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P-SPECTRUM AND COLLAPSING OF CONNECTED SUMS CALCULUS OF THE LIMIT

COLETTE ANNÉ AND JUNYA TAKAHASHI

ABSTRACT. The goal of the following is to calculate the limit spectrum of the Hodge-Laplace operator under the perturbation of collapse of one part of a connected sum. This gives some new results concerning the 'conformal spectrum' on differential forms.

1. INTRODUCTION

The second author has shown in [T02] that if the compact manifold M , of dimension m is the connected sum of M_1 and M_2 around the common point p_0 , endowed with riemannian metrics g_1, g_2 , then, in the situation of collapse of one part of the connected sum :

$$M(\varepsilon) = (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$$

the spectrum of the Laplace-Beltrami operator acting on functions converges, as ε goes to 0, to the spectrum of the Laplace-Beltrami operator on M_1 .

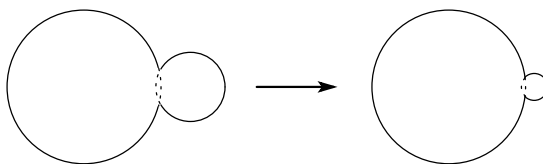


Figure 1: collapsing of $M(\varepsilon)$

To make this construction, we can suppose that the two metrics are flat around the point p_0 , then the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric, and can be identified. One can then define geometrically $M(\varepsilon)$ as a riemannian manifold C^∞ by part.

Later on Colbois and El Soufi have introduced in [CE03] the notion of 'conformal spectrum' as the supremum, for each integer k , of the value of the k^{th} eigenvalue on a conformal class of metrics with fixed volume. Using the result of [T02] they could show that the conformal spectrum of a compact manifold is

always bounded from below by the conformal spectrum of the standart sphere of same dimension.

It is then natural to ask whether such a result holds also for differential forms. In fact Colbois and El Soufi have shown in [CE06] that, in dimension $m \geq 4$, the positive conformal spectrum with fixed volume, on p-forms, $2 \leq p \leq m - 2$, is unbounded. We study here the limit spectrum of the Hodge-Laplace operator under the perturbation of collapse of one part of a connected sum. This study has been started in [T03].

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1.1. The results. As before the manifold M , of dimension $m \geq 3$ (there is no problem in dimension 2), is the connected sum of two riemannian manifolds (M_1, g_1) and (M_2, g_2) around the commun point p_0 , and we suppose that the metrics are such that the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric for all ε small enough. As a consequence, (M_1, g_1) is flat in a neighbourhood of p_0 and $\partial(M_2 - B(p_0, 1))$ is a standard sphere. Indeed one can write $g_1 = dr^2 + r^2 h(r)$ in polar coordinates around $p_0 \in M_1$ and the metric $h(r)$ on the sphere converges, as $r \rightarrow 0$, to the standard metric. But if the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric for all ε small enough, then $h(r)$ is constant for r small enough. The conclusion follows.

One can then define geometrically $M(\varepsilon) = (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$ as the connected sum obtained by the collapse of M_2 (the question of the metric on $M(\varepsilon)$ is discusted below). On such a manifold, a Gauß-Bonnet operator D_ε , Sobolev spaces and also a Hodge-Laplace operator Δ_ε can be defined as follows (the details are given in [AC95]): on a manifold $X = X_1 \cup X_2$, which is the union of two riemannian manifolds with isometric boundaries, if D_1 and D_2 are the Gauß-Bonnet “ $d + d^*$ ” operators acting on the differential forms of each part, the quadratic form $q(\varphi) = \int_{X_1} |D_1(\varphi|_{X_1})|^2 + \int_{X_2} |D_2(\varphi|_{X_2})|^2$ is well defined and closed on the domain

$$\mathcal{D}(q) = \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2), (\varphi_1)|_{\partial X_1} \stackrel{L_2}{=} (\varphi_2)|_{\partial X_2}\}$$

and on this space the total Gauß-Bonnet operator $D(\varphi) = (D_1(\varphi_1), D_2(\varphi_2))$ is defined and selfadjoint. For this definition, we have, in particular, to identify $(\Lambda T^* X_1)|_{\partial X_1}$ and $(\Lambda T^* X_2)|_{\partial X_2}$. This can be done by decomposing the forms in tangential and normal part (with inner normal), the equality above means then that the tangential parts are equal and the normal part opposite. This definition generalizes the definition in the smooth case.

The Hodge-de Rham operator $(d + d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q . This gives compatibility conditions between φ_1 and φ_2 on the commun boundary. We don't give details

on these facts because, as remarked in the next section, it is sufficient to work with smooth metrics on M .

The multiplicity of 0 in the spectrum of Δ_ε is given by the cohomology, it is then independent on ε and can be related on the cohomology of each part by the Mayer-Vietoris argument. The point is to study the convergence of the other eigenvalues, the so-called *positive spectrum*, as $\varepsilon \rightarrow 0$. The second author has shown in [T03] a result of boundedness

Proposition A (Takahashi). *The limit sup of the k^{th} positive eigenvalue on p -forms of $M(\varepsilon)$ is bounded, as $\varepsilon \rightarrow 0$, by the k^{th} positive eigenvalue on p -forms of M_1 .*

We show here that it is also true for the lower bound. The difficulty is to identify a good limit problem, namely a good boundary condition for $M_2 - B(p_0, 1)$ at the limit. It appears that this condition is of the type of Atiyah-Patodi-Singer (APS) as defined in [APS75].

Let φ_ε be a family of eigenforms on $M(\varepsilon)$ of degree p for the Hodge-Laplace operator:

$$\Delta_\varepsilon \varphi_\varepsilon = \lambda_\varepsilon \varphi_\varepsilon ; \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lambda < +\infty.$$

Proposition B. *If $\lambda_\varepsilon \neq 0$ then $\lambda \neq 0$ and λ belongs to the spectrum of the Hodge-Laplace operator of M_1 .*

The first point is a consequence of the application of the so called McGowan's lemma; indeed $M(\varepsilon)$ has no small eigenvalues as is shown in Proposition 1 below. To prove the convergent part of the proposition we shall decompose the eigenforms using the good control of the APS-boundary term. More precisely there exists an elliptic extension \mathcal{D}_2 of the Gauß-Bonnet operator D_2 on $M_2(1) = M_2 - B(p_0, 1)$ and a family ψ_ε bounded in $H^1(M_1) \times \text{dom}(\mathcal{D}_2)$ such that $\|\varphi_\varepsilon - \psi_\varepsilon\| \rightarrow 0$ with ε .

If we make this construction for an orthonormal family of the k first eigenforms we obtain, with the help of Proposition A, the full theorem

Theorem C. *Let $M(\varepsilon) = (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$ be the connected sum of the two riemannian manifolds M_1 and $\varepsilon.M_2$ of dimension $m = n + 1$. For $p \in \{1, \dots, n\}$ let $0 < \lambda_1^p \leq \lambda_2^p, \dots$ be the positive spectrum of the Hodge-Laplace operator on the p -forms of M_1 and $0 < \lambda_1^p(\varepsilon) \leq \lambda_2^p(\varepsilon), \dots$ the positive spectrum of the Hodge-Laplace operator on the p -forms of $M(\varepsilon)$, then*

$$\forall k \geq 1, \quad \lim_{\varepsilon \rightarrow 0} \lambda_k^p(\varepsilon) = \lambda_k^p.$$

More over the multiplicity of 0 is given by the cohomology and

$$H^p(M(\varepsilon)) \simeq H^p(M_1) \oplus H^p(M_2).$$

Remark 1. A. *The result of convergence of the positive spectrum is also true for $p = 0$ and has been shown in [T02]. Naturally $H^0(M(\varepsilon)) \simeq H^0(M_1) = \mathbb{R}$. By Hodge-duality this solves also the case $p = m$.*

Applying the Theorem C to the case $M_1 = \mathbb{S}^m$ and $M_2 = M$, we obtain

Corollary D. *Let (M, g) be a compact riemannian manifold of dimension m , for any degree p , any integer $N \geq 1$ and any $\varepsilon > 0$ there exists on M a metric \bar{g} conformal to g such that the N first positive eigenvalues on the p -forms are ε -close to the N first positive eigenvalues on the p -forms of the standard sphere with same dimension and same volume as (M, g) .*

Remark 1.B. *For the completion of the panorama let's recall that Pierre Jammes has shown, in [J07], that in dimension $m \geq 4$ the infimum of the p -spectrum in a conformal class, with fixed volume, is 0 for $2 \leq p \leq m - 2$ and $p \neq \frac{m}{2}$ but has a positive minoration for $p = \frac{m}{2}$.*

We now proceed to the proof of the theorems. Let's first precise the metrics.

1.2. Choice of the metric. From now on, we denote

$$M_2(1) = M_2 - B(p_0, 1).$$

It is supposed here that the ball $B(p_0, 1)$ can really be inbedded in the manifold M_2 , this can always be satisfied by a scaling of the metric g_2 on M_2 .

Recall that Dodziuk has proved in [D82, Prop. 3.3] that if two metrics g, \bar{g} on the same compact manifold satisfy

$$e^{-\eta}g \leq \bar{g} \leq e^{\eta}g. \quad (1)$$

then, the corresponding eigenvalues of the Hodge-Laplace operator acting on p -forms satisfy

$$e^{-(n+2p)\eta}\lambda_k^p(g) \leq \lambda_k^p(\bar{g}) \leq e^{(n+2p)\eta}\lambda_k^p(g).$$

This result is based on the fact that the multiplicity of 0 is given by the cohomology and the positive spectrum by exact forms, hence the minimax formula does not involve derivatives of the metric; it stays valid if one of the two metric is only smooth by part, because in this last case the Hodge decomposition is still true.

Then, for a metric g_1 on M_1 there exists, for each $\eta > 0$ a metric \bar{g}_1 on M_1 which is flat on a ball B_η centered in p_0 and such that

$$e^{-\eta}g_1 \leq \bar{g}_1 \leq e^{\eta}g_1.$$

Then our result can be extended to any other construction which does not suppose that the metric g_1 is flat in a neighbourhood of p_0 .

Now, we regard $M(\varepsilon)$ as the union of $M_1 - B(p_0, 3\varepsilon)$ and $\varepsilon \cdot \bar{M}_2(1)$ where $\bar{M}_2(1) = \left(B_{\mathbb{R}^m}(0, 3) - B_{\mathbb{R}^m}(0, 1) \right) \cup M_2(1)$ is endowed with a metric only smooth by part: the euclidean metric on the first part and the restriction of g_2 on the second part, but this metric can be approached, as close as we want, by a smooth metric which is still flat on $B_{\mathbb{R}^m}(0, 3) - B_{\mathbb{R}^m}(0, 3/2)$ and these two metrics will satisfy the estimate (1). Thus, replacing 3ε by ε for simplicity, we can suppose, without loss of generality, that we are in the following situation:

$M_2(1)$ is endowed with a metric which is conical (flat) near the boundary, namely $g_2 = ds^2 + (1-s)^2h$, h being the canonical metric of the sphere $\mathbb{S}^n = \partial(M_2(1))$, and $s \in [0, 1/2[$ being the distance to the boundary ($M_2(1)$ looks like a trumpet) and $M_1(\varepsilon) = M_1 - B(p_0, \varepsilon)$ with a conical metric $g_1 = dr^2 + r^2h$ around the point p_0 . Thus, $M(\varepsilon) = M_1(\varepsilon) \cup \varepsilon.M_2(1)$ is smooth.

Let $\mathcal{C}_{a,b}$ be the cone $]a, b[\times \mathbb{S}^n$ endowed with the (conical) metric $dr^2 + r^2h$.

Remark 1.C. *The idea of this work is rather close to [Row06], see also [Maz06], in spite of different geometric approach: from the point of view of the shrunk manifold $M_2(1)$, ie. by a blow up of this part, the situation converges to the adjonction of an euclidean space to the boundary (more precisely of the exterior of a ball) and this gives the intuition of the limit problem for this part, see subsection 3.5 below. In [Row06] the study is also made by blowing up a neighbourhood of the singularity and considering an infinite cone. We insist that in fact our result is not included in these refered papers. Indeed, [Row06] is only concerned by the scalar Laplacian and in the 'spectral convergence theorem' of [Maz06], it is supposed that 0 is not in the spectrum of the operator L_Z which corresponds, in our situation, to the hypothesis that the non compact manifold \widetilde{M}_2 does not admit L_2 -harmonic forms.*

2. SMALL EIGENVALUES

Let's show that $M(\varepsilon)$ has no small eigenvalues.

Proposition 1. *If $1 \leq p \leq n$, There is a constant $\lambda_0 > 0$ such that, if $1 \leq p \leq n$,*

$$\lambda_\varepsilon \neq 0 \Rightarrow \lambda_\varepsilon \geq \lambda_0.$$

Proof. We shall use the McGowan's lemma as enonciated in [GP95]. Recall that this lemma, in the spirit of Mayer Vietoris theorem, gives control of positive eigenvalues in terms of positive eigenvalues of certain covers with certain boundary conditions. We use the cover $M_\varepsilon = M_1(\varepsilon) \cup \varepsilon.(M_2(1) \cup \mathcal{C}_{1,2})$. Let

$$U_1 = M_1(\varepsilon) \text{ and } U_2 = \varepsilon.(M_2(1) \cup \mathcal{C}_{1,2})$$

then $U_{1,2} = U_1 \cap U_2 = \varepsilon.\mathcal{C}_{1,2}$ and $H^{p-1}(U_1 \cap U_2) = 0$ for $1 < p \leq n$.

The lemma 1 of [GP95] asserts that, in this case and for these values of p , the first positive eigenvalue of the Hodge-Laplace operator on exact p -forms of M_ε is, up to a power of 2, bounded from below by

$$\lambda_0(\varepsilon) = \left(\left(\frac{1}{\mu^p(U_1)} + \frac{1}{\mu^p(U_2)} \right) \left(\frac{\omega_{p,m}c_p}{\mu^{p-1}(U_{1,2})} + 1 \right) \right)^{-1}$$

where $\mu^k(U)$ is the first positive eigenvalue of the Laplacian acting on exact k -forms of U and satisfying absolute boundary conditions, $\omega_{p,m}$ is a combinatorial constant and c_p is the square of an upper bound of the first derivative of a partition of 1 subordinate to the cover.

For us $c_p, \mu^p(U_2)$ and $\mu^{p-1}(U_{1,2})$ are all of order ε^{-2} , but $\mu^p(U_1)$ is bounded for $p \leq n$ as was shown in [AC93] (remark that the small eigenvalue exhibited here in degree $m-1$ is in the coexact spectrum). This give a uniform bound for the exact spectrum of degree p with $1 < p \leq n$ but the exact spectrum for 1-forms comes from the spectrum on function which has been studied in [T02], thus the exact spectrum is controled for $1 \leq p \leq n$, by Hodge duality it gives a control for all the positive spectrum in these degrees. Finally we can assert that there exists $\lambda_0 > 0$ such that $\forall \varepsilon, \lambda_0(\varepsilon) > \lambda_0$. \square

The proof of the main Proposition B needs some useful notations and estimates, it is the goal of the following section.

3. ESTIMATES AND TOOLS

As in [ACP07] we use the following change of variables : with

$$\varphi_{\varepsilon|M_1(\varepsilon)} = \varphi_{1,\varepsilon} \quad \text{and} \quad \varphi_{\varepsilon|M_2(1)} = \varepsilon^{p-m/2} \varphi_{2,\varepsilon}$$

we write on the cone

$$\varphi_{1,\varepsilon} = dr \wedge r^{-(n/2-p+1)} \beta_{1,\varepsilon} + r^{-(n/2-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$.

On the other part, it is more convenient to define $r = 1 - s$ for $s \in [0, 1/2]$ and write $\varphi_{2,\varepsilon} = (dr \wedge r^{-(n/2-p+1)} \beta_{2,\varepsilon} + r^{-(n/2-p)} \alpha_{2,\varepsilon})$ near the boundary. Then we can define, for $r \in [1/2, 1]$ (the boundary of $M_2(1)$ corresponds to $r = 1$)

$$\sigma_2(r) = (\beta_2(r), \alpha_2(r)) = U(\varphi_2).$$

The L_2 norm, for a form supported on M_1 in the cone $\mathcal{C}_{\varepsilon,1}$, has the expression

$$\|\varphi\|^2 = \int_{M_1} |\sigma_1|^2 dr \wedge d\text{vol}_{\mathbb{S}^n} + \int_{M_2} |\varphi_2|^2 d\text{vol}_{M_2}$$

and the quadratic form on study is

$$q(\varphi) = \int_{M(\varepsilon)} |(d + d^*)\varphi|^2 = \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_1)|^2 + \frac{1}{\varepsilon^2} \int_{M_2(1)} |D_2(\varphi_2)|^2 \quad (2)$$

where D_1 , resp. D_2 , are the Gauß-Bonnet operator of M_1 , resp. M_2 , namely $D_j = d + d^*$ acting on differential forms. In terms of σ_1 , which, a priori, belongs to $C^\infty([\varepsilon, 1[, C^\infty(\Lambda^{p-1}T^*\mathbb{S}^n) \oplus C^\infty(\Lambda^pT^*\mathbb{S}^n))$ the operator has, on the cone of M_1 , the expression

$$UD_1U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r}A \right) \quad \text{with} \quad A = \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix}$$

where P is the operator of degree which multiplies by p a p -form, and D_0 is the Gauß-Bonnet operator of the sphere \mathbb{S}^n .

While the Hodge-deRham operator has, in these coordinates, the expression

$$U\Delta_1U^* = -\partial_r^2 + \frac{1}{r^2}A(A+1). \quad (3)$$

The same expressions are valid for UD_2U^* and $U\Delta_2U^*$ near the boundary of $M_2(1)$ but we shall not use them because we need global estimates on this part.

The compatibility condition is, for the quadratic form, $\varepsilon^{1/2}\alpha_1(\varepsilon) = \alpha_2(1)$ and $\varepsilon^{1/2}\beta_1 = \beta_2(1)$ or

$$\sigma_2(1) = \varepsilon^{1/2}\sigma_1(\varepsilon). \quad (4)$$

The compatibility condition for the Hodge-deRham operator, of first order, is obtained by expressing that $D\varphi \sim (UD_1U^*\sigma_1, \frac{1}{\varepsilon}UD_2U^*\sigma_2)$ belongs to the domain of D . In terms of σ it gives

$$\sigma'_2(1) = \varepsilon^{3/2}\sigma'_1(\varepsilon). \quad (5)$$

Let ξ_1 be a cut-off function on M_1 around p_0 :

$$0 \leq r \leq 1/2 \Rightarrow \xi_1(r) = 1 \text{ and } r \geq 1 \Rightarrow \xi_1(r) = 0.$$

Proposition 2. *For our given family φ_ε satisfying $\Delta(\varphi_\varepsilon) = \lambda_\varepsilon\varphi_\varepsilon$ with λ_ε bounded, the family $(1 - \xi_1) \cdot \varphi_{1,\varepsilon}$ is bounded in $H^1(M_1)$.*

Then it remains to study $\xi_1 \cdot \varphi_{1,\varepsilon}$ which can be expressed with the polar coordinates, this is the goal of the next section.

Remark 3. *The same cannot be done with the component on M_2 or more precisely this does not give what we want to prove, namely that this component goes to 0 with ε . To do so we have first to consider $\varphi_{2,\varepsilon}$ in the domain of an elliptic operator, this is the main difficulty, in contrast with the case concerning functions. In fact we will decompose $\varphi_{2,\varepsilon}$ in a part which clearly goes to 0 and an other part which belongs to the domain of an elliptic operator, this operator is naturally D_2 but the point is to determine the boundary conditions.*

3.1. Expression of the quadratic form. — For any φ such that the component φ_1 is supported in the cone $\mathcal{C}_{1,\varepsilon}$, one has, with $\sigma_1 = U\varphi_1$ and by the same calculus as in [ACP07] :

$$\begin{aligned} \int_{\mathcal{C}_{\varepsilon,1}} |D_1\varphi|^2 d\text{vol}_{g_\varepsilon} &= \int_\varepsilon^1 \left| \left(\partial_r + \frac{1}{r}A \right) \sigma_1 \right|^2 dr \\ &= \int_\varepsilon^1 \left[|\sigma'_1|^2 + \frac{2}{r} \langle \sigma'_1, A\sigma_1 \rangle + \frac{1}{r^2} |A\sigma_1|^2 \right] dr \\ &= \int_\varepsilon^1 \left[|\sigma'_1|^2 + \partial_r \left(\frac{1}{r} \langle \sigma_1, A\sigma_1 \rangle \right) + \frac{1}{r^2} (\langle \sigma_1, A\sigma_r \rangle + |A\sigma_1|^2) \right] dr \\ &= \int_\varepsilon^1 \left[|\sigma'_1|^2 + \frac{1}{r^2} \langle \sigma_1, (A + A^2)\sigma_1 \rangle \right] dr - \frac{1}{\varepsilon} \langle \sigma_1(\varepsilon), A\sigma_1(\varepsilon) \rangle. \end{aligned}$$

we then have

$$q(\varphi) = \int_{\varepsilon}^1 \left[|\sigma'_1|^2 + \frac{1}{r^2} \langle \sigma_1, (A + A^2)\sigma_1 \rangle \right] dr - \frac{1}{\varepsilon} \langle \sigma_1(\varepsilon), A\sigma_1(\varepsilon) \rangle + \frac{1}{\varepsilon^2} \int_{M_2(1)} |D_2\varphi_2|^2 \quad (6)$$

On the other hand we have, as well,

$$\begin{aligned} \int_{\mathcal{C}_{1/2,1}} |D_2\varphi|^2 d\text{vol}_{g_\varepsilon} &= \int_{1/2}^1 \left| \left(\partial_r + \frac{1}{r}A \right) \sigma_2 \right|^2 dr \\ &= \int_{1/2}^1 \left[|\sigma'_2|^2 + \frac{1}{r^2} \langle \sigma_2, (A + A^2)\sigma_2 \rangle \right] dr \\ &\quad + \langle \sigma_2(1), A\sigma_2(1) \rangle - \langle \sigma_2(1/2), A\sigma_2(1/2) \rangle. \end{aligned}$$

Thus the first boundary terms annihilate, and one has also

$$q(\varphi) = \int_{\varepsilon}^1 \left[|\sigma'_1|^2 + \frac{1}{r^2} \langle \sigma_1, (A + A^2)\sigma_1 \rangle \right] dr + \frac{1}{\varepsilon^2} \int_{1/2}^1 \left[|\sigma'_2|^2 + \frac{1}{r^2} \langle \sigma_2, (A + A^2)\sigma_2 \rangle \right] dr - \frac{1}{\varepsilon^2} \langle \sigma_2(1/2), A\sigma_2(1/2) \rangle. \quad (7)$$

We remark that the boundary term $-\langle \sigma_2(1/2), A\sigma_2(1/2) \rangle$ is positif if σ_2 belongs to the eigenspace of A with negative eigenvalues. In fact we know the spectrum of A :

3.2. Spectrum of A . — It has been calculated in [BS88]. By their result, we have that the spectrum of A is given by the values $\gamma = \pm \frac{1}{2} \pm \sqrt{\mu^2 + (\frac{n-1}{2} - p)^2}$ for μ^2 covering the spectrum of $\Delta_{\mathbb{S}^n}$ acting on the coclosed p -forms.

Now the spectrum for the standard sphere has been calculated in [GM75] and as a consequence one has $\mu^2 \geq (n-p)(p+1)$ on coclosed p -forms, unless $p = 0$ for which we have in fact $\mu^2 \geq (n-p)(p+1)$ on coexact p -forms (*ie.* non constant functions). As a consequence

$$\mu^2 + \left(\frac{n-1}{2} - p \right)^2 \geq (n-p)(p+1) + \left(\frac{n+1}{2} - (p+1) \right)^2 = \left(\frac{n+1}{2} \right)^2$$

and then

$$|\gamma| \geq \frac{n}{2}. \quad (8)$$

For $p = 0$, the eigenvalues of A corresponding to the constant function are in fact $\pm \frac{n}{2}$ as we can see with the expression of A , so the minoration (8) is allways valid and, in particular, $0 \notin \text{Spec}(A)$.

consequence. — The elliptic operator $A(A+1)$ is non negative (and positive if $n \geq 3$). Indeed $A(A+1) = (A+1/2)^2 - 1/4$ and the values of the eigenvalues of A give the conclusion.

3.3. Equations satisfied. — On the cones, $\sigma = (\sigma_1, \sigma_2)$ satisfies the equations

$$\left(-\partial_r^2 + \frac{1}{r^2}A(A+1)\right)\sigma_1 = \lambda_\varepsilon\sigma_1 \quad (9)$$

$$\Delta_2 U^* \sigma_2 = \varepsilon^2 \lambda_\varepsilon U^* \sigma_2 \quad (10)$$

and the compatibility conditions have been given in (4) and (5):

$$\sigma_2(1) = \varepsilon^{1/2}\sigma_1(\varepsilon), \quad \sigma_2'(1) = \varepsilon^{3/2}\sigma_1'(\varepsilon). \quad (11)$$

We decompose σ_1 along a base of eigenvectors of A : $\sigma_1 = \sum \sigma_1^\gamma$ and $A\sigma_1^\gamma = \gamma\sigma_1^\gamma$.

3.4. Boundary control. — We know that $\int_\varepsilon^1 |(\partial_r + \frac{A}{r})\sigma_1|^2 \leq \lambda + 1$ for ε small enough. This inequality stays valid for $\xi_1\sigma_1$ with a bigger constant: there exists $\Lambda > 0$ such that for any $\varepsilon > 0$

$$\sum_{\gamma \in \text{Spec}(A)} \int_\varepsilon^1 |\partial_r(\xi_1\sigma_1^\gamma) + \frac{\gamma}{r}(\xi_1\sigma_1^\gamma)|^2 \leq \Lambda.$$

Then, if we remark that $\partial_r\sigma + \frac{\gamma}{r}\sigma = r^{-\gamma}\partial_r(r^\gamma\sigma)$ we can write, for $\gamma < 0 \Rightarrow \gamma \leq -\frac{n}{2}$,

$$(\varepsilon^\gamma\sigma_1^\gamma(\varepsilon))^2 = \left(\int_\varepsilon^1 \partial_r(r^\gamma\xi_1\sigma_1^\gamma)\right)^2 \leq \int_\varepsilon^1 r^{2\gamma} \int_\varepsilon^1 |\partial_r(\xi_1\sigma_1^\gamma) + \frac{\gamma}{r}(\xi_1\sigma_1^\gamma)|^2 \quad (12)$$

So $\sigma_1^\gamma(\varepsilon) = O(\varepsilon^{1/2}/\sqrt{|2\gamma+1|})$. This suggests that the limit σ is harmonic on $M_2(1)$ with boundary condition $\Pi_{<0}\sigma_2 = 0$, if $\Pi_{<0}$ denote the spectral projector of A on the total eigenspace of negative eigenvalues. The limit problem appearing here has a boundary condition of Atiyah-Patodi-Singer type [APS75]. Indeed we have

Proposition 4. *There exists a constant C such that the boundary value satisfies, for all $\varepsilon > 0$*

$$\|\Pi_{<0}(\sigma_{1,\varepsilon}(\varepsilon))\|^2 \leq C\varepsilon.$$

Proof. We know that $q(\xi_1\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon})$ is bounded by Λ , on the other hand the expression of the quadratic form (6) can be done with respect to the decomposition

along $\text{Im } \Pi_{>0}$ and $\text{Im } \Pi_{<0}$. Namely:

$$\begin{aligned}
q(\xi_1 \varphi_{1,\varepsilon}, \varphi_{2,\varepsilon}) &= \int_{\varepsilon}^1 \left| \left(\partial_r + \frac{1}{r} A \right) \Pi_{<0}(\xi_1 \sigma_{1,\varepsilon}) \right|^2 dr \\
&\quad + \int_{\varepsilon}^1 \left| \left(\partial_r + \frac{1}{r} A \right) \Pi_{>0}(\xi_1 \sigma_{1,\varepsilon}) \right|^2 dr + \frac{1}{\varepsilon^2} \int_{M_2(1)} |D_2 \varphi_2|^2 \\
&\geq \int_{\varepsilon}^1 \left| \left(\partial_r + \frac{1}{r} A \right) \Pi_{<0}(\xi_1 \sigma_{1,\varepsilon}) \right|^2 dr \\
&\geq \int_{\varepsilon}^1 \left[|\Pi_{<0}(\xi_1 \sigma_{1,\varepsilon})'|^2 + \frac{1}{r^2} \langle \Pi_{<0}(\xi_1 \sigma_{1,\varepsilon}), (A + A^2) \Pi_{<0}(\xi_1 \sigma_{1,\varepsilon}) \rangle \right] dr \\
&\quad - \frac{1}{\varepsilon} \langle \Pi_{<0} \sigma_1(\varepsilon), A \circ \Pi_{<0} \sigma_1(\varepsilon) \rangle \\
&\geq \frac{n}{2\varepsilon} \|\Pi_{<0} \sigma_{1,\varepsilon}(\varepsilon)\|^2
\end{aligned}$$

because $A(A+1)$ is non negative and $-A \circ \Pi_{<0} \geq \frac{n}{2}$. \square

3.5. Limit problem. — We study here good candidates for the limit Gauß-Bonnet operator. On M_1 the problem is clear, the question here is to identify the boundary conditions on $M_2(1)$.

- On M_1 the natural problem is the Friedrich extension of D_1 on the cone, it is not a real conical singularity and $\Delta_1 = D_1^* \circ D_1$ is the usual Hodge-de Rham operator (we can see with the expression of φ_1 using the Bessel functions, see appendix, that $\sum_{\gamma} |d_{\varepsilon,\bar{\gamma}}|^2 \varepsilon^{-2\bar{\gamma}+1} / 2\bar{\gamma} - 1$ is bounded so $\lim_{\varepsilon \rightarrow 0} \sum_{\gamma} |d_{\varepsilon,\bar{\gamma}}|^2 = 0$ and the limit $U\varphi$ has only regular components, *ie.* in terms of $f_{\bar{\gamma}}(r)$).
- For $n \geq 2$ the forms on $M_2(1)$ satisfying $D_2(\varphi) = 0$ $\Pi_{<0} \circ U(\varphi) = 0$ on the boundary are precisely the L_2 forms in $\text{Ker}(D_2)$ on the large manifold \widetilde{M}_2 obtained from $M_2(1)$ by gluing a conic cylinder $[1, \infty[\times \mathbb{S}^n$ with metric $dr^2 + r^2 h$, *ie.* the exterior of the sphere in \mathbb{R}^{n+1} .

Indeed, these L_2 forms must satisfy $(\partial_r + \frac{1}{r} A)\sigma = 0$ or, $\forall \gamma \in \text{Spec}(A)$, $\exists \sigma_0^\gamma \in \text{ker}(A - \gamma)$ such that $\sigma^\gamma = r^{-\gamma} \sigma_0^\gamma \in L_2$ which is possible only for $\gamma > 1/2$. This limit problem is of the category *non parabolic at infinity* in the terminology of Carron [C01], see particularly the theorem 2.1 there, then as a consequence of theorem 0.4 of the same paper we know that its kernel is finite dimensional, more precisely it gives:

Proposition 5. *The operator D_2 acting on the forms of $M_2(1)$, with the boundary condition $\Pi_{<0} \circ U = 0$, is elliptic in the sens that the H_1 norm of elements of the domain is controled by the norm of the graph. Let's \mathcal{D}_2 denote this operator.*

Corollary 6. *The kernel of \mathcal{D}_2 is of finite dimension and can be identify with a subspace of the total space $\sum_p H^p(M_2(1))$ of absolute cohomology.*

We shall see in Corollary 15 below that this kernel is in fact the total space $\sum_p H^p(M_2)$.

Proof. We show that there exists a constant $C > 0$ such that for each $\varphi \in H^1(\Lambda T^*M_2(1))$ satisfying $\Pi_{<0} \circ U(\varphi) = 0$, then

$$\|\varphi\|_{H^1} \leq C(\|\varphi\|_{L_2} + \|D_2(\varphi)\|_{L_2}).$$

Thus \mathcal{D}_2 is closable.

Denote, for such a φ , by $\tilde{\varphi}$ its harmonic prolongation on \widetilde{M}_2 . Then $\tilde{\varphi}$ is in the domain of the Dirac operator on \widetilde{M}_2 which is elliptic, it means that for each smooth function f with compact support there exists a constant $C_f > 0$ such that

$$\forall \psi \in \text{dom}(D_2) \quad \|f \cdot \psi\|_{H^1} \leq C_f(\|\psi\|_{L_2} + \|D_2(\psi)\|_{L_2})$$

(it is the fact that an operator 'non parabolic at infinity' is continue from its domain to H_{loc}^1 , Theorem 1.2 of Carron)

If we apply this inequality for some $f = 1$ on $M_2(1)$ and $\psi = \tilde{\varphi}$ we obtain in particular that

$$\|\varphi\|_{H^1(M_2(1))} \leq C(\|\tilde{\varphi}\|_{L_2} + \|D_2(\tilde{\varphi})\|_{L_2})$$

with $C = C_f$. We remark first that

$$\|D_2(\tilde{\varphi})\|_{L_2(\widetilde{M}_2)} = \|D_2(\varphi)\|_{L_2(M_2(1))}.$$

Now we can write, by the use of cut-off functions, $\varphi = \varphi_0 + \bar{\varphi}$ with φ_0 null near the boundary and $\bar{\varphi}$ supported in $1/2 \leq r \leq 1$. Then $\tilde{\varphi}_0 = 0$ so, for the control of $\|\tilde{\varphi}\|_{L_2}$, we can suppose that $\varphi = \bar{\varphi}$. We write $U\varphi = \sigma$ and $\sigma = \sum_{\gamma} \sigma^{\gamma}$ on the eigenspaces of A . We have

$$\|\tilde{\varphi}\|_{L_2(\mathbb{R}^m - B(0,1))}^2 = \sum_{\gamma > 0} \frac{1}{2\gamma - 1} |\sigma^{\gamma}(1)|^2,$$

now $\gamma \geq 1$ and $\sigma^{\gamma}(1/2) = 0$, so one has $\sigma^{\gamma}(1) = \int_{1/2}^1 \partial_r(r^{\gamma}\sigma^{\gamma})$ and by Cauchy-Schwarz inequality

$$|\sigma^{\gamma}(1)|^2 \leq \int_{1/2}^1 (r^{-\gamma} \partial_r(r^{\gamma}\sigma^{\gamma}))^2 \int_{1/2}^1 r^{2\gamma}$$

or

$$|\sigma^{\gamma}(1)|^2 \leq \|(\partial_r + \frac{1}{r}A)(\sigma^{\gamma})\|^2 \frac{1}{2\gamma + 1}$$

as a consequence

$$\sum_{\gamma > 0} \frac{1}{2\gamma - 1} |\sigma^{\gamma}(1)|^2 \leq \sum_{\gamma > 0} \|(\partial_r + \frac{1}{r}A)(\sigma^{\gamma})\|^2 \frac{1}{4\gamma^2 - 1} \leq \|D_2(\varphi)\|^2$$

then, changing the constant, we have also

$$\|\varphi\|_{H^1(M_2(1))} \leq C(\|\varphi\|_{L_2(M_2(1))} + \|D_2(\varphi)\|_{L_2(M_2(1))}).$$

□

alternative proof of the proposition, in the spirit of [APS75]. — To study this boundary condition it is better to write again the p -form near the boundary as $\varphi_2 = dr \wedge r^{-(n/2-p+1)}\beta_2 + r^{-(n/2-p)}\alpha_2$ with, as before, $U(\varphi_2) = \sigma_2 = (\beta_2, \alpha_2)$. On the cone $r \in [1/2, 1]$, $UD_2U^* = \partial_r + \frac{1}{r}A$ and we can construct, as in [APS75] a parametrix of D_2 by gluing an interior parametrix with one constructed on the 'long' cone $r \in]0, 1]$ as follows :

Given a form ψ on $M_2(1)$, if we look for a form φ such that $D_2\varphi = \psi$, we write ψ as the sum of two terms, the first one with support in the neighborhood of the boundary and the second one nul near the boundary. On the second term we apply an interior parametrix Q_0 of the elliptic operator D_2 . Let's now suppose that φ is supported in the cone $r \in [1/2, 1]$. We decompose $U\psi$ along the eigenspaces of $A : U\psi = \sum_\gamma \psi^\gamma$ and if also $U\varphi = \sum_\gamma \varphi^\gamma$, then φ^γ must satisfy

$$\partial_r \varphi^\gamma + \frac{\gamma}{r} \varphi^\gamma = r^{-\gamma} \partial_r (r^\gamma \varphi^\gamma) = \psi^\gamma.$$

We take the solution

$$\varphi^\gamma = r^{-\gamma} \int_1^r \rho^\gamma \psi^\gamma(\rho) d\rho \text{ if } \gamma < 0 \quad (13)$$

$$\varphi^\gamma = r^{-\gamma} \int_0^r \rho^\gamma \psi^\gamma(\rho) d\rho \text{ if } \gamma > 0 \quad (14)$$

Thus $\gamma < 0 \Rightarrow \varphi^\gamma(1) = 0$. It is now easy to verify that \mathcal{D}_2 satisfies the property (SE) of [L97] p. 54 (with $\rho(x) = \sqrt{x}$).

This fact and the vacuity of $\text{Spec}(A) \cap]-1, +1[$ assure the construction of the parametrix on the cone, see [L97] and also [BS88] who make this construction. In fact the parametrix on the cone gives only H^1 regularity with weight function, but we will cut the singular point for $M_2(1)$, these results are in [L97] Proposition 1.3.12 and following.

3.6. Boundedness. — Recall that $A(A+1)$ is non negative.

Proposition 7. *Let χ be a cut-off function supported in $[3/4, 1[$ equal to 1 on $[7/8, 1[$ and $\sigma_{2,\varepsilon} = U(\varphi_{2,\varepsilon})$. The family $\psi_{2,\varepsilon} = \varphi_{2,\varepsilon} - U^*\left(\Pi_{<0}(\chi\sigma_{2,\varepsilon})\right)$ belongs to the domain of \mathcal{D}_2 , is bounded in $H^1(M_2)$ and $\lim_{\varepsilon \rightarrow 0} \|\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}\| = 0$.*

As a consequence of this result, there exists a subsequence of $\varphi_{2,\varepsilon}$, which converge in L_2 to an harmonic form satisfying the boundary conditions of \mathcal{D}_2 .

Proof. We write in the following $\sigma_{2,\varepsilon} = \sigma_2$. It is clear that $\psi_{2,\varepsilon}$ belongs to the domain of \mathcal{D}_2 , and is a bounded family for the operator norm. Thus, by ellipticity it is also a bounded family in $H^1(M_2)$. Now

$$\|\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}\|^2 \leq \int_{3/4}^1 |\Pi_{<0}\sigma_2(r)|^2 dr$$

but as a consequence of (7)

$$|\Pi_{<0}\sigma_2(r)|^2 = 2 \int_{1/2}^r \langle \Pi_{<0}\sigma_2'(t), \Pi_{<0}\sigma_2(t) \rangle dt + |\Pi_{<0}\sigma_2(1/2)|^2 \leq 2\varepsilon\Lambda + \varepsilon^2 \frac{2}{n}\Lambda$$

using the inequality of Cauchy-Schwarz, the fact that the L_2 -norm of φ_ε is 1 and that $(-A \circ \Pi_{<0}) \geq \frac{n}{2}$. \square

Corollary 8. *The family $\Pi_{>0}\sigma_2(1)$ is bounded in $H^{1/2}(\mathbb{S}^n)$ as the boundary value of $\psi_{2,\varepsilon}$.*

We now define a better prolongation of $\Pi_{>0}\sigma_2(1)$ on $M_1(\varepsilon)$. More generally let

$$P_\varepsilon : \Pi_{>0}(H^{1/2}(\mathbb{S}^n)) \rightarrow H^1(\mathcal{C}_{\varepsilon,1}) \quad (15)$$

$$\sigma = \sum_{\gamma \in \text{Spec}(A), \gamma > 0} \sigma_\gamma \mapsto P_\varepsilon(\sigma) = \sum_{\gamma \in \text{Spec}(A), \gamma > 0} \varepsilon^{\gamma-1/2} r^{-\gamma} \sigma_\gamma. \quad (16)$$

We remark that there exists a constant C such that

$$\|P_\varepsilon(\sigma)\|_{L_2(M_1(\varepsilon))}^2 \leq C \sum |\sigma_\gamma|^2 = C \|\Pi_{>0}\sigma_2(1)\|_{L_2(\mathbb{S}^n)}^2 \quad (17)$$

and also that, if $\psi_2 \in \text{Dom}\mathcal{D}_2$ and with the same cut-off function ξ_1 , which has value 1 for $0 \leq r \leq 1/2$ and 0 for $r \geq 1$, then $(\xi_1 P_\varepsilon(\psi_{2|\mathbb{S}^n}), \psi_2)$ defines through the isometries U an element of $H^1(M_\varepsilon)$. Let

$$\tilde{\psi}_1 := \xi_1 P_\varepsilon(\psi_{2|\mathbb{S}^n}).$$

We now decompose $\varphi_{1,\varepsilon}$ as follows. Let

$$\xi_1 \varphi_{1,\varepsilon} = \varphi_\varepsilon^+ + \varphi_\varepsilon^-$$

according to the decomposition of σ_1 along the positive or negative spectrum of A . Then $\tilde{\psi}_1$ and φ_ε^+ have the same values on the boundary so the difference $\varphi_\varepsilon^+ - \tilde{\psi}_1$ can be viewed in $H^1(M_1)$ by a prolongation by 0 on the ball, while the boundary value of φ_ε^- is small. We introduce for this term the cut-off function taken in [ACP07]

$$\xi_\varepsilon(r) = \begin{cases} 1 & \text{if } r \geq 2\sqrt{\varepsilon}, \\ \frac{\log(2\varepsilon) - \log r}{\log(\sqrt{\varepsilon})} & \text{if } r \in [2\varepsilon, 2\sqrt{\varepsilon}], \\ 0 & \text{if } r \leq 2\varepsilon. \end{cases}$$

Lemma 9. $\lim_{\varepsilon \rightarrow 0} \|(1 - \xi_\varepsilon)\varphi_\varepsilon^-\|_{L_2} = 0$.

This is a consequence of the estimate of the Proposition 4.

Proposition 10. *The forms $\psi_{1,\varepsilon} = (1 - \xi_1)\varphi_{1,\varepsilon} + (\varphi_\varepsilon^+ - \tilde{\psi}_1) + \xi_\varepsilon\varphi_\varepsilon^-$ belong to $H^1(M_1)$ and define a bounded family.*

Proof. We will show that each term is bounded. For the first one it is already done in Proposition 1. For the second one, we remark that

$$\left(\partial_r + \frac{A}{r}\right)(\varphi_\varepsilon^+ - \tilde{\psi}_1) = \left(\partial_r + \frac{A}{r}\right)(\varphi_\varepsilon^+) + \partial_r(\xi)P_\varepsilon(\psi_{2|\mathbb{S}^n}) := f_\varepsilon \quad (18)$$

and f_ε is uniformly bounded in $L_2(M_1)$ because of (17). This estimate shows also that the L_2 -norm of $(\varphi_\varepsilon^+ - \tilde{\psi}_1)$ is bounded. Thus the family $(\varphi_\varepsilon^+ - \tilde{\psi}_1)$ is bounded for the q -norm in $H^1(M_1)$ which is equivalent to the H^1 -norm.

For the third one we use the estimate due to the expression of the quadratic form. Expressing that $\int_{\mathcal{C}_{r,1}} |D_1\varphi^-|^2$ is bounded by Λ gives that

$$\frac{1}{r} \langle \sigma_1^-(r), |A|\sigma_1^-(r) \rangle \leq \Lambda \quad (19)$$

by the same argument as used for the Proposition 4. Now

$$\|D_1(\xi_\varepsilon\varphi_\varepsilon^-)\| \leq \|\xi_\varepsilon D_1(\varphi_\varepsilon^-)\| + \|d\xi_\varepsilon|\varphi_\varepsilon^-\| \leq \|D_1(\varphi_\varepsilon^-)\| + \|d\xi_\varepsilon|\varphi_\varepsilon^-\|$$

the first term is bounded and, with $|A| \geq \frac{n}{2}$ and the estimate (19), we have

$$\begin{aligned} \|d\xi_\varepsilon|\varphi_\varepsilon^-\|^2 &\leq \frac{8\Lambda}{n \log^2 \varepsilon} \int_\varepsilon^{\sqrt{\varepsilon}} \frac{dr}{r} \\ &\leq \frac{4\Lambda}{n|\log \varepsilon|}. \end{aligned}$$

This complete the proof. □

In fact the decomposition used here is almost orthogonal:

Lemma 11.

$$\langle (\varphi_\varepsilon^+ - \tilde{\psi}_1), \tilde{\psi}_1 \rangle = O(\varepsilon).$$

proof of lemma 11. — If we decompose the terms under the eigenspaces of A we see that only the positive eigenvalues are involved and, with $f_\varepsilon = \sum_{\gamma>0} f^\gamma$ and $(\varphi_\varepsilon^+ - \tilde{\psi}_1) = \sum_{\gamma>0} \varphi_0^\gamma$, the equation (18) and the fact that $(\varphi_\varepsilon^+ - \tilde{\psi}_1)(\varepsilon) = 0$ give

$$\varphi_0^\gamma(r) = r^{-\gamma} \int_\varepsilon^r \rho^\gamma f^\gamma(\rho) d\rho.$$

Then for each positive eigenvalue γ of A

$$\begin{aligned}
\langle \varphi_0^\gamma, \tilde{\psi}_1^\gamma \rangle &= \varepsilon^{\gamma-1/2} \sigma_\gamma \int_\varepsilon^1 r^{-2\gamma} \int_\varepsilon^r \rho^\gamma f^\gamma(\rho) d\rho \\
&= \varepsilon^{\gamma-1/2} \sigma_\gamma \int_\varepsilon^1 \frac{r^{-2\gamma+1}}{2\gamma-1} r^\gamma f^\gamma(r) dr \\
&= \varepsilon^{\gamma-1/2} \sigma_\gamma \int_\varepsilon^1 \frac{r^{-\gamma+1}}{2\gamma-1} f^\gamma(r) dr \\
&\leq C \varepsilon^{\gamma-1/2} \|\sigma_\gamma\| \frac{\varepsilon^{(-2\gamma+3)/2}}{(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\| \\
&\leq C \varepsilon \|\sigma_\gamma\| \|f^\gamma\|.
\end{aligned}$$

This estimate gives the lemma.

Remark: For $\gamma = 3/2$ the bound is in fact $\varepsilon \sqrt{|\log(\varepsilon)|}$ and for $\gamma = 1$ it is $\sqrt{\varepsilon}$.

4. PROOF OF THEOREM B

Lemma 12. *If $\lambda \neq 0$, then $\lambda_\varepsilon \neq 0$ for all ε and*

$$\lim_{\varepsilon \rightarrow 0} (L_2) \tilde{\psi}_{1,\varepsilon} = 0$$

and also

$$\lim_{\varepsilon \rightarrow 0} (L_2) \psi_{2,\varepsilon} = 0$$

as well as in q -norm.

Proof. We know, by the Proposition 1, that there is a universal lower bound for positive eigenvalues on $M(\varepsilon)$, so if $\lambda = \lim \lambda_\varepsilon$ is positive, it means that all the λ_ε are also positive! Let

$$\psi_\varepsilon = (\tilde{\psi}_{1,\varepsilon}, \psi_{2,\varepsilon}) \in \text{dom}(q).$$

The L_2 -norm of ψ_ε is bounded and

$$\begin{aligned}
q(\psi_\varepsilon) &\leq O(\varepsilon^2) + \int_\varepsilon^1 |\xi_1'(r) P_\varepsilon(\sigma_2)|^2 \\
&\leq O(\varepsilon^2) + C \int_{1/2}^1 |P_\varepsilon(\sigma_2)|^2 \\
&\leq O(\varepsilon^2) + O(\varepsilon^{n-1})
\end{aligned}$$

due to the expression of P_ε and the fact that $\text{spec}(|A|) \geq \frac{n}{2}$. Because $n \geq 2$ and Proposition 1 is true, we conclude that the distance of ψ_ε to $\text{Ker } \Delta_\varepsilon$ is $O(\varepsilon)$. But we know that $\lambda_\varepsilon \neq 0$, so φ_ε is orthogonal to $\text{Ker } \Delta_\varepsilon$ and, with the previous result

$$\langle \varphi_\varepsilon, \psi_\varepsilon \rangle = O(\sqrt{\varepsilon}).$$

Now we use Proposition 7 and Lemma 11, the conclusion is

$$\lim_{\varepsilon \rightarrow 0} \|\tilde{\psi}_{1,\varepsilon}\|^2 + \|\psi_{2,\varepsilon}\|^2 = 0.$$

□

As a consequence of this result and Proposition 7, we obtain

Corollary 13. $\lim_{\varepsilon \rightarrow 0} (L_2)\varphi_{2,\varepsilon} = 0$.

Recall now that $\psi_{1,\varepsilon} = \varphi_{1,\varepsilon} - \tilde{\psi}_{1,\varepsilon} - (1 - \xi_\varepsilon)\varphi_\varepsilon^-$ and that we know, by the last Lemma and Lemma 9, that the two last terms converge to 0.

Corollary 14. *We can extract from $\psi_{1,\varepsilon}$ a subsequence which converge in L_2 and weakly in H^1 , and any such subsequence defines at the limit a form $\varphi \in H^1(M)$ such that*

$$\|\varphi\|_{L_2} = 1 \text{ and } \Delta\varphi = \lambda\varphi \text{ weakly.}$$

5. PROOF OF THEOREM C

5.1. multiplicity of 0. The dimension of the kernel of Δ_ε is given by the cohomology of M which can be calculated with the Mayer-Vietoris sequence associated to the covering U_1, U_2 introduced at the beginning, see Proposition 1. If we remember also that $H^p(M_j - B, \mathbb{R}) \sim H^p(M_j, \mathbb{R})$ for $p < m$, we obtain that $H^p(M, \mathbb{R}) \sim H^p(M_1, \mathbb{R}) \oplus H^p(M_2, \mathbb{R})$ for $1 \leq p \leq (m - 1)$ while $H^{0,m}(M, \mathbb{R}) \sim H^{0,m}(M_1, \mathbb{R}) \sim H^{0,m}(M_2, \mathbb{R})$.

The transplantation of the harmonic forms of M_1 in M has been described in [AC93]. With the previous calculation, we have good candidates for transplantation of the cohomology of M_2 : for each $\sigma_2 \in \text{Ker } \mathcal{D}_2$ with L_2 -norm equal to 1, let

$$\tilde{\psi}_\varepsilon = (\tilde{\psi}_1, \psi_2) = U^* \left(\xi_1 P_\varepsilon(\sigma_2|_{\mathbb{S}^n}), \sigma_2 \right).$$

Now let $\varphi_\varepsilon \in \text{Ker } \Delta_\varepsilon$. We apply to φ_ε the preceding estimates: there exists a subsequence which gives at the limit $\psi_1 \in \text{Ker } \Delta_1$ and $\psi_2 \in \text{Ker } \mathcal{D}_2$; and only one of these two terms can be zero. The conclusion is that all the harmonic forms of M_ε can be approached by forms like $\tilde{\psi}_\varepsilon$ or $\chi_\varepsilon \varphi_1$, with $\varphi_1 \in \text{Ker } \Delta_1$. As a consequence on has

Corollary 15. *For $1 \leq p \leq (m - 1)$ the two spaces $H^p(M_2, \mathbb{R})$ and $\text{Ker } \mathcal{D}_2$ are isomorphic.*

5.2. convergence of the positive spectrum. The proof is made by induction. We show first that $\lim \lambda_1(\varepsilon) = \lambda_1$:

Proof. We know by the Proposition A that $\limsup \lambda_1(\varepsilon) \leq \lambda_1$ and by Proposition B that $\liminf \lambda_1(\varepsilon)$ is in the positive spectrum of Δ_1 , and as a consequence $\liminf \lambda_1(\varepsilon) \geq \lambda_1$. □

Now suppose that for all j , $1 \leq j \leq k$ one has $\lim \lambda_j(\varepsilon) = \lambda_j$, we have to show that $\lim \lambda_{k+1}(\varepsilon) = \lambda_{k+1}$.

Proof. We know by Proposition A that $\limsup \lambda_{k+1}(\varepsilon) \leq \lambda_{k+1}$; let $\varphi_\varepsilon^{(1)}, \dots, \varphi_\varepsilon^{(k+1)}$ be an orthonormal family of eigenforms on $M(\varepsilon)$:

$$\Delta_\varepsilon \varphi_\varepsilon^{(j)} = \lambda_j(\varepsilon) \varphi_\varepsilon^{(j)}$$

and choose a sequence $\varepsilon_l \rightarrow 0$ such that

$$\lim_{l \rightarrow \infty} \lambda_{k+1}(\varepsilon_l) = \liminf \lambda_{k+1}(\varepsilon).$$

We apply to each $\varphi_\varepsilon^{(j)}$ the same decomposition as in Proposition 10, this gives a family $\psi_\varepsilon^{(1)}, \dots, \psi_\varepsilon^{(k+1)}$ bounded in $H^1(M_1)$ and such that for each indice j

$$\lim_{\varepsilon \rightarrow 0} \|\varphi_{1,\varepsilon}^{(j)} - \psi_\varepsilon^{(j)}\| = 0$$

while, as in Corollary 13

$$\lim_{\varepsilon \rightarrow 0} (L_2) \varphi_{2,\varepsilon}^{(j)} = 0.$$

So, by extraction of a subsequence, we can suppose that $\psi_{\varepsilon_l}^{(1)}, \dots, \psi_{\varepsilon_l}^{(k+1)}$ converge in $L_2(M_1)$ and weakly in $H^1(M_1)$, the limit $\varphi^{(1)}, \dots, \varphi^{(k+1)}$ is orthonormal and satisfies

$$\forall j, 1 \leq j \leq k \Delta_1 \varphi^{(j)} = \lambda_j \varphi^{(j)} \text{ and } \Delta_1 \varphi^{(k+1)} = \liminf \lambda_{k+1}(\varepsilon) \varphi^{(k+1)}.$$

This shows that $\liminf \lambda_{k+1}(\varepsilon) \geq \lambda_{k+1}$ and finishes the proof. \square

APPENDIX

Bessel functions. — Following the calculations of [ACP07] we write

$$\sigma_1^\gamma = c_{\varepsilon,\gamma} f_{\bar{\gamma}}(r) + d_{\varepsilon,\gamma} g_{\bar{\gamma}}(r) \quad (20)$$

with

$$f_{\bar{\gamma}}(t) = t^{\bar{\gamma}+1} F_{\bar{\gamma}}(\lambda_m t^2)$$

$$g_{\bar{\gamma}}(t) = \begin{cases} t^{-\bar{\gamma}} G_{\bar{\gamma}}(\lambda_m t^2) & \text{if } \bar{\gamma} + 1/2 \notin \mathbb{N} \\ t^{-\bar{\gamma}} G_{\bar{\gamma}}(\lambda_m t^2) + a \log(t) f_{\bar{\gamma}}(t) & \text{if } \bar{\gamma} + 1/2 \in \mathbb{N}. \end{cases}$$

and

$$\begin{cases} \bar{\gamma} = \gamma & \text{if } \gamma \geq 0 \\ \bar{\gamma} = -\gamma - 1 & \text{if } \gamma < 0 \end{cases}$$

We know by [BS88] that $\bar{\gamma} \geq 0$ and $\bar{\gamma} = 0$ is only possible for $n = 2$.

Saying that the family $(\|\varphi_\varepsilon\|_{L_2})$ is bounded gives that the sum $\sum_\gamma |c_{\varepsilon,\gamma}|^2 / (2\bar{\gamma} + 3)$ and $\sum_\gamma |d_{\varepsilon,\gamma}|^2 \varepsilon^{-2\bar{\gamma}+1} / (2\bar{\gamma} - 1)$ are bounded, uniformly in ε . As a consequence of this and the fact that $\|(\partial_r + \frac{1}{r}A)\sigma_1\|^2 = O(\lambda_\varepsilon)$ we obtain

$$\sum_{\gamma < 0} |2\gamma + 1| d_{\varepsilon,\gamma}^2 \varepsilon^{2\gamma+1} = O(\lambda_\varepsilon)$$

But

$$\left\| \sum_{\gamma < 0} \sigma_1^\gamma(\varepsilon) \right\|^2 \leq 2 \sum_{\gamma < 0} (c_{\varepsilon, \gamma}^2 \varepsilon^{-2\gamma} + d_{\varepsilon, \bar{\gamma}}^2 \varepsilon^{2\gamma+2})$$

we conclude that $\left\| \sum_{\gamma < 0} \sigma_1^\gamma(\varepsilon) \right\|^2$ is bounded in ε and with the compatibility condition (4) that

$$\|\Pi_{<0}\sigma_2(1)\| = O(\sqrt{\varepsilon}).$$

Before studying the limit problem, let's look at the condition we have obtained for the term corresponding to $\gamma > 0$. The condition (5) gives that

$$D_2(\sigma_2)(0) = \varepsilon^{3/2} D_1(\sigma_1)(\varepsilon) = \varepsilon^{1/2} (\varepsilon \sigma_1'(\varepsilon) + A \sigma_1(\varepsilon))$$

But the expression of σ_1 on the cone gives, for $\gamma > 0$,

$$\begin{aligned} \varepsilon \sigma_1^{\gamma'}(\varepsilon) + A \sigma_1(\varepsilon)^\gamma &= (\gamma + 1) c_\varepsilon \varepsilon^{\gamma+1} F_\gamma(\lambda_m \varepsilon^2) + 2 c_\varepsilon \varepsilon^{\gamma+3} \lambda_m F_\gamma'(\lambda_m \varepsilon^2) \\ &\quad + 2 d_\varepsilon \varepsilon^{-\gamma+2} \lambda_m G_\gamma'(\lambda_m \varepsilon^2) + \gamma c_\varepsilon \varepsilon^{\gamma+1} F_\gamma(\lambda_m \varepsilon^2). \end{aligned} \quad (21)$$

ie. the term with $\varepsilon^{-\bar{\gamma}}$ disappeared. Thus

$$\|\Pi_{>0}(\varepsilon \sigma_1'(\varepsilon) + A \sigma_1(\varepsilon))\|^2 = O\left(\sum_{\gamma > 0} (2\gamma + 1)^2 c_\varepsilon^2 \varepsilon^{2\gamma+2} + d_\varepsilon^2 |G_\gamma'(\lambda_m \varepsilon^2)|^2 \varepsilon^{-2\gamma+4} \lambda_m^2 \right)$$

because $F_\gamma(\lambda_m \varepsilon^2)$ is bounded uniformly in γ .

To control the second part of this bound we use the fact that σ_1 satisfies (9), and by the definition of the Bessel functions one has in particular

$$2(2\gamma - 1) \lambda_m t^{-\gamma} G_\gamma'(\lambda_m t^2) - 4 \lambda_m^2 t^{-\gamma+2} G_\gamma''(\lambda_m t^2) = \lambda_m t^{-\gamma} G_\gamma(\lambda_m t^2)$$

as a consequence

$$\lambda_m \varepsilon^{-\gamma+2} G_\gamma'(\lambda_m \varepsilon^2) = \frac{4 \lambda_m^2 \varepsilon^{-\gamma+4} G_\gamma''(\lambda_m \varepsilon^2)}{2(2\gamma - 1)} + \lambda_m \varepsilon^{-\gamma+2} G_\gamma(\lambda_m \varepsilon^2)$$

from what we conclude that

$$\sum_{\gamma > 0} d_\varepsilon^2 \varepsilon^{-2\gamma+4} \lambda_m^2 = O\left(\sum_{\gamma > 0} d_\varepsilon^2 \varepsilon^{-2\gamma+4} / (2\gamma - 1)^2 \right)$$

which is convergent (it is in fact $O(\varepsilon^3)$).

Thus $\|\Pi_{>0}(\varepsilon \sigma_1'(\varepsilon) + A \sigma_1(\varepsilon))\|$ is bounded and

$$\|\Pi_{>0}(\sigma_2'(1) + A \sigma_2(1))\| = O(\varepsilon^{1/2})$$

which gives at the limit the first order condition corresponding to the condition founded above, if 0 does not belong to the spectrum of A .

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