Parry conditions*)

$$F_0 = 1, F_n = a_1 F_{n-1} + \dots + a_n F_0 + 1, \forall 1 \leq n \leq d.$$

It is well-known (using the greedy algorithm) that for every integer n we have $n = \sum_{i=0}^N c_i F_i$ with $(c_i)_{0 \leq i \leq N} \in \mathcal{D}_{a_1, \dots, a_d}$, where $\mathcal{D}_{a_1, \dots, a_d}$ is the set of sequences $(\varepsilon_i)_{l \leq i \leq k}$, $l, k \in \mathbb{Z}$, such that for all $l \leq i \leq k$:

- (1) $\varepsilon_i \in \{0, 1, \dots, a_1\}$;
- (2) $\varepsilon_i \varepsilon_{i-1} \dots \varepsilon_{i-d} <_{lex} a_1 a_2 \dots a_d 1$ when $i \geq l + d$.
- (3) $\varepsilon_i \varepsilon_{i-1} \dots \varepsilon_l 0^{d-i+l} <_{lex} a_1 a_2 \dots a_d 1$ when $l \leq i \leq l + d$,

where $<_{lex}$ is the usual lexicographic ordering. We set

$$\mathcal{D}_{a_1, \dots, a_d}^\infty = \{(\varepsilon_i)_{i \geq l}; l \in \mathbb{Z}, (\varepsilon_i)_{l \leq i \leq n} \in \mathcal{D}_{a_1, \dots, a_d}, \forall n \geq l\}.$$

Now, consider the following polynomial

$$P_{a_1, \dots, a_d}(x) = x^{d+1} - a_1 x^d - a_2 x^{d-1} - \dots - a_d x - 1.$$

It can be checked that P has a root $\beta = \beta_1 \in]1, +\infty[$ and d roots with modulus less than 1. Let $\beta_1, \beta_2, \beta_3, \dots, \beta_r$ be the roots of P belonging to \mathbb{R} and $\beta_{r+1}, \dots, \beta_{r+s}, \bar{\beta}_{r+1}, \dots, \bar{\beta}_{r+s}$ its complex roots. For all $i \in \mathbb{Z}$, we set

$$\alpha^i = (\beta_2^i, \dots, \beta_r^i, \beta_{r+1}^i, \dots, \beta_{r+s}^i).$$

We also put $\alpha^0 = 1 = (1, \dots, 1)$. Then, the Rauzy fractal associated to σ is the set $\mathcal{E}_{a_1, \dots, a_d} \subset \mathbb{R}^{r-1} \times \mathbb{C}^s \approx \mathbb{R}^d$ defined by

$$\mathcal{E}_{a_1, \dots, a_d} = \left\{ \sum_{i=d+1}^{+\infty} \varepsilon_i \alpha^i; (\varepsilon_i)_{i \geq d+1} \in \mathcal{D}_{a_1, \dots, a_d}^\infty \right\}.$$

The set $\mathcal{E}_{1,1} = X$ is the classical two-dimensional Rauzy fractal.

The structure of the boundary of Rauzy fractals has been first investigated by Ito and M. Kimura in [IK91]. They showed that the boundary of $\mathcal{E}_{1,1}$ is a Jordan curve generated by the Dekking method [D82] and they calculated its Hausdorff dimension. Relating the boundary of $\mathcal{E}_{a_1,1}$ to the complex numbers having at least two expansions in base α , A. Messaoudi [M00, M05] constructed a finite automaton characterizing and generating this boundary. As a consequence it permitted to parameterize the boundary of $\mathcal{E}_{a_1,1}$, to compute its the Hausdorff dimension and to show it is a quasi circle.

In [T06], J. M. Thuswaldner studied the set \mathcal{E}_{a_1, a_2} . In particular, he gave an explicit formula for the fractal dimension of the boundary of this set.

In this paper we propose to study the boundary of the Rauzy fractal set $\mathcal{E}_{1,1,1} \subset \mathbb{R} \times \mathbb{C}$. We construct a finite automaton characterizing its boundary. As a consequence we prove that $\mathcal{E}_{1,1,1}$ has 18 neighborhoods where 6 of them intersect the central tile $\mathcal{E}_{1,1,1}$ in a point. We also prove that the boundary can be generated by two subregions. More precisely, the boundary of $\mathcal{E}_{1,1,1}$ is $\bigcup_{i=1}^{18} X_i$ where X_i , $i = 1, \dots, 6$ are singletons, and for all $i \in [7, 18]$, there exist affine functions f_{ij} , $j = 1, \dots, m_i$ and g_{ij} , $j = 1, \dots, n_i$ from $\mathbb{R} \times \mathbb{C}$ to itself such that

$$X_i = \bigcup_{j=1}^{m_i} f_{ij}(X_7) \bigcup_{j=1}^{n_i} g_{ij}(X_8).$$

2. NOTATIONS, DEFINITIONS AND BACKGROUND

2.1. β -expansions. Let $\beta > 1$ be a real number. A β -representation of a non-negative real number x is an infinite sequence $(x_i)_{i \leq k}$, $x_i \in \mathbb{Z}^+ = [0, +\infty[$, such that

$$x = x_k \beta^k + x_{k-1} \beta^{k-1} + \dots + x_1 \beta + x_0 + x_{-1} \beta^{-1} + x_{-2} \beta^{-2} + \dots.$$

where k is an integer. It is denoted by

$$x = x_k x_{k-1} \dots x_1 x_0 . x_{-1} x_{-2} \dots$$

A particular β -representation, called the β -expansion, is computed by the "greedy algorithm" (see [P60]): denote by $\lfloor y \rfloor$ and $\{y\}$ respectively the integer part and the fractional part of a number y . There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. We get

$$x = x_k \beta^k + x_{k-1} \beta^{k-1} + \dots$$

If $k < 0$ ($x < 1$), we put $x_0 = x_{-1} = \dots = x_{k+1} = 0$. If an expansion ends by infinitely many zeros, it is said to be finite, and the ending zeros are omitted.

The digits x_i belong to the set $A = \{0, \dots, \beta - 1\}$ if β is an integer, or to the set $A = \{0, \dots, \lfloor \beta \rfloor\}$ if β is not an integer. The β -expansion of every positive real number x is the lexicographically greatest among all β -representations of x .

We denote by $\text{Fin}(\beta)$ the set of numbers which have finite greedy β -expansion. Let $N \in \mathbb{Z}$, we denote by $\text{Fin}_N(\beta)$ the set of numbers x such that in their β -expansion $(x_i)_{i \leq k}$, $x_i = 0$ for all $i < N$. We will sometimes denote a β -expansion $x_n \dots x_k$, $n \geq k$ by $(x_i)_{k \leq i \leq n}$. We put

$$E_\beta = \{(x_i)_{i \geq k}; k \in \mathbb{Z}, \forall n \geq k, (x_i)_{k \leq i \leq n} \text{ is a finite } \beta\text{-expansion}\}.$$

In the case where β is the dominant root of the polynomial P_{a_1, \dots, a_d} , it is known (see [FS92]) that $E_\beta = \mathcal{D}_{a_1, \dots, a_d}^\infty$. We will need the two following classical lemmas.

Lemma 1 ([P60]). *Let $x_n \dots x_0$ and $y_m \dots y_0$ be two β -expansions. Then, the following are equivalent*

- $\sum_{i=0}^n x_i \beta^i < \sum_{i=0}^m y_i \beta^i$,
- $x_n \dots x_0 <_{lex} y_m \dots y_0$,

where $<_{lex}$ is the lexicographical order.

Lemma 2 ([FS92]). *If $\beta = \beta_1$, then $\mathbb{Z}[\beta] \cap [0, +\infty[\subset \text{Fin}(\beta)$.*

2.2. Boundary of $\mathcal{E}_{a_1, \dots, a_d}$. The coordinates of α have modulus strictly less than 1. Moreover, 0 belongs to the interior of $\mathcal{E}_{a_1, \dots, a_d}$ ([A00], see also [R82] for $\mathcal{E}_{1,1}$). Hence, for all $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ there exists $k \in \mathbb{N}$ such that $\alpha^k z \in \mathcal{E}_{a_1, \dots, a_d}$. Then, all $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ can be written as follows $z = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$, where $l \in \mathbb{Z}$ and $(\varepsilon_i)_{i \geq l} \in \mathcal{D}_{a_1, \dots, a_d}^{\infty}$. The sequence $(\varepsilon_i)_{i \geq l}$ is called α -expansion of z . We should remark that these α -expansions are not unique : some z can have many different α -expansions. In [M05] it is proven that the points belonging to the boundary of $\mathcal{E}_{a_1, \dots, a_d}$ have at least two different α -expansions. These points are characterized by the following theorem which is a straightforward consequence of a result due to W. Thurston [T90] (see also [M05]).

Theorem 3. *There exists a finite automaton B such that for all distinct elements of $\mathcal{D}_{a_1, \dots, a_d}^{\infty}$, $(b_i)_{i \geq l}$ and $(c_i)_{i \geq l}$, the following are equivalent :*

- $\sum_{i=l}^{\infty} b_i \alpha^i = \sum_{i=l}^{\infty} c_i \alpha^i$
- $((b_i, c_i))_{i \geq l}$ is recognizable by B (i.e an infinite path in B beginning in the initial state).

The proof of this result does not give explicitly the states of the automaton. In [M98] is given an algorithm that gives these states for $\mathcal{E}_{1,1}$. In [M06], they were given for $\mathcal{E}_{a_1,1}$ where $a_1 \geq 2$.

3. CHARACTERIZATION OF THE BOUNDARY OF $\mathcal{E}_{1,1,1}$

In the sequel we suppose $d = 3$ and $a_1 = a_2 = a_3 = 1$, and $P(x) = P_{1,1,1}(x) = x^4 - x^3 - x^2 - x - 1 = (x - \beta_1)(x - \beta_2)(x - \beta_3)(x - \overline{\beta_3})$ where $\beta_1, \beta_2, \beta_3$ are defined in Section 1. Approximations of these numbers are $\beta = \beta_1 = 1.9275\dots$, $\beta_2 = -0.7748\dots$ and $\beta_3 = -0.0763\dots + i0.8147\dots$. We recall that we defined for all $i \in \mathbb{Z}$, $\alpha^i = (\beta_2^i, \beta_3^i)$.

In this situation

$$\begin{aligned} \mathcal{D} &= \mathcal{D}_{1,1,1} = \{(\varepsilon_i)_{l \leq i \leq n}; l, n \in \mathbb{Z}, \varepsilon_i \in \{0, 1\}, \varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} \neq 1111, l \leq i \leq n\}, \\ \mathcal{D}^{\infty} &= \mathcal{D}_{1,1,1}^{\infty} = \{(\varepsilon_i)_{i \geq l}; l \in \mathbb{Z}, (\varepsilon_i)_{l \leq i \leq n} \in \mathcal{D}_{1,1,1}, n \geq l\} \text{ and} \\ \mathcal{E} &= \mathcal{E}_{1,1,1} = \left\{ \sum_{i=4}^{+\infty} \varepsilon_i \alpha^i; (\varepsilon_i)_{i \geq 4} \in \mathcal{D}^{\infty} \right\} \\ &= \left\{ \sum_{i=4}^{+\infty} \varepsilon_i \alpha^i; \varepsilon_i \in \{0, 1\}, \varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} \neq 1111, i \geq 4 \right\}. \end{aligned}$$

An important and known result is:

Proposition 4. *The set \mathcal{E} is compact, connected and generates a periodic tiling of $\mathbb{R} \times \mathbb{C}$ with group periods $G = \mathbb{Z}\alpha^0 + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2$:*

$$\mathbb{R} \times \mathbb{C} = \bigcup_{p \in G} (\mathcal{E} + p),$$

and the Lebesgue measure of $(\mathcal{E} + p) \cap (\mathcal{E} + q)$ is zero whenever $p \neq q$, $p, q \in G$.

Proof. The proof can easily be deduced from [R82] or [AI01]. □

3.1. Definition of the automaton recognizing the points with at least two expansions. In the sequel we proceed to the construction of the automaton \mathcal{A} that characterize the boundary of \mathcal{E} . This characterization will be proven in Section 3.2. The set of states of the automaton \mathcal{A} is

$$S = \left\{ \pm \sum_{i=0}^3 c_i \alpha^i; c_0 c_1 c_2 c_3 \neq 1111, c_i \in \{0, 1\}, 0 \leq i \leq 3 \right\} \cup \left\{ \pm(\alpha^{-1} + 1 + \alpha^2), \pm(\alpha^{-2} + \alpha^{-1} + \alpha), \pm(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3) \right\}.$$

Let s and t be two states. The set of edges is the set of $(s, (a, b), t) \in S \times \{0, 1\}^2 \times S$ satisfying $t = \frac{s}{\alpha} + (a - b)\alpha^3$. The set of initial states is $\{0\}$ and the set of states is S . A path (resp. infinite path) of \mathcal{A} is a sequence $(a_n, b_n)_{k \leq n \leq l}$ (resp. $(a_n, b_n)_{n \geq k}$) such that there exists a sequence $(e_n)_{k \leq n \leq l+1}$ (resp. $(e_n)_{n \geq k}$) of elements of S for which $(e_n, (a_n, b_n), e_{n+1})$ belongs to S for all $n \in \{k, k+1, \dots, l+1\}$ (resp. $n \geq k$). We say it starts in the initial state when $e_k = 0$. The automaton is explicitly defined in the Annexe at the end of this paper.

Let us explain the behavior of this automaton. Let $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belonging to \mathcal{D}^∞ , $x = \sum_{i=l}^\infty \varepsilon_i \alpha^i$ and $y = \sum_{i=l}^\infty \varepsilon'_i \alpha^i$. For all $k \geq l$ we set

$$(1) \quad A_k(\varepsilon, \varepsilon') = \alpha^{-k+3} \sum_{i=l}^k (\varepsilon_i - \varepsilon'_i) \alpha^i$$

In Subsection 3.2 we will prove that $x = y$ if and only if all the A_k , $k \geq l$, belong to S . But as, for all $k \geq l$, we have

$$(2) \quad A_{k+1}(\varepsilon, \varepsilon') = \frac{A_k(\varepsilon, \varepsilon')}{\alpha} + (\varepsilon_{k+1} - \varepsilon'_{k+1})\alpha^3,$$

this means that $x = y$ if and only if

$$(0, (\varepsilon_l, \varepsilon'_l), A_l(\varepsilon, \varepsilon')) \left((A_k(\varepsilon, \varepsilon'), (\varepsilon_{k+1}, \varepsilon'_{k+1}), A_{k+1}(\varepsilon, \varepsilon')) \right)_{k \geq l}$$

is an infinite sequence of edges of S starting in the initial state. And, this is equivalent to say that $(\varepsilon_i, \varepsilon'_i)_{i \geq l}$ is an infinite path of \mathcal{A} starting in the initial state. Let us give an example on how we can use this automaton to obtain information about the digits of x and y . Let s be the smallest integer such that $\varepsilon_s \neq \varepsilon'_s$. Hence $A_i(\varepsilon, \varepsilon') = 0$ for $i \in \{l, \dots, s-1\}$. Suppose $\varepsilon_s > \varepsilon'_s$, that is $\varepsilon_s = 1$ and $\varepsilon'_s = 0$. Then, $A_s = \alpha^3$. From (2) we deduce $A_{s+1}(\varepsilon, \varepsilon') = \alpha^2 + (\varepsilon_{s+1} - \varepsilon'_{s+1})\alpha^3$ which should belong to S . Hence $A_{s+1}(\varepsilon, \varepsilon') = \alpha^2 \in S$ if $\varepsilon_{s+1} = \varepsilon'_{s+1}$, and, $A_{s+1}(\varepsilon, \varepsilon') = \alpha^2 + \alpha^3 \in S$ if $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (1, 0)$. Hence, $(\alpha^3, (1, 0), \alpha^2 + \alpha^3)$, $(\alpha^3, (0, 0), \alpha^2)$ and $(\alpha^3, (1, 1), \alpha^2)$ are edges coming from the state α^3 . Let us explain why $(\alpha^3, (0, 1), \alpha^2 - \alpha^3)$ is not an edge, and hence why we cannot have $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (0, 1)$. We should have that $\alpha^2 - \alpha^3 = -\alpha^{-1} - 1 - \alpha$ belongs to S . Then β should satisfies the same equality. Hence $\beta^{-1} + 1 + \beta$ should belong to

$$\left\{ \sum_{i=0}^3 c_i \beta^i; c_0 c_1 c_2 c_3 \neq 1111, c_i \in \{0, 1\}, 0 \leq i \leq 3 \right\} \cup \left\{ (\beta^{-1} + 1 + \beta^2), (\beta^{-2} + \beta^{-1} + \beta), (\beta^{-3} + \beta^{-2} + 1 + \beta^3) \right\},$$

which is not possible by Lemma 1.

3.2. Characterization of the points with at least two expansions.

Lemma 5. *Let $(\varepsilon_i)_{i \geq 0}, (\varepsilon'_i)_{i \geq 0} \in \mathcal{D}^\infty$. Then,*

$$\left| \sum_{i=0}^{+\infty} (\varepsilon_i - \varepsilon'_i) \beta_2^i \right| \leq \frac{1}{1 + \beta_2}, \quad \left| \sum_{i=0}^{+\infty} (\varepsilon_i - \varepsilon'_i) \beta_3^i \right| \leq \frac{C}{1 - |\beta_3|^6}.$$

where $C = \max \left\{ \left| \sum_{i=0}^5 (c_i - d_i) \beta_3^i \right|; (c_i)_{0 \leq i \leq 5} \in \mathcal{D}, (d_i)_{0 \leq i \leq 5} \in \mathcal{D} \right\}$.

Proof. The second inequality is easy to establish. For the first inequality, as $-1 < \beta_2 < 0$, all sequences $(c_i)_{i \geq 0}$ which terms are 0 or 1 satisfy the following inequality :

$$\frac{\beta_2}{1 - \beta_2^2} = \sum_{i=0}^{+\infty} \beta_2^{2i+1} \leq \sum_{i=0}^{+\infty} c_i \beta_2^i \leq \sum_{i=0}^{+\infty} \beta_2^{2i} = \frac{1}{1 - \beta_2^2}.$$

This achieves the proof. \square

For all $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belonging to \mathcal{D}^∞ , we set

$$S(\varepsilon, \varepsilon') = \{A_k(\varepsilon, \varepsilon'); k \geq l\} = \left\{ \alpha^{-k+3} \sum_{i=l}^k (\varepsilon_i - \varepsilon'_i) \alpha^i; k \geq l \right\}.$$

Proposition 6. *Let $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$, $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$, where $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belong to \mathcal{D}^∞ . Then, $x = y$ if and only if the set $S(\varepsilon, \varepsilon')$ is finite. Moreover*

$$S(\varepsilon, \varepsilon') \subset S = \left\{ \pm \sum_{i=0}^3 c_i \alpha^i; (c_i)_{0 \leq i \leq 3} \in \mathcal{D} \right\} \\ \cup \left\{ \pm(\alpha^{-1} + 1 + \alpha^2), \pm(\alpha^{-2} + \alpha^{-1} + \alpha), \pm(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3) \right\}$$

and

$$S = \bigcup_{(\varepsilon, \varepsilon') \in \Delta} S(\varepsilon, \varepsilon'),$$

where $\Delta = \{((\varepsilon_i)_{i \geq l}, (\varepsilon'_i)_{i \geq l}) \in \mathcal{D}^\infty \times \mathcal{D}^\infty; \sum_{i=l}^{\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i\}$.

Proof. It is easy to establish that if $S(\varepsilon, \varepsilon')$ is finite then $x = y$. Let us prove the reciprocal. Let $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i = y$ with $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belonging to \mathcal{D}^∞ . Let us prove that $A_k = A_k(\varepsilon, \varepsilon')$ belongs to S for all $k \geq l$. As $x = y$, for all $k \geq l$, we have

$$(3) \quad A_k = \sum_{i=k+1}^{\infty} (\varepsilon'_i - \varepsilon_i) \alpha^{i-k+3} = \sum_{i=4}^{\infty} (\varepsilon'_{i+k-3} - \varepsilon_{i+k-3}) \alpha^i.$$

Let us fix $k \geq l$ and assume $A_k \neq 0$. From (1), we deduce there exist $n, p, q, r \in \mathbb{Z}$ such that

$$(4) \quad A_k = n\alpha^3 + p\alpha^2 + q\alpha + r.$$

But $n\beta^3 + p\beta^2 + q\beta + r$ or $-(n\beta^3 + p\beta^2 + q\beta + r)$ belongs to $\mathbb{Z}[\beta] \cap \mathbb{R}^+$, which is contained in $\text{Fin}(\beta)$ (see Lemma 2). We deduce there exists $(c_i)_{s \leq i \leq m} \in \mathcal{D}$ such that $c_m = 1$ and

$$(5) \quad n\beta^3 + p\beta^2 + q\beta + r = \pm \sum_{i=s}^m c_i \beta^i.$$

We suppose it is equal to $\sum_{i=s}^m c_i \beta^i$. The other case can be treated in the same way. As β , β_2 and β_3 are algebraically conjugate, from (1), (4) and (5) we have

$$(6) \quad \beta^{-k+3} \sum_{i=l}^k \varepsilon_i \beta^i = \beta^{-k+3} \sum_{i=l}^k \varepsilon'_i \beta^i + \sum_{i=s}^m c_i \beta^i.$$

From Lemma 1, $\beta^{-k+3} \sum_{i=l}^k \varepsilon_i \beta^i < \beta^4$, consequently $m \leq 3$. Setting $c_i = 0$ for $i > m$, we have

$$(7) \quad A_k = \sum_{i=s}^3 c_i \alpha^i.$$

Remark that if $s \geq 0$ then A_k belongs to S . Hence we suppose $s \leq -1$. Suppose $s = -1$ and $c_{-1} = 1$. Let us show that A_k is equal to $\alpha^{-1} + 1 + \alpha^2$ and consequently belongs to S . In order to do so, we show that the other cases are not possible. Using Lemma 5 and (3), the first entry of (A_k) , $(A_k)_1$, should satisfy $|(A_k)_1| \leq \beta_2^4 (1 + \beta_2)^{-1}$ which is less than $a = 1.6004$. This excludes the following points : $\alpha^{-1} + \alpha + \alpha^2 + \alpha^3$, $\alpha^{-1} + \alpha + \alpha^3$, $\alpha^{-1} + \alpha$ and $\alpha^{-1} + \alpha^3$ because the absolute value of their first entries is greater than the value below it in the following array :

$\beta_2^{-1} + \beta_2 + \beta_2^2 + \beta_2^3$	$\beta_2^{-1} + \beta_2 + \beta_2^3$	$\beta_2^{-1} + \beta_2$	$\beta_2^{-1} + \beta_2^3$
1.9	2.5	2.0	1.7

In the same way we should have $|(A_k)_2| \leq C|\beta_3|^4(1 - |\beta_3|^6)^{-1}$ which is less than $b = 1,8120$. This excludes the following points : $\alpha^{-1} + 1 + \alpha^3$, $\alpha^{-1} + \alpha^2 + \alpha^3$ and $\alpha^{-1} + 1 + \alpha^2 + \alpha^3$, because the absolute value of their second entries is greater than the value below it in the following array :

$\beta_3^{-1} + 1 + \beta_3^3$	$\beta_3^{-1} + \beta_3^2 + \beta_3^3$	$\beta_3^{-1} + 1 + \beta_3^2 + \beta_3^3$
2.0	1.9	1.9

In order to exclude the other cases, except $\frac{1}{\alpha} + 1 + \alpha^2$, we used (2) to compute A_{k+i} , $i \geq 1$. Let us explain the strategy. Suppose neither $(A_k)_1$ nor $(A_k)_2$ is greater than respectively a and b . Then, we compute A_{k+1} using (2). We have three possible values : $\frac{A_k}{\alpha}$, $\frac{A_k}{\alpha} + \alpha^3$ and $\frac{A_k}{\alpha} - \alpha^3$. To check that A_k does not belong to S , it suffices to show that for all these values, either the first entry or the second is respectively greater than a or b . If it is not the case, for each value that does not satisfy this (both entries are less than, respectively, a and b) we apply again this strategy. Applying this just once we show that $\frac{1}{\alpha} + 1 + \alpha + \alpha^3$ does not belong to S . The values of the relevant entries are in the following array and should be read in the following way : The value (1.9 for example) below a relevant entry of A_{k+1} (resp. $\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1 + \beta_2^2$) is greater than the absolute value of the relevant entry : $|\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1 + \beta_2^2| > 1.9$.

A_k	$\frac{1}{\alpha} + 1 + \alpha + \alpha^3$		
A_{k+1}	$\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1 + \beta_2^2$	$\frac{1}{\beta_3^2} + \frac{1}{\beta_3} + 1 + \beta_3^2 + \beta_3^3$	$\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1 + \beta_2^2 - \beta_2^3$
	1.9	1.9	2.4

For the following case, $\frac{1}{\alpha} + 1$, we need to apply the strategy twice because for $A_{k+1} = \frac{1}{\beta_2^2} + \frac{1}{\beta_3} - \beta_3^3$ both entries are respectively less than a and b .

A_k	$\frac{1}{\alpha} + 1$		
A_{k+1}	$\frac{1}{\beta_2^2} + \frac{1}{\beta_3}$	$\frac{1}{\beta_2^2} + \frac{1}{\beta_3} + \beta_3^3$	
	1.83	2.0	
A_{k+2}	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} - \beta_3^2$	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} - \beta_3^2 + \beta_3^3$	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} - \beta_3^2 - \beta_3^3$
	2.1	1.63	2.7

For the case $A_k = \frac{1}{\alpha} + 1 + \alpha$ we need two steps because at the first one both $\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1$ and $\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1 + \beta_3^3$ have entries less than, respectively, a and b .

A_k	$\frac{1}{\alpha} + 1 + \alpha$		
A_{k+1}			$\frac{1}{\beta_2^2} + \frac{1}{\beta_2} + 1 - \beta_2^3$
			1.84
A_{k+2}	$\frac{1}{\beta_2^2} + \frac{1}{\beta_2^2} + \frac{1}{\beta_2}$	$\frac{1}{\beta_2^2} + \frac{1}{\beta_2^2} + \frac{1}{\beta_2} + \beta_2^3$	$\frac{1}{\beta_2^2} + \frac{1}{\beta_2^2} + \frac{1}{\beta_2} - \beta_3^3$
	1.77	2.23	1.818
	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + \frac{1}{\beta_3} + \beta_3^2$	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + \frac{1}{\beta_3} + \beta_2^2 + \beta_2^3$	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + \frac{1}{\beta_3} + \beta_2^2 - \beta_3^3$
	1.86	1.63	2.24

For the three following cases, $\frac{1}{\alpha} + \alpha^2$, $\frac{1}{\alpha}$ and $\frac{1}{\alpha} + \alpha + \alpha^2$, we need three steps.

A_k	$\frac{1}{\alpha} + \alpha^2$		
A_{k+1}	$\frac{1}{\beta_2^2} + \beta_3$		$\frac{1}{\beta_2^2} + \beta_3 - \beta_3^3$
	1.89		2.34
A_{k+2}	$\frac{1}{\beta_3^3} + 1 + \beta_3^2$		$\frac{1}{\beta_3^3} + 1 + \beta_3^2 - \beta_3^3$
	1.83		2.26
A_{k+3}	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3} + \beta_3 + \beta_3^2$	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3} + \beta_3 + \beta_3^2 + \beta_3^3$	$\frac{1}{\beta_2^4} + \frac{1}{\beta_2} + \beta_2 + \beta_2^2 - \beta_2^3$
	1.818	2.32	1.77

A_k	$\frac{1}{\alpha}$		
A_{k+1}	$\frac{1}{\beta_2^2}$		$\frac{1}{\beta_2^2} - \beta_2^3$
	1.66		2.13
A_{k+2}		$\frac{1}{\beta_2^2} + \beta_2^2 + \beta_2^3$	$\frac{1}{\beta_3^3} + \beta_3^2 - \beta_3^3$
		2.01	2.17
A_{k+3}	$\frac{1}{\beta_3^4} + \beta_3$	$\frac{1}{\beta_3^4} + \beta_3 + \beta_3^3$	$\frac{1}{\beta_3^4} + \beta_3 - \beta_3^3$
	2.00	2.21	1.92

A_k	$\frac{1}{\alpha} + \alpha + \alpha^2$		
A_{k+1}	$\frac{\alpha}{\beta_2^2} + 1 + \beta_2$		$\frac{1}{\beta_2^2} + 1 + \beta_2 - \beta_2^3$
	1.89		2.35
A_{k+2}	$\frac{1}{\beta_2^3} + \frac{1}{\beta_2} + 1 + \beta_2^2$	$\frac{1}{\beta_2^3} + \frac{1}{\beta_2} + 1 + \beta_2^2 + \beta_2^3$	
	1.84	2.30	
A_{k+3}	$\frac{1}{\beta_2^4} + \frac{1}{\beta_2^2} + \frac{1}{\beta_2} + \beta_2$ $-\beta_2^2$	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^2} + \frac{1}{\beta_3} + \beta_3$ $-\beta_3^2 + \beta_3^3$	$\frac{1}{\beta_2^4} + \frac{1}{\beta_2^2} + \frac{1}{\beta_2} + \beta_2$ $-\beta_2^2 - \beta_2^3$
	1.77	1.818	2.23

Hence the only possible A_k (with $c_{-1} = 1$) is $\frac{1}{\alpha} + 1 + \alpha^2$.

Suppose now $s \leq -2$ and $c_s = 1$. It is useful for the sequel to remark that $U = (u_i)_{i \geq s} = (c_s, c_{s+1}, \dots, c_2, c_3, \varepsilon_{k+1}, \varepsilon_{k+2}, \varepsilon_{k+3}, \dots)$ belongs to \mathcal{D}^∞ . Indeed, if $c_3 = 0$, it is clear. If $c_3 = 1$ and $c_2 = 0$ then by (6), $\varepsilon_k = 1$. Hence $\varepsilon_{k+1}\varepsilon_{k+2}\varepsilon_{k+3} \neq 111$ and U belongs to \mathcal{D}^∞ . The other cases can be treated in the same way.

Using (3) and (7) we obtain

$$u = \sum_{i=s}^3 c_i \alpha^i + \sum_{i=4}^{\infty} \varepsilon_{i+k-3} \alpha^i = \sum_{i=4}^{\infty} \varepsilon'_{i+k-3} \alpha^i = v.$$

We set $V = (\varepsilon'_{i+k-3})_{i \geq s}$ where $\varepsilon'_{i+k-3} = 0$ when $s \leq i \leq 3$. Then (U, V) belongs to Δ , $A_{s+4}(U, V) = c_s \alpha^{-1} + c_{s+1} + c_{s+2} \alpha + c_{s+3} \alpha^2 + c_{s+4} \alpha^3$ and $A_3(U, V) = \sum_{i=s}^3 c_i \alpha^i$. Doing what we did for x and y to u and v we obtain that $A_{s+4}(U, V) = \alpha^{-1} + 1 + \alpha^2$. Let us show that for all $n \geq s + 5$, $A_n(U, V)$ belongs to

$$\mathcal{C} = \{\pm(\alpha^{-2} + \alpha^{-1} + \alpha), \pm(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3)\}.$$

This will imply that A_k belongs to S for all $k \geq l$. We have that $A_{s+5}(U, V)$ belongs to

$$\left\{ \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha, \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha + \alpha^3, \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha - \alpha^3 \right\}.$$

The third one can be excluded because $\frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3 - \beta_3^3 \geq 1.85$. We proceed as before to exclude the second element :

A_k	$\frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha + \alpha^3$		
A_{k+1}	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 + \beta_3^2$		$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 + \beta_3^2 - \beta_3^3$
	2.00		2.55
A_{k+2}	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3$	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3 + \beta_3^3$	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3 - \beta_3^3$
	2.45	2.54	2.47

Consequently, $A_{s+5}(U, V) = \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha$. We deduce $A_{s+6}(U, V) = \frac{1}{\alpha^3} + \frac{1}{\alpha^2} + 1 + \alpha^3$ because $\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 > 2.03$ and $\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 - \beta_3^3 > 2.56$. Once again, $A_{s+7}(U, V) = \frac{1}{\alpha^4} + \frac{1}{\alpha^3} + \frac{1}{\alpha} + \alpha^2 - \alpha^3$ because $\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3^2 > 1.85$ and $\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3^2 + \beta_3^3 > 2.16$. But an easy computation leads to $\frac{1}{\alpha^4} + \frac{1}{\alpha^3} + \frac{1}{\alpha} + \alpha^2 - \alpha^3 = -(\frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha) = -A_{s+5}(U, V)$. Then continuing in the same way we can check $A_n(U, V) = -A_{n+2}(U, V)$ and $A_n \in \mathcal{C}$ for $n \geq s + 5$. As $3 \geq s + 5$, we obtain that $A_k(\varepsilon, \varepsilon') = A_3(U, V)$ belongs to \mathcal{C} . Thus $S(\varepsilon, \varepsilon')$ is included in S .

To complete the proof we should show that each element of S belongs to $\Gamma = \cup_{(\varepsilon, \varepsilon') \in \Delta} S(\varepsilon, \varepsilon')$.

Remark that if A_k belongs to Γ then $-A_k$ also belongs to Γ . Consequently it is sufficient to consider the cases where $A_k = \sum_{i=0}^3 c_i \alpha^i$ with $(c_i)_{0 \leq i \leq 3} \in \mathcal{D}$ or $A_k = \alpha^{-1} + 1 + \alpha^2$, $\alpha^{-2} + \alpha^{-1} + \alpha$ or $\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3$.

Notice that we have

$$\begin{aligned} -\alpha^3 &= \sum_{i=1}^{+\infty} (\alpha^{4i} + \alpha^{4i+1} + \alpha^{4i+2}) = 1 + \alpha + \alpha^2 + \sum_{i=1}^{+\infty} (\alpha^{4i+1} + \alpha^{4i+2} + \alpha^{4i+3}) \\ &= \alpha + \alpha^2 + \alpha^4 + \sum_{i=1}^{+\infty} (\alpha^{4i+2} + \alpha^{4i+3} + \alpha^{4i+4}) \\ &= \alpha^2 + \alpha^4 + \alpha^5 + \sum_{i=1}^{+\infty} (\alpha^{4i+3} + \alpha^{4i+4} + \alpha^{4i+5}). \end{aligned}$$

Hence, $1 + \alpha + \alpha^2$, $\alpha + \alpha^2$ and α^2 belong to Γ . Multiplying by α we deduce $\alpha + \alpha^2 + \alpha^3$, $\alpha^2 + \alpha^3$ and α^3 belong to Γ . Now subtracting $-\alpha^2$ we obtain $1 + \alpha$ and α belong to Γ . We have that 1 belongs to Γ because $\sum_{i=1}^{+\infty} \alpha^{4i} = 1 + \sum_{i=1}^{+\infty} \alpha^{4i+1}$. Now, $1 + \alpha^2$ belongs to Γ because

$$\sum_{i=2}^{\infty} \alpha^{2i} = 1 + \alpha^2 + \sum_{i=2}^{\infty} \alpha^{2i+1}.$$

Multiplying by α we deduce $\alpha + \alpha^3$ belongs to Γ . Because

$$(8) \quad \alpha^{-3} + \alpha^{-2} + 1 + \alpha^3 + \sum_{i=1}^{\infty} (\alpha^{4i+2} + \alpha^{4i+3}) = \sum_{i=1}^{\infty} (\alpha^{4i} + \alpha^{4i+1}),$$

we obtain that $\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3$ belongs to Γ . Multiplying (8) by, respectively, α and α^2 we obtain, respectively, that $\alpha^{-2} + \alpha^{-1} + \alpha$ and $\alpha^{-1} + 1 + \alpha^2$ belong to Γ . From

$$\alpha^4 + \sum_{i=1}^{\infty} \alpha^{4i+3} = 1 + \alpha^3 + \sum_{i=1}^{\infty} \alpha^{4i+1} = 1 + \alpha + \alpha^3 + \sum_{i=1}^{\infty} \alpha^{4i+2}$$

it is clear $1 + \alpha^3$ and $1 + \alpha + \alpha^3$ belong to Γ .

The equality

$$\alpha^4 + \sum_{i=1}^{\infty} \alpha^{4i+2} = 1 + \alpha^2 + \alpha^3 + \sum_{i=1}^{\infty} \alpha^{4i+1}$$

implies that $1 + \alpha^2 + \alpha^3$ belongs to Γ and achieves the proof. \square

Corollary 7. *Let \mathcal{A} be the automaton defined in Subsection 3.1. Then, for all $(\varepsilon_i)_{i \geq l}$ and $(\varepsilon'_i)_{i \geq l}$ belonging to \mathcal{D}^∞ the following assertions are equivalent :*

- $\sum_{i \geq l} \varepsilon_i \alpha^i = \sum_{i \geq l} \varepsilon'_i \alpha^i$;
- $(\varepsilon_i, \varepsilon'_i)_{i \geq l}$ is an infinite path in \mathcal{A} beginning in the initial state.

3.3. Neighbors of \mathcal{E} . Here we prove that the set \mathcal{E} has 18 neighbors where 6 of them have an intersection with \mathcal{E} reduced to a singleton, and that the boundary can be generated by just 2 subregions.

Lemma 8. *Let $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq l}$ be two elements of \mathcal{D}^∞ such that $\sum_{i=4}^\infty \varepsilon_i \alpha^i = \sum_{i=l}^\infty \varepsilon'_i \alpha^i$, where $l < 4$ and $\varepsilon'_l = 1$, then $\varepsilon'_l \alpha^l + \varepsilon'_{l+1} \alpha^{l+1} \dots + \varepsilon'_3 \alpha^3$ belongs to S . In particular $l \geq -3$ and*

$$\begin{aligned} \varepsilon'_l \alpha^l + \dots + \varepsilon'_3 \alpha^3 &= \alpha^{-3} + \alpha^{-2} + 1 + \alpha^3 && \text{if } l = -3, \\ \varepsilon'_l \alpha^l + \dots + \varepsilon'_3 \alpha^3 &= \alpha^{-2} + \alpha^{-1} + \alpha && \text{if } l = -2, \\ \varepsilon'_l \alpha^l + \dots + \varepsilon'_3 \alpha^3 &= \alpha^{-1} + 1 + \alpha^2 && \text{if } l = -1. \end{aligned}$$

Proof. Let $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq l}$ be two elements of \mathcal{D}^∞ such that $\sum_{i=4}^\infty \varepsilon_i \alpha^i = \sum_{i=l}^\infty \varepsilon'_i \alpha^i$ where $l < 4$ and $\varepsilon'_l = 1$. From Proposition 6, for all $l \leq i \leq 3$, $\varepsilon'_l \alpha^l + \varepsilon'_{l+1} \alpha^{l+1} \dots + \varepsilon'_{i-3} \alpha^3$ belongs to S . In particular, for $i = l$, we obtain the result. \square

Lemma 9. *Let $u \in S$. Then, there exists $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ belonging to \mathcal{D}^∞ such that $\sum_{i=4}^\infty \varepsilon_i \alpha^i = u + \sum_{i=4}^\infty \varepsilon'_i \alpha^i$.*

Proof. This comes from Proposition 6 and the identity (3). \square

In our context, Lemma 2 in [M05] can be formulated in the following way.

Lemma 10. *Let $x \in \mathbb{R} \times \mathbb{C}$, then x belongs to the boundary of \mathcal{E} if and only if there exists $l \leq 3$ such that $x = \sum_{i=4}^{+\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{+\infty} \varepsilon'_i \alpha^i$, where $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq l}$ belong to \mathcal{D}^∞ , and, $\varepsilon'_l \neq 0$.*

Proposition 11. *The boundary of \mathcal{E} is the union of the 18 non empty regions $\mathcal{E}(u)$, $u \in \{a, -a; a \in A\}$, whose pairwise intersections have measure zero, where*

$$\mathcal{E}(u) = \mathcal{E} \cap (\mathcal{E} + u) \text{ and}$$

$$A = \{1, 1 + \alpha, 1 + \alpha^2, 1 + \alpha + \alpha^2, \alpha^{-3} + \alpha^{-2} + 1 + \alpha^3, \alpha, \alpha + \alpha^2, \alpha^2, \alpha^{-2} + \alpha^{-1} + \alpha\}.$$

Proof. Let u be an element of A , then u is a state of the automaton \mathcal{A} . From Lemma 9, there exist $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ belonging to \mathcal{D}^∞ such that $\sum_{i=4}^\infty \varepsilon_i \alpha^i = u + \sum_{i=4}^\infty \varepsilon'_i \alpha^i$. Thus, from Proposition 4, $\mathcal{E} \cap (\mathcal{E} + u)$ is not empty and with measure zero. It will be useful to check that

$$\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3 = 1 + 2\alpha + \alpha^2 \text{ and } \alpha^{-2} + \alpha^{-1} + \alpha = -1 + \alpha^2.$$

Consequently, Proposition 4 implies $\bigcup_{u \in A} \mathcal{E}(u) \cup \mathcal{E}(-u)$ is contained in the boundary of \mathcal{E} .

Now, let z be an element of the boundary of \mathcal{E} , then by Lemma 10 there exist two elements of \mathcal{D}^∞ , $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq l}$, $l \in \mathbb{Z}$, $l < 4$ such that $z = \sum_{i=4}^{+\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{+\infty} \varepsilon'_i \alpha^i$. We can suppose $\varepsilon_l = 1$. Let us consider the following four cases.

Suppose $l = -3$. From Lemma 8, we deduce that $z \in \mathcal{E}(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3)$.

Suppose $l = -2$. From Lemma 8, we deduce that $z \in \mathcal{E}(\alpha^{-2} + \alpha^{-1} + \alpha)$.

Suppose $l = -1$. From Lemma 8, we deduce that $z = \sum_{i=4}^{+\infty} \varepsilon_i \alpha^i = \alpha^{-1} + 1 + \alpha^2 + \sum_{i=4}^{+\infty} \varepsilon'_i \alpha^i$. Corollary 7 implies that $t = (0, 1)(0, 1)(0, 0)(0, 1)(0, 0)(\varepsilon_4, \varepsilon'_4) \dots$ is an infinite path of the automaton \mathcal{A} starting at the initial state. Using the automaton, we see that $t = (0, 1)(0, 1), (0, 0)(0, 1)(0, 0)awww \dots$, where $a = (1, 1)$ or $(0, 0)$ and $w = (0, 1)(1, 0)(1, 0)(0, 1)$. Consequently, $z = \alpha^{-1} + 1 + \alpha^2 + \alpha^4 + \alpha^5 + \sum_{i=2}^{\infty} (\alpha^{4i} + \alpha^{4i+1})$ or $z = \alpha^{-1} + 1 + \alpha^2 + \alpha^5 + \sum_{i=2}^{\infty} (\alpha^{4i} + \alpha^{4i+1})$. Thus, $z = -\alpha - \alpha^2 + \alpha^6 + \sum_{i=2}^{\infty} (\alpha^{4i} + \alpha^{4i+1})$ or $z = -1 - 2\alpha - \alpha^2 + \sum_{i=1}^{\infty} (\alpha^{4i} + \alpha^{4i+1})$, and, $z \in \mathcal{E}(-\alpha - \alpha^2) \cup \mathcal{E}(-1 - 2\alpha - \alpha^2)$.

Suppose $l \geq 0$. Then $z = \sum_{i=4}^{+\infty} \varepsilon_i \alpha^i = \varepsilon'_0 + \varepsilon'_1 \alpha + \varepsilon'_2 \alpha^2 + \varepsilon'_3 \alpha^3 + \sum_{i=4}^{+\infty} \varepsilon'_i \alpha^i$.

If $\varepsilon'_3 = 0$, then $z \in \mathcal{E}(u)$ where $u = \varepsilon'_0 + \varepsilon'_1 \alpha + \varepsilon'_2 \alpha^2$.

If $\varepsilon'_3 = 1$ and $\varepsilon'_4 = 0$, then $z = (\varepsilon'_0 - 1) + (\varepsilon'_1 - 1)\alpha + (\varepsilon'_2 - 1)\alpha^2 + \alpha^4 + \sum_{i=5}^{+\infty} \varepsilon'_i \alpha^i \in \mathcal{E}(u)$ where $u = (\varepsilon'_0 - 1) + (\varepsilon'_1 - 1)\alpha + (\varepsilon'_2 - 1)\alpha^2$.

Now suppose $\varepsilon'_3 = \varepsilon'_4 = 1$ and $\varepsilon'_5 = 0$. Then $\varepsilon'_1 = 0$ or $\varepsilon'_2 = 0$, and, $z = \varepsilon'_0 + (\varepsilon'_1 - 1)\alpha + (\varepsilon'_2 - 1)\alpha^2 + \alpha^5 + \sum_{i=6}^{+\infty} \varepsilon'_i \alpha^i$. Hence :

- If $\varepsilon'_0 = 0$, then $z \in \mathcal{E}(u)$ where $u = (\varepsilon'_1 - 1)\alpha + (\varepsilon'_2 - 1)\alpha^2$.
- If $\varepsilon'_0 = 1$, then $t = (0, 1)(0, \varepsilon'_1)(0, \varepsilon'_2)(0, 1)(\varepsilon_4, 1)(\varepsilon_5, 0) \dots$ is an infinite path in the automaton beginning in the initial state. This implies that $t = (0, 1)(0, 1)(0, 0)(0, 1)(1, 1)(0, 0)ww \dots$ where $w = (0, 1)(1, 0)(1, 0)(0, 1)$. Hence $z = 1 + \alpha + \alpha^3 + \alpha^4 + \alpha^6 + \sum_{i=2}^{\infty} (\alpha^{4i+1} + \alpha^{4i+2})$. Thus $z + \alpha^{-2} + \alpha^{-1} + \alpha = \alpha^5 + \alpha^6 + \sum_{i=2}^{\infty} (\alpha^{4i+1} + \alpha^{4i+2})$ and z belongs to $\mathcal{E}(-\alpha^{-2} - \alpha^{-1} - \alpha)$.

If $\varepsilon'_3 = \varepsilon'_4 = \varepsilon'_5 = 1$, then $\varepsilon'_2 = \varepsilon'_6 = 0$. Hence $z = \varepsilon'_0 + \varepsilon'_1 \alpha - \alpha^2 + \alpha^6 + \sum_{i=7}^{+\infty} \varepsilon'_i \alpha^i$.

- If $\varepsilon'_0 = \varepsilon'_1 = 0$, then $z \in \mathcal{E}(-\alpha^2)$.
- When $\varepsilon'_0 + \varepsilon'_1 = 1$, there is no infinite path in the automaton starting in the initial state and beginning with $(0, \varepsilon'_0)(0, \varepsilon'_1)(0, 0)(0, 1)(\varepsilon_4, 1)(\varepsilon_5, 1)$.
- Hence it remains to consider the case : $\varepsilon'_0 + \varepsilon'_1 = 2$. But it is easy to check that this implies $\varepsilon'_6 = 1$ which is not possible.

This ends the proof. \square

Using the automaton given in the Annexe, we deduce the following result.

Proposition 12. *Let $X = \mathcal{E}(1 + \alpha + \alpha^2)$ and $Y = \mathcal{E}(1 + \alpha)$. Then,*

$$\begin{aligned} a) \mathcal{E}(1) &= 1 + \alpha X, & b) \mathcal{E}(\alpha^2) &= -\frac{1}{\alpha} - 1 - \alpha + \frac{X}{\alpha}, \\ c) \mathcal{E}(1 + \alpha^2) &= \left\{ \frac{\alpha^4}{1 - \alpha^2} \right\}, & d) \mathcal{E}(\alpha^{-2} + \alpha^{-1} + \alpha) &= \left\{ \frac{\alpha^5 + \alpha^6}{1 - \alpha^4} \right\}, \\ e) \mathcal{E}(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3) &= \left\{ \frac{\alpha^4 + \alpha^5}{1 - \alpha^4} \right\} & f) \mathcal{E}(\alpha) &= f_0(X) \cup f_1(X) \cup f_1(Y) \end{aligned}$$

$$g) \mathcal{E}(\alpha + \alpha^2) = g_0(X) \cup g_1(X) \cup g_1(Y) \cup g_2(Y) \cup g_3(Y), \text{ where}$$

$$\begin{aligned} f_0(z) &= \alpha + \alpha^2 z, & f_1(z) &= \alpha + \alpha^4 + \alpha^2 z, & g_0(z) &= \alpha^5 + \alpha^4 z, \\ g_1(z) &= \alpha^5 + \alpha^6 + \alpha^4 z, & g_2(z) &= \alpha z, & g_3(z) &= \alpha^4 + \alpha z, \end{aligned}$$

$$h) X = \bigcup_{i=0}^4 h_i(X) \cup h_1(Y) \cup h_3(Y) \text{ and } i) Y = \bigcup_{i=5}^{11} h_i(Y) \cup \bigcup_{i=12}^{17} h_i(X), \text{ where}$$

$$\begin{aligned}
h_0(z) &= \alpha^4 + \alpha^4 z, & h_1(z) &= \alpha^4 + \alpha^6 + \alpha^4 z \\
h_2(z) &= \alpha^4 + \alpha^5 + \alpha^4 z, & h_3(z) &= \alpha^4 + \alpha^5 + \alpha^6 + \alpha^4 z, \\
h_4(z) &= 1 + \alpha + \alpha^2 + \alpha^7 + \alpha^5 z, & h_5(z) &= h_2(z) \\
h_6(z) &= \alpha^4 + \alpha^7 + \alpha^4 z, & h_7(z) &= \alpha^4 + \alpha^8 + \alpha^9 + \alpha^7 z, \\
h_8(z) &= h_0(z), & h_9(z) &= \alpha^4 + \alpha^5 + \alpha^7 + \alpha^4 z, \\
h_{10}(z) &= \alpha^4 + \alpha^5 + \alpha^8 + \alpha^9 + \alpha^7 z, & h_{11}(z) &= 1 + \alpha + \alpha^6 + \alpha^7 + \alpha^5 z \\
h_{12}(z) &= \alpha^4 + \alpha^8 + \alpha^7 z, & h_{13}(z) &= h_7(z), \\
h_{14}(z) &= \alpha^4 + \alpha^5 + \alpha^8 + \alpha^7 z, & h_{15}(z) &= h_{10}(z) \\
h_{16}(z) &= 1 + \alpha + \alpha^6 + \alpha^5 z, & h_{17}(z) &= h_{11}(z),
\end{aligned}$$

Proof. a) The set $1 + \alpha X$ is clearly included in $1 + \alpha \mathcal{E}$. Moreover it is easy to check that $1 + \alpha X$ is a subset of $\alpha^4 + \alpha \mathcal{E}$ which is included in \mathcal{E} . Hence $1 + \alpha X \subset \mathcal{E}(1)$. On the other hand, let $z \in \mathcal{E}(1)$. Then, there exist $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ in \mathcal{D}^∞ such that $z = 1 + \sum_{i \geq 4} \varepsilon_i \alpha^i = \sum_{i \geq 4} \varepsilon'_i \alpha^i$. From Corollary 7, $(1, 0)(0, 0)(0, 0)(\varepsilon_4, \varepsilon'_4)(\varepsilon_5, \varepsilon'_5)$ is a finite path in the automaton \mathcal{A} starting at the initial state. Following this path in the automaton we deduce $(\varepsilon_4, \varepsilon'_4) = (0, 1)$ and $(\varepsilon_5, \varepsilon'_5) = (1, 0)$. It gives $z = 1 + \alpha^5 + \alpha^2 w = \alpha^4 + \alpha^2 w'$ where $w, w' \in \mathcal{E}$. Consequently $\mathcal{E}(1) \subset (1 + \alpha \mathcal{E}) \cap (\alpha^4 + \alpha \mathcal{E}) = 1 + \alpha(\mathcal{E} \cap (1 + \alpha + \alpha^2 + \mathcal{E})) = 1 + \alpha X$.

b) We have $1 + \alpha + \alpha^2 + \alpha \mathcal{E}(\alpha^2) = (\alpha^4 + \alpha \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha \mathcal{E}) \subset \mathcal{E} \cap (1 + \alpha + \alpha^2 + \mathcal{E}) = X$. Hence $\mathcal{E}(\alpha^2) \subset -\frac{1}{\alpha} - 1 - \alpha + \frac{X}{\alpha}$. To prove the other inclusion, let $z \in X$. Then by the automaton we deduce that $z = 1 + \alpha + \alpha^2 + \alpha w = \alpha^4 + \alpha w'$, $w, w' \in \mathcal{E}$. Hence $-\frac{1}{\alpha} - 1 - \alpha + \frac{z}{\alpha} = w = \alpha^2 + w'$ and $-\frac{1}{\alpha} - 1 - \alpha + \frac{X}{\alpha} \subset \mathcal{E}(\alpha^2)$.

c) Let $z \in \mathcal{E}(1 + \alpha^2)$: $z = \sum_{i \geq 4} \varepsilon_i \alpha^i = 1 + \alpha^2 + \sum_{i \geq 4} \varepsilon'_i \alpha^i$, $(\varepsilon_i)_{i \geq 4}$, $(\varepsilon'_i)_{i \geq 4} \in \mathcal{D}^\infty$. Corollary 7 and the automaton show that $(0, 1)(0, 0)(0, 1)(0, 0)(\varepsilon_4, \varepsilon'_4) \dots$ is a infinite path starting in the initial state and $(\varepsilon_i, \varepsilon'_i)_{i \geq 4}$ is equal to $uuu \dots$ where $u = (1, 0)(0, 1)$. Then, $z = \alpha^4 + \alpha^6 + \alpha^8 + \dots = 1 + \alpha^2 + \alpha^5 + \alpha^7 + \alpha^9 + \dots$ and $\mathcal{E}(1 + \alpha^2) = \alpha^4(1 - \alpha^2)^{-1}$.

d) Let $z \in \mathcal{E}(\alpha^{-2} + \alpha^{-1} + \alpha)$. From Corollary 7 and using the automaton we deduce that $z = \alpha^{-2} + \alpha^{-1} + \alpha + \alpha^4 + \sum_{i=2}^\infty (\alpha^{4i-1} + \alpha^{4i}) = \sum_{i=1}^\infty (\alpha^{4i+1} + \alpha^{4i+2})$. Hence $\mathcal{E}(\alpha^{-2} + \alpha^{-1} + \alpha) = (\alpha^5 + \alpha^6)(1 - \alpha^4)^{-1}$.

e) Corollary 7 and the automaton give the result.

f) Let $z \in \mathcal{E}(\alpha)$. Then, there exist $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ in \mathcal{D}^∞ such that $z = \sum_{i \geq 4} \varepsilon_i \alpha^i = \alpha + \sum_{i \geq 4} \varepsilon'_i \alpha^i$. From Corollary 7, $(0, 0)(0, 1)(0, 0)(0, 0)(\varepsilon_4, \varepsilon'_4) \dots$ is a path in the automaton starting in the initial state. Hence, $(\varepsilon_4, \varepsilon'_4)(\varepsilon_5, \varepsilon'_5)(\varepsilon_6, \varepsilon'_6)$ belongs to $\{(0, 0), (1, 1), (0, 1)\}(1, 0)(0, 1)$. Consequently, z belongs to the union of $(\alpha^5 + \alpha^2 \mathcal{E}) \cap (\alpha + \alpha^2 \mathcal{E})$, $(\alpha^4 + \alpha^5 + \alpha^2 \mathcal{E}) \cap (\alpha + \alpha^4 + \alpha^2 \mathcal{E})$ and $(\alpha^5 + \alpha^2 \mathcal{E}) \cap (\alpha + \alpha^4 + \alpha^2 \mathcal{E})$ which is equal to $f_0(X) \cup f_1(X) \cup f_1(Y)$. Hence $\mathcal{E}(\alpha) = f_0(X) \cup f_1(X) \cup f_1(Y)$.

g) Let $z \in \mathcal{E}(\alpha + \alpha^2)$. Then, there exist $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ in \mathcal{D}^∞ such that $z = \sum_{i \geq 4} \varepsilon_i \alpha^i = \alpha + \alpha^2 + \sum_{i \geq 4} \varepsilon'_i \alpha^i$. From Corollary 7, $(0, 0)(0, 1)(0, 1)(0, 0)(\varepsilon_4, \varepsilon'_4) \dots$ is a path in the automaton starting in the initial state. Hence, we either have

- (1) $((\varepsilon_i, \varepsilon'_i))_{4 \leq i \leq 7} \in (0, 1)(1, 0)\{(0, 0), (1, 1), (1, 0)\}(0, 1)$,
- (2) $(\varepsilon_4, \varepsilon'_4) \in \{(0, 0), (1, 1)\}$, or
- (3) $((\varepsilon_i, \varepsilon'_i))_{i \geq 4} \in (0, 1)\{(0, 0)(0, 0), (0, 0)(1, 1), (1, 1)(0, 0)\}ww \dots$,

where $w = (0, 1)(1, 0)(1, 0)(0, 1)$. This means that z belongs to

$$\begin{aligned}
& ((\alpha^5 + \alpha^4 \mathcal{E}) \cap (\alpha + \alpha^2 + \alpha^4 + \alpha^7 + \alpha^4 \mathcal{E})) \\
\cup & ((\alpha^5 + \alpha^6 + \alpha^4 \mathcal{E}) \cap (\alpha + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^7 + \alpha^4 \mathcal{E})) \\
\cup & ((\alpha^5 + \alpha^6 + \alpha^4 \mathcal{E}) \cap (\alpha + \alpha^2 + \alpha^4 + \alpha^7 + \alpha^4 \mathcal{E})) \\
\cup & ((\alpha \mathcal{E} \cap (\alpha + \alpha^2 + \alpha \mathcal{E})) \cup ((\alpha^4 + \alpha \mathcal{E}) \cap (\alpha + \alpha^2 + \alpha^4 + \alpha \mathcal{E}))) \\
\cup & \{z_1, z_2, z_3\} \\
= & g_0(X) \cup g_1(X) \cup g_1(Y) \cup g_2(Y) \cup g_3(Y) \cup \{z_1, z_2, z_3\}
\end{aligned}$$

where $z_1 = \sum_{i=2}^{+\infty} (\alpha^{4i} + \alpha^{4i+1}) = \alpha + \alpha^2 + \alpha^4 + \alpha^7 + \sum_{i=2}^{+\infty} (\alpha^{4i+2} + \alpha^{4i+3})$, $z_2 = \alpha^6 + z_1$ and $z_3 = \alpha^5 + z_1$. We can also check that $(1, 0)(1, 0)(0, 0)(0, 0)uuu \dots$, where $u = (0, 1)(1, 1)(1, 0)(1, 0)$, is an infinite path of the automaton starting in the initial state. Consequently, $z_1 = \alpha^4 + \alpha^5 + \sum_{i=2}^{+\infty} (\alpha^{4i+1} + \alpha^{4i+2})$ and

$$z_1 \in ((\alpha^4 + \alpha \mathcal{E}) \cap (\alpha + \alpha^2 + \alpha^4 + \alpha \mathcal{E})) = g_3(Y).$$

Moreover, it shows that z_2 belongs to $g_3(Y)$. In the same way, $z_1 = \alpha^5 + \alpha^6 + \alpha^8 + \sum_{i=2}^{+\infty} (\alpha^{4i+3} + \alpha^{4i+4})$. Thus, $z_3 = 2\alpha^5 + \alpha^6 + \alpha^8 + \sum_{i=2}^{+\infty} (\alpha^{4i+3} + \alpha^{4i+4}) = \alpha^5 + \sum_{i=2}^{+\infty} (\alpha^{4i} + \alpha^{4i+1})$. But $2\alpha^5 + \alpha^6 = \alpha + \alpha^2 + \alpha^7$, consequently z_3 belongs to $(\alpha \mathcal{E}) \cap (\alpha + \alpha^2 + \alpha \mathcal{E}) = g_2(Y)$.

h) Let $z \in X = \mathcal{E}(1 + \alpha + \alpha^2)$. Then, there exist $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ in \mathcal{D}^∞ such that $z = \sum_{i \geq 4} \varepsilon_i \alpha^i = 1 + \alpha + \alpha^2 + \sum_{i \geq 4} \varepsilon'_i \alpha^i$. From Corollary 7, we necessarily have $(\varepsilon_4, \varepsilon'_4) = (1, 0)$ and one of the following situations :

- (1) $((\varepsilon_i, \varepsilon'_i)_{i \geq 5} \in (1, 0)\{(0, 0), (1, 1)\}ww \dots$ where $w = (0, 1)(1, 0)$;
- (2) $((\varepsilon_i, \varepsilon'_i)_{i \geq 5} \in (0, 1)\{(0, 0), (1, 1)\}ww \dots$ where $w = (0, 1)(1, 0)(1, 0)(0, 1)$;
- (3) $(\varepsilon_i, \varepsilon'_i)_{5 \leq i \leq 8} \in \{(0, 0), (1, 1)\}^2(0, 1)(1, 0)$;
- (4) $(\varepsilon_i, \varepsilon'_i)_{5 \leq i \leq 9} = (1, 0)(1, 0)(0, 1)(1, 0)(0, 1)$;
- (5) $(\varepsilon_i, \varepsilon'_i)_{5 \leq i \leq 8} \in \{(0, 0), (1, 1)\}(1, 0)(0, 1)(1, 0)$.

This means z belongs to $\bigcup_{i=0}^4 h_i(X) \cup h_1(Y) \cup h_3(Y) \cup \{x_1, x_2, x_3, x_4\}$ where

$$\begin{aligned}
x_1 &= \alpha^4 + \alpha^5 + \sum_{i=4}^{+\infty} \alpha^{2i} = 1 + \alpha + \alpha^2 + \sum_{i=3}^{+\infty} \alpha^{2i+1}, \\
x_2 &= x_1 + \alpha^6, \\
x_3 &= \alpha^4 + \sum_{i=2}^{+\infty} (\alpha^{4i} + \alpha^{4i+1}) = 1 + \alpha + \alpha^2 + \alpha^5 + \alpha^7 + \sum_{i=2}^{+\infty} (\alpha^{4i+2} + \alpha^{4i+3}), \\
x_4 &= x_3 + \alpha^6, \\
h_0(X) &= (\alpha^4 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^7 + \alpha^4 \mathcal{E}), \\
h_1(X) &= (\alpha^4 + \alpha^6 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^6 + \alpha^7 + \alpha^4 \mathcal{E}), \\
h_2(X) &= (\alpha^4 + \alpha^5 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^5 + \alpha^7 + \alpha^4 \mathcal{E}), \\
h_3(X) &= (\alpha^4 + \alpha^5 + \alpha^6 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^4 \mathcal{E}), \\
h_4(X) &= (\alpha^4 + \alpha^5 + \alpha^6 + \alpha^8 + \alpha^5 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^7 + \alpha^5 \mathcal{E}), \\
h_1(Y) &= (\alpha^4 + \alpha^6 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^7 + \alpha^4 \mathcal{E}) \text{ and} \\
h_3(Y) &= (\alpha^4 + \alpha^5 + \alpha^6 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^2 + \alpha^5 + \alpha^7 + \alpha^4 \mathcal{E}).
\end{aligned}$$

We easily can check (using Corollary 7 and the automaton) that

$$x_1 = x_3 = \alpha^4 + \sum_{i=2}^{+\infty} (\alpha^{4i} + \alpha^{4i+1}),$$

and thus $x_1 \in h_0(X)$, $x_2 \in h_1(X)$, and $x_2 = x_4$, which concludes the proof of h).

i) Let $z \in X = \mathcal{E}(1 + \alpha)$. Then, there exist $(\varepsilon_i)_{i \geq 4}$ and $(\varepsilon'_i)_{i \geq 4}$ in \mathcal{D}^∞ such that $z = \sum_{i \geq 4} \varepsilon_i \alpha^i = 1 + \alpha + \sum_{i \geq 4} \varepsilon'_i \alpha^i$. From Corollary 7, we necessarily have $(\varepsilon_4, \varepsilon'_4) = (1, 0)$ and one of the following situations :

- (1) $((\varepsilon_i, \varepsilon'_i))_{i \geq 5} \in \{(0, 0), (1, 1)\}(0, 1)^2\{(0, 0), (1, 1)\}^2 ww \dots$;
- (2) $((\varepsilon_i, \varepsilon'_i))_{i \geq 5} \in (1, 0)\{(0, 0), (1, 1)\}^2(1, 0)(0, 1)ww \dots$;
- (3) $((\varepsilon_i, \varepsilon'_i))_{5 \leq i \leq 7} \in \{(0, 0), (1, 1)\}(0, 1)\{(0, 0), (1, 1)\}$;
- (4) $((\varepsilon_i, \varepsilon'_i))_{5 \leq i \leq 8} \in \{(0, 0), (1, 1)\}(0, 1)^2(1, 0)\{(0, 0), (1, 1), (1, 0)\}(0, 1)(1, 0)$;
- (5) $((\varepsilon_i, \varepsilon'_i))_{5 \leq i \leq 8} \in (1, 0)(0, 1)\{(0, 0), (1, 1), (0, 1)\}(1, 0)(0, 1)$.

where $w = (0, 1)(1, 0)(1, 0)(0, 1)$. Hence z belongs to

$$\left(\bigcup_{i=5}^{11} h_i(Y) \right) \cup \left(\bigcup_{i=12}^{17} h_i(X) \right) \cup \{y_i; 1 \leq i \leq 8\},$$

where

$$y_1 = \alpha^4 + \sum_{i=2}^{+\infty} (\alpha^{4i+3} + \alpha^{4i+4}) = 1 + \alpha + \alpha^6 + \alpha^7 + \alpha^{10} + \sum_{i=3}^{+\infty} (\alpha^{4i+1} + \alpha^{4i+2}),$$

$$y_2 = y_1 + \alpha^9, \quad y_3 = y_1 + \alpha^8, \quad y_4 = y_1 + \alpha^5, \quad y_5 = y_1 + \alpha^5 + \alpha^9.$$

$$y_6 = y_1 + \alpha^5 + \alpha^8 = 1 + \alpha + \sum_{i=2}^{+\infty} (\alpha^{4i+1} + \alpha^{4i+2}), \quad y_7 = y_6 + \alpha^7, \quad y_8 = y_6 + \alpha^6,$$

$$h_5(Y) = (\alpha^4 + \alpha^5 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^4 \mathcal{E}),$$

$$h_6(Y) = (\alpha^4 + \alpha^7 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^4 \mathcal{E}),$$

$$h_7(Y) = (\alpha^4 + \alpha^8 + \alpha^9 + \alpha^7 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7 \mathcal{E}),$$

$$h_8(Y) = (\alpha^4 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^4 \mathcal{E}),$$

$$h_9(Y) = (\alpha^4 + \alpha^5 + \alpha^7 + \alpha^4 \mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^4 \mathcal{E}),$$

$$h_{10}(Y) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^9 + \alpha^7 \mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7 \mathcal{E}),$$

$$h_{11}(Y) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^5 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^5 \mathcal{E}),$$

$$h_{12}(Y) = (\alpha^4 + \alpha^8 + \alpha^7 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7 \mathcal{E}),$$

$$h_{13}(X) = (\alpha^4 + \alpha^8 + \alpha^9 + \alpha^7 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^7 \mathcal{E}),$$

$$h_{14}(X) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^7 \mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7 \mathcal{E}),$$

$$h_{15}(X) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^9 + \alpha^7 \mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^7 \mathcal{E}),$$

$$h_{16}(X) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^5 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^5 \mathcal{E}),$$

$$h_{17}(X) = (\alpha^4 + \alpha^5 + \alpha^7 + \alpha^8 + \alpha^5 \mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^5 \mathcal{E}),$$

Let us prove that for each integer $i \in \{1, \dots, 8\}$, there exists $j \in \{5, \dots, 11\}$ or $k \in \{12, \dots, 17\}$ such that y_i belongs to $h_j(X)$ or to $h_k(Y)$.

Indeed, since $y_1 = \alpha^4 + \alpha^3 z_1$ (see case g)), then $y_1 \in (\alpha^4 + \alpha^7 + \alpha^4 \mathcal{E}) \cap (2\alpha^4 + \alpha^5 + \alpha^7 + \alpha^4 \mathcal{E}) = h_6(Y)$. We deduce that y_2 and y_3 belong also to $h_6(Y)$, and, y_4 and y_5 belong to $h_9(Y)$.

Using the automaton we can verify that $y_6 = 1 + \alpha + \alpha^5 + \alpha^6 + \sum_{i=2}^{+\infty} (\alpha^{4i+2} + \alpha^{4i+3})$. Hence, y_6 belongs to $1 + \alpha + \alpha^5 + \alpha^6 + \alpha^4 \mathcal{E}$. But it also belongs to $\alpha^4 + \alpha^5 + \alpha^4 \mathcal{E}$. Thus $y_6 \in h_5(Y)$ and $y_7 \in h_9(Y)$.

We have $y_8 = y_6 + \alpha^6 \in (1 + \alpha + \alpha^6 + \alpha^5 \mathcal{E})$. On the other hand we can check using the automaton that $y_8 = \alpha^4 + \alpha^5 + \alpha^8 + \sum_{i=5}^{+\infty} \alpha^{2i}$, hence $y_8 \in (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^5 \mathcal{E})$ and y_8 belongs to $h_{16}(X)$. \square

Remarks and comments. There are points which has at least 6 expansions in base α . For example:

$$\begin{aligned} \alpha + \sum_{i=2}^{+\infty} \alpha^{2i} &= \sum_{i=1}^{+\infty} (\alpha^{4i} + \alpha^{4i+1}) \\ &= 1 + \alpha + \alpha^2 + \sum_{i=2}^{\infty} \alpha^{2i+1} \\ &= 1 + \alpha + \sum_{i=1}^{\infty} (\alpha^{4i+1} + \alpha^{4i+2}) \\ &= \alpha + \alpha^2 + \alpha^4 + \sum_{i=1}^{\infty} (\alpha^{4i+3} + \alpha^{4i+4}) \\ &= \alpha^{-3} + \alpha^{-2} + 1 + \alpha^3 + \sum_{i=1}^{\infty} (\alpha^{4i+2} + \alpha^{4i+3}). \end{aligned}$$

We address the two following questions :

- (1) Can you parameterize the boundary of $\mathcal{E}_{1,1,1}$?
- (2) Does this boundary be homeomorphic to the sphere ?

The technics used in this work can be used to study $\mathcal{E}_{a_1, a_2, \dots, a_d}$ with the assumption that $a_1 \geq a_2 \geq \dots \geq a_d \geq 1$.

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