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Abstract

We consider the continuous model of log-infinitely divisible multifractal random measures (MRM) introduced in [1]. If M is a non degenerate multifractal measure with associated metric $\rho(x,y) = M([x,y])$ and structure function ζ, we show that we have the following relation between the (Euclidian) Hausdorff dimension $\dim_H$ of a measurable set K and the Hausdorff dimension $\dim_H^\rho$ with respect to ρ of the same set: $\zeta(\dim_H^\rho(K)) = \dim_H(K)$.

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1 Introduction

Multiplicative cascades are random measures that were introduced by Mandelbrot in [11] to model the energy dissipation of a turbulent flow. This model, which arises as the limit of discrete random multipliers, has been the object of numerous studies in probability theory (see for instance [9] for an account on the achieved results). In the note [4], the authors related the Hausdorff dimension $\dim_H$ of a measurable set K to the Hausdorff dimension of the same

set in the random metric induced by the multiplicative cascade: this gave the so called KPZ formula in analogy with a similar formula in quantum gravity.

In this work, we derive a similar formula in the context of log-infinitely divisible multifractal random measures (MRM) introduced by the authors in [1]. MRM are scale invariant generalisations of the log normal model introduced in [10] (and rigorously defined mathematically by Kahane in [8]) and the log Poisson model studied in [3]. MRM have been used as models of the energy dissipation in a turbulent flow (see [7]) and of the volatility of a financial asset (see [2], [5]); as such, MRM are much more realistic models than multiplicative cascades whose construction relies on a discrete dyadic decomposition of the unit interval. In particular, this dyadic dependent construction entails that multiplicative cascades have non stationary increments which is not the case of MRM.

The following note is organized as follows: section 2 reminds the definition and main properties of MRM. Section 3 reminds the background on Hausdorff dimensions needed in the proof of the main theorem. In section 4, we state the main theorem: theorem 4.1. In section 5, we give the detailed proof of theorem 4.1: our proof follows tightly the one given in [4] for multiplicative cascades. Nevertheless, the main estimates needed to carry out the proof are more difficult for MRM (the use of scale invariance is crucial).

2 Introductory background about MRM

The reader is referred to [1] for all the proofs of the results stated in this section.

Independently scattered infinitely divisible random measure. Let S^+ be the half-plane

$$S^+ = \{(t, y); t \in \mathbb{R}, y \in \mathbb{R}_+^*\}$$

with which we associate the measure (on the Borel σ-algebra $\mathcal{B}(S^+)$)

$$\theta(dt, dy) = y^{-2} dt\, dy.$$

The characteristic function of an infinitely divisible random variable X can be written as $\mathbb{E}[e^{iqX}] = e^{\varphi(q)}$, where φ is characterized by the Lévy-Khintchine formula

$$\varphi(q) = imq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}^*} (e^{iqx} - 1 - iq\sin(x))\, \nu(dx)$$

and $\nu(dx)$ is the so-called Lévy measure. It satisfies $\int_{\mathbb{R}^*} \min(1, x^2)\, \nu(dx) < +\infty$.

Following [1], we consider an independently scattered infinitely divisible random measure μ associated to (φ, θ) and distributed on the half-plane S^+ (see [12]). More precisely, μ satisfies:

1) For every sequence of disjoint sets $(A_n)_n$ in $\mathcal{B}(S^+)$, the random variables $(\mu(A_n))_n$ are independent and

$$\mu\big(\bigcup_n A_n\big) = \sum_n \mu(A_n) \text{ a.s.},$$

2) for any measurable set A in $\mathcal{B}(S^+)$, $\mu(A)$ is an infinitely divisible random variable whose characteristic function is

$$\mathbb{E}(e^{iq\mu(A)}) = e^{\varphi(q)\theta(A)}.$$

We stress the fact that μ is not necessarily a random signed measure. Let us additionnally mention that there exists a convex function ψ defined on $\mathbb{R}$ such that for all non empty subset A of S^+:

- $\psi(q) = +\infty$, if $\mathbb{E}(e^{q\mu(A)}) = +\infty$,

-$\mathbb{E}(e^{q\mu(A)}) = e^{\psi(q)\theta(A)}$ otherwise.

Let q_c be defined as $q_c = \sup\{q \geq 0; \psi(q) < +\infty\}$. For any $q \in [0, q_c[$, $\psi(q) < +\infty$ and $\psi(q) = \varphi(-iq)$.

Multifractal Random Measures (MRM). We consider an independently scattered infinitely divisible random measure μ associated to (φ, θ) such that $q_c > 1$, namely that:

$$\exists \epsilon > 0,\ \psi(1+\epsilon) < +\infty,$$

and $\psi(1) = 0$.

Definition 2.1. Filtration $\mathcal{F}_l$. *Let Ω be the probability space on which μ is defined. $\mathcal{F}_l$ is defined as the σ-algebra generated by $\{\mu(A); A \subset S^+, \mathrm{dist}(A, \mathbb{R}^2 \setminus S^+) \geq l\}$.*

Let us now define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(l) = \begin{cases} l, & \text{if } l \leq T \\ T & \text{if } l \geq T \end{cases}$$

The cone-like subset $A_l(t)$ of S^+ is defined by

$$A_l(t) = \{(s, y) \in S^+; y \geq l, -f(y)/2 \leq s - t \leq f(y)/2\}.$$

For forthcoming computations, we stress that $\theta(A_l(t)) = \int_l^{+\infty} f(y) y^{-2}\, dy < +\infty$ and, for $l \leq T$, $\theta(A_l(t)) = \ln(T/l) + 1$.

Definition 2.2. $\omega_l(\mathbf{t})$ process. *The process $\omega_l(t)$ is defined as $\omega_l(t) = \mu(A_l(t))$.*

Definition 2.3. $\mathbf{M_l(t)}$ **measure.** *For any $l > 0$, we define the measure $M_l(dt) = e^{\omega_l(t)}\,dt$, that is*

$$M_l(I) = \int_I e^{\omega_l(r)}\,dr$$

for any Lebesgue measurable subset $I \subset \mathbb{R}$.

Definition 2.4. Multifractal Random Measure (MRM). *With probability one, there exists a limit measure (in the sense of weak convergence of measures)*

$$M(dt) = \lim_{l \to 0^+} M_l(dt).$$

This limit is called the Multifractal Random Measure. The scaling exponent of M is defined by

$$\forall q \geq 0, \quad \zeta(q) = q - \psi(q).$$

Proposition 2.5. Main properties of the MRM.

1. *the measure M has no atoms in the sense that $M(\{t\}) = 0$ for any $t \in \mathbb{R}$.*
2. *The measure M is different from 0 if and only if there exists $\epsilon > 0$ such that $\zeta(1+\epsilon) > 1$; in that case, $\mathbb{E}(M([0,t])) = t$.*
3. *if $\zeta(q) > 1$ then $\mathbb{E}[M([0,t])^q] < +\infty$.*
4. *For any fixed $\lambda \in]0,1]$ and $l \leq T$, the two processes $(\omega_{\lambda l}(\lambda t))_{0 \leq t \leq T}$ and $(\Omega_\lambda + \omega_l(t))_{0 \leq t \leq T}$ have the same law, where Ω_λ is an infinitely divisible random variable independent from the process $(\omega_l(t))_{0 \leq t \leq T}$ and its law is characterized by $\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}$.*
5. *For any $\lambda \in]0,1]$, the law of the process $(M([0,\lambda t]))_{0 \leq t \leq T}$ is equal to the law of $(W_\lambda M([0,t]))_{0 \leq t \leq T}$, where $W_\lambda = \lambda e^{\Omega_\lambda}$ and Ω_λ is an infinitely divisible random variable (independent of $(M([0,t]))_{0 \leq t \leq T}$) and its characteristic function is*

$$\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}.$$

6. *If $\zeta(q) \neq -\infty$ then*

$$\mathbb{E}\big[M([0,t])^q\big] = (t/T)^{\zeta(q)}\mathbb{E}\big[M([0,T])^q\big].$$

Proposition 2.6. Main properties of the scaling exponent. *If there is $\epsilon > 0$ such that $\zeta(1+\epsilon) > 1$, the function $q \in [0,1] \mapsto \zeta(q)$ is continuous, strictly monotone increasing and maps $[0,1]$ onto $[0,1]$.*

3 Hausdorff dimension

In this section, we just set out the minimal required background about the Hausdorff dimension to understand our main result and its proof. We refer to [6] for an account on Hausdorff dimensions.

Definition 3.1. *Let (X, d) be a metric space. If $K \subset X$ and $s \in [0, +\infty[$, the s-dimensional Hausdorff content of K is defined by*

$$C_H^s(K) = \inf \left\{ \sum_i r_i^n ; \textit{there is a cover of K by balls with radii } r_i > 0 \right\}.$$

Using the standard convention $\inf \emptyset = +\infty$, the Hausdorff dimension of K is defined by

$$\dim_H(K) = \inf \{ s \geq 0; C_H^s(K) = 0 \}.$$

Lemma 3.2. (Frostman) *Let (X, d) be a metric space.The s-capacity of a Borelian set $K \subset X$*

$$\mathrm{Cap}_s(K) = \inf \left\{ \Big(\int_{K \times K} |y - x|^{-s} \gamma(dx)\gamma(dy) \Big)^{-1}; \ \gamma \textit{ is a Borel measure such that } \gamma(K) = 1 \right\}$$

is linked to the Hausdorff dimension of K by the relation

$$\dim_H(K) = \sup \{ s \geq 0; \mathrm{Cap}_s(K) > 0 \}.$$

4 Main result

If we define for $x, y \in \mathbb{R}$, $\rho(x, y) = M([x, y])$, then $\mathbb{P}$ a.s. ρ is a random metric on $\mathbb{R}$. The interval $[0, T]$ can be seen as a metric space when it is equipped either with the Euclidean metric $|\cdot|$ or with the random metric ρ. The main purpose of this paper is to establish a relation between the Hausdorff dimension of a measurable set $K \subset [0, T]$ equipped with the Euclidean metric and its Hausdorff dimension with respect to the (random) metric space $([0, T], \rho)$.

Theorem 4.1. *Assume there is $\epsilon > 0$ such that $\zeta(1 + \epsilon) > 1$ and that for all $q \in [0, 1]$ we have $\psi(-q) < \infty$. Let $K \subset [0, T]$ be some deterministic and measurable nonempty set and δ_0 its Hausdorff dimension with respect to the Euclidian metric. Then the Hausdorff dimension $\dim_H^\rho(K)$ of K with respect to the random metric ρ coincides $\mathbb{P}$ a.s. with the unique solution δ in $[0, 1]$ of the equation $\delta_0 = \zeta(\delta)$.*

5 Proof of Theorem 4.1

Lemma 5.1. *Let $x < y \in \mathbb{R}$. If $q \in [0,1]$ then*

$$\mathbb{E}[\rho(x,y)^q] \leq C(T,q)|x-y|^{\zeta(q)},$$

where $C(T,q)$ is a positive constant only depending on T, q. As a consequence, if K, δ, δ_0 are defined as in Theorem 4.1, then a.s. $\zeta(\dim_H^\rho(K)) \leq \delta_0$.

Proof. By stationarity of the measure M and Proposition 2.5, we have

$$\mathbb{E}[\rho(x,y)^q] = \mathbb{E}[M([x,y])^q] = \mathbb{E}[M([0,y-x])^q] = |y-x|^{\zeta(q)} T^{-\zeta(q)} \mathbb{E}[M([0,T])^q].$$

So we can choose $C(T,q) = T^{-\zeta(q)}\mathbb{E}[M([0,T])^q] < +\infty$.

Let $\alpha > 0$ and $q \in [0,1]$ such that $\zeta(q) > \delta_0$. There exists a covering of K by a countable family $([x_n, y_n])_n$ such that $\sum_n |x_n - y_n|^{\zeta(q)} < \alpha$. Hence

$$\mathbb{E}\Big[\sum_n \rho(x_n,y_n)^q\Big] = \sum_n \mathbb{E}\big[\rho(x_n,y_n)^q\big] \leq C(T,q)\sum_n |y_n - x_n|^{\zeta(q)} \leq C(T,q)\alpha.$$

By the Markov inequality, $\mathbb{P}\big(\sum_n \rho(x_n,y_n)^q \leq C(T,q)\sqrt{\alpha}\big) \geq 1-\sqrt{\alpha}$. Put in other words,with probability $1-\sqrt{\alpha}$, we have a covering of K with balls whose ρ-radii satisfy $\sum_n \rho(x_n,y_n)^q \leq C(T,q)\sqrt{\alpha}$. Thus $q \geq \dim_H^\rho(K)$ a.s. and the lemma follows. □

Proposition 5.2. *Let $K, \delta, \delta_0, \dim_H^\rho(K)$ be as in Theorem 4.1 and let $q \in [0,1]$ be such that $\zeta(q) < \delta_0$. Then a.s. $q \leq \dim_H^\rho(K)$, that is $\delta_0 \leq \zeta(\dim_H^\rho(K))$.*

Proof. Since $\zeta(q) < \delta_0$, by the Frostman Lemma, there is a Borel probability measure γ_0 supported by K such that $\gamma_0(K) = 1$ and

$$\int_{[0,T]^2} |x-y|^{-\zeta(q)}\, \gamma_0(dx)\, \gamma_0(dy) < +\infty.$$

Let us define, for any $0 < l < T$, the measure on $[0,T]$:

$$\nu_l(dr) = e^{q\omega_l(r) - \psi(q)(\ln(T/l)+1)}\, \gamma_0(dr)$$

and its associated metric on $\mathbb{R}$:

$$\forall x, y \in \mathbb{R}, \quad \rho_l(x,y) = \nu_l([x,y]).$$

We now investigate the quantity:

$$\begin{aligned}\phi(l,\gamma_0) &\equiv \mathbb{E}\Big[\int_{[0,T]^2}\rho_l(x,y)^{-q}\,\nu_l(dx)\,\nu_l(dy)\Big]\\ &= \int_{[0,T]^2}\mathbb{E}\Big[\rho_l(x,y)^{-q}e^{q\omega_l(x)+q\omega_l(y)-2\psi(q)(\ln(T/l)+1)}\Big]\,\gamma_0(dx)\gamma_0(dy)\\ &= 2\int_{y\geq x}\mathbb{E}\Big[\rho_l(0,y-x)^{-q}e^{q\omega_l(0)+q\omega_l(y-x)-2\psi(q)(\ln(T/l)+1)}\Big]\,\gamma_0(dx)\gamma_0(dy)\end{aligned}$$

by stationarity of the process ω_l. To this purpose, we split the above integral in two terms as

$$\begin{aligned}\phi(l,\gamma_0) =& 2\int_{0\leq y-x<l}\mathbb{E}\Big[\rho_l(0,y-x)^{-q}e^{q\omega_l(0)+q\omega_l(y-x)-2\psi(q)(\ln(T/l)+1)}\Big]\,\gamma_0(dx)\gamma_0(dy)\\ &+2\int_{y-x\geq l}\mathbb{E}\Big[\rho_l(0,y-x)^{-q}e^{q\omega_l(0)+q\omega_l(y-x)-2\psi(q)(\ln(T/l)+1)}\Big]\,\gamma_0(dx)\gamma_0(dy)\\ \equiv&\phi_1(l,\gamma_0)+\phi_2(l,\gamma_0).\end{aligned}$$

We first estimate $\phi_1(l,\gamma_0)$. Using the Jensen inequality and the decrease of the mapping $x\mapsto x^{-q}$ yields

$$\begin{aligned}&\phi_1(l,\gamma_0)\\ =&2\int_{0\leq y-x<l}\mathbb{E}\Big[\Big(\int_0^{y-x}e^{\omega_l(r)\,dr}\Big)^{-q}e^{q\omega_l(0)+q\omega_l(y-x)-2\psi(q)(\ln(T/l)+1)}\Big]\,\gamma_0(dx)\gamma_0(dy)\\ =&\frac{2e^{-2\psi(q)}l^{2\psi(q)}}{T^{2\psi(q)}}\int_{0\leq y-x<l}\mathbb{E}\Big[\Big(\int_0^{y-x}e^{\omega_l(r)-\omega_l(0)-\omega_l(y-x)}\,dr\Big)^{-q}\Big]\,\gamma_0(dx)\gamma_0(dy)\\ \leq&\int_{0\leq y-x<l}\frac{2e^{-2\psi(q)}l^{2\psi(q)}}{T^{2\psi(q)}|y-x|^q}\mathbb{E}\Big[e^{\int_0^{y-x}(q\omega_l(0)+q\omega_l(y-x)-q\omega_l(r))\frac{dr}{y-x}}\Big]\,\gamma_0(dx)\gamma_0(dy).\end{aligned}$$

Given $0\leq x<y\leq T$ such that $y-x<l$, define $A_l^i\equiv A_l(0)\cap A_l(y-x)\neq\emptyset$. Each cone-like subset $A_l(r)$ ($0\leq r\leq y-x$) can be split into three terms as $A_l(r)=A_l^g(r)\cup A_l^i\cup A_l^d(r)$, where $A_l^g(r)$ (resp. $A_l^d(r)$) denotes the part of $A_l(r)$ located on the left (resp. right) of A_l^i. It is worth emphasizing that:

$$(\omega_l^d(r))_{0\leq r\leq y-x}=(\mu(A_l^d(y-x)\setminus A_l^d(y-x-r))-\psi'(0)\theta(A_l^d(y-x)\setminus A_l^d(y-x-r)))_{0\leq r\leq y-x}$$

is a right-continuous martingale, as well as $(\omega_l^g(r))_{0\leq r\leq y-x}$ where:

$$\omega_l^g(r)=\mu(A_l^g(0)\setminus A_l^g(r))-\psi'(0)\theta(A_l^g(0)\setminus A_l^g(r)).$$

By using the fact that $\psi'(0) < 0$, we get:

$$q\omega_l(0) + q\omega_l(y-x) - q\omega_l(r) = q\omega_l^i + q\mu(A_l^d(y-x) \setminus A_l^d(r)) + q\mu(A_l^g(0) \setminus A_l^g(r))$$
$$\leq q\omega_l^i + q\omega_l^d(y-x-r) + q\omega_l^g(r).$$

Since $(\omega_l^d(r))_r$, $(\omega_l^g(r))_r$ and $w_l^i = \mu(A_l^i)$ are independent, the last expression is estimated as:

$$\begin{aligned}
&\phi_1(l,\gamma_0)\\
&\leq \int_{0\leq y-x<l} \frac{2e^{-2\psi(q)}l^{2\psi(q)}}{T^{2\psi(q)}|y-x|^q}\mathbb{E}[e^{q\omega_l^i}]\mathbb{E}[\sup_{0\leq r\leq y-x} e^{q\omega_l^d(y-x-r)}]\mathbb{E}[\sup_{0\leq r\leq y-x} e^{q\omega_l^g(r)}]\,\gamma_0(dx)\gamma_0(dy)\\
&\leq \int_{0\leq y-x<l} \frac{2C_q^2e^{-2\psi(q)}l^{2\psi(q)}}{T^{2\psi(q)}|y-x|^q}\mathbb{E}[e^{q\omega_l^i}]\mathbb{E}[e^{q\omega_l^d(y-x)}]\mathbb{E}[e^{q\omega_l^g(y-x)}]\,\gamma_0(dx)\gamma_0(dy),
\end{aligned}$$

the last inequality resulting from the Doob inequality applied to the function $x \to e^x$ (C_q is a constant only depending on q). It remains to compute $\theta(A_l^i)$, $\theta(A_l^g(0))$ and $\theta(A_l^d(y-x))$. It is plain to see that

$$\theta(A_l^i) = \ln(T/l) + 1 - (y-x)/l, \quad \theta(A_l^d(y-x)) = \theta(A_l^g(0)) = (y-x)/l,$$

in such a way that (we use that $\psi(q) < 0$ for all q in $]0,1[$):

(1)

$$\begin{aligned}
\phi_1(l,\gamma_0) \leq& \int_{0\leq y-x<l} \frac{2C_q^2e^{-2\psi(q)}l^{2\psi(q)}}{T^{2\psi(q)}|y-x|^q} e^{\psi(q)\left(\ln(T/l)+1+(y-x)/l\right)} e^{2(\psi(q)\frac{y-x}{l} - \psi'(0)\frac{y-x}{l})}\gamma_0(dx)\gamma_0(dy)\\
\leq& 2e^{-2\psi'(0)}C_q^2(eT)^{-\psi(q)} \int_{0\leq y-x<l} \frac{1}{|y-x|^{\zeta(q)}}\gamma_0(dx)\gamma_0(dy).
\end{aligned}$$

Let us now focus on $\phi_2(l,\gamma_0)$. In what follows, we make a change of variable $u = Tr/(y-x)$:

$$\begin{aligned}
&\phi_2(l,\gamma_0)\\
&= 2\int_{y-x\geq l} \mathbb{E}\Big[\frac{e^{q\omega_l(0)+q\omega_l(y-x)-2\psi(q)(\ln(T/l)+1)}}{\left(\int_0^{y-x} e^{\omega_l(r)\,dr}\right)^q}\Big]\,\gamma_0(dx)\gamma_0(dy)\\
&= \int_{y-x\geq l} \frac{2T^q}{|y-x|^q}\mathbb{E}\Big[\frac{e^{q\omega_l(0)+q\omega_l(y-x)-2\psi(q)(\ln(T/l)+1)}}{\left(\int_0^{T} e^{\omega_l((y-x)uT^{-1})}\,du\right)^q}\Big]\,\gamma_0(dx)\gamma_0(dy)
\end{aligned}$$

We remind the reader of the following property: the process $(\omega_{l'\alpha}(\alpha t))_{0\le t\le T}$ has the same law as the process $(\Omega_\alpha+\omega_{l'}(t))_{0\le t\le T}$, where $\alpha\in]0,1]$, $l'\le T$ and Ω_α is an infinitely divisible random variable independent from the process $(\omega_{l'}(t))_{0\le t\le T}$ such that $\mathbb{E}[e^{iq\Omega_\alpha}]=\alpha^{-\varphi(q)}$. In particular, choosing $l'=lT/(y-x)$ and $\alpha=(y-x)/T$, the process $\big(\omega_l\big((y-x)t/T\big)\big)_{0\le t\le T}$ has the same law as the process $(\Omega_{(y-x)/T}+\omega_{lT/(y-x)}(t))_{0\le t\le T}$. Plugging this relation into the above estimate of $\phi_2(l,\gamma_0)$ yields

$$\begin{aligned}
&\phi_2(l,\gamma_0)\\
&=\int_{y-x\ge l}\frac{2T^q}{|y-x|^q}\mathbb{E}\Big[\frac{e^{q\Omega_{(y-x)/T}+q\omega_{\frac{lT}{y-x}}(0)+q\omega_{\frac{lT}{y-x}}(T)-2\psi(q)(\ln(T/l)+1)}}{\Big(\int_0^T e^{\omega_{\frac{lT}{y-x}}(u)}\,du\Big)^q}\Big]\,\gamma_0(dx)\gamma_0(dy)\\
&=\int_{y-x\ge l}\frac{2T^{\zeta(q)}}{|y-x|^{\zeta(q)}}\mathbb{E}\Big[\frac{e^{q\omega_{\frac{lT}{y-x}}(0)+q\omega_{\frac{lT}{y-x}}(T)-2\psi(q)(\ln(\frac{y-x}{l})+1)}}{\Big(\int_0^T e^{\omega_{\frac{lT}{y-x}}(u)}\,du\Big)^q}\Big]\,\gamma_0(dx)\gamma_0(dy)
\end{aligned}$$

Thus it just remains to show that there exists $C>0$ such that for all l' in $[0,T]$:

$$\mathbb{E}\Big[\frac{e^{q\omega_{l'}(0)+q\omega_{l'}(T)-2\psi(q)(\ln(T/l')+1)}}{\Big(\int_0^T e^{\omega_{l'}(u)}\,du\Big)^q}\Big]\le C$$

In the above inequality, we will restrict to the (non obvious) case $l'\in[0,T/4]$. We have:

$$\begin{aligned}
&\mathbb{E}\Big[\frac{e^{q\omega_{l'}(0)+q\omega_{l'}(T)-2\psi(q)(\ln(T/l')+1)}}{\Big(\int_0^T e^{\omega_{l'}(u)}\,du\Big)^q}\Big]\\
&\le\mathbb{E}\Big[\frac{e^{q\omega_{l'}(0)+q\omega_{l'}(T)-2\psi(q)(\ln(T/l')+1)}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]
\end{aligned}$$

It is worth mentioning that the sets $A_{l'}(0)$, $A_{l'}(T)$ are disjoint. We then define

$$\begin{aligned}
B^g_{l'}&=A_{l'}(0)\setminus A_{l'}(T/4)\\
B^d_{l'}&=A_{l'}(T)\setminus A_{l'}(3T/4)
\end{aligned}$$

We stress that for any u in $[T/4,3T/4]$:

$$A_{l'}(u)\cap B^g_{l'}=\emptyset,\quad A_{l'}(u)\cap B^d_{l'}=\emptyset$$

Using the relation $\theta(B^g_{l'}) = \theta(B^g_{l'}) = \ln(T/l') + 1 - \ln(4)$ and the independence of $\mu(B^g_{l'})$, $\mu(B^d_{l'})$, $(\mu(A_{l'}(u)))_{T/4\leq u\leq 3T/4}$, we get:

$$\begin{aligned}
&\mathbb{E}\Big[\frac{e^{q\omega_{l'}(0)+q\omega_{l'}(T)-2\psi(q)(\ln(T/l')+1)}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]\\
&= e^{-2\psi(q)(\ln(T/l')+1)}\mathbb{E}\Big[e^{q\mu(B^g_{l'})}\Big]\mathbb{E}\Big[e^{q\mu(B^d_{l'})}\Big]\mathbb{E}\Big[\frac{e^{q\mu(A_{l'}(0)\cap A_{l'}(T/4))+q\mu(A_{l'}(T)\cap A_{l'}(3T/4))}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]\\
&= e^{-2\ln(4)\psi(q)}\mathbb{E}\Big[\frac{e^{q\mu(A_{l'}(0)\cap A_{l'}(T/4))+q\mu(A_{l'}(T)\cap A_{l'}(3T/4))}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]
\end{aligned}$$

Let us denote $\mathcal{A}^g_{l'}(u), \mathcal{A}^d_{l'}(u)$ the following sets for $u \in [T/4, 3T/4]$:

$$\begin{aligned}
\mathcal{A}^g_{l'}(u) &= (A_{l'}(0)\cap A_{l'}(u))\setminus A_{l'}(3T/4)\\
\mathcal{A}^d_{l'}(u) &= (A_{l'}(T)\cap A_{l'}(u))\setminus A_{l'}(T/4)
\end{aligned}$$

We have the following decompositions:

$$\begin{aligned}
\mu\Big(A_{l'}(0)\cap A_{l'}(T/4)\Big) &= \mu\Big(\mathcal{A}^g_{l'}(T/4)\Big) + \mu\Big(A_{l'}(0)\cap A_{l'}(3T/4)\Big),\\
\mu\Big(A_{l'}(T)\cap A_{l'}(3T/4)\Big) &= \mu\Big(\mathcal{A}^d_{l'}(3T/4)\Big) + \mu\Big(A_{l'}(T)\cap A_{l'}(T/4)\Big).
\end{aligned}$$

We also have for all u in $[T/4, 3T/4]$:

$$\begin{aligned}
\mu\Big(A_{l'}(u)\Big) = &\mu\Big(\mathcal{A}^g_{l'}(u)\Big)\\
&+\mu\Big(A_{l'}(0)\cap A_{l'}(3T/4)\Big)\\
&+\mu\Big(\mathcal{A}^d_{l'}(u)\Big)\\
&+\mu\Big(A_{l'}(T)\cap A_{l'}(T/4)\Big)\\
&+\mu\Big(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T)\Big).
\end{aligned}$$

Therefore, we get:

$$
\begin{aligned}
&\mathbb{E}\Big[\frac{e^{q\omega_{l'}(0)+q\omega_{l'}(T)-2\psi(q)(\ln(T/l')+1)}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]\\
&= e^{-2\ln(4)\psi(q)}\mathbb{E}\Big[\frac{e^{q\mu(A_{l'}(0)\cap A_{l'}(T/4))+q\mu(A_{l'}(T)\cap A_{l'}(3T/4))}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]\\
&= e^{-2\ln(4)\psi(q)}\mathbb{E}\Big[\frac{e^{q\mu(\mathcal{A}^g_{l'}(T/4))+q\mu(\mathcal{A}^d_{l'}(3T/4))}}{\Big(\int_{T/4}^{3T/4} e^{\mu(\mathcal{A}^g_{l'}(u))+\mu(\mathcal{A}^d_{l'}(u))+\mu(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T)))}\,du\Big)^q}\Big]\\
&\leq e^{-2\ln(4)\psi(q)}\mathbb{E}\Big[e^{q\mu(\mathcal{A}^g_{l'}(T/4))-q\inf_u\mu(\mathcal{A}^g_{l'}(u))}\Big]\\
&\times\mathbb{E}\Big[e^{q\mu(\mathcal{A}^d_{l'}(3T/4))-q\inf_u\mu(\mathcal{A}^d_{l'}(u))}\Big]\mathbb{E}\Big[\frac{1}{\Big(\int_{T/4}^{3T/4} e^{\mu(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T))}\,du\Big)^q}\Big]\\
&= e^{-2\ln(4)\psi(q)}\mathbb{E}\Big[e^{q\sup_u(\mu(\mathcal{A}^g_{l'}(T/4))-\mu(\mathcal{A}^g_{l'}(u)))}\Big]\\
&\times\mathbb{E}\Big[e^{q\sup_u(\mu(\mathcal{A}^d_{l'}(3T/4))-\mu(\mathcal{A}^d_{l'}(u)))}\Big]\mathbb{E}\Big[\frac{1}{\Big(\int_{T/4}^{3T/4} e^{\mu(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T))}\,du\Big)^q}\Big]\\
&= e^{-2\ln(4)\psi(q)}\mathbb{E}\Big[e^{q\sup_u(\mu(\mathcal{A}^g_{l'}(T/4)\setminus\mathcal{A}^g_{l'}(u)))}\Big]\\
&\times\mathbb{E}\Big[e^{q\sup_u(\mu(\mathcal{A}^d_{l'}(3T/4)\setminus\mathcal{A}^d_{l'}(u)))}\Big]\mathbb{E}\Big[\frac{1}{\Big(\int_{T/4}^{3T/4} e^{\mu(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T))}\,du\Big)^q}\Big]
\end{aligned}
$$

The process

$$\mu(\mathcal{A}^g_{l'}(T/4)\setminus\mathcal{A}^g_{l'}(u))-\psi'(0)\theta(\mathcal{A}^g_{l'}(T/4)\setminus\mathcal{A}^g_{l'}(u))$$

is a martingale for u in $[T/4,3T,4]$ and we have $\theta(\mathcal{A}^g_{l'}(T/4))$ bounded independently from l'. By applying Doob's inequality, there exists some constant $C>0$ independent from l' such that:

$$\mathbb{E}\Big[e^{q\sup_u(\mu(\mathcal{A}^g_{l'}(T/4)\setminus\mathcal{A}^g_{l'}(u)))}\Big]\leq C.$$

Similarly, we have:

$$\mathbb{E}\Big[e^{q\sup_u(\mu(\mathcal{A}^d_{l'}(3T/4)\setminus\mathcal{A}^d_{l'}(u)))}\Big]\leq C$$

Therefore, we get:

$$\mathbb{E}\Big[\frac{e^{q\omega_{l'}(0)+q\omega_{l'}(T)-2\psi(q)(\ln(T/l')+1)}}{\Big(\int_{T/4}^{3T/4} e^{\omega_{l'}(u)}\,du\Big)^q}\Big]$$
$$\leq C\mathbb{E}\Big[\frac{1}{\Big(\int_{T/4}^{3T/4} e^{\mu(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T))}\,du\Big)^q}\Big]$$

Since $\psi(-q) < \infty$, by using the same argument than the proof of theorem 3 (Moments of negative orders) in [3], one can show that:

$$\sup_{l'}\mathbb{E}\Big[\frac{1}{\Big(\int_{T/4}^{3T/4} e^{\mu\Big(A_{l'}(u)\setminus(A_{l'}(0)\cup A_{l'}(T)\Big)}\,du\Big)^q}\Big] < \infty.$$

To sum up, gathering the estimates of $\phi_1(l,\gamma_0)$ and $\phi_2(l,\gamma_0)$, we have proved the existence of some constant $C > 0$ such that:

$$\phi(l,\gamma_0) \leq C\int_{[0,T]^2}\frac{1}{|y-x|^{\zeta(q)}}\gamma_0(dx)\gamma_0(dy) < +\infty.$$

Let us now define the measure $\nu(dt) = \lim_{l\to 0^+}\nu_l(dt)$ (see Lemma 5.3 below). From Lemma 5.3 and the Fatou lemma, we obtain

$$\begin{aligned}\mathbb{E}\Big[\int_{[0,T]^2}\rho(x,y)^{-q}\,\nu(dx)\,\nu(dy)\Big] &\leq \mathbb{E}\Big[\liminf_{l\to 0^+}\int_{[0,T]^2}\rho_l(x,y)^{-q}\,\nu_l(dx)\,\nu_l(dy)\Big]\\ &\leq \liminf_{l\to 0^+}\mathbb{E}\Big[\int_{[0,T]^2}\rho_l(x,y)^{-q}\,\nu_l(dx)\,\nu_l(dy)\Big]\\ &\leq C\int_{[0,T]^2}\frac{1}{|y-x|^{\zeta(q)}}\gamma_0(dx)\gamma_0(dy) < +\infty.\end{aligned}$$

As a consequence, $\mathbb{P}$ a.s. the integral $\int_{[0,T]^2}\rho(x,y)^{-q}\,\nu(dx)\,\nu(dy)$ is finite. We complete the proof with the Frostman Lemma. □

Lemma 5.3. *Assume that we are given $q \in [0,1]$ such that*

$$\int_{[0,T]^2}\frac{\gamma_0(dx)\gamma_0(dy)}{|y-x|^{\zeta(q)}} < +\infty.$$

We consider, for any $l > 0$, the measure on $[0,T]$:

$$\nu_l(dt) = e^{q\omega_l(t) - \psi(q)\left(\ln(T/l)+1\right)}\, \gamma_0(dt).$$

Then the weak limit (in the sense of measures)

$$\nu(dt) = \lim_{l \to 0^+} \nu_l(dt)$$

exists $\mathbb{P}$-a.s., is finite, supported by K $\mathbb{P}$-a.s., and we have

$$\int_{[0,T]^2} \rho(x,y)^{-q}\, \nu(dx)\, \nu(dy) \leq \liminf_{l \to 0^+} \int_{[0,T]^2} \rho_l(x,y)^{-q}\, \nu_l(dx)\, \nu_l(dy).$$

Proof. According to the proof of Proposition 5.2, we have

$$\phi(l, \gamma_0) \leq C \int_{[0,T]^2} \frac{\gamma_0(dx)\gamma_0(dy)}{|y-x|^{\zeta(q)}} < +\infty.$$

Furthermore, $\rho_l(x,y) \leq \rho_l(0,T)$ for any $0 \leq x \leq y \leq T$, in such a way that

$$\mathbb{E}[\nu_l(A)^2 \rho_l(0,T)^{-\zeta(q)}] \leq \phi(l, \gamma_0) \leq C \int_{[0,T]^2} \frac{\gamma_0(dx)\gamma_0(dy)}{|y-x|^{\zeta(q)}} < +\infty$$

for any Lebesgue measurable subset A of $[0,T]$. Moreover, if the Lebesgue measure of A is strictly positive then the Hölder inequality yields

$$\begin{aligned} \mathbb{E}[\nu_l(A)^{2/(1+\zeta(q))}] \leq & \mathbb{E}[\nu_l(A)^2 M_l([0,T])^{-\zeta(q)}]^{1/(1+\zeta(q))} \mathbb{E}[M_l([0,T])]^{\zeta(q)/(1+\zeta(q))} \\ & \leq C' \int_{[0,T]^2} \frac{\gamma_0(dx)\gamma_0(dy)}{|y-x|^{\zeta(q)}} < +\infty. \end{aligned} \tag{2}$$

We remind the reader that $(\nu_l(A))_l$ is martingale for any Lebesgue measurable subset A of $[0,T]$. From (2), this martingale is bounded in $L^{1+\epsilon}$ for some $\epsilon > 0$. As a consequence, it converges $\mathbb{P}$-a.s. towards a limit denoted by $\nu(A)$ as $l \to 0$. It is readily seen that ν is a measure on $[0,T]$ $\mathbb{P}$-a.s. Since $\nu_l(K^c) = 0$, it is clear that $\nu(K^c) = 0$ $\mathbb{P}$-a.s.

Finally, $\mathbb{E}[\nu([0,T])] = \lim_{l \to 0} \mathbb{E}[\nu_l([0,T])] = \gamma_0([0,T]) \geq 1$. Moreover $\{\nu([0,T]) > 0\}$ is an event of the asymptotic σ-field generated by the random variables $(\nu_l(A))_l$ and has therefore probability 0 or 1. As a consequence, the event $\{\nu([0,T]) > 0\}$ has probability 1.

The last inequality of the lemma results from Lemma 5.4 below and the weak convergence of measures. □

Lemma 5.4. *$\mathbb{P}$ a.s., the metric $(\rho_l)_l$ uniformly converges towards the metric ρ as $l \to 0$, that is*

$$\mathbb{P}\ \textit{a.s.}, \quad \lim_{l\to 0} \sup_{0\le x\le y\le T} |\rho_l(x,y) - \rho(x,y)| = 0.$$

Proof. The mapping $x \mapsto \rho(0,x)$ is continuous because of the non-degeneracy of ρ (see Proposition 2.5). Moreover, for each $l > 0$, the mapping $x \mapsto \rho_l(0,x)$ is increasing and the sequence $(\rho_l(0,x)$ converges pointwise $\mathbb{P}$ a.s. towards $\rho(0,x)$ (see Definition 2.4). The uniform convergence then results from the Dini theorem. □

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