Gaussian Multiplicative Chaos revisited

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GAUSSIAN MULTIPLICATIVE CHAOS REVISITED

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Abstract. In this article, we extend the theory of multiplicative chaos for positive definite functions in $\mathbb{R}^d$ of the form $f(x) = \lambda^2 \ln^+ \frac{T}{|x|} + g(x)$ where g is a continuous and bounded function. The construction is simpler and more general than the one defined by Kahane in 1985. As main application, we give a rigorous mathematical meaning to the Kolmogorov-Obukhov model of energy dissipation in a turbulent flow.

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1. Introduction

The theory of multiplicative chaos was first defined rigorously by Kahane in 1985 in the article [12]. More specifically, Kahane built a theory relying on the notion of σ-positive type kernel: a generalized function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$ is of σ-positive type if there exists a sequence $K_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ of continuous positive and positive definite kernels such that:

$$K(x,y) = \sum_{k \geqslant 1} K_k(x,y). \tag{1.1}$$

If K is a σ-positive type kernel with decomposition (1.1), one can consider a sequence of gaussian processes $(X_n)_{n \geqslant 1}$ of covariance given by $\sum_{k=1}^{n} K_k$. It is proven in [12] that the sequence of random measures m_n given by:

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad m_n(A) = \int_A e^{X_n(x) - \frac{1}{2}E[X_n(x)^2]} dx \tag{1.2}$$

converges almost surely in the space of Radon measures (equiped with the topology of weak convergence) towards a random measure m and that the limit measure m obtained does not depend on the sequence $(K_k)_{k \geqslant 1}$ used in the decomposition

(1.1) of K. Thus, the theory enables to give a unique and mathematically rigorous definition to a random measure m in $\mathbb{R}^d$ defined formally by:

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad m(A) = \int_A e^{X(x)-\frac{1}{2}E[X(x)^2]}dx \tag{1.3}$$

where $(X(x))_{x\in\mathbb{R}^d}$ is a "gaussian field" whose covariance K is a σ-positive type kernel. As it will appear later the σ-positive type condition is not easy to check in practice. From where the need to avoid this hypothesis.

The main application of the theory is to give a meaning to the "limit-lognormal" model introduced by Mandelbrot in [15]. In the sequel, we define $\ln^+ x$ for $x > 0$ by the following formula:

$$\ln^+ x = \max(\ln(x), 0).$$

The "limit-lognormal" model corresponds to the choice of a stationnary K given by:

$$K(s,t) = \lambda^2 \ln^+(R/|x-y|) + O(1) \tag{1.4}$$

where λ^2, R are positive parameters and $O(1)$ is a bounded quantity as $|x-y| \to 0$. This model has many applications that we now review in the following subchapters.

1.1. **Multplicative chaos in dimension 1: a model for the volatility of a financial asset.** If $(X(t))_{t\geqslant 0}$ is the logarithm of the price of a financial asset, the volatility m of the asset on the interval $[0,t]$ is by definition equal to the quadratic variation of X:

$$m[0,t] = \lim_{n\to\infty} \sum_{k=1}^{n} (X(tk/n) - X(t(k-1)/n))^2$$

The volatility m can be viewed as a random measure on $\mathbb{R}$. The choice for m of multiplicative chaos associated to the kernel $K(s,t) = \lambda^2 \ln^+ \frac{T}{|t-s|}$ satisfies many empirical properties measured on financial markets: lognormality of the volatility, long range correlations (see [5] for a study of the SP500 index and components and [6] for a general review). Note that K is indeed of σ-positive type (see example 2.3 below) so m is well defined. In the context of finance, λ^2 is called the intermittency parameter in analogy with turbulence and T is the correlation length. Volatility modeling and forecasting is an important field of finance since it is related to option pricing and risk forecasting: we refer to [8] for the problem of forecasting volatility with this choice of m.

Given the volatility m, the most natural way to construct a model for the (log) price X is to set:

$$X(t) = B_{m[0,t]} \tag{1.5}$$

where $(B_t)_{t\geqslant 0}$ is a brownian motion independent of m. Formula (1.5) defines the Multifractal Random Walk (MRW) first introduced in [1] (see [2] for a recent review of the financial applications of the MRW model).

1.2. **Multiplicative chaos in dimension 3: a model for the energy dissipation in a turbulent fluid.** We refer to [9] for an introduction to the statistical theory of 3 dimensional turbulence. Consider a stationnary flow at high Reynolds number; it is believed that at small scales the velocity field of the flow is homogeneous and isotropic in space . By small scales we mean scales much smaller than the integral scale R charactersistic of the time stationnary force driving the flow. In the work [13] and [17], Kolmogorov and Obukhov propose to model the mean energy dissipation per unit mass in a ball $B(x,l)$ of center x and radius $l << R$ by a random variable ϵ_l such that $\ln(\epsilon_l)$ is normal with variance σ_l^2 given by:

$$\sigma_l^2 = \lambda^2 \ln(\frac{R}{l}) + A$$

where A is a constant and λ^2 is the intermittency parameter. As noted by Mandelbrot ([15]), the only way to define such a model is to construct a random measure ϵ by a limit procedure. Then, one can define ϵ_l by the formula:

$$\epsilon_l = \frac{3 < \epsilon >}{4\pi l^3} \epsilon(B(x,l))$$

where $< \epsilon >$ is the average mean energy disspation per unit mass. Formally, one is looking for a random measure ϵ such that:

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \epsilon(A) = \int_A e^{X(x) - \frac{1}{2}E[X(x)^2]} dx \tag{1.6}$$

where $(X(x))_{x \in \mathbb{R}^d}$ is a "gaussian field" whose covariance K is given by $K(x,y) = \lambda^2 \ln^+ \frac{R}{|x-y|}$. The kernel $\lambda^2 \ln^+ \frac{R}{|x-y|}$ is positive definite considered as a tempered distribution (see (2.1) for a definition of positive definite distributions and lemma 3.2 for a proof). Therefore, one can give a rigorous meaning to (1.6) by using theorem-definition 2.1 below.

However, it is not clear if $\lambda^2 \ln^+ \frac{R}{|x-y|}$ is of σ-positive type in $\mathbb{R}^3$ and therefore, in [12], Kahane considers the σ-positive type kernel $K(x,y) = \int_{1/T}^{\infty} \frac{e^{-u|x-y|}}{u} du$ as an approximation of $\lambda^2 \ln^+ \frac{R}{|x-y|}$: indeed, one can show that $\int_{1/T}^{\infty} \frac{e^{-u|x-y|}}{u} du = \ln^+ \frac{R}{|x-y|} + g(|x-y|)$ where g is a bounded continuous function. Nevertheless, it is important to work with $\lambda^2 \ln^+ \frac{R}{|x-y|}$ since this choice leads to measures which exhibit generalized scale invariance properties (see proposition 3.3).

1.3. **Organization of the paper.** In section 2, we remind the definition of positive definite tempered distributions and we state theorem-definition 2.1 where we define multiplicative chaos m associated to kernels of the type $\ln^+ \frac{R}{|x|} + O(1)$. In section 3, we review the main properties of the measure m: existence of moments and density with respect to Lebesgue measure, multifractality and generalized scale invariance. In section 4 and 5, we give respectively the proofs of section 2 and 3.

2. Definition of multiplicative chaos

2.1. **Positive definite tempered distributions.** Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of smooth rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions (see [19]). A distribution f in $\mathcal{S}'(\mathbb{R}^d)$ is of positive definite if:

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi(x)\overline{\varphi(y)}dxdy \geqslant 0. \tag{2.1}$$

On $\mathcal{S}'(\mathbb{R}^d)$, one can define the Fourier transform $\hat{f}$ of a tempered distribution by the formula:

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \hat{f}(\xi)\varphi(\xi)d\xi = \int_{\mathbb{R}^d} f(x)\hat{\varphi}(x)dx \tag{2.2}$$

where $\hat{\varphi}(x) = \int_{\mathbb{R}^d} e^{-2i\pi x.\xi}\varphi(\xi)d\xi$ is the Fourier transform of φ. An extension of Bochner's theorem (Schwartz, [19]) states that a tempered distribution f is positive definite if and only if it's Fourier transform is a tempered positive measure.

By definition, a function f in $\mathcal{S}'(\mathbb{R}^d)$ is of σ-positive type if the associated kernel $K(x,y) = f(x-y)$ is of σ-positive type. As mentioned in the introduction, Kahane's theory of multiplicative chaos is defined for σ-positive type functions f. The main problem stems from the fact that definition (1.1) is not practical. Indeed, is there a simple characterization (like the computation of a Fourier transform) of functions whose associated kernel can be decomposed along (1.1)? If such a characterization exists, how does one find the kernels K_n explicitely?

Finally, we recall the following simple implication: If f belongs to $\mathcal{S}'(\mathbb{R}^d)$and is of σ-positive type, f is positive and positive definite. However, the converse statement is not clear.

2.2. **A generalized theory of multiplicative chaos.** In this subsection, we contruct a theory of multiplicative chaos for positive definite functions of type $\lambda^2 \ln^+ \frac{R}{|x|} + O(1)$ without the assumption of σ-positivity for the underlying function. The theory is therefore much easier to use.

We consider in $\mathbb{R}^d$ a positive definite function f such that

$$f(x) = \lambda^2 \ln^+ \frac{R}{|x|} + g(x) \tag{2.3}$$

where $\lambda^2 \neq 2d$ and $g(x)$ is a bounded continuous function. Let $\theta : \mathbb{R}^d \to \mathbb{R}$ be some continuous function with the following properties:

(1) θ is positive definite
(2) $|\theta(x)| \leqslant \frac{1}{1+|x|^{d+\gamma}}$ for some $\gamma > 0$.
(3) $\int_{\mathbb{R}^d} \theta(x)dx = 1$

Here is the main theorem of the article:

Theorem 2.1. *(Definition of multiplicative chaos)*

For all $\epsilon > 0$, we consider the centered gaussian field $(X_\epsilon(x))_{x\in\mathbb{R}^d}$ defined by the convolution:

$$E[X_\epsilon(x)X_\epsilon(y)] = (\theta^\epsilon * f)(y-x),$$

where $\theta^\epsilon = \frac{1}{\epsilon^d}\theta(\frac{\cdot}{\epsilon})$. Then the associated random measure $m_\epsilon(dx) = e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]}dx$ converges in law in the space of Radon mesures (equiped with the topology of weak convergence) as ϵ goes to 0 towards a random measure m independent of the choice of the regularizing function θ with the properties (1), (2), (3). We call the measure m the multiplicative chaos associated to the function f.

We review below two possible choices of the underlying function f. The first example is a d-dimensional generalization of the cone construction considered in [3]. The second example is $\lambda^2 \ln^+ \frac{R}{|x|}$ for $d = 1, 2, 3$ (the case $d = 2, 3$ seems to have never been considered in the litterature). Both examples are in fact of σ-positive type (except perhaps the crucial example of $\lambda^2 \ln^+ \frac{R}{|x|}$ in dimension $d = 3$) and it is easy to show that in these cases theorem-definition 2.1 and Kahane's theory lead to the same limit measure m.

Example 2.2. *One can construct a positive definite function f with decomposition (2.3) by generalizing to dimension d the cone construction of [3]. This was performed in [4]. For all x in $\mathbb{R}^d$, we define the cone $C(x)$ in $\mathbb{R}^d \times \mathbb{R}_+$:*

$$C(x) = \{(y,t) \in \mathbb{R}^d \times \mathbb{R}_+;\ |y - x| \leqslant \frac{t \wedge T}{2}\}.$$

The function f is given by:

$$f(x) = \lambda^2 \int_{C(0)\cap C(x)} \frac{dydt}{t^{d+1}} \tag{2.4}$$

One can show that f has decomposition (2.3) (see [4]). The function f is of σ-positive type in the sense of Kahane since one can write $f = \sum_{n\,\geqslant\,1} f_n$ with f_n given by:

$$f_n(x) = \lambda^2 \int_{C(0)\cap C(x);\ \frac{1}{n}\leqslant t < \frac{1}{n-1}} \frac{dydt}{t^{d+1}}.$$

In dimension $d = 1$, we get the simple formula $f(x) = \ln^+ \frac{R}{|x|}$.

Example 2.3. *In dimension $d = 1, 2$, the function $f(x) = \ln^+ \frac{R}{|x|}$ is of σ-positive type in the sense of Kahane and in particular positive definite. Indeed, one has by straightforward calculations:*

$$\ln^+ \frac{T}{|x|} = \int_0^\infty (t - |x|)_+ \nu_T(dt)$$

where $\nu_T(dt) = 1_{[0,T[}(t)\frac{dt}{t^2} + \frac{\delta_T}{T}$. For all $\mu > 0$, we have:

$$\ln^+ \frac{T}{|x|} = \frac{1}{\mu} \ln^+ \frac{T^\mu}{|x|^\mu} = \frac{1}{\mu}\int_0^\infty (t - |x|^\mu)_+ \nu_{T^\mu}(dt).$$

We are therefore led to considering the $\mu > 0$ such that $(1 - |x|^\mu)_+$ is positive definite (the so called Kuttner-Golubov problem: see [10] for an introduction).

For $d = 1$, it is straightforward to show that $(1-|x|)_+$ is of positive type. One can thus write $f = \sum_{n\geqslant 1} f_n$ with f_n given by:

$$f_n(x) = \int_{\frac{T}{n}}^{\frac{T}{n-1}} (t-|x|)_+ \nu_T(dt).$$

For $d = 2$, the function $(1-|x|^{1/2})$ is positive definite (Pasenchenko, [18]). One can thus write $f = \sum_{n\geqslant 1} f_n$ with f_n given by:

$$f_n(x) = \int_{\frac{T^{1/2}}{n}}^{\frac{T^{1/2}}{n-1}} (t-|x|^{1/2})_+ \nu_{T^{1/2}}(dt).$$

In dimension $d = 3$, the function $\ln^+ \frac{R}{|x|}$ is positive definite (see lemma 3.2 below) but it is an open question whether it is of σ-positive type.

3. Main properties of multiplicative chaos

In the sequel, we will consider the structure functions ζ_p defined for all p in $\mathbb{R}$ by:

$$\zeta_p = (d+\frac{\lambda^2}{2})p - \frac{\lambda^2 p^2}{2}. \tag{3.1}$$

3.1. **Multiplicative chaos is equal to 0 for $\lambda^2 > 2d$.** The following proposition can be seen as a phase transition and shows that the logarithmic kernel is crucial in the theory of multiplicative chaos:

Proposition 3.1. *If $\lambda^2 > 2d$, the limit measure is equal to 0.*

3.2. **Generalized scale invariance.** In this subsection and the following, in view of proposition 3.1, we will suppose that $\lambda^2 < 2d$:

Let m be a homogeneous random measure on $\mathbb{R}^d$. We note $B(0,R)$ the ball of center 0 and radius R in $\mathbb{R}^d$. We say m has the generalized scale invariance property with integral scale $R > 0$ if for all c in $]0,1]$ the following equality in law holds:

$$(m(cA))_{A\subset B(0,R)} \overset{(Law)}{=} e^{\Omega_c}(m(A))_{A\subset B(0,R)} \tag{3.2}$$

where Ω_c is a random variable independent from m. If m is different from 0, then it is immediate to prove that $(\Omega_{e^{-t}})_{t\geqslant 0}$ is a Levy process. In the context of gaussian multiplicative chaos, the process $(\Omega_{e^{-t}})_{t\geqslant 0}$ will be Brownian motion with drift.

In order to get scale invariance with integral scale R, one can choose $f = \ln^+ \frac{R}{|.|}$. This is possible if and only if $\ln^+ \frac{R}{|.|}$ is positive defnite. This motivates the following lemma:

Lemma 3.2. *Let $d \geqslant 1$ be the dimension of the space and $R > 0$ the integral scale. We consider the function $f : \mathbb{R}^d \to \mathbb{R}_+$ defined by:*

$$f(x) = \ln^+ \frac{R}{|x|}.$$

The function f is positive definite if and only if $d \leqslant 3$.

The above choice of f leads to measures that have the generalized scale invariance property.

Proposition 3.3. *Let d be less or equal to 3 and m be the gaussian multplicative chaos with kernel $\lambda^2 \ln^+ \frac{R}{|x|}$. Then m is scale invariant; for all c in $]0,1]$, the following equality holds:*

$$(m(cA))_{A\subset B(0,R)} \overset{(Law)}{=} e^{\Omega_c}(m(A))_{A\subset B(0,R)}, \tag{3.3}$$

where Ω_c is a gaussian random variable independent of m with mean $-(d+\frac{\lambda^2}{2})\ln(1/c)$ and variance $\lambda^2 \ln(1/c)$

The proof of the proposition is straightforward.

Remark 3.4. *It remains an open question to construct homogeneous measures in dimension greater or equal to 4 which are scale invariant.*

3.3. **Existence of moments and multifractality.** We remind that we suppose that $\lambda^2 < 2d$: this ensures the existence of $\epsilon > 0$ such that $\zeta_{1+\epsilon} > d$. Therefore, there exists a unique $p_* > 1$ such that $\zeta_{p_*} = d$. The following two propositions establish the existence of positive and negative moments for the limit measure.

Proposition 3.5. *(Positive Moments)*
Let p belong to $]0,p_*[$ and m be the gaussian multiplicative chaos associated to the function f given by (2.3). For all bounded A in $\mathcal{B}(\mathbb{R}^d)$,

$$E[m(A)^p] < \infty \tag{3.4}$$

Let θ be some function satisfying the conditions (1), (2), (3) of section 2.2. With the notations of theorem 2.1, we consider the random measure m_ϵ associated to θ. We have the following convergence for all bounded A in $\mathcal{B}(\mathbb{R}^d)$:

$$E[m_\epsilon(A)^p] \underset{\epsilon\to 0}{\to} E[m(A)^p]. \tag{3.5}$$

Proposition 3.6. *(Negative Moments)*
Let p belong to $]-\infty,0]$ and m be the gaussian multiplicative chaos associated to the function f given by (2.3). For all $c>0$,

$$E[m(B(0,c))^p] < \infty \tag{3.6}$$

Let θ be some function satisfying the conditions (1), (2), (3) of section 2.2. With the notations of theorem 2.1, we consider the random measure m_ϵ associated to θ. We have the following convergence for all $c>0$:

$$E[m_\epsilon(B(0,c))^p] \underset{\epsilon\to 0}{\to} E[m(B(0,c))^p]. \tag{3.7}$$

The following proposition states the existence of the structure functions.

Proposition 3.7. *Let p belong to $]-\infty,p_*[$. Let m be the gaussian multiplicative chaos associated to the function f given by (2.3). There exists some $C_p>0$ (independent of g and R in decomposition (2.3): $C_p = C_p(\lambda^2)$) such that we have the following multifractal behaviour:*

$$E[m([0,c]^d)^p] \underset{c\to 0}{\sim} e^{\frac{p(p-1)g(0)}{2}} C_p(\frac{c}{R})^{\zeta_p}. \tag{3.8}$$

In the next proposition, we will suppose that $d \leqslant 3$ and that $f(x) = \lambda^2 \ln^+ \frac{R}{|x|}$. In this case, we can prove the existence of a C^∞ density.

Proposition 3.8. *Let d be less or equal to 3 and m be the gaussian multiplicative chaos with kernel $\lambda^2 \ln^+ \frac{R}{|x|}$. For all $c < R$, the variable $m(B(0,c))$ has a C^∞ density with respect to the Lebesgue measure.*

4. Proof of theorem 2.1

4.1. **A few intermediate lemmas.** In order to prove the theorem, we start by giving some lemmas we will need in the proof.

Lemma 4.1. *Let θ be some function on $\mathbb{R}^d$ such that there exists $\gamma, C > 0$ with $|\theta(x)| \leqslant \frac{C}{1+|x|^{d+\gamma}}$. Then we have the following convergence:*

$$\sup_{|z|>A} \Big| \int_{\mathbb{R}^d} |\theta(v)| \ln \Big|\frac{z}{z-v}\Big| dv \Big| \underset{A\to\infty}{\to} 0. \tag{4.1}$$

Proof. We have:

$$\int_{\mathbb{R}^d} |\theta(v)| \ln \Big|\frac{z}{z-v}\Big| dv = \int_{|v| \leqslant \sqrt{|z|}} |\theta(v)| \ln \Big|\frac{z}{z-v}\Big| dv + \int_{|v|>\sqrt{|z|}} |\theta(v)| \ln \Big|\frac{z}{z-v}\Big| dv.$$

Consider the first term: we have $1 - \frac{|v|}{|z|} \leqslant \frac{|z-v|}{|z|} \leqslant 1 + \frac{|v|}{|z|}$ so that for $|v| \leqslant \sqrt{|z|}$:

$$1 - \frac{1}{\sqrt{|z|}} \leqslant \frac{|z-v|}{|z|} \leqslant 1 + \frac{1}{\sqrt{|z|}},$$

thus we get $|\ln \frac{|z-v|}{|z|}| \leqslant \ln(\frac{1}{1-\frac{1}{\sqrt{|z|}}}) \leqslant \frac{1}{\sqrt{|z|}-1}$. We conclude:

$$\int_{|v| \leqslant \sqrt{|z|}} |\theta(v)| \ln \Big|\frac{z}{z-v}\Big| dv \leqslant \frac{1}{\sqrt{|z|}-1} \int_{\mathbb{R}^d} |\theta(v)| dv$$

Consider the second term: we have:

$$\int_{|v|>\sqrt{|z|}} |\theta(v)| \ln \Big|\frac{z}{z-v}\Big| dv \leqslant \ln|z| \int_{|v|>\sqrt{|z|}} |\theta(v)| dv + \int_{|v|>\sqrt{|z|}} |\theta(v)| |\ln|z-v|| dv$$

The first term above is obvious; we decompose the second:

$$\int_{|v|>\sqrt{|z|}} |\theta(v)| |\ln|z-v|| dv = \int_{\sqrt{|z|}<|v|<|z|+1} |\theta(v)| |\ln|z-v|| dv + \int_{|v| \geqslant |z|+1} |\theta(v)| |\ln|z-v|| dv$$

For $|v| \geqslant |z| + 1$, we have $1 \leqslant |z-v| \leqslant |z||v|$ and thus

$$0 \leqslant \ln|z-v| \leqslant \ln|z| + \ln|v|$$

which enables to handle the corresponding integral. Let us now estimate the remaining term $I = \int_{\sqrt{|z|}<|v|<|z|+1} |\theta(v)| |\ln|z-v|| dv$. Applying Cauchy Schwarz gives:

$$I \leqslant \Big(\int_{\sqrt{|z|}<|v|<|z|+1} |\theta(v)|^2 dv\Big)^{1/2} \Big(\int_{\sqrt{|z|}<|v|<|z|+1} |\ln|z-v||^2 dv\Big)^{1/2},$$

from which we get straightforwardly:

$$I \leqslant \frac{C \ln|z|}{|z|^{d/2+\gamma/2-d/4}} \underset{|z|\to\infty}{\to} 0.$$

□

We will also use the following lemma:

Lemma 4.2. *Let λ be a positive number such that $\lambda^2 \neq 2$ and $(X_i)_{1 \leqslant i \leqslant n}$ an i.i.d. sequence of centered gaussian variables with variance $\lambda^2 \ln(n)$. For all positive p such that $p < \max(\frac{2}{\lambda^2}, 1)$, there exists $0 < x < 1$ such that:*

$$E[\sup_{1 \leqslant i \leqslant n} e^{pX_i - p\frac{\lambda^2}{2}\ln(n)}] = O(n^{xp}) \tag{4.2}$$

Proof. By Fubini we get:

$$\begin{aligned}
&E[\sup_{1 \leqslant i \leqslant n} e^{pX_i - p\frac{\lambda^2}{2}\ln(n)}] \\
&= \int_0^\infty P(\sup_{1 \leqslant i \leqslant n} e^{pX_i - p\frac{\lambda^2}{2}\ln(n)} > v)dv \\
&= \int_0^\infty P(\sup_{1 \leqslant i \leqslant n} X_i > \frac{\ln(v)}{p} + \frac{\lambda^2}{2}\ln(n))dv \\
&= \int_{-\infty}^\infty pe^{pu} P(\sup_{1 \leqslant i \leqslant n} X_i > u + \frac{\lambda^2}{2}\ln(n))du \\
&\leqslant 1 + \int_0^\infty pe^{pu} P(\sup_{1 \leqslant i \leqslant n} X_i > u + \frac{\lambda^2}{2}\ln(n))du,
\end{aligned} \tag{4.3}$$

where we performed the change of variable $u = \frac{\ln(v)}{p}$ in the above identities. If we define $\bar{F}(u) = P(X_1 > u)$ then we have:

$$P(\sup_{1 \leqslant i \leqslant n} X_i > u + \frac{\lambda^2}{2}\ln(n)) = 1 - e^{n\ln(1-\bar{F}(u+\frac{\lambda^2}{2}\ln(n)))}.$$

Let x be some positive number such that $0 < x < 1$. Using (4.3), we get:

$$\begin{aligned}
E[\sup_{1 \leqslant i \leqslant n} e^{pX_i - p\frac{\lambda^2}{2}\ln(n)}] &\leqslant n^{xp} + p\int_{x\ln(n)}^\infty e^{pu}(1 - e^{n\ln(1-\bar{F}(u+\frac{\lambda^2}{2}\ln(n)))})du \\
&\leqslant n^{xp} + pn^{xp}\int_0^\infty e^{p\widetilde{u}}(1 - e^{n\ln(1-\bar{F}(\widetilde{u}+(\frac{\lambda^2}{2}+x)\ln(n)))})d\widetilde{u}
\end{aligned} \tag{4.4}$$

We have:

$$\begin{aligned}
\bar{F}(\widetilde{u} + (\frac{\lambda^2}{2} + x)\ln(n))) &= \frac{1}{\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_{\widetilde{u}+(\frac{\lambda^2}{2}+x)\ln(n)}^\infty e^{-\frac{v^2}{2\lambda^2\ln(n)}}dv \\
&= \frac{n^{-\frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_{\widetilde{u}}^\infty e^{-(\frac{1}{2}+\frac{x}{\lambda^2})\widetilde{v} - \frac{\widetilde{v}^2}{2\lambda^2\ln(n)}}d\widetilde{v},
\end{aligned}$$

where we performed the change of variable $\widetilde{v} = v - (\frac{\lambda^2}{2} + x)\ln(n)$. Thus, we get:

$$\begin{aligned}
& n^{xp} \int_0^\infty e^{p\widetilde{u}} (1 - e^{n \ln(1 - \bar{F}(\widetilde{u} + (\frac{\lambda^2}{2} + x)\ln(n)))}) d\widetilde{u} \\
& \leqslant n^{xp+1} \int_0^\infty e^{p\widetilde{u}} \bar{F}(\widetilde{u} + (\frac{\lambda^2}{2} + x)\ln(n)) d\widetilde{u} \\
& \leqslant \frac{n^{xp+1-\frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_0^\infty e^{p\widetilde{u}} (\int_{\widetilde{u}}^\infty e^{-(\frac{1}{2}+\frac{x}{\lambda^2})\widetilde{v} - \frac{\widetilde{v}^2}{2\lambda^2 \ln(n)}} d\widetilde{v}) d\widetilde{u} \\
& \leqslant \frac{n^{xp+1-\frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{p\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_0^\infty e^{p\widetilde{v} - (\frac{1}{2}+\frac{x}{\lambda^2})\widetilde{v} - \frac{\widetilde{v}^2}{2\lambda^2 \ln(n)}} d\widetilde{v} \\
& \leqslant \frac{n^{xp+1-\frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{p\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_{-\infty}^\infty e^{p\widetilde{v} - (\frac{1}{2}+\frac{x}{\lambda^2})\widetilde{v} - \frac{\widetilde{v}^2}{2\lambda^2 \ln(n)}} d\widetilde{v} \\
& = \frac{n^{xp+\alpha(x,\lambda,p)}}{p},
\end{aligned} \tag{4.5}$$

with $\alpha(x,\lambda^2,p) = 1 - \frac{(\lambda^2/2+x)^2}{2\lambda^2} + (p - \frac{1}{2} - \frac{x}{\lambda^2})^2 \frac{\lambda^2}{2}$. We have by combining (4.4) and (4.5):

$$E[\sup_{1 \leqslant i \leqslant n} e^{pX_i - p\frac{\lambda^2}{2}\ln(n)}] \leqslant n^{xp} + n^{xp+\alpha(x,\lambda,p)},$$

We focus on the case $p \in]\frac{1}{2} + \frac{1}{\lambda^2}, \max(\frac{2}{\lambda^2}, 1)[$ (This implies inequality (4.2) for $p \leqslant \frac{1}{2} + \frac{1}{\lambda^2}$ by Holders inequality).

First case: $\lambda^2 < 2$.

Note that $\alpha(1, \lambda^2, \frac{2}{\lambda^2}) = 0$ so if $p < \frac{2}{\lambda^2}$ then there exists $0 < x < 1$ such that $\alpha(x, \lambda^2, p) < 0$.

Second case: $\lambda^2 > 2$.

Note that $\alpha(1, \lambda^2, 1) = 0$ so if $p < 1$ then there exists $0 < x < 1$ such that $\alpha(x, \lambda^2, p) < 0$.

□

4.2. **Proof of theorem 2.1.** For sake of simplicity, we give the proof in the case $d = 1$, $R = 1$ and the function $f(x) = \lambda^2 \ln^+ \frac{1}{|x|}$. This is no restriction; indeed, the proof in the general case is an immediate adaptation of the following proof.

Uniqueness. Let $\alpha \in]0, 1/2[$. We consider θ and $\widetilde{\theta}$ two continuous functions satisfying properties (1), (2) and (3). We note:

$$m(dt) = e^{X(t) - \frac{1}{2}E[X(t)^2]} dt = \lim_{\epsilon \to 0} e^{X_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]} dt,$$

where $(X_\epsilon(t))_{t\in\mathbb{R}}$ is a gaussian process of covariance $q_\epsilon(|t-s|)$ with:

$$q_\epsilon(x) = (\theta^\epsilon * f)(x) = \lambda^2 \int_{\mathbb{R}} \theta(v) \ln^+(\frac{1}{|x - \epsilon v|}) dv$$

We define similarly the measure $\widetilde{m}$, $\widetilde{X}_\epsilon$ and $\widetilde{q}_\epsilon$ associated to the function $\widetilde{\theta}$. Note that we suppose that the random measures $m_\epsilon(dt) = e^{X_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]}dt$ and $\widetilde{m}_\epsilon(dt) = e^{\widetilde{X}_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]}dt$ converge in law in the space of Radon measures: this is no restriction since the equality $E[m_\epsilon(A)] = E[\widetilde{m}_\epsilon(A)] = |A|$ for all bounded A in $\mathcal{B}(\mathbb{R})$ implies the measures are tight.

We will show that:

$$E[m[0,1]^\alpha] = E[\widetilde{m}[0,1]^\alpha]$$

for α in the interval $]0,1/2[$. If we define $Z_\epsilon(t)(u) = \sqrt{t}\widetilde{X}_\epsilon(u) + \sqrt{1-t}X_\epsilon(u)$ with $X_\epsilon(u)$ and $\widetilde{X}_\epsilon(u)$ independent, we get by using a continuous version of lemma 6.1:

$$E[\widetilde{m}_\epsilon[0,1]^\alpha] - E[m_\epsilon[0,1]^\alpha] = \frac{\alpha(\alpha-1)}{2}\int_0^1 \varphi_\epsilon(t)dt, \tag{4.6}$$

with $\varphi_\epsilon(t)$ defined by:

$$\varphi_\epsilon(t) = \int_{[0,1]^2} (\widetilde{q}_\epsilon(|t_2-t_1|) - q_\epsilon(|t_2-t_1|)E[\mathcal{X}_\epsilon(t,t_1,t_2)]dt_1dt_2,$$

where $\mathcal{X}_\epsilon(t,t_1,t_2)$ is given by:

$$\mathcal{X}_\epsilon(t,t_1,t_2) = \frac{e^{Z_\epsilon(t)(t_1)+Z_\epsilon(t)(t_2)-\frac{1}{2}E[Z_\epsilon(t)(t_1)^2]-\frac{1}{2}E[Z_\epsilon(t)(t_2)^2]}}{(\int_0^1 e^{Z_\epsilon(t)(u)-\frac{1}{2}E[Z_\epsilon(t)(u)^2]}du)^{2-\alpha}}.$$

We state and prove the following short lemma we will need in the sequel.

Lemma 4.3. *For $A > 0$, we denote $C_A^\epsilon = \sup_{|x| \geqslant A\epsilon} |q_\epsilon(x) - \widetilde{q}_\epsilon(x)|$. We have:*

$$\lim_{A\to\infty}(\overline{\lim_{\epsilon\to 0}} C_A^\epsilon) = 0.$$

Proof. Let $|x| \geqslant A\epsilon$. If $|x| \geqslant 1/2$ then $q_\epsilon(x)$ and $\widetilde{q}_\epsilon$ converge uniformly towards $\ln^+ \frac{1}{|x|}$ thus $q_\epsilon(x) - \widetilde{q}_\epsilon$ converges uniformly to 0. If $|x| < 1/2$, we write:

$$q_\epsilon(x) = \ln\frac{1}{\epsilon} + Q(x/\epsilon) + R_\epsilon(x),$$

where $Q(x) = \int_{\mathbb{R}} \ln\frac{1}{|x-z|}\theta(z)dz$ and $R_\epsilon(x)$ converges uniformly to 0 (for $|x| < 1/2$) as $\epsilon \to 0$. This follows from straightforward calculations. Applying lemma 4.1, we get that $Q(x) = \ln\frac{1}{|x|} + \Sigma(x)$ with $\Sigma(x) \to 0$ for $|x| \to \infty$. Thus $Q(x) - \widetilde{Q}(x)$ is a continuous function such that for $|x| \geqslant A\epsilon$ and $|x| \leqslant 1/2$ we have:

$$|q_\epsilon(x) - \widetilde{q}_\epsilon(x)| \leqslant \sup_{|y| \geqslant A} |Q(y) - \widetilde{Q}(y)| + \sup_{|x| \leqslant 1/2} |R_\epsilon(x) - \widetilde{R}_\epsilon(x)|$$

The result follows. □

One can decompose expression (4.6) in the following way:

$$E[\widetilde{m}_\epsilon[0,1]^\alpha] - E[m_\epsilon[0,1]^\alpha] = \frac{\alpha(\alpha-1)}{2}\int_0^1 \varphi_\epsilon^A(t)dt + \frac{\alpha(\alpha-1)}{2}\int_0^1 \bar{\varphi}_\epsilon^A(t)dt \tag{4.7}$$

where:

$$\varphi_\epsilon^A(t)=\int_{[0,1]^2,|t_2-t_1|\leqslant A\epsilon}(\widetilde{q}_\epsilon(|t_2-t_1|)-q_\epsilon(|t_2-t_1|)E[\mathcal{X}_\epsilon(t,t_1,t_2)]dt_1dt_2$$

and

$$\bar{\varphi}_\epsilon^A(t)=\int_{[0,1]^2,|t_2-t_1|>A\epsilon}(\widetilde{q}_\epsilon(|t_2-t_1|)-q_\epsilon(|t_2-t_1|)E[\mathcal{X}_\epsilon(t,t_1,t_2)]dt_1dt_2.$$

With the notations of lemma 4.3, we have:

$$\begin{aligned}|\bar{\varphi}_\epsilon^A(t)|&\leqslant \lambda^2C_A^\epsilon\int_{[0,1]^2,|t_2-t_1|>A\epsilon}E[\mathcal{X}_\epsilon(t,t_1,t_2)]dt_1dt_2\\&\leqslant \lambda^2C_A^\epsilon\int_{[0,1]^2}E[\mathcal{X}_\epsilon(t,t_1,t_2)]dt_1dt_2\\&=\lambda^2C_A^\epsilon E[(\int_0^1 e^{Z_\epsilon(t)(u)-\frac{1}{2}E[Z_\epsilon(t)(u)^2]}du)^\alpha]\\&\leqslant \lambda^2C_A^\epsilon.\end{aligned}$$

Thus, taking the limit as ϵ goes to 0 in (4.7) gives:

$$\overline{\lim_{\epsilon\to0}}|E[\widetilde{m}_\epsilon[0,1]^\alpha]-E[m_\epsilon[0,1]^\alpha]|\leqslant\frac{\alpha(1-\alpha)}{2}\lambda^2\overline{\lim_{\epsilon\to0}}C_A^\epsilon+\frac{\alpha(1-\alpha)}{2}\overline{\lim_{\epsilon\to0}}\int_0^1|\varphi_\epsilon^A(t)|dt$$

We will show that $\lim\limits_{\epsilon\to0}\varphi_\epsilon^A(0)=0$ (the general case $\varphi_\epsilon^A(t)$ is similar). There exists a constant $\widetilde{C}_A>0$ independent of ϵ such that:

$$\sup_{|x|\leqslant A\epsilon}|\widetilde{q}_\epsilon(x)-q_\epsilon(x)|\leqslant\widetilde{C}_A.$$

Therefore, we have:

$$\begin{aligned}|\varphi_\epsilon^A(0)|&\leqslant\widetilde{C}_A\int_0^1\int_{t_1-A\epsilon}^{t_1+A\epsilon}E[\mathcal{X}_\epsilon(0,t_1,t_2)]dt_2dt_1\\&=\widetilde{C}_AE\Big[\frac{\int_0^1\int_{t_1-A\epsilon}^{t_1+A\epsilon}e^{X_\epsilon(t_1)+X_\epsilon(t_2)-\frac{1}{2}E[X_\epsilon(t_1)^2]-\frac{1}{2}E[X_\epsilon(t_2)^2]}dt_1dt_2}{(\int_0^1e^{X_\epsilon(u)-\frac{1}{2}E[X_\epsilon(u)^2]}du)^{2-\alpha}}\Big]\end{aligned}\tag{4.8}$$

Now we have:

$$\begin{aligned}&\int_0^1\int_{t_1-A\epsilon}^{t_1+A\epsilon}e^{X_\epsilon(t_1)+X_\epsilon(t_2)-\frac{1}{2}E[X_\epsilon(t_1)^2]-\frac{1}{2}E[X_\epsilon(t_2)^2]}dt_1dt_2\\&\leqslant(\sup_{t_1}\int_{t_1-A\epsilon}^{t_1+A\epsilon}e^{X_\epsilon(t_2)-\frac{1}{2}E[X_\epsilon(t_2)^2]}dt_2)\int_0^1e^{X_\epsilon(t_1)-\frac{1}{2}E[X_\epsilon(t_1)^2]}dt_1\\&\leqslant2(\sup_{0\leqslant i<\frac{1}{2A\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon}e^{X_\epsilon(t_2)-\frac{1}{2}E[X_\epsilon(t_2)^2]}dt_2)\int_0^1e^{X_\epsilon(t_1)-\frac{1}{2}E[X_\epsilon(t_1)^2]}dt_1\end{aligned}$$

In view of (4.8), this implies:

$$\begin{aligned}|\varphi_\epsilon^A(0)| &\leqslant 2\widetilde{C}_A E\Big[\big(\sup_{0\leqslant i<\frac{1}{2A\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{X_\epsilon(t_2)-\frac{1}{2}E[X_\epsilon(t_2)^2]}dt_2\big)\big(\int_0^1 e^{X_\epsilon(t_1)-\frac{1}{2}E[X_\epsilon(t_1)^2]}dt_1\big)^{\alpha-1}\Big]\\
&\leqslant 2\widetilde{C}_A E\Big[\big(\sup_{0\leqslant i<\frac{1}{2A\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{X_\epsilon(t_2)-\frac{1}{2}E[X_\epsilon(t_2)^2]}dt_2\big)^\alpha\Big],\end{aligned}$$

where we used the inequality $\frac{\sup_i a_i}{(\sum_i a_i)^{1-\alpha}} \leqslant (\sup_i a_i)^\alpha$. For sake of simplicity, we now replace $2A$ by A.

The idea to study the above supremum is to make the approximation $X_\epsilon(t) \approx X_\epsilon(Ai\epsilon)$ for t in $[Ai\epsilon, A(i+1)\epsilon]$. If we define $\mathcal{C}_\epsilon$ by:

$$\mathcal{C}_\epsilon = \sup_{\substack{0\leqslant i<\frac{1}{A\epsilon}\\ Ai\epsilon\leqslant u\leqslant A(i+1)\epsilon}} (X_\epsilon(u)-X_\epsilon(Ai\epsilon)), \tag{4.9}$$

then we have:

$$\begin{aligned}&E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}}\int_{Ai\epsilon}^{A(i+1)\epsilon} e^{X_\epsilon(t)-\frac{1}{2}E[X_\epsilon(t)^2]}dt\big)^\alpha\Big]\\
&\leqslant E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}}\int_{Ai\epsilon}^{A(i+1)\epsilon} e^{X_\epsilon(Ai\epsilon)-\frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]}dt\big)^\alpha e^{\alpha\mathcal{C}_\epsilon}\Big]\\
&= E\Big[\big(\epsilon A\sup_{0\leqslant i<\frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon)-\frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]}\big)^\alpha e^{\alpha\mathcal{C}_\epsilon}\Big]\\
&\leqslant (\epsilon A)^\alpha E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon)-\frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]}\big)^{2\alpha}\Big]^{1/2} E\Big[e^{2\alpha\mathcal{C}_\epsilon}\Big]^{1/2}.\end{aligned} \tag{4.10}$$

It is straightforward to see that there exists some $c\geqslant 0$ (independent of ϵ) such that for all s,t in $[0,1]$:

$$E[X_\epsilon(s)X_\epsilon(t)]\geqslant -c$$

We introduce a centered gaussian random variable Z independent of X_ϵ and such that $E[Z^2]=c$. Let $(R_i^\epsilon)_{1\leqslant i<\frac{1}{A\epsilon}}$ be a sequence of i.i.d gaussian random variables such that $E[(R_i^\epsilon)^2]=E[X_\epsilon(Ai\epsilon)^2]+c$. By applying corollary 6.3, we get:

$$\begin{aligned}E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon)-\frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]}\big)^{2\alpha}\Big] &= \frac{1}{e^{2\alpha^2c-\alpha c}}E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon)+Z-\frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]-\frac{c}{2}}\big)^{2\alpha}\Big]\\
&\leqslant \frac{1}{e^{2\alpha^2c-\alpha c}}E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}} e^{R_i^\epsilon-\frac{1}{2}E[(R_i^\epsilon)^2]}\big)^{2\alpha}\Big]\end{aligned}$$

We have $E[(R_i^\epsilon)^2]=\lambda^2\ln\frac{1}{\epsilon}+C(\epsilon)$ with $C(\epsilon)$ converging to some constant as ϵ goes to 0. Since $2\alpha<1$, by appling lemma 4.2, there exists $0<x<1$ such that:

$$E\Big[\big(\sup_{0\leqslant i<\frac{1}{A\epsilon}} e^{R_i^\epsilon-\frac{1}{2}E[(R_i^\epsilon)^2]}\big)^{2\alpha}\Big]\leqslant C\big(\frac{1}{\epsilon}\big)^{2\alpha x}$$

and therefore we have:

$$|\varphi_\epsilon^A(0)| \leqslant C\epsilon^\gamma E\Big[e^{2\alpha\mathcal{C}_\epsilon}\Big]^{1/2}$$

with $\gamma = \alpha(1-x) > 0$.

One can write $\mathcal{C}_\epsilon = \sup_{\substack{0 \leqslant i < \frac{1}{A\epsilon} \\ 0 \leqslant v \leqslant 1}} W_\epsilon^i(v)$ where $W_\epsilon^i(v) = X_\epsilon(Ai\epsilon + A\epsilon v) - X_\epsilon(Ai\epsilon)$. We have:

$$E[W_\epsilon^i(v)W_\epsilon^i(v')] = g_\epsilon(v-v')$$

where g_ϵ is a continuous function bounded by M independently of ϵ. Let Y be a centered Gaussian random variable independent of W_ϵ^i such that: $E[Y^2] = M$. Thus, we can write:

$$E\Big[e^{2\alpha\mathcal{C}_\epsilon}\Big] = \frac{E\Big[e^{2\alpha\sup\limits_{i,v} W_\epsilon^i(v)}\Big]}{e^{2\alpha^2M^2}}.$$

Now let us consider a family $(\overline{W}_\epsilon^i)_{1 \leqslant i < \frac{1}{A\epsilon}}$ of centered i.i.d. Gaussian processes of law $(W_\epsilon^0(v) + Y)_{0 \leqslant v \leqslant 1}$. Applying corollary 6.3 of the appendix, we get:

$$E\Big[e^{2\alpha\mathcal{C}_\epsilon}\Big] \leqslant \frac{E\Big[e^{2\alpha\sup\limits_{i,v} \overline{W}_\epsilon^i(v)}\Big]}{e^{2\alpha^2M^2}}$$

We now estimate $E\Big[e^{2\alpha\sup\limits_{i,v} \overline{W}_\epsilon^i(v)}\Big]$. Let us denote $\mathcal{X}_i = \sup\limits_{0 \leqslant v \leqslant 1} \overline{W}_\epsilon^i$. Applying Corollary 3.2 of [14] to the continuous gaussian process $(W_\epsilon^0(v) + Y)_{0 \leqslant v \leqslant 1}$, we get that the random variable has a Gaussian tail:

$$P(\mathcal{X}_i > z) \leqslant Ce^{-\frac{z^2}{2\sigma^2}}, \quad \forall z > 0$$

for some C and σ. The above tail inequality gives the existence of some constant $C > 0$ such that:

$$E\Big[e^{2\alpha \sup\limits_{0 \leqslant i < \frac{1}{A\epsilon}} \mathcal{X}_i}\Big] \leqslant Ce^{C\sqrt{\ln(\frac{1}{\epsilon})}}.$$

Therefore we have $E\Big[e^{2\alpha\mathcal{C}_\epsilon}\Big] \leqslant Ce^{C\sqrt{\ln(\frac{1}{\epsilon})}}$ and then:

$$|\varphi_\epsilon^A(0)| \leqslant C\epsilon^\gamma e^{C\sqrt{\ln(\frac{1}{\epsilon})}}.$$

It follows that $\overline{\lim\limits_{\epsilon\to 0}}|\varphi_\epsilon^A(0)| = 0$ so that for $\alpha < 1/2$:

$$\overline{\lim_{\epsilon\to 0}}|E[\widetilde{m}_\epsilon[0,1]^\alpha] - E[m_\epsilon[0,1]^\alpha]| \leqslant \frac{\alpha(1-\alpha)}{2}\lambda^2\overline{\lim_{\epsilon\to 0}}C_A^\epsilon.$$

Since $\overline{\lim\limits_{\epsilon\to 0}}C_A^\epsilon \to 0$ as A goes to infinity (lemma 4.3), we conclude that:

$$\overline{\lim_{\epsilon\to 0}}|E[\widetilde{m}_\epsilon[0,1]^\alpha] - E[m_\epsilon[0,1]^\alpha]| = 0.$$

It is straightforward to check that the above proof can be generalized to show that for all positive $\lambda_1,\dots,\lambda_n$ and intervals $I_1,\dots,I_n$ we have:

$$E[(\sum_{k=1}^{n}\lambda_k m(I_k))^{\alpha}]=E[(\sum_{k=1}^{n}\lambda_k \widetilde{m}(I_k))^{\alpha}]$$

This implies that the random measures m and $\widetilde{m}$ are equal (see [7]).

Existence. Let $f(x)$ be a real positive definite function on $\mathbb{R}^d$ (note that this implies that f is symmetric). Let us recall that the centered Gaussian field of correlation $f(x-y)$ is given by:

$$X(x)=\int_{\mathbb{R}^d}\zeta(x,\xi)\sqrt{\hat{f}(\xi)}W(d\xi),$$

where $\zeta(x,\xi)=\cos(2\pi x.\xi)-\sin(2\pi x.\xi)$ and $W(d\xi)$ is the standard white noise on $\mathbb{R}^d$. This can also be written:

$$X(x)=\int_{]0,\infty[\times\mathbb{R}^d}\zeta(x,\xi)\sqrt{\hat{f}(\xi)}g(t,\xi)W(dt,d\xi),\tag{4.11}$$

where $W(dt,d\xi)$ is the white noise on $]0,\infty[\times\mathbb{R}^d$ and $\int_0^\infty g(t,\xi)^2dt=1$ for all ξ. The interest of the expression (4.11) appears in what follows. Let the function θ be radially symmetric and $\hat{\theta}$ be a decreasing function of $|\xi|$ (for instance take $\theta(x)=\frac{e^{-|x|^2/2}}{(2\pi)^{d/2}}$). Let us consider $g(t,\xi)=\sqrt{-\hat{\theta}'(t|\xi|)|\xi|}$ so that $\int_\epsilon^\infty g(t,\xi)^2dt=\hat{\theta}(\epsilon|\xi|)$ for $|\xi|\neq 0$. Then if we consider the fields X_ϵ defined by:

$$X_\epsilon(x)=\int_{]\epsilon,\infty[\times\mathbb{R}^d}\zeta(x,\xi)\sqrt{\hat{f}(\xi)}g(t,\xi)W(dt,d\xi)\tag{4.12}$$

we will find:

$$\begin{aligned}E[X_\epsilon(x)X_\epsilon(y)]&=\int_{\mathbb{R}^d}\cos(2\pi(x-y).\xi)\hat{f}(\xi)\hat{\theta}(\epsilon|\xi|)d\xi\\&=(f*\theta^\epsilon)(x-y).\end{aligned}$$

The interest of (4.12) is to make the approximation process appear as a martingale. Indeed, if we define the filtration $\mathcal{F}_\epsilon=\sigma\{W(A,x),A\subset]\epsilon,\infty[,x\in\mathbb{R}^d\}$, we have that for all $A\in\mathcal{B}(\mathbb{R}^d)$, $(m_\epsilon(A))_{\epsilon>0}$ is a positive $\mathcal{F}_\epsilon$-martingale of expectation $|A|$ so it converges almost surely to a random variable $m(A)$ such that:

$$E[m(A)]\leqslant|A|.\tag{4.13}$$

This defines a collection $(m(A))_{A\in\mathcal{B}(\mathbb{R}^d)}$ of random variables such that:

(1) for all disjoint and bounded sets A_1,A_2 in $\mathcal{B}(\mathbb{R}^d)$,

$$m(A_1\cup A_2)=m(A_1)+m(A_2)\quad a.s.$$

(2) For any bounded sequence $(A_n)_{n\geqslant 1}$ decreasing to $\emptyset$:

$$m(A_n)\underset{n\to\infty}{\longrightarrow}0\quad a.s.$$

By theorem 6.1. VI. in [7], one can consider a version of the collection $(m(A))_{A\in\mathcal{B}(\mathbb{R}^d)}$ such that m is a random measure. It is straightforward that m_ϵ converges almost surely towards m in the space of Radon measures (equiped with the weak topology).

5. Proofs of section 3

5.1. **Proof of proposition 3.1.**

Proof. Since $\zeta_1 = d$, note that $\lambda^2 > 2d$ is equivalent to the existence of $\alpha < 1$ such that $\zeta_\alpha > d$. Let α be fixed and such that $\zeta_\alpha > d$. We will show that $m[[0,1]^d] = 0$. We partition the cube $[0,1]^d$ into $\frac{1}{\epsilon^d}$ subcubes $(I_j)_{1\leqslant j\leqslant \frac{1}{\epsilon^d}}$ of size ϵ. One has by subadditivity and homogeneity:

$$\begin{aligned} &E[(\int_{[0,1]^d} e^{X_\epsilon(x)-\frac{1}{2}E[X_\epsilon(x)^2]}dx)^\alpha] \\ &= E[(\sum_{1\leqslant j\leqslant \frac{1}{\epsilon^d}} \int_{I_j} e^{X_\epsilon(x)-\frac{1}{2}E[X_\epsilon(x)^2]}dx)^\alpha] \\ &\leqslant E[\sum_{1\leqslant j\leqslant \frac{1}{\epsilon^d}} (\int_{I_j} e^{X_\epsilon(x)-\frac{1}{2}E[X_\epsilon(x)^2]}dx)^\alpha] \\ &= \frac{1}{\epsilon^d} E[(\int_{[0,\epsilon]^d} e^{X_\epsilon(x)-\frac{1}{2}E[X_\epsilon(x)^2]}dx)^\alpha] \end{aligned}$$

Let Y_ϵ be a centered gaussian random variable of variance $\lambda^2\ln(\frac{1}{\epsilon}) + \lambda^2 c$ where c is such that:

$$\theta^\epsilon * \ln^+\frac{1}{|x|} \geqslant \ln\frac{1}{\epsilon} + c$$

for $|x|\leqslant\epsilon$ and ϵ small enough. By definition of c, we have

$$\forall t,t'\in[0,\epsilon],\quad E[X_\epsilon(t)X_\epsilon(t')] \geqslant E[Y_\epsilon^2].$$

Using corollary (6.2) in a continuous version, this implies:

$$\begin{aligned} &E[(\int_{[0,1]^d} e^{X_\epsilon(t)-\frac{1}{2}E[X_\epsilon(t)^2]}dt)^\alpha] \\ &\leqslant \frac{1}{\epsilon^d} E[(\int_{[0,\epsilon]^d} e^{Y_\epsilon-\frac{1}{2}E[Y_\epsilon^2]}dt)^\alpha] \\ &= e^{\frac{\alpha^2-\alpha}{2}c}\epsilon^{\zeta_\alpha - d} \end{aligned}$$

Taking the limit as ϵ goes to 0 gives $m[[0,1]^d]=0$. □

5.2. **Proof of lemma 3.2.**

Proof. One has the following general formula for the Fourier transform of radial functions:

$$\hat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}}\int_0^\infty \rho^{\frac{d}{2}} J_{\frac{d-2}{2}}(2\pi|\xi|\rho) f(\rho)d\rho, \tag{5.1}$$

where J_ν is the Bessel function of order ν.

First case: $d \leqslant 3$.

It is enough to consider the case $d = 3$ (Indeed, this implies that the same holds for smaller dimensions). Using the explicit formula $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$, we conclude by integrating by parts:

$$\begin{aligned}
\hat{f}(\xi) &= \frac{2}{|\xi|}\int_0^T \rho \sin(2\pi|\xi|\rho)\ln(\frac{T}{\rho})d\rho \\
&= \frac{1}{\pi|\xi|^2}\int_0^T \cos(2\pi|\xi|\rho)(\ln(\frac{T}{\rho})-1)d\rho \\
&= \frac{1}{2\pi^2|\xi|^3}(\int_0^T \frac{\sin(2\pi|\xi|\rho)}{\rho}d\rho - \sin(2\pi|\xi|T)) \\
&= \frac{1}{2\pi^2|\xi|^3}(\mathrm{sinc}(2\pi|\xi|T) - \sin(2\pi|\xi|T)),
\end{aligned}$$

where sinc is the sinus cardinal function:

$$\mathrm{sinc}(x) = \int_0^x \frac{\sin(\rho)}{\rho}d\rho.$$

We introduce for $x \geqslant 0$ the function $l(x) = \mathrm{sinc}(x) - \sin(x)$. We have $l'(x) = \frac{\sin(x) - x\cos(x)}{x}$. Thus, there exists α in $]\pi, 2\pi[$ such that l is increasing on $]0,\alpha[$ and decreasing on $]\alpha, 2\pi[$. Since $l(0) = 0$ and $l(2\pi) = \int_0^{2\pi}\frac{\sin(\rho)}{\rho}d\rho \geqslant 0$, we conclude that for all x in $[0, 2\pi]$, $l(x) \geqslant 0$. A classical computation (Dirichlet integral) gives $\int_0^\infty \frac{\sin(\rho)}{\rho}d\rho = \frac{\pi}{2}$. Thus, we have by an integration by parts:

$$\begin{aligned}
\int_0^{2\pi}\frac{\sin(\rho)}{\rho}d\rho &= \frac{\pi}{2} - \int_{2\pi}^\infty \frac{\sin(\rho)}{\rho}d\rho \\
&= \frac{\pi}{2} - \int_{2\pi}^\infty \frac{1-\cos(\rho)}{\rho^2}d\rho \\
&\geqslant \frac{\pi}{2} - \frac{1}{2\pi} \\
&\geqslant 1
\end{aligned}$$

Therefore, if $x \geqslant 2\pi$, we have:

$$\begin{aligned}
l(x) &= \int_0^x \frac{\sin(\rho)}{\rho}d\rho - \sin(x) \\
&\geqslant \int_0^{2\pi}\frac{\sin(\rho)}{\rho}d\rho - \sin(x) \\
&\geqslant 0.
\end{aligned}$$

Second case: $d \geqslant 4$. Combining (5.1) with the identity $\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$, we get:

$$\begin{aligned}\hat{f}(\xi) &= \frac{2\pi}{|\xi|^{\frac{d-2}{2}}}\int_0^T \rho^{\frac{d}{2}} J_{\frac{d-2}{2}}(2\pi|\xi|\rho)\ln(\frac{T}{\rho})d\rho \\ &= \frac{1}{(2\pi)^{d/2}|\xi|^d}\int_0^{2\pi|\xi|T} x^{\frac{d}{2}} J_{\frac{d-2}{2}}(x)\ln(\frac{2\pi|\xi|T}{x})dx \\ &= \frac{1}{(2\pi)^{d/2}|\xi|^d}\int_0^{2\pi|\xi|T} x^{\frac{d}{2}-1} J_{\frac{d}{2}}(x)dx \qquad (5.2)\end{aligned}$$

One has the following asymptotic expansion as x goes to ∞ ([11]):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos(x - \frac{(1+2\nu)\pi}{4}) - \frac{(4\nu^2-1)\sqrt{2}}{8\sqrt{\pi}x^{3/2}}\sin(x - \frac{(1+2\nu)\pi}{4}) + O(\frac{1}{x^{5/2}}). \quad (5.3)$$

Combining (5.2) with (5.3), we therefore get the following expansion as $|\xi|$ goes to infinity:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}|\xi|^d}\left(\sqrt{\frac{2}{\pi}}(2\pi|\xi|T)^{\frac{d-3}{2}}\sin(2\pi|\xi|T - \frac{(1+2\nu)\pi}{4}) + o(|\xi|^{\frac{d-3}{2}})\right).$$

Thus $\overline{\lim}_{|\xi|\to\infty} |\xi|^d \hat{f}(\xi) = -\underline{\lim}_{|\xi|\to\infty} |\xi|^d \hat{f}(\xi) = +\infty$. □

5.3. **Proofs of section 3.3.**

Proof of proposition 3.5 and proposition 3.6. Let θ be some function satisfying the conditions (1), (2), (3) of section 2.2 and m_ϵ be the random measure associated to $\theta^\epsilon * f$. We consider $\widetilde{m}_\epsilon$ the random measure associated to $\widetilde{f}_\epsilon$ where $\widetilde{f}_\epsilon$ is the function of example 2.2:

$$\widetilde{f}_\epsilon(x) = \lambda^2 \int_{C(0)\cap C(x);\ \epsilon<t<\infty} \frac{dydt}{t^{d+1}}.$$

One can show that there exists $c, C > 0$ such that for all x we have:

$$\widetilde{f}_\epsilon(x) - c \leqslant (\theta^\epsilon * f)(x) \leqslant \widetilde{f}_\epsilon(x) + C$$

By using corollary 6.2 of the appendix in a continuous version, we conclude that there exists $c, C > 0$ such that for all ϵ and all bounded A in $\mathcal{B}(\mathbb{R}^d)$:

$$cE[\widetilde{m}_\epsilon(A)^p] \leqslant E[m_\epsilon(A)^p] \leqslant CE[\widetilde{m}_\epsilon(A)^p].$$

First case: p belongs to $]0, p_*[$.

Proposition 3.5 is therefore established if we can show that:

$$\sup_{\epsilon>0} E[\widetilde{m}_\epsilon(A)^p] < \infty.$$

The above bound can be proved by adapting the proof of theorem 3 in [3] .

Second case: p belongs to $]-\infty, 0[$.

Proposition 3.5 is therefore established if we can show that for all $c > 0$:

$$\sup_{\epsilon>0} E[\widetilde{m}_\epsilon(B(0,c))^p] < \infty.$$

The above bound can be proved by adapting the proof of the corresponding result in [16].

Proof of proposition 3.7. For the sake of simplicity, we consider the case $R = 1$ and we will consider the case $p \in [1, p_*[$. We consider θ a continuous and positive function with compact support $B(0, A)$ satisfying properties (1), (2) and (3) of section 2.2. We note:

$$m_\epsilon(dx) = e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]}dx,$$

where $(X_\epsilon(x))_{x\in\mathbb{R}^d}$ is a gaussian field of covariance $q_\epsilon(x-y)$ with:

$$q_\epsilon(x) = (\theta^\epsilon * f)(x) = \int_{\mathbb{R}^d} \theta(z)(\lambda^2 \ln^+ \frac{1}{|x-\epsilon z|} + g(x - \epsilon z))dz.$$

Let c, c' be two positive numbers in $]0, \frac{1}{2}[$ such that $c < c'$. If ϵ is sufficiently small and u, v belong to $[0,1]^d$, we get:

$$\begin{aligned}
q_{c\epsilon}(c(v-u)) &= \int_{\mathbb{R}^d} \theta(z) \left(\lambda^2 \ln \frac{1}{|c(v-u) - c\epsilon z|} + g(c(v-u) - c\epsilon z)\right) dz \\
&= \lambda^2 \ln(\frac{c'}{c}) + \int_{\mathbb{R}^d} \theta(z) \left(\lambda^2 \ln \frac{1}{|c'(v-u) - c'\epsilon z|} + g(c(v-u) - c\epsilon z)\right) dz \\
&\leqslant \lambda^2 \ln(\frac{c'}{c}) + q_{c'\epsilon}(c'(v-u)) + C_{c,c',\epsilon},
\end{aligned}$$

where

$$C_{c,c',\epsilon} = \sup_{\substack{|z| \leqslant A \\ |v-u| \leqslant 1}} |g(c(v-u) - c\epsilon z) - g(c'(v-u) - c'\epsilon z)|.$$

Let $Y_{c,c',\epsilon}$ be some centered gaussian variable with variance $C_{c,c',\epsilon} + \lambda^2 \ln(\frac{c'}{c})$. By using corollary 6.2 of the appendix in a continuous version, we conclude that:

$$\begin{aligned}
E[m_{c\epsilon}([0,c]^d)^p] &= E[(\int_{[0,c]^d} e^{X_{c\epsilon}(x) - \frac{1}{2}E[X_{c\epsilon}(x)^2]}dx)^p] \\
&= c^{dp} E[(\int_{[0,1]^d} e^{X_{c\epsilon}(cu) - \frac{1}{2}E[X_{c\epsilon}(cu)^2]}du)^p] \\
&\leqslant c^{dp} E[(\int_{[0,1]^d} e^{X_{c'\epsilon}(c'u) + Y_{c,c',\epsilon} - \frac{1}{2}E[(X_{c'\epsilon}(c'u) + Y_{c,c',\epsilon})^2]}du)^p] \\
&= c^{dp} (\frac{c'}{c})^{\frac{p(p-1)\lambda^2}{2}} e^{\frac{p(p-1)C_{c,c',\epsilon}}{2}} E[(\int_{[0,1]^d} e^{X_{c'\epsilon}(c'u) - \frac{1}{2}E[X_{c'\epsilon}(c'u)^2]}du)^p] \\
&= (\frac{c}{c'})^{dp - \frac{p(p-1)\lambda^2}{2}} e^{\frac{p(p-1)C_{c,c',\epsilon}}{2}} E[(\int_{[0,c']^d} e^{X_{c'\epsilon}(x) - \frac{1}{2}E[X_{c'\epsilon}(x)^2]}dx) \\
&= (\frac{c}{c'})^{\zeta_p} e^{\frac{p(p-1)C_{c,c',\epsilon}}{2}} E[m_{c'\epsilon}([0,c']^d)^p]
\end{aligned}$$

Taking the limit $\epsilon \to 0$ in the above inequality leads to:

$$\frac{E[m([0,c]^d)^p]}{c^{\zeta_p}} \leqslant e^{\frac{p(p-1)C_{c,c'}}{2}} \frac{E[m([0,c']^d)^p]}{c'^{\zeta_p}}, \tag{5.4}$$

where $C_{c,c'} = \sup_{|v-u| \leqslant 1} |g(c(v-u)) - g(c'(v-u))|$. Similarly, we have:

$$\frac{E[m([0,c']^d)^p]}{c'^{\zeta_p}} \leqslant e^{\frac{p(p-1)C_{c,c'}}{2}} \frac{E[m([0,c]^d)^p]}{c^{\zeta_p}}. \tag{5.5}$$

Since $C_{c,c'}$ goes to 0 as $c, c' \to 0$,we conclude by inequality (5.4) and (5.5) that $(\frac{E[m([0,c]^d)^p]}{c^{\zeta_p}})_{c>0}$ is a Cauchy sequence as $c \to 0$ bounded from below and above by positive constants. Therefore, there exists some $c_p > 0$ such that:

$$E[m([0,c]^d)^p] \underset{c\to 0}{\sim} c_p c^{\zeta_p}.$$

The same method can be applied to show that $\frac{c_p}{e^{\frac{p(p-1)g(0)}{2}}}$ is independent of g. This concludes the proof by setting $C_p = \frac{c_p}{e^{\frac{p(p-1)g(0)}{2}}}$.

Proof of proposition 3.8. We use the scaling relation (3.3) to compute the characteristic function of $m(B(0,c))$ for all ξ in $\mathbb{R}$:

$$\begin{aligned} E[e^{i\xi m(B(0,c))}] &= E[e^{i\xi e^{\Omega_c} m(B(0,R))}] \\ &= E[\mathcal{F}(\xi m(B(0,R)))], \end{aligned}$$

where $\mathcal{F}$ is the characteristic function of e^{Ω_c}. It is easy to show that for all $n \in \mathbb{N}$ there exists $C > 0$ such that:

$$|\mathcal{F}(\xi)| \leqslant \frac{C}{|\xi|^n}.$$

From this, we conclude by proposition 3.6 that:

$$E[e^{i\xi m(B(0,c))}] \leqslant \frac{C}{|\xi|^n} E[\frac{1}{m(B(0,R))^n}] \leqslant \frac{C'}{|\xi|^n}.$$

This implies the existence of a C^∞ density.

6. Appendix

We give the following classical lemma first derived in [12].

Lemma 6.1. *Let $(X_i)_{1 \leqslant i \leqslant n}$ and $(Y_i)_{1 \leqslant i \leqslant n}$ be two independent centered gaussian vectors and $(p_i)_{1 \leqslant i \leqslant n}$ a sequence of positive numbers. If $\phi : \mathbb{R}_+ \to \mathbb{R}$ is some smooth function with polynomial growth at infinity, we define:*

$$\varphi(t) = E[\phi(\sum_{i=1}^n p_i e^{Z_i(t) - \frac{1}{2}E[Z_i(t)^2]})],$$

with $Z_i(t)=\sqrt{t}X_i+\sqrt{1-t}Y_i$. Then, we have the following formula for the derivative:

$$\varphi'(t)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j(E[X_iX_j]-E[Y_iY_j])E[e^{Z_i(t)+Z_j(t)-\frac{1}{2}E[Z_i(t)^2]-\frac{1}{2}E[Z_j(t)^2]}\phi''(W_{n,t})], \tag{6.1}$$

where:

$$W_{n,t}=\sum_{k=1}^{n}p_ke^{Z_k(t)-\frac{1}{2}E[Z_k(t)^2]}$$

As a consequence of the above lemma, one can derive the following classical comparaison principle:

Corollary 6.2. *Let $(p_i)_{1\leqslant i\leqslant n}$ be a sequence of positive numbers. Consider $(X_i)_{1\leqslant i\leqslant n}$ and $(Y_i)_{1\leqslant i\leqslant n}$ two centered gaussian vectors such that:*

$$\forall i,j \quad E[X_iX_j]\leqslant E[Y_iY_j].$$

Then, for all convex function $F:\mathbb{R}\to\mathbb{R}_+$, we have:

$$E[F(\sum_{i=1}^{n}p_ie^{X_i-\frac{1}{2}E[X_i^2]})],\ \leqslant E[F(\sum_{i=1}^{n}p_ie^{Y_i-\frac{1}{2}E[Y_i^2]})]. \tag{6.2}$$

We will also use another corollary:

Corollary 6.3. *Let $(X_i)_{1\leqslant i\leqslant n}$ and $(Y_i)_{1\leqslant i\leqslant n}$ be two centered gaussian vectors such that:*

- $\forall i, \qquad E[X_i^2]=E[Y_i^2]$.
- $\forall i\neq j, \quad E[X_iX_j]\leqslant E[Y_iY_j]$.

Then, for all increasing function $F:\mathbb{R}\to\mathbb{R}_+$, we have:

$$E[F(\sup_{1\leqslant i\leqslant n}Y_i)]\leqslant E[F(\sup_{1\leqslant i\leqslant n}X_i)]. \tag{6.3}$$

Proof. It is enough to show inequality (6.3) for $F=1_{]x,+\infty[}$ for some $x\in\mathbb{R}$. Let β be some positive parameter. Integrating equality (6.1) applied to the convex function $\phi:u\to e^{-e^{-\beta x}u}$ and the sequences (βX_i), (βY_i), $p_i=e^{\frac{\beta^2}{2}E[X_i^2]}$, we get:

$$E[e^{-\sum_{i=1}^{n}e^{\beta(X_i-x)}}]\leqslant E[e^{-\sum_{i=1}^{n}e^{\beta(Y_i-x)}}]$$

By letting $\beta\to\infty$, we conclude:

$$P(\sup_{1\leqslant i\leqslant n}X_i<x)\leqslant P(\sup_{1\leqslant i\leqslant n}Y_i<x).$$

□

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