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# Asymptotic Mean Time To Failure and Higher Moments for Large, Recursive Networks

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## Abstract

This paper deals with asymptotic expressions of the Mean Time To Failure (MTTF) and higher moments for large, recursive, and non-repairable systems in the context of two-terminal reliability. Our aim is to extend the well-known results of the series and parallel cases. We first consider several exactly solvable configurations of identical components with exponential failure-time distribution functions to illustrate different (logarithmic or power-law) behaviors as the size of the system, indexed by an integer  $n$ , increases. The general case is then addressed: it provides a simple interpretation of the origin of the power-law exponent and an efficient asymptotic expression for the total reliability of large, recursive systems. Finally, we assess the influence of the non-exponential character of the component reliability on the  $n$ -dependence of the MTTF.

*Key words:* network reliability, mean time to failure, generating function, moments, cumulants

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## 1 Introduction

In non-repairable systems, the Mean Time To Failure (MTTF), i.e., the average value  $\langle t \rangle$  of the occurrence of failure is a key parameter of the corresponding reliability [1,2,3]. Analytical expressions of the MTTF have long been well known for simple configurations such as series, parallel, and  $k$ -out-of- $n$  systems, where each element has a reliability described by the exponential distribution  $p(t) = \exp(-\lambda t)$  — the asymptotic dependence of the MTTF for a total number  $n$  of elements is  $1/n$  (series) and  $\ln n$  (parallel) — or by more complex time distributions [1,3].

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In this work, we show that these results can be extended to recursive, meshed systems of arbitrary size, the latter being indexed by an integer  $n$ . Our paper is organized as follows. In Section 2, we briefly survey definitions and simple results that apply for series, parallel and  $k$ -out-of- $n$  systems. General expressions for the higher moments  $\mu_m = \langle t^m \rangle$  and for the associated cumulants  $\kappa_m$  are also given for  $k$ -out-of- $n$  systems. Section 3 is devoted to a few exactly solvable configurations [4,5,6,7] of identical components with exponential failure-time distribution functions, which give rise to different behaviors from those of the series-parallel ones as  $n$  goes to infinity. These examples allow us to derive the asymptotic expansion of the MTTF when  $n$  is large, and pave the way to the general case addressed in Section 4, which can be roughly divided into “series-like” and “parallel-like” configurations. Section 5 provides a simple, approximate expression for the corresponding total reliability and contributes to an understanding of the behavior of large systems, which are an important issue [8]. We finally investigate in Section 6 how the asymptotic  $n$ -dependence of the MTTF and higher moments is modified by taking non-exponential failure-time distribution functions into account.

## 2 Results for simple systems

### 2.1 Definitions

As explained in many textbooks [1,2,3]), if  $R(t)$  is the system’s reliability, the MTTF is defined by

$$\text{MTTF} = \langle t \rangle = \int_0^{\infty} t (-R'(t)) dt = \int_0^{\infty} dt R(t); \quad (1)$$

The higher moments  $\mu_m$  are

$$\mu_m = \langle t^m \rangle = \int_0^{\infty} t^m (-R'(t)) dt = m \int_0^{\infty} dt t^{m-1} R(t). \quad (2)$$

For instance, the standard deviation  $\sigma$  is given by  $\sigma^2 = \langle t^2 \rangle - \langle t \rangle^2$ . The moment generating function  $f(z)$  is a formal series defined by

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{\langle t^k \rangle}{k!} z^k; \quad (3)$$

by construction,  $\langle t^m \rangle = f^{(m)}(0)$  (the  $m$ th derivative of  $f(z)$  taken at  $z = 0$ ). Since  $f(0) = 1$ , the cumulant generating function  $g(z)$  is subsequently defined by

$$f(z) = \exp\left(\sum_{k=m}^{\infty} \frac{\kappa_m}{m!} z^m\right) = \exp g(z), \quad (4)$$

where  $\kappa_m$  is the cumulant of order  $m$ ;  $\kappa_2 = \mu_2 - \mu_1^2$  is the variance,  $\kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$ , and so on. Since  $g(z) = \ln\langle e^{zt} \rangle$ ,  $\kappa_m = g^{(m)}(0)$ . Using the same integration by parts as in eq. (1) leads to

$$f(z) = \langle e^{zt} \rangle = 1 + z \int_0^{\infty} dt e^{zt} R(t) = 1 + z \tilde{R}(-z), \quad (5)$$

where  $\tilde{R}$  is the Laplace transform of the reliability.

Calculations are often performed by replacing  $R(t)$  with an exponential function,  $R_{\text{exp}}(t) = \exp(-\lambda t)$ . We have then

$$\langle t^m \rangle^{(\text{exp})} = m \int_0^{\infty} dt e^{-\lambda t} t^{m-1} = \frac{m!}{\lambda^m} \quad (6)$$

and  $\langle e^{zt} \rangle = \left(1 - \frac{z}{\lambda}\right)^{-1}$ . Consequently,

$$g^{(\text{exp})}(z) = -\ln\left(1 - \frac{z}{\lambda}\right), \quad (7)$$

from which

$$\kappa_m^{(\text{exp})} = \frac{(m-1)!}{\lambda^m}. \quad (8)$$

When  $p(t) = \exp(-\lambda t)$  is the common reliability of the system's elements, the total reliability is

$$R(t) = \mathcal{R}(p(t)) = \mathcal{R}(e^{-\lambda t}), \quad (9)$$

where  $\mathcal{R}(p)$  is the reliability polynomial; using the change of variable  $t = -\frac{\ln p}{\lambda}$ , we get

$$\begin{aligned}
\text{MTTF} &= \int_0^{\infty} dt R(t) = \frac{1}{\lambda} \int_0^{\infty} d(\lambda t) \mathcal{R}(e^{-\lambda t}) \\
&= \frac{1}{\lambda} \int_0^1 \frac{dp}{p} \mathcal{R}(p), \tag{10}
\end{aligned}$$

and

$$\langle t^m \rangle = \frac{m}{\lambda^m} \int_0^1 \frac{dp}{p} (-\ln p)^{m-1} \mathcal{R}(p). \tag{11}$$

We make full use of eqs. (10) and (11) in the next sections. The moment generating function also reads

$$\begin{aligned}
\langle e^{zt} \rangle &= 1 + z \int_0^{\infty} dt e^{zt} \mathcal{R}(e^{-\lambda t}) \\
&= 1 + \frac{z}{\lambda} \int_0^1 \frac{dp}{p^{1+z/\lambda}} \mathcal{R}(p). \tag{12}
\end{aligned}$$

## 2.2 A few results

- For  $n$  elements in series,  $\mathcal{R}_n(p) = p^n$ , so that  $R_n(t) = \exp(-n\lambda t)$ , and  $\text{MTTF}_n^{(\text{series})} = \frac{1}{n\lambda}$  [1,3]. Equations (6) and (8) become

$$\mu_m^{(\text{series})} = \frac{m!}{n^m \lambda^m} \tag{13}$$

$$\kappa_m^{(\text{series})} = \frac{(m-1)!}{n^m \lambda^m}. \tag{14}$$

- If the  $n$  elements are in parallel, then  $R_n(t) = 1 - (1 - \exp(-\lambda t))^n$ , and [1,3]

$$\text{MTTF}_n^{(\text{parallel})} = \frac{1}{\lambda} \sum_{i=1}^n \frac{1}{i} = \frac{1}{\lambda} (\psi(n+1) - \psi(1)) \rightarrow \frac{1}{\lambda} \left( \ln n + \mathbf{C} + \frac{1}{2n} + \dots \right) \tag{15}$$

for  $n$  large, where  $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$  is the digamma function and  $\mathbf{C} \approx 0.577216$  is the Euler gamma constant (see eq. 8.362.1 of [9]). In this case, the MTTF diverges as  $n$  goes to infinity.

- For  $k$ -out-of- $n$ :G systems, the reliability polynomial is [1,3]

$$\mathcal{R}_{k,n}(p) = \sum_{i=k}^n \frac{n!}{(n-i)! i!} p^i (1-p)^{n-i}. \quad (16)$$

All the moments could be obtained from eq. (11). However, the formula for the cumulants being eventually extremely simple, we focus on these parameters. Starting from eq. (12), we find

$$\begin{aligned} \langle e^{zt} \rangle &= 1 + \frac{z}{\lambda} \int_0^1 \frac{dp}{p^{1+z/\lambda}} \sum_{i=k}^n \frac{n!}{(n-i)! i!} p^i (1-p)^{n-i} \\ &= 1 + \frac{z}{\lambda} \sum_{i=k}^n \frac{n!}{(n-i)! i!} \frac{\Gamma\left(i - \frac{z}{\lambda}\right) \Gamma(n+1-i)}{\Gamma\left(n+1 - \frac{z}{\lambda}\right)} \\ &= 1 + \frac{z}{\lambda} \frac{n!}{\Gamma\left(n+1 - \frac{z}{\lambda}\right)} \sum_{i=k}^n \frac{\Gamma\left(i - \frac{z}{\lambda}\right)}{i!} \\ &= \frac{n!}{\Gamma\left(n+1 - \frac{z}{\lambda}\right)} \frac{\Gamma\left(k - \frac{z}{\lambda}\right)}{(k-1)!}; \end{aligned} \quad (17)$$

the last equality is proven by induction. Consequently,

$$g(z) = \ln \frac{\Gamma\left(k - \frac{z}{\lambda}\right)}{\Gamma(k)} \frac{\Gamma(n+1)}{\Gamma\left(n+1 - \frac{z}{\lambda}\right)}, \quad (18)$$

whence (see eq. 8.363.8 of [9])

$$\begin{aligned} \kappa_m &= g^{(m)}(0) = \left(\frac{-1}{\lambda}\right)^m \left(\psi^{(m-1)}(k) - \psi^{(m-1)}(n+1)\right) \\ &= \frac{(m-1)!}{\lambda^m} \sum_{i=k}^n \frac{1}{i^m}; \end{aligned} \quad (19)$$

Equation (19) generalizes the expressions obtained for the series ( $k = n$ ) and parallel ( $k = 1$ ) cases, and a similar expression for the variance [10].

When  $m > 1$ ,  $\kappa_m$  is bounded by the finite value  $\frac{(m-1)!}{\lambda^m} \zeta(m)$ , where  $\zeta(m)$  is the Zeta function.

Let us now investigate more complex systems.

### 3 Simple recursive architectures

We consider in this section configurations that are not reducible to series-parallel ones. These systems are studied in the two-terminal reliability context, in which the source and the destination are the large, colored nodes in Figs. 1,5,6,9, and 10. The size of the system is indexed by an integer  $n$ , which counts the number of elementary building blocks of the whole structure. All the nodes are assumed perfect (they never fail), while the reliability of all the edges is described by  $p(t) = \exp(-\lambda t)$  (we may omit in the following the explicit reference to time and note this reliability  $p$ , for greater generality). These recursive architectures have been solved recently. Our aim is to show that the associated MTTFs, as well as moments and cumulants of higher order, can be calculated exactly and that their quite distinct asymptotic expansions in  $n$  give good approximations to the exact results.

#### 3.1 $K_4$ ladder

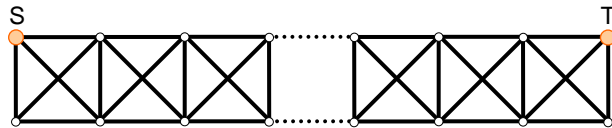


Fig. 1.  $K_4$  ladder architecture.

This architecture, displayed in Fig. 1, is constituted by the repetition of perfect graphs  $K_4$  (the fully-connected graph with four nodes). This configuration is exactly solvable even when edges and nodes have distinct reliabilities [6]. For perfect nodes, and with all edge reliabilities equal to  $p$ , the two-terminal reliability has the simple form

$$\mathcal{R}_{n \geq 1}(p) = \alpha_+(p) \zeta_+^n(p) + \alpha_-(p) \zeta_-^n(p), \quad (20)$$

$$\zeta_{\pm}(p) = \frac{p}{2} \left[ 2 + 4p - 14p^2 + 13p^3 - 4p^4 \pm \sqrt{\mathcal{A}(p)} \right], \quad (21)$$

$$\alpha_{\pm}(p) = \frac{1+p}{4} \pm \frac{2+2p+10p^2-27p^3+19p^4-4p^5}{4\sqrt{\mathcal{A}(p)}}, \quad (22)$$

$$\mathcal{A}(p) = 4 + 32p^2 - 204p^3 + 452p^4 - 516p^5 + 329p^6 - 112p^7 + 16p^8. \quad (23)$$

The two eigenvalues are displayed in Fig. 2 as functions of  $p$ . As  $n$  increases, the contribution of  $\zeta_+$  prevails over that of  $\zeta_-$ . Therefore, when  $n$  is large,

$$\mathcal{R}_n(p) \approx \alpha_+(p) \zeta_+^n(p), \quad (24)$$

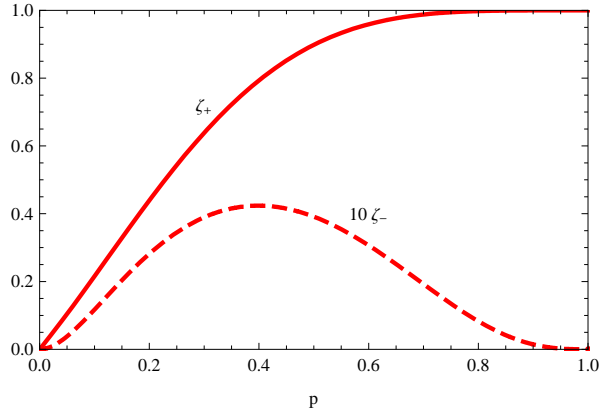


Fig. 2.  $\zeta_{\pm}$  for the  $K_4$  ladder.

because the contribution of  $\zeta_-^n(p)$  vanishes exponentially. The approximation given by eq. (24) lies at the heart of our method for deriving the MTF's asymptotic expansion. Inserting it into eq. (10) gives

$$\langle t \rangle \approx \frac{1}{\lambda} \int_0^1 \frac{dp}{p} \alpha_+(p) \zeta_+^n(p). \quad (25)$$

Here, the lower bound (zero) does not play a significant role because  $\zeta_{\pm}(0) = 0$ . As  $n$  increases,  $\zeta_+^n(p)$  is negligible except close to  $p = 1$ , as illustrated in Fig. 3.

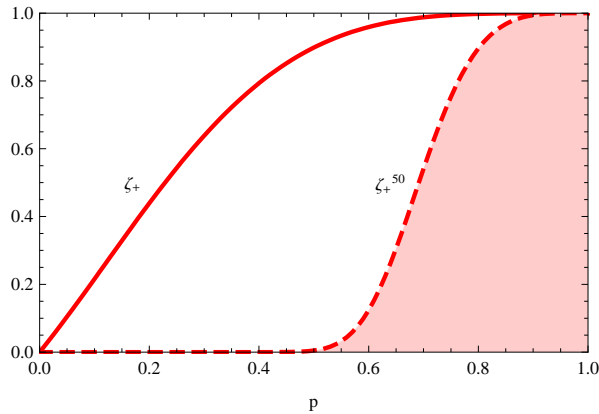


Fig. 3. Different powers of  $\zeta_+$  as a function of  $p$ .

The gist of the asymptotic expansion is therefore to consider  $\alpha_+$  and  $\zeta_+$  in the vicinity of unity. Setting  $q = 1 - p$ , we have from eqs. (21)–(22)

$$\alpha_+(1 - q) = 1 - 2q^3 + 4q^5 - 3q^6 + 6q^7 + \dots, \quad (26)$$

$$-\ln \zeta_+(1 - q) = q^4 + 2q^5 - 4q^7 + \frac{9}{2}q^8 + \dots. \quad (27)$$



Note that  $\zeta_+(1) = \alpha_+(1) = 1$  because  $\mathcal{R}_n(p=1) = 1$ . We can write

$$\langle t \rangle \approx \frac{1}{\lambda} \int_0^1 \frac{dq}{1-q} \alpha_+(1-q) \exp[-n(-\ln \zeta_+(1-q))]. \quad (28)$$

At this point, we have to rescale the variable  $q$  in order to extract the asymptotic behavior of the integral. Equation (27) gives

$$\exp[-n(-\ln \zeta_+(1-q))] = \exp[-n(q^4 + 2q^5 - 4q^7 + \dots)]; \quad (29)$$

this suggests setting  $\tau = nq^4$ , or equivalently  $q = \tau^{1/4} n^{-1/4}$ , so that

$$\exp[-n(-\ln \zeta_+(1-q))] = e^{-\tau} \exp[-n(2\frac{\tau^{5/4}}{n^{5/4}} - 4\frac{\tau^{7/4}}{n^{7/4}} + \dots)]. \quad (30)$$

In the last exponential, the argument is  $-2\tau^{5/4} n^{-1/4} + 4\tau^{7/4} n^{-3/4} + \dots$ . Because of the  $\exp(-\tau)$  factor, we can neglect the contribution of large  $\tau$ 's, so that when  $n$  is large, we only need to expand the second exponential and all other factors in the limit  $\tau \rightarrow 0$ :

$$\langle t \rangle \approx \frac{1}{\lambda} \int_0^n \frac{1}{4n^{1/4}} \frac{d\tau \tau^{-3/4}}{1 - \frac{\tau^{1/4}}{n^{1/4}}} (1 - 2\frac{\tau^{3/4}}{n^{3/4}} + 4\frac{\tau^{5/4}}{n^{5/4}} + \dots) e^{-\tau} \exp[-(2\frac{\tau^{5/4}}{n^{1/4}} - 4\frac{\tau^{7/4}}{n^{3/4}} + \dots)]. \quad (31)$$

The error made by replacing the upper bound of the integral by  $+\infty$  vanishes exponentially as  $n$  goes to infinity. Keeping only the prevailing terms in each factor of eq. (31) leads to

$$\begin{aligned} \langle t \rangle &\rightarrow \frac{1}{\lambda} \int_0^\infty \frac{1}{4n^{1/4}} d\tau \tau^{-3/4} e^{-\tau} \\ &= \frac{1}{\lambda} \frac{1}{4n^{1/4}} \Gamma(1/4) = \frac{1}{\lambda} \frac{\Gamma(5/4)}{n^{1/4}} \approx \frac{0.906402}{\lambda n^{1/4}}. \end{aligned} \quad (32)$$

For the leading-order term (and this term only),  $\alpha_+$  does not play any role since it may safely be replaced with 1. The asymptotic  $n$ -dependence is not  $1/n$  or  $\ln n$  anymore as in the series and parallel cases, but a power-law,  $n^{-1/4}$ , which slowly decreases with  $n$ . The following terms of the expansion may be derived easily by expanding all the factors in eq. (31):

$$\lambda \langle t \rangle = \frac{\Gamma(5/4)}{n^{1/4}} + \frac{17}{32} \frac{\Gamma(3/4)}{n^{3/4}} - \frac{3}{4n} - \frac{293}{512} \frac{\Gamma(5/4)}{n^{5/4}} + \dots \quad (33)$$

The exact MTTF is obtained straightforwardly by using eq. (10) and the value  $\mathcal{R}_n(p)$  deduced from the three-term recursion relation at the origin of eq. (20):

$$\begin{aligned}\mathcal{R}_n(p) &= (\zeta_+(p) + \zeta_-(p)) \mathcal{R}_{n-1}(p) - \zeta_+(p) \zeta_-(p) \mathcal{R}_{n-2}(p) \\ &= p(2 + 4p - 14p^2 + 13p^3 - 4p^4) \mathcal{R}_{n-1}(p) \\ &\quad - p^3(4 - 18p + 36p^2 - 42p^3 + 30p^4 - 12p^5 + 2p^6) \mathcal{R}_{n-2}(p),\end{aligned}\quad (34)$$

with  $\mathcal{R}_0(p) = 1$  and  $\mathcal{R}_1(p) = p(1 + 2p - 7p^3 + 7p^4 - 2p^5)$ . The exact values of the MTTF are then obtained by a simple integration of  $\mathcal{R}_n(p)$ . The exact and the asymptotic (limited to the first three terms of the expansion) results for the MTTF are plotted in Fig. 4. Even for moderate values of  $n$ , the agreement between the two is good.

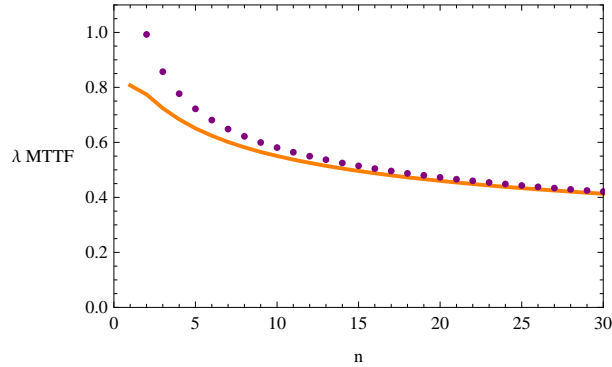


Fig. 4. Comparison between exact and asymptotic (see eq. (33)) MTTF.

Following this method, we also find the asymptotic expansion of  $\langle t^2 \rangle$  by adding the factor  $-2 \ln\left(1 - \frac{\tau^{1/4}}{n^{1/4}}\right)$  and another  $1/\lambda$  in eq. (31). As regards the leading term of the expansion, a mere factor  $2 \frac{\tau^{1/4}}{n^{1/4}} \lambda^{-1}$  is added in the integral. Finally

$$\lambda^2 \langle t^2 \rangle = \frac{\sqrt{\pi}}{2\sqrt{n}} + \frac{17}{12n} - \frac{13\Gamma(9/4)}{10n^{5/4}} - \frac{221\sqrt{\pi}}{480n^{3/2}} + \dots \quad (35)$$

Further terms can be routinely obtained using mathematical software such as MATHEMATICA.

After simplification, the variance of the distribution is therefore deduced to behave as

$$\langle t^2 \rangle - \langle t \rangle^2 = \frac{1}{\lambda^2} \left[ \left( \frac{\sqrt{\pi}}{2} - \Gamma(5/4)^2 \right) \frac{1}{\sqrt{n}} + \frac{17}{12n} \left( 1 - \frac{3\pi\sqrt{2}}{16} \right) + \dots \right]. \quad (36)$$

We could perform similar calculations for higher moments or cumulants, and again would find asymptotic power-law behaviors.

### 3.2 Generalized fan

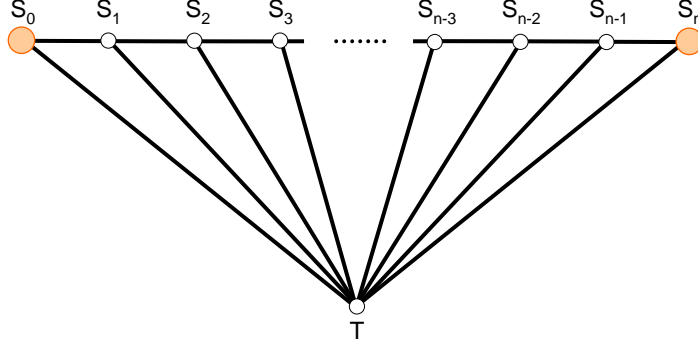


Fig. 5. Generalized fan: the source is  $S_0$ , the destination  $S_n$ .

This architecture, displayed in Fig. 5, has been considered in previous studies [11,12,13] and recently solved for the two-terminal reliability  $\mathcal{R}_n$  between  $S_0$  and  $S_n$  [5]. For perfect nodes,

$$\mathcal{R}_n = \frac{p^2}{(1-p(1-p))^2} + p^n (1-p)^{n+2} \left[ \frac{np}{(1-p(1-p))} + \frac{1+p^2}{(1-p(1-p))^2} \right]. \quad (37)$$

When  $n \rightarrow \infty$ ,  $\mathcal{R}_n$  clearly tends to the constant  $\mathcal{R}_\infty = \frac{p^2}{(1-p(1-p))^2} \neq 1$  ( $p(1-p)$  is always less than  $1/4$ , so the last contribution in eq. (37) decreases faster than  $4^{-n}$ ). This stems from the existence of one path with a finite number of hops, namely  $S_0 \rightarrow T \rightarrow S_n$ . The MTTF's asymptotic behavior is therefore different from that of the preceding section: it does not vary with  $n$ . This is also true for higher moments, with

$$\langle t^m \rangle_\infty = \frac{m}{\lambda^m} \int_0^1 dp (-\ln p)^{m-1} \frac{p}{(1-p(1-p))^2}, \quad (38)$$

where  $\langle t^m \rangle_\infty = \lim_{n \rightarrow \infty} \langle t^m \rangle_n$ . The first of these integrals are

$$\lambda \langle t \rangle_\infty = \frac{9 + 2\pi\sqrt{3}}{27} \approx 0.736400, \quad (39)$$

$$\lambda^2 \langle t^2 \rangle_\infty = \frac{2}{9} \psi'(1/3) - \frac{4}{27} \pi^2 \approx 0.781302, \quad (40)$$

where  $\psi'$  is the derivative of the digamma function  $\psi$ . From the first values of  $\langle t^m \rangle$ , we can infer the general result

$$\lambda^m \langle t^m \rangle_\infty = (-1)^m \frac{m}{3^{m+1}} \left( 1 + \frac{1}{2^{m-1}} \right) (\psi^{(m-1)}(1/3) - \psi^{(m-1)}(2/3))$$

$$-\frac{m!}{3^{m-1}} \left(1 - \frac{1}{2^{m-2}}\right) (3^{m-2} - 1) \zeta(m-1). \quad (41)$$

Depending on the parity of  $m$ , the difference  $\psi^{(m-1)}(1/3) - \psi^{(m-1)}(2/3)$  may actually be further simplified (leaving only  $\psi^{(m-1)}(1/3)$  for  $m$  even, or powers of  $\pi$  for  $m$  odd). It is easy to prove that, asymptotically,

$$\langle t^m \rangle_\infty \sim \frac{m!}{2^m \lambda^m}. \quad (42)$$

In that limit, it looks as if only the  $S_0 \rightarrow T \rightarrow S_n$  connection exists.

### 3.3 Double fan

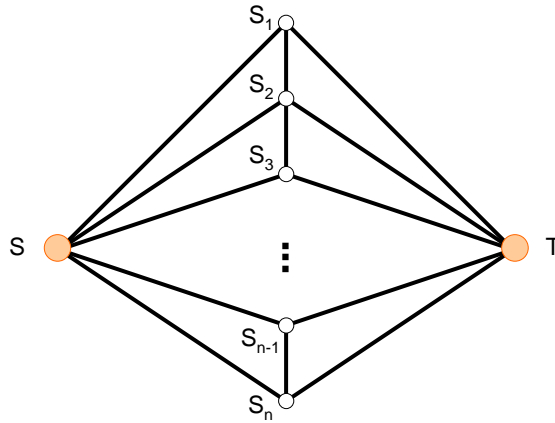


Fig. 6. Double fan: the source is  $S$ , the destination  $T$ .

This configuration, displayed in Fig. 6, is a slight generalization of  $n$  double links in parallel. As  $n \rightarrow \infty$ , there is an infinity of paths of finite length connecting  $S$  to  $T$ ; for perfect nodes, the associated two-terminal reliability  $\mathcal{R}_n$  is [7]

$$\mathcal{R}_n = 1 - \alpha_+ \zeta_+^n - \alpha_- \zeta_-^n, \quad (43)$$

with

$$\zeta_\pm = \frac{1-p}{2} \left(1 + 2p(1-p) \pm \sqrt{1 + 4p^2(1-p)^2}\right), \quad (44)$$

$$\alpha_\pm = \frac{1}{2} \pm \frac{1}{2} \frac{1 + 2p^2}{\sqrt{1 + 4p^2(1-p)^2}}. \quad (45)$$

Here again — if we forget that 1 is a third eigenvalue — we have two eigenvalues  $\zeta_\pm$ . However, the situation is different from that of the  $K_4$  ladder, because

$\zeta_{\pm} \rightarrow 0$  when  $p \rightarrow 1$ , while  $\zeta_+ \rightarrow 1$  and  $\alpha_+ \rightarrow 1$  when  $p \rightarrow 0$ . We also expect that

$$\mathcal{R}_n \geq 1 - (1 - p^2)^n, \quad (46)$$

the right-hand side of eq. (46) corresponding to  $n$  elements of reliability  $p^2$  in parallel.

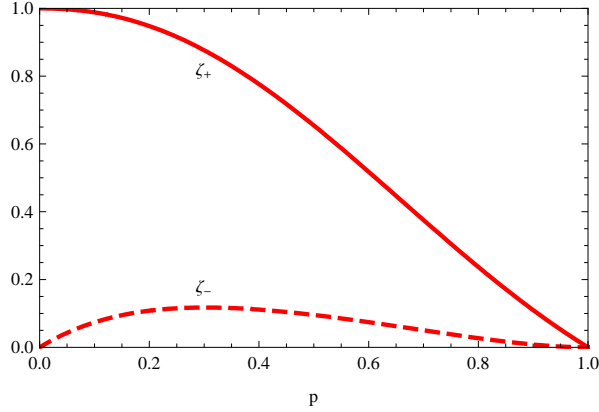


Fig. 7.  $\zeta_{\pm}$  for the double fan.

As  $n$  increases, the contribution of  $\zeta_+$  prevails over that of  $\zeta_-$  (see Fig. 7), so that  $\mathcal{R}_n \approx 1 - \alpha_+ \zeta_+^n$ . Consequently,

$$\langle t \rangle_n \approx \frac{1}{\lambda} \int_0^1 \frac{dp}{p} (1 - \alpha_+ \zeta_+^n). \quad (47)$$

Because of the  $1/p$  factor, the asymptotic expansion of  $\langle t \rangle_n$  is now controlled by the behaviors of  $\zeta_+$  and  $\alpha_+$  for  $p \rightarrow 0$ :

$$\zeta_+ = 1 - p^2 - 2p^3 + 2p^4 + 4p^5 - 8p^6 - 4p^7 + \dots, \quad (48)$$

$$\alpha_+ = 1 + 2p^3 - 8p^5 + 12p^6 + 24p^7 + \dots. \quad (49)$$

We can write

$$\begin{aligned} 1 - \alpha_+ \zeta_+^n &= 1 - (1 - p^2)^n \\ &\quad + (1 - p^2)^n (1 - \exp[-n(-\ln \zeta_+ + \ln(1 - p^2))]) \\ &\quad + (1 - \alpha_+) \exp[-n(-\ln \zeta_+)] \end{aligned} \quad (50)$$

Each term on the right-hand side of eq. (50) vanishes for  $p \rightarrow 0$ , so that the  $1/p$  factor does not lead to a diverging integral in eq. (47). The first term of eq. (50) gives the prevailing contribution, namely

$$\begin{aligned}
\frac{1}{\lambda} \int_0^1 \frac{dp}{p} (1 - (1 - p^2)^n) &= \frac{1}{2\lambda} \int_0^1 \frac{dr}{r} (1 - (1 - r)^n) \\
&= \frac{1}{2\lambda} \int_0^1 \frac{ds}{1-s} (1 - s^n) = \frac{1}{2\lambda} \sum_{i=0}^n \frac{1}{i}. \tag{51}
\end{aligned}$$

This contribution is — unsurprisingly — half the usual result for  $n$  elements in parallel, because the reliability  $p^2$  translates into a  $2\lambda$  failure rate. For the two other contributions, the change of variable  $\tau = np^2$  gives a factor  $\exp(-\tau)$ ; the remaining factors in eq. (50) must be expanded in the vicinity of  $\tau \rightarrow 0$ , as in section 3.1. Summing the three contributions gives

$$\lambda \langle t \rangle_n \rightarrow \frac{\ln n + \mathbf{C}}{2} + \frac{\sqrt{\pi}}{2\sqrt{n}} - \frac{11}{4n} + \frac{95\sqrt{\pi}}{16n^{3/2}} - \frac{1321}{24n^2} + \dots \tag{52}$$

A comparison of the exact results with the asymptotic expansion in which we have kept the first three terms of eq. (52) is plotted in Fig. 8. The agreement is good, even for moderate values of  $n$ .

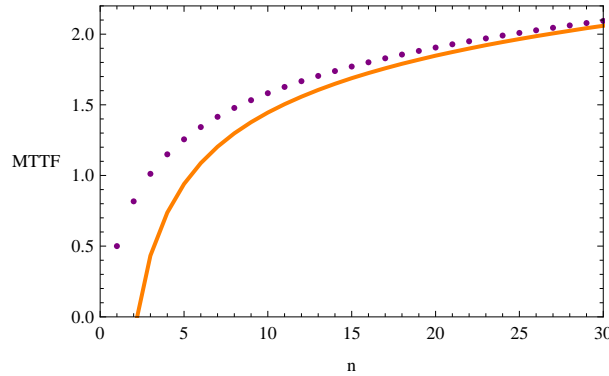


Fig. 8. Comparison between exact (dots) and asymptotic  $\frac{\ln n + \mathbf{C}}{2} + \frac{\sqrt{\pi}}{2\sqrt{n}} - \frac{11}{4n}$  MTTF's for the double fan, in units of  $\lambda^{-1}$ .

### 3.4 Street $3 \times n$

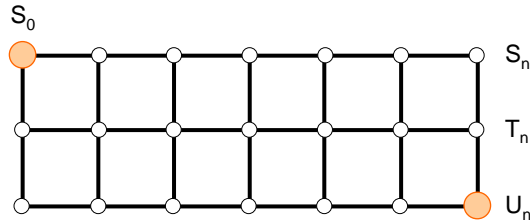


Fig. 9. Street  $3 \times n$ : the source is  $S_0$ , the destination  $U_n$ .

In the preceding subsections, we have considered architectures for which  $\mathcal{R}_n$  is exactly known through the analytic expressions of two eigenvalues  $\zeta_{\pm}$ . In complex systems, more than two eigenvalues may coexist, and be known only as roots of polynomial equations. However, the MTTF's asymptotic expression can still be derived from the knowledge of the generating function  $\mathcal{G}(z)$  [14] of the  $\mathcal{R}_n$ 's, namely

$$\mathcal{G}(z) = \sum_n \mathcal{R}_n z^n. \quad (53)$$

Such is the case of the Street  $3 \times n$ , displayed in Fig. 9. This configuration has been studied for the two-terminal reliability  $\mathcal{R}_n$  between  $S_0$  and  $U_n$  [6,15,16,17,18,19,20,21] (there is actually an offset of 1 between our  $n$  and these references'  $n$  because our source is  $S_0$ ). For perfect nodes,  $\mathcal{G}$  is given by  $\mathcal{N}/(\mathcal{D}_1 \mathcal{D}_2)$ , with [6]

$$\begin{aligned} \mathcal{N} = & p^2 - (1-p)p^4(3+3p-4p^2)z \\ & + (1-p)^3 p^6(2+11p-3p^2-2p^3)z^2 \\ & + (1-p)^3 p^8(2-4p+3p^2+11p^3-13p^4+3p^5)z^3 \\ & - (1-p)^4 p^{10}(3+6p-12p^2+10p^3-10p^4+4p^5)z^4 \\ & + (1-p)^6 p^{12}(1+8p-p^2-5p^3-p^4+p^5)z^5 \\ & - (1-p)^8 p^{15}(2+5p-4p^2)z^6 + (1-p)^{10} p^{18} z^7, \end{aligned} \quad (54)$$

$$\begin{aligned} \mathcal{D}_1 = & 1 - (1-p^2)p(1+p-p^2)z \\ & + (1-p)^2 p^3(1+p+p^2-2p^3)z^2 - (1-p)^4 p^6 z^3, \end{aligned} \quad (55)$$

$$\begin{aligned} \mathcal{D}_2 = & 1 - p(2+2p+p^2-9p^3+5p^4)z \\ & + (1-p)p^2(1+5p+5p^2-6p^3-15p^4 \\ & \quad + 13p^5 + p^6 - 2p^7)z^2 \\ & - (1-p)^2 p^4(2+6p+6p^2-26p^3+17p^4 \\ & \quad - 18p^5 + 27p^6 - 16p^7 + 3p^8)z^3 \\ & + (1-p)^4 p^6(1+6p+4p^2-p^3-17p^4 \\ & \quad + 9p^5 + 3p^6 - 2p^7)z^4 \\ & - (1-p)^6 p^9(2+4p+p^2-7p^3+3p^4)z^5 \\ & + (1-p)^8 p^{12} z^6. \end{aligned} \quad (56)$$

$\mathcal{N}$ ,  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are polynomials in both  $z$  and  $p$ .

The eigenvalue of greatest modulus, named  $\zeta_+$  again, actually obeys  $\mathcal{D}_2 = 0$  for  $z = 1/\zeta_+$  (in the limit  $p \rightarrow 1$ ,  $\mathcal{D}_2 \rightarrow 1 - z$  and  $\zeta_+ \rightarrow 1$ ); all other eigenvalues tend to zero in that limit. Even though it is not possible to get an analytic expression for  $\zeta_+$  as a function of  $p$  ( $\mathcal{D}_2$  is of degree 6 in  $z$ ), we can readily compute it numerically. We can also deduce from the constraint  $\mathcal{D}_2(z = 1/\zeta_+) = 0$  the expansion of  $\zeta_+$  as a function of  $q$  for small  $q$ 's, starting with  $\zeta_+ = 1$ :

$$\zeta_+ = 1 - q^3 - 4q^4 - 4q^5 + 14q^6 + \dots, \quad (57)$$

$$-\ln \zeta_+ = q^3 + 4q^4 + 4q^5 - \frac{27}{2}q^6 + \dots. \quad (58)$$

The scaling variable  $\tau$  should be  $n$  times the leading term of eq. (58), namely  $\tau = nq^3$ , so that

$$\begin{aligned} \zeta_+^n &= e^{-\tau} \exp -n \left( 4 \left( \frac{\tau}{n} \right)^{4/3} + 4 \left( \frac{\tau}{n} \right)^{5/3} + \dots \right) \\ &= e^{-\tau} \exp \left( -4\tau^{4/3} n^{-1/3} - 4\tau^{5/3} n^{-2/3} + \dots \right). \end{aligned} \quad (59)$$

$\alpha_+$  is deduced from  $p$  and the numerical value of  $\zeta_+$  through the residue of  $\mathcal{G}$  at  $z = 1/\zeta_+$ . The general result is

$$\alpha_+ = \frac{-\zeta_+ \mathcal{N} \left( \frac{1}{\zeta_+} \right)}{\mathcal{D}'_z \left( \frac{1}{\zeta_+} \right)}, \quad (60)$$

where  $\mathcal{D}$  is the denominator of  $\mathcal{G}$  and  $\mathcal{D}'_z = \frac{\partial \mathcal{D}}{\partial z}$ . Here,  $\mathcal{D} = \mathcal{D}_1 \mathcal{D}_2$ , leading to

$$\alpha_+ \rightarrow 1 - 2q^2 - 4q^3 + 7q^4 + 22q^5 + 20q^6 + \dots. \quad (61)$$

After inserting eqs. (59) and (61) in eq. (29), and further series expansions and integration similar to those of Section 3.1, the final asymptotic expansion reads

$$\lambda \langle t \rangle_n \rightarrow \frac{\Gamma(4/3)}{n^{1/3}} - \frac{5}{9} \frac{\Gamma(2/3)}{n^{2/3}} + \frac{7}{3n} + \dots. \quad (62)$$

Here again, it exhibits a power-law behavior, but  $n^{-1/3}$  this time. In the next section, we relate the exponent to a specific property of the network.



## 4 General case

In the preceding section, we have derived the asymptotic MTTF when the two-terminal reliability  $\mathcal{R}_n$  is known, at least implicitly through a recursion relation. Here, we want to show that the leading terms of the MTTF and other moments may be obtained for arbitrary, recursive configurations. As shown in [6],  $\mathcal{R}_n$  can be generally expressed as a product of transfer matrices — whose size may be large but remains finite. For identical edge reliabilities  $p$ , the asymptotic behavior is controlled by the largest eigenvalue  $\zeta_+$  of the (now unique) transfer matrix. We consider in the following architectures that look like some kind of “series-like” system, albeit more complex than those of the  $K_4$  ladder and the Street  $3 \times n$ , or to a “parallel-like” one, like the double fan.

### 4.1 “Series-like” configuration

This happens when the shortest path connecting the source to the destination has a length equivalent to  $n$  as  $n \rightarrow \infty$ . Here, we use again  $\mathcal{R}_n \approx \alpha_+ \zeta_+^n$ , and the MTTF is still controlled by the behavior of  $\zeta_+$  and  $\alpha_+$  in the vicinity of  $p = 1$ . The relevant expansions for  $q \rightarrow 0$  have the form

$$-\ln \zeta_+(1-q) = \alpha_i q^i + \alpha_{i+1} q^{i+1} + \dots, \quad (63)$$

$$\alpha_+(1-q) = 1 + \alpha'_1 q + \alpha'_2 q^2 + \dots, \quad (64)$$

from which

$$\langle t \rangle_n = \frac{1}{\lambda} \int_0^1 \frac{dq}{1-q} (1 + \alpha'_1 q + \alpha'_2 q^2 + \dots) \exp \left[ -n (-\ln \zeta_+(1-q) - \alpha_i q^i) \right] e^{-n \alpha_i q^i}. \quad (65)$$

The adequate change of variable is now  $\tau = n \alpha_i q^i$ , or equivalently  $q = \frac{\tau^{1/i}}{(n \alpha_i)^{1/i}}$ ; the upper bound of the integral,  $n \alpha_i$ , may again be replaced by  $+\infty$ . The first term in the asymptotic expansion is therefore

$$\begin{aligned} \langle t \rangle_n &\rightarrow \frac{1}{\lambda} \int_0^\infty \frac{1}{i} \frac{1}{(n \alpha_i)^{1/i}} \tau^{1/i-1} d\tau e^{-\tau} \\ &= \frac{1}{\lambda} \frac{\Gamma(1 + 1/i)}{(n \alpha_i)^{1/i}}. \end{aligned} \quad (66)$$

Likewise, in order to calculate the first term of the asymptotic expansion of  $\lambda^2 \langle t^2 \rangle_n$ , we have an additional factor  $-2 \ln(1 - q)$ , which is equivalent to  $2q$  when  $q \rightarrow 0$  and brings an extra  $\tau^{1/i}$ . Finally,

$$\langle t^2 \rangle_n \rightarrow \frac{1}{\lambda^2} \frac{\Gamma(1 + 2/i)}{(n \alpha_i)^{2/i}}, \quad (67)$$

so that the variance goes as

$$\langle t^2 \rangle_n - \langle t \rangle_n^2 \rightarrow \frac{1}{\lambda^2} \frac{\Gamma(1 + 2/i) - \Gamma(1 + 1/i)^2}{(n \alpha_i)^{2/i}}, \quad (68)$$

from which

$$\frac{\sqrt{\langle t^2 \rangle_n - \langle t \rangle_n^2}}{\langle t \rangle_n} \rightarrow \sqrt{\frac{\Gamma(1 + 2/i)}{\Gamma(1 + 1/i)^2} - 1}, \quad (69)$$

which is independent of  $n$ . Actually, it only depends on  $i$ , which is the lowest order of the  $q$ -dependence of  $1 - \zeta_+$  when  $q \rightarrow 0$  (see eq. (63)).

For the higher moments  $\langle t^m \rangle_n$ , the generalization is straightforward; we can also go beyond the first order in the expansion, following the recipe of the preceding section. Setting  $\eta = (n \alpha_i)^{-1/i}$ , we find

$$\begin{aligned} \lambda^m \langle t^m \rangle_n = \eta^m \left\{ \right. & \Gamma\left(1 + \frac{m}{i}\right) \\ & + \eta \frac{m}{i} \left[ \frac{1}{2} (1 + m + 2 \alpha'_1) \Gamma\left(\frac{1 + m}{i}\right) - \frac{\alpha_{i+1}}{\alpha_i} \Gamma\left(1 + \frac{1 + m}{i}\right) \right] \\ & + \eta^2 \frac{m}{i} \left[ \frac{10 + 11m + 3m^2 + 12(1 + m)\alpha'_1 + 24\alpha'_2}{24} \Gamma\left(\frac{2 + m}{i}\right) \right. \\ & \quad \left. - \left( \frac{\alpha_{i+2}}{\alpha_i} + \frac{\alpha_{i+1}}{\alpha_i} \frac{1 + m + 2\alpha'_1}{2} \right) \Gamma\left(1 + \frac{2 + m}{i}\right) \right. \\ & \quad \left. + \frac{1}{2} \left( \frac{\alpha_{i+1}}{\alpha_i} \right)^2 \Gamma\left(2 + \frac{2 + m}{i}\right) \right] + \dots \left. \right\}. \quad (70) \end{aligned}$$

We may wonder: is this result merely formal, or is it actually possible to determine  $\lambda^m \langle t^m \rangle_n$  for an arbitrary, recursive network? The answer to this question is yes. Even though we do not know the exact value of  $\mathcal{R}_n$  or the associated greatest eigenvalue  $\zeta_+$ , we can still infer the  $\alpha_k$ 's and  $\alpha'_j$ 's appearing in eqs. (63)–(64) because for large  $n$ ,  $\mathcal{R}_n \approx \alpha_+ \zeta_+^n$ . These parameters can be

deduced from the expansion of the unavailability  $\mathcal{U}_n = 1 - \mathcal{R}_n$  for  $q \rightarrow 0$ . We have

$$\begin{aligned} \mathcal{U}_n &\approx 1 - \alpha_+ e^{-n(-\ln \zeta_+)} \\ &= 1 - (1 + \alpha'_1 q + \alpha'_2 q^2 + \dots) \exp \left[ -n \left( \alpha_i q^i + \alpha_{i+1} q^{i+1} + \dots \right) \right] \end{aligned} \quad (71)$$

and must keep track of the successive powers of  $q$ , along with their dependence with  $n$ . Let us illustrate this claim with the Street  $3 \times n$  case. A simple cut enumeration gives (when  $n$  is large, so as to avoid “boundary” effects)

$$\mathcal{U}_n = 2q^2 + (n+4)q^3 + (4n-7)q^4 + \dots \quad (72)$$

The first term of the right-hand side of eq. (72) is easy to obtain. There are only two cuts of order 2 preventing a connection between source and destination (see the top of Fig. 10), hence the  $2q^2$  term. For the cuts of order 3, different possibilities occur as displayed at the bottom of Fig. 10. Firstly, three parallel links may fail; there are  $n$  such instances. Secondly, close to the source or the destination, there are four triple failures (only two are represented by green stars in Fig. 10, the remaining ones can be deduced by symmetry). This gives the  $(n+4)q^3$  term. A comparison between eqs. (71) and (72) gives  $\alpha'_1 = 0$ ,  $\alpha'_2 = -2$ ,  $\alpha'_3 = -4$ , and  $\alpha_3 = 1$ . From these values, we get

$$\lambda^m \langle t^m \rangle_n = \frac{1}{n^{1/3}} \left( \Gamma \left( 1 + \frac{m}{3} \right) - \frac{5}{6} \frac{1}{n^{1/3}} \Gamma \left( 1 + \frac{m+1}{3} \right) + \dots \right), \quad (73)$$

which agrees with eq. (62) when  $m = 1$ .

In conclusion, the exponent of the power-law behavior in  $n$  is nothing but the inverse of the number of necessary cuts to isolate each elementary cell from its neighbors. Note that  $\alpha_i$  is not necessarily equal to 1, since it represents the number of independent cuts of order  $i$ .

#### 4.2 “Parallel-like” configuration

We can also consider configurations where the reliability is asymptotically equal to 1 when  $n \rightarrow \infty$ , as in section 3.3. In the vicinity of  $p \rightarrow 0$  (the relevant domain here),  $\mathcal{R}_n \approx 1 - \alpha_+ \zeta_+^n$  and the needed expansions are

$$\begin{aligned} -\ln \zeta_+ &= \beta_i p^i + \beta_{i+1} p^{i+1} + \dots, \\ \alpha_+ &= 1 + \beta'_1 p + \beta'_2 p^2 + \dots, \end{aligned} \quad (74)$$

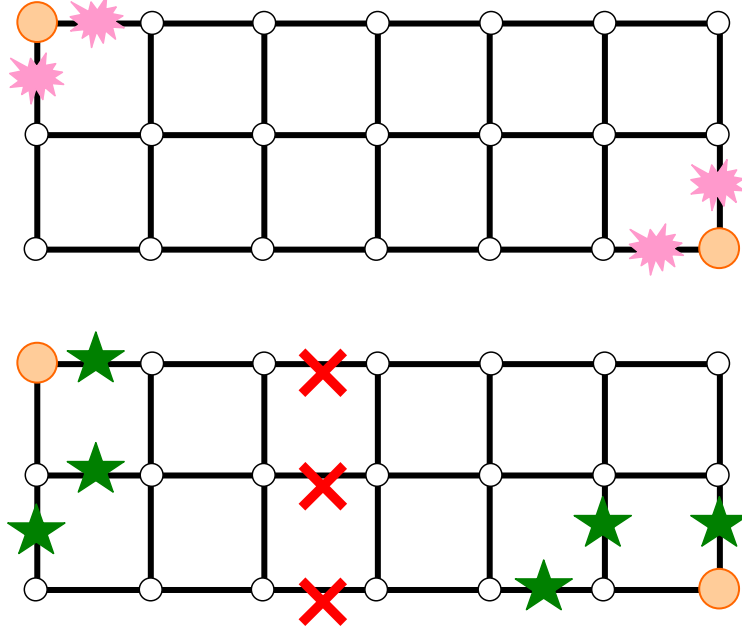


Fig. 10. Cuts of order 2 (top) and 3 (bottom) for the Street  $3 \times n$ . Failed edges are indicated by stars and crosses.

so that

$$1 - \alpha_+ \zeta_+^n = \underbrace{\left(1 - (1 - p^i)^{\beta_i n}\right)}_A + \underbrace{\left((1 - p^i)^{\beta_i n} - \zeta_+^n\right)}_B + \underbrace{(1 - \alpha_+) \zeta_+^n}_C. \quad (76)$$

The contribution of A is (omitting the  $\lambda^{-1}$  factor)

$$A \rightarrow \int_0^1 \frac{dp}{p} \left(1 - (1 - p^i)^{\beta_i n}\right) = \frac{1}{i} \sum_{k=1}^{\beta_i n} \frac{1}{k} \rightarrow \frac{1}{i} \left(\ln \beta_i n + \mathbf{C} + \frac{1}{2\beta_i n} + \dots\right), \quad (77)$$

The contribution of B depends on the value of  $i$  because

$$-\ln \zeta_+ + \beta_i \ln(1 - p^i) = \beta_{i+1} p^{i+1} + \dots + \beta_i \left(-\frac{1}{2} p^{2i} + \dots\right). \quad (78)$$

If  $i = 1$ , we must take the two terms of degree 2 into account; otherwise, only the  $\beta_{i+1} p^{i+1}$  term needs be kept. After asymptotic expansions similar to those performed in the preceding sections, we have

$$B \rightarrow \left(\beta_2 - \frac{\beta_1}{2}\right) \frac{1}{\beta_1^2} \frac{1}{n} + \dots \quad (i = 1) \quad (79)$$

$$\rightarrow \frac{\beta_{i+1}}{i} \Gamma\left(1 + \frac{1}{i}\right) \frac{1}{\beta_i^{1+1/i}} n^{-1/i} + \dots \quad (i > 1). \quad (80)$$

The contribution of  $C$  is easier to compute because  $1 - \alpha_+$  vanishes as  $p \rightarrow 0$ , thereby compensating the singular term  $1/p$  in the integral. With the change of variable  $\tau = n \beta_i p^i$ , we get

$$C \rightarrow \frac{-\beta'_1}{(n \beta_i)^{1/i}} \Gamma\left(1 + \frac{1}{i}\right) + \dots \quad (81)$$

Note that the  $n$ -dependence of  $C$  is  $n^{-1/i}$ , which decreases less rapidly than  $1/n$  if  $i > 1$ . The sum of contributions  $A$ ,  $B$ , and  $C$  finally expands as (for  $i \geq 1$ )

$$\lambda \text{MTTF}_n \rightarrow \frac{1}{i} (\ln(\beta_i n) + \mathbf{C}) + \left(\frac{\beta_{i+1}}{i \beta_i} - \beta'_1\right) \frac{\Gamma\left(1 + \frac{1}{i}\right)}{(n \beta_i)^{1/i}} + \dots \quad (82)$$

As in the preceding subsection 4.1, the coefficients  $\beta_j$ ,  $\beta'_k$ , etc. may be deduced by evaluating the availability through a path enumeration in the limit  $p \rightarrow 0$ . For instance, in the case of the double fan

$$\mathcal{R}_n = n p^2 + 2(n-1) p^3 + \dots \quad (83)$$

This must be compatible with the expansions of  $\alpha_+$  and  $-\ln \zeta_+$  in eqs. (74)–(75). Because eq. (83) has no linear term,  $\beta_1 = \beta'_1 = 0$ . The coefficient of  $p^2$  being equal to  $n$ , we deduce  $i = 2$ ,  $\beta_2 = 1$ , and  $\beta'_2 = 0$ ; the coefficient of  $p^3$  then implies  $\beta_3 = \beta'_3 = 2$ . Inserting these values in eq. (82) gives back the first two terms of eq. (52).

## 5 Approximate reliability of large, recursive, “series-like” systems

We have seen that the asymptotic expansion of the MTTF and the higher moments for a large, recursive system can be obtained with minimal effort. In the same line of thought, is it possible to find an approximate reliability such that all its moments give the same value as the true reliability, at least for the first terms of the expansion in  $n$ . The first term of eq. (70), namely

$$\langle t^m \rangle_n = \frac{\Gamma(1 + m/i)}{\lambda^m (n \alpha_i)^{m/i}}, \quad (84)$$

reminds us of what would be obtained for a Weibull distribution [1,3]. Indeed, it is straightforward to show that

$$R_n^{(0)}(t) = \exp\left(-n \alpha_i \lambda^i t^i\right), \quad (85)$$

would give eq. (84) exactly.

Is it possible to improve this expression, i.e., propose an effective reliability leading to the correct first two terms in the asymptotic expansion of each moment  $\langle t^m \rangle_n$ ? Provided that  $\alpha'_1 = 0$ , i.e., that there is no cut of order one (it would then be easy to factor out this link contribution, and proceed with the remaining parts of the system), the answer is again positive.

Calculating the moments of

$$R_n^{(1)}(t) = \exp \left[ -n (\alpha_i \lambda^i t^i + \tilde{\alpha}_{i+1} \lambda^{i+1} t^{i+1}) \right], \quad (86)$$

we find that

$$\lambda^m \langle t^m \rangle_n = \frac{1}{(n \alpha_i)^{m/i}} \left( \Gamma \left( 1 + \frac{m}{i} \right) + \frac{\tilde{\alpha}_{i+1}}{\alpha_i} \frac{1}{(n \alpha_i)^{m/i}} \left( -\frac{m}{i} \Gamma \left( 1 + \frac{m+1}{i} \right) \right) + \dots \right). \quad (87)$$

Equations (70) and (87) match if

$$\frac{\tilde{\alpha}_{i+1}}{\alpha_i} = \frac{\alpha_{i+1}}{\alpha_i} - \frac{1}{2} \frac{m+1 + 2\alpha'_1}{\frac{m+1}{i}}, \quad (88)$$

so that for  $\alpha'_1 = 0$ , the constraint is satisfied when

$$\tilde{\alpha}_{i+1} = \alpha_{i+1} - \frac{i}{2} \alpha_i. \quad (89)$$

This finally gives

$$R_n^{(1)}(t) = \exp \left[ -n \left( \alpha_i \lambda^i t^i + \left( \alpha_{i+1} - \frac{i}{2} \alpha_i \right) \lambda^{i+1} t^{i+1} \right) \right]. \quad (90)$$

This expression slightly improves over the Weibull distribution of eq. (85). In the case of the Street  $3 \times n$ ,

$$R_n^{(1)}(t) = \exp \left[ -n \left( \lambda^3 t^3 + \frac{5}{2} \lambda^4 t^4 \right) \right]. \quad (91)$$

This expression and the true reliability are plotted in Fig. 11; the agreement is already satisfying for  $n = 30$ .

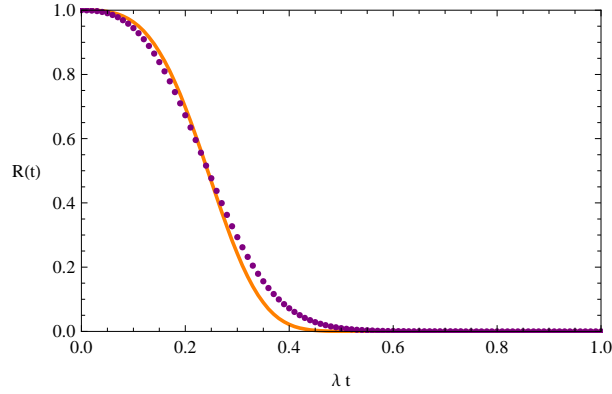


Fig. 11. Exact (dots) and asymptotic (full line) reliability for the Street  $3 \times n$  architecture ( $n = 30$ ).

## 6 Non-exponential distribution functions

In the preceding sections, we have considered elements whose reliability is  $p(t) = \exp(-\lambda t)$ . Although this distribution is often chosen because calculations are simpler, other models may be used: Weibull, gamma, lognormal, etc. [1,3]. We investigate here the influence of the true  $p(t)$  on the  $n$ -dependence of the MTTF and higher moments for “series-like” systems.

We can invert  $p(t)$  as  $t = \chi(p)$ , so that eq. (10) transforms into

$$\text{MTTF}_n = \langle t \rangle_n = \int_0^1 dp \mathcal{R}_n(p) (-\chi'(p)). \quad (92)$$

For higher moments, we would get

$$\langle t^m \rangle_n = m \int_0^1 dp \chi^{m-1}(p) \mathcal{R}_n(p) (-\chi'(p)). \quad (93)$$

For “series-like” configurations, we have again to consider what happens for  $p \rightarrow 1$ , or equivalently for  $t \rightarrow 0$ . Assuming that asymptotically

$$-\chi'(p) \rightarrow a_\beta (1-p)^\beta, \quad (94)$$

we have (because  $\chi(1) = 0$ )

$$\chi(p) = t \rightarrow \frac{a_\beta}{\beta+1} (1-p)^{\beta+1}, \quad (95)$$

so that

$$p \rightarrow 1 - \left( \frac{(\beta + 1)t}{a_\beta} \right)^{1/(\beta+1)}. \quad (96)$$

Keeping  $-\ln \zeta_+ \rightarrow \alpha_i q^i$  and  $\alpha_+ \rightarrow 1$ , we get to lowest order

$$\begin{aligned} \langle t^m \rangle_n &\rightarrow m \int_0^1 dq \left( \frac{a_\beta}{\beta + 1} \right)^{m-1} q^{(\beta+1)(m-1)} a_\beta q^\beta \exp(-n \alpha_i q^i) \\ &\rightarrow \left( \frac{a_\beta}{\beta + 1} \right)^m \frac{1}{(n \alpha_i)^{(\beta+1)m/i}} \Gamma \left( 1 + \frac{(\beta + 1)m}{i} \right). \end{aligned} \quad (97)$$

The  $n$ -dependence of  $\langle t^m \rangle_n$  is therefore affected by the structure of the graph (through  $i$  and  $\alpha_i$ ) and by the true failure-time distribution of each element (through  $a_\beta$  and  $\beta$ ). The exponent of the power-law asymptotic behavior is  $\frac{(\beta + 1)m}{i}$ , and the total reliability goes asymptotically as

$$R_n(t) \approx \exp \left[ -n \alpha_i \left( \frac{(\beta + 1)t}{a_\beta} \right)^{1/(\beta+1)} \right]. \quad (98)$$

## 7 Conclusion

We have shown that very simple asymptotic expansions may be obtained for the mean time to failure (and higher moments) for general recursive, meshed networks. By contrast with the simple series and parallel systems considered in many textbooks, the size-dependence of the MTTF of a “series-like” system follows a power-law behavior, whose exponent is linked to the number of cuts disconnecting an elementary cell to its neighbors. Comparison with the exact results for various architectures show that the agreement is often reached when the system contains a few dozens of the (repeated) pattern structure. A simple, approximate expression for the effective global reliability of the system has also been proposed, which is very simple to derive by a mere enumeration of cut-sets or path-sets.

The calculations have been performed in the context of the two-terminal reliability of general systems; they obviously apply to all-terminal reliability, and would belong to the “series-like” category.



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