



HAL
open science

On exponentials of exponential generating series

Roland Bacher

► **To cite this version:**

| Roland Bacher. On exponentials of exponential generating series. 2008. hal-00292997v1

HAL Id: hal-00292997

<https://hal.science/hal-00292997v1>

Preprint submitted on 3 Jul 2008 (v1), last revised 18 Oct 2010 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On exponentials of exponential generating series

Roland Bacher

July 3, 2008

Abstract¹: Identifying the algebra of exponential generating series with the shuffle algebra of formal power series, one can define an exponential map $\exp_! : X\mathbb{K}[[X]] \rightarrow 1 + X\mathbb{K}[[X]]$ for the associated Lie group formed by exponential generating series with constant coefficient 1 over an arbitrary field \mathbb{K} . The main result of this paper states that the map $\exp_!$ (and its inverse map $\log_!$) induces a bijection between rational, respectively algebraic, series in $X\mathbb{K}[[X]]$ and $1 + X\mathbb{K}[[X]]$ if the field \mathbb{K} is a subfield of the algebraically closed field $\overline{\mathbb{F}}_p$ of characteristic p .

1 Introduction

The equality

$$\left(\sum_{n=0}^{\infty} \alpha_n \frac{X^n}{n!} \right) \left(\sum_{n=0}^{\infty} \beta_n \frac{X^n}{n!} \right) = \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n+m}{n} \alpha_n \beta_m \frac{X^{n+m}}{(n+m)!}$$

shows that we can define an algebra structure on the vector space

$$\mathcal{E}(\mathbb{K}) = \left\{ \sum_{n=0}^{\infty} \alpha_n \frac{X^n}{n!} \mid \alpha_0, \alpha_1, \dots \in \mathbb{K} \right\}$$

of exponential generating series with coefficients $\alpha_0, \alpha_1, \dots$ in an arbitrary field or ring \mathbb{K} . For the sake of simplicity we work in the sequel only over fields. The expression $\alpha_n/n!$ should be considered formally since the numerical value of $n!$ is zero over a field of positive characteristic $p \leq n$.

We denote by

$$\mathfrak{m}_{\mathcal{E}} = \left\{ \sum_{n=1}^{\infty} \alpha_n \frac{X^n}{n!} \mid \alpha_1, \alpha_2, \dots \in \mathbb{K} \right\} \subset \mathcal{E}(\mathbb{K})$$

the maximal ideal of the local algebra $\mathcal{E}(\mathbb{K})$. A straightforward computation shows that $a^n/n!$ is well-defined for $a \in \mathfrak{m}_{\mathcal{E}}$ over an arbitrary field.

¹Keywords: Bell numbers, exponential function, shuffle product, formal power series, rational series, algebraic series, homogeneous form, automaton sequence, Math. class: 11B73, 11B85, 11E08, 11E76, 22E65

Endowing \mathbb{K} with the discrete topology and $\mathcal{E}(\mathbb{K})$ with the topology given by coefficientwise convergency, the functions

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \text{ and } \log(1+a) = - \sum_{n=1}^{\infty} \frac{(-a)^n}{n}$$

are always defined for $a \in \mathfrak{m}_{\mathcal{E}}$.

Switching back to ordinary generating series

$$A = \sum_{n=1}^{\infty} \alpha_n X^n, \quad B = \sum_{n=1}^{\infty} \beta_n X^n \in \mathfrak{m}$$

contained in the maximal ideal $\mathfrak{m} = X\mathbb{K}[[X]]$, of (ordinary) formal power series, we write

$$\exp_!(A) = 1 + B$$

if

$$\exp \left(\sum_{n=1}^{\infty} \alpha_n \frac{X^n}{n!} \right) = 1 + \sum_{n=1}^{\infty} \beta_n \frac{X^n}{n!} .$$

It is easy to see that $\exp_!$ defines a one-to-one map from \mathfrak{m} onto $1 + \mathfrak{m}$ with inverse map

$$1 + B \longmapsto A = \log_!(1 + B) .$$

It satisfies

$$\exp_!(A + B) = \exp_!(A) \sqcup \exp_!(B)$$

for all $A, B \in \mathfrak{m}$ where the shuffle product

$$\left(\sum_{n=0}^{\infty} \alpha_n X^n \right) \sqcup \left(\sum_{n=0}^{\infty} \beta_n X^n \right) = \sum_{n,m=0}^{\infty} \binom{n+m}{n} \alpha_n \beta_m X^{n+m}$$

corresponds to the ordinary product of the associated exponential generating series. The map $\exp_!$ defines thus an isomorphism from the additive group $(\mathfrak{m}, +)$ onto the *special shuffle-group* $(1 + \mathfrak{m}, \sqcup)$ with group-law given by the shuffle-product. It coincides with the familiar exponential map from the Lie algebra \mathfrak{m} into the special shuffle group, considered as an infinite-dimensional Lie group.

Theorem 1.1. *Let \mathbb{K} be a subfield of the algebraically closed field $\overline{\mathbb{F}}_p$ of positive characteristic p . Given a series $A \in \mathfrak{m} = X\mathbb{K}[[X]]$ the following two assertions are equivalent:*

- (i) *A is rational.*
- (ii) *$\exp_!(A)$ is rational.*

Example 1.2. The Bell numbers B_0, B_1, B_2, \dots , see pages 45,46 in [5] or Example 5.2.4 in [6], are the natural integers defined by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}$$

and have combinatorial interpretations.

Reducing the associated ordinary generating series $\sum_{n=0}^{\infty} B_n x^n = \exp_1(x/(1-x))$ modulo a prime p defines the series expansion of a rational fraction. A few reductions of the ordinary generating series

$$\sum_{n=0}^{\infty} B_n x^n = 1 + x + 2x^2 + 5x^3 + 15x^4 + 52x^5 + 203x^6 + 877x^7 + 4140x^8 + \dots$$

of Bell numbers modulo small primes are

$$\frac{1}{1+x+x^2} \pmod{2}, \quad \frac{1+x+x^2}{1-x^2-x^3} \pmod{3}, \quad \frac{1+x+2x^2-x^4}{1-x^4-x^5} \pmod{5}.$$

Theorem 1.3. Let \mathbb{K} be a subfield of the algebraically closed field $\overline{\mathbb{F}}_p$ of positive characteristic p . Given a series $A \in \mathfrak{m} = X\mathbb{K}[[X]]$ the following two assertions are equivalent:

- (i) A is algebraic.
- (ii) $\exp_1(A)$ is algebraic.

Corollary 1.4. Over a subfield $\mathbb{K} \subset \overline{\mathbb{F}}_p$, the group isomorphism

$$\exp_1 : (\mathfrak{m}, +) \longrightarrow (1 + \mathfrak{m}, \sqcup)$$

restricts to an isomorphism between the subgroups of rational elements in $(\mathfrak{m}, +)$ and in $(1 + \mathfrak{m}, \sqcup)$.

It restricts also to an isomorphism between the subgroups of algebraic elements in $(\mathfrak{m}, +)$ and in $(1 + \mathfrak{m}, \sqcup)$.

Theorem 1.1 and 1.3 can be made effective:

Given a rational series $A \in \mathbb{K}[[X]]$ represented by a reduced fraction f/g where f, g with $g \neq 0$ are two coprime polynomials of degree $\deg(f)$ and $\deg(g)$, we set $\|A\| = \max(1 + \deg(f), \deg(g))$.

Theorem 1.5. We have

$$\|\exp_1(A)\| \leq p^{q^{\|A\|}} \quad \text{and} \quad \|\log_1(1+A)\| \leq 1 + \|1+A\|^p$$

for a rational series A in $\mathfrak{m} \subset \overline{\mathbb{F}}_p[[X]]$ having all its coefficients in a finite subfield $\mathbb{F}_q \subset \overline{\mathbb{F}}_p$ containing $q = p^e$ elements.

Given a prime p and a formal power series $C = \sum_{n=0}^{\infty} \gamma_n X^n$ in $\mathbb{K}_p[[X]]$ with coefficients in a subfield \mathbb{K} of $\overline{\mathbb{F}}_p$, we define for $f \in \mathbb{N}$, $k \in \mathbb{N}$, $k < p^f$ the series

$$C_{k,f} = \sum_{n=0}^{\infty} \gamma_{k+np^f} X^n .$$

We denote by $\kappa(C)$ the dimension $\dim(\sum_{k,f} \mathbb{K}C_{k,f}) \in \mathbb{N} \cup \{\infty\}$ of the vector space spanned by all series of the form $C_{k,f}$.

Algebraic series in $\mathbb{K}[[X]]$ for \mathbb{K} a subfield of $\overline{\mathbb{F}}_p$ are characterised by a Theorem of Christol (see Theorem 12.2.5 in [1]) stating that a series C in $\overline{\mathbb{F}}_p[[X]]$ is algebraic if and only if $\kappa(C)$ is finite. We have $\kappa(A+B) \leq \kappa(A) + \kappa(B)$ and an algebraic series $A \in \overline{\mathbb{F}}_p[[X]]$ has a minimal polynomial of degree at most $p^{\kappa(A)}$ with respect to A .

Theorem 1.6. *We have*

$$\kappa(\exp_!(A)) \leq q^{\kappa(A)} p^{q^{\kappa(A)}} \text{ and } \kappa(\log_!(1+A)) \leq 1 + 4(\kappa(1+A))^p$$

for an algebraic series A in $\mathfrak{m} \subset \overline{\mathbb{F}}_p[[X]]$ having all its coefficients in a finite subfield $\mathbb{F}_q \subset \overline{\mathbb{F}}_p$ containing $q = p^e$ elements.

A map $\mu : \mathcal{V} \rightarrow \mathcal{W}$ between two \mathbb{K} -vector spaces is a homogeneous form of degree d if $l \circ \mu : \mathcal{V} \rightarrow \mathbb{K}$ is homogeneous of degree d for every linear form $l : \mathcal{W} \rightarrow \mathbb{K}$.

A useful ingredient for proving Theorems 1.1, 1.3 and their effective versions is the following characterisation of $\log_!$:

Proposition 1.7. *Over a field $\mathbb{K} \subset \overline{\mathbb{F}}_p$, the application $\log_! : 1 + \mathfrak{m} \rightarrow \mathfrak{m}$ extends to a homogeneous form of degree p from $\mathbb{K}[[X]]$ into \mathfrak{m} .*

Example 1.8. *In characteristic 2, we have*

$$\log_! \left(\sum_{n=0}^{\infty} \alpha_n X^n \right) = \sum_{n=0}^{\infty} \alpha_{2^n}^2 X^{2^{n+1}} + \sum_{0 \leq i < j} \binom{i+j}{i} \alpha_i \alpha_j X^{i+j}$$

for $\sum_{n=0}^{\infty} \alpha_n X^n$ in $1 + X\overline{\mathbb{F}}_2[[X]]$.

Notice that Theorems 1.1 and 1.3 fail in characteristic zero: We have $\log_!(1-X) = -\sum_{n=1}^{\infty} (n-1)! X^n$ which is obviously transcendental.

The rest of the paper has two parts. In a first part we recall a few definitions and well-known results and prove all results mentioned above.

In a second part, we generalise Theorems 1.1 and 1.5 to formal power series in several non-commuting variables.

2 Rational and algebraic elements in $\mathbb{K}[[X]]$

This section recalls a few well-known facts concerning rational and algebraic elements in the algebra $\mathbb{K}[[X]]$ of formal power series.

We denote by $\tau : \mathbb{K}[[X]] \longrightarrow \mathbb{K}[[X]]$ the shift operator

$$\tau \left(\sum_{n=0}^{\infty} \alpha_n X^n \right) = \sum_{n=0}^{\infty} \alpha_{n+1} X^n$$

acting on formal power series. The following well-known result characterises rational series:

Proposition 2.1. *A formal power series $A = \sum_{n=0}^{\infty} \alpha_n X^n$ of $\mathbb{K}[[X]]$ is rational if and only if the series $A, \tau(A), \tau^2(A), \dots, \tau^k(A) = \sum_{n=0}^{\infty} \alpha_{n+k} X^n, \dots$ span a finite-dimensional vector-space in $\mathbb{K}[[X]]$.*

More precisely, the vector space spanned by $A, \tau(A), \tau^2(A), \dots, \tau^i(A), \dots$ has dimension $\|A\| = \max(1 + \deg(f), \deg(g))$ if f/g with $f, g \in \mathbb{K}[X]$ is a reduced expression of a rational series A .

The function $A \longrightarrow \|A\|$ satisfies the inequality

$$\|A + B\| \leq \|A\| + \|B\|$$

for rational series A, B in $\mathbb{K}[[X]]$. As a particular case we have

$$\|A\| - 1 \leq \|1 + A\| \leq \|A\| + 1.$$

Given a prime p and a formal power series $C = \sum_{n=0}^{\infty} \gamma_n X^n$ in $\overline{\mathbb{F}}_p[[X]]$ we denote by $\kappa(C) \in \mathbb{N} \cup \{\infty\}$ the dimension of the “ p -kernel”

$$\mathcal{K}(C) = \sum_{f,k} \overline{\mathbb{F}}_p C_{k,f}$$

spanned by all series

$$C_{k,f} = \sum_{n=0}^{\infty} \gamma_{k+npf} X^n$$

for $f \in \mathbb{N}$, $k \in \mathbb{N}$, $k < p^f$.

Algebraic series of $\mathbb{K}[[X]]$ for \mathbb{K} a subfield of the algebraic closure $\overline{\mathbb{F}}_p$ of finite prime characteristic p are characterised by the following Theorem of Christol (see [4] or Theorem 12.2.5 in [1]):

Theorem 2.2. *A formal power series $C = \sum_{n=0}^{\infty} \gamma_n X^n$ of $\overline{\mathbb{F}}_p[[X]]$ is algebraic if and only if $\kappa(C)$ is finite.*

We state without proof the following well-known consequence.

Corollary 2.3. *An algebraic series of $\overline{\mathbb{F}}_p[[X]]$ has all its coefficients in a finite subfield of $\overline{\mathbb{F}}_p$.*

Proposition 2.4. *Let $C = \sum_{n=0}^{\infty} \gamma_n X^n$ be an algebraic series with coefficients in a subfield $\mathbb{K} \subset \overline{\mathbb{F}}_p$.*

(i) *We have*

$$\mathcal{K}(\tau(C)) \subset \mathcal{K}(C) + \tau(\mathcal{K}(C))$$

which implies

$$\kappa(\tau(C)) \leq 2\kappa(C) .$$

(ii) *We have*

$$\mathcal{K}(C) \subset \mathbb{K} + \mathcal{K}(\tau(C)) + X\mathcal{K}(\tau(C))$$

which implies

$$\kappa(C) \leq 1 + 2\kappa(\tau(C)) .$$

Proof Assertion (i) follows from an iterated application of the easy computations

$$(\tau(C))_{k,1} = C_{k+1,1}$$

if $0 \leq k < p - 1$ and

$$(\tau(C))_{p-1,1} = \tau(C_{0,1}) .$$

The proof of assertion (ii) is similar. □

3 The shuffle algebra

This section recalls mostly well-known results concerning shuffle products of elements in the set $\mathbb{K}[[X]]$ of formal power series over a commutative field \mathbb{K} which is arbitrary unless specified otherwise.

The *shuffle product*

$$A \sqcup B = C = \sum_{n=0}^{\infty} \gamma_n X^n$$

of $A = \sum_{n=0}^{\infty} \alpha_n X^n$ and $B = \sum_{n=0}^{\infty} \beta_n X^n$ is defined by

$$\gamma_n = \sum_{k=0}^n \binom{n}{k} \alpha_k \beta_{n-k}$$

and corresponds to the usual product $ab = c$ of the associated exponential generating series

$$a = \sum_{n=0}^{\infty} \alpha_n \frac{X^n}{n!}, \quad b = \sum_{n=0}^{\infty} \beta_n \frac{X^n}{n!}, \quad c = \sum_{n=0}^{\infty} \gamma_n \frac{X^n}{n!} .$$

The *shuffle algebra* is the algebra $(\mathbb{K}[[X]], \sqcup)$ obtained by endowing the vector space $\mathbb{K}[[X]]$ of ordinary generating series with the shuffle product. By construction, the shuffle algebra is isomorphic to the algebra $\mathcal{E}(\mathbb{K})$ of exponential generating series. In characteristic zero, the trivial identity

$$\sum_{n=0}^{\infty} \alpha_n X^n = \sum_{n=0}^{\infty} (n! \alpha_n) \frac{X^n}{n!}$$

gives an isomorphism between the usual algebra $\mathbb{K}[[X]]$ of ordinary generating series and the shuffle algebra $(\mathbb{K}[[X]], \sqcup)$.

The identity $\left(\sum_{n \geq 0} \lambda^n X^n\right) \sqcup \left(\sum_{n \geq 0} \mu^n X^n\right) = \sum_{n \geq 0} (\lambda + \mu)^n X^n$, equivalent to $e^{\lambda X} e^{\mu X} = e^{(\lambda + \mu)X}$ implies that the convergency radius of the shuffle product of two complex series with strictly positive convergency radii ρ_1, ρ_2 is at least the harmonic mean $1 / \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$ of ρ_1 and ρ_2 .

Proposition 3.1. *The shift operator $\tau(\sum_{n=0}^{\infty} \alpha_n X^n) = \sum_{n=0}^{\infty} \alpha_{n+1} X^n$ acts as a derivation on the shuffle algebra.*

Proof The map τ is clearly linear. The computation

$$\begin{aligned} \tau \left(\sum_{i,j \geq 0} \binom{i+j}{i} \alpha_i \beta_j X^{i+j} \right) &= \sum_{i,j \geq 0} \binom{i+j}{i} \alpha_i \beta_j X^{i+j-1} = \\ &= \sum_{i,j \geq 0} \left(\binom{i+j-1}{i-1} + \binom{i+j-1}{j-1} \right) \alpha_i \beta_j X^{i+j-1} \end{aligned}$$

shows that τ satisfies the Leibniz rule $\tau(A \sqcup B) = \tau(A) \sqcup B + A \sqcup \tau(B)$. \square

Proposition 3.1 is trivial in characteristic zero: the usual derivation d/dX acts obviously as the shift operator on the algebra $\mathcal{E}(\mathbb{K})$ of exponential generating series over a field of characteristic zero.

Proposition 3.2. *Shuffle products of rational power series are rational.*

More precisely, we have

$$\| A \sqcup B \| \leq \| A \| \| B \|$$

for two rational series A, B in $\mathbb{K}[[X]]$.

Proof Proposition 3.1 implies $\tau^n(A \sqcup B) = \sum_{k=0}^n \binom{n}{k} \tau^k(A) \sqcup \tau^{n-k}(B)$. The series $\tau^n(A \sqcup B)$ belongs thus to the vector space spanned by shuffle products with factors in the vector-spaces $\sum_{n \geq 0} \mathbb{K} \tau^n(A)$ and $\sum_{n \geq 0} \mathbb{K} \tau^n(B)$. This implies the inequality. Proposition 2.1 ends the proof. \square

Proposition 3.3. *Shuffle products of algebraic series in $\overline{\mathbb{F}}_p[[X]]$ are algebraic.*

More precisely, we have

$$\kappa(A \sqcup B) \leq \kappa(A) \kappa(B) .$$

Proof Denoting as in Section 2 by $C_{k,f}$ the series

$$C_{k,f} = \sum_{n=0}^{\infty} \gamma_{k+npf} X^n$$

associated to a series $C = \sum_{n=0}^{\infty} \gamma_n X^n$ and by $\kappa(C)$ the dimension of the vector space $\mathcal{K}(C) = \sum_{k,f} \overline{\mathbb{F}}_p C_{k,f}$, Lucas's identity

$$\binom{n}{k} \equiv \prod_{i \geq 0} \binom{\nu_i}{\kappa_i} \pmod{p}$$

for $n = \sum_{i \geq 0} \nu_i p^i$, $k = \sum_{i \geq 0} \kappa_i p^i$ with $\nu_i, \kappa_i \in \{0, \dots, p-1\}$ implies

$$(A \sqcup B)_{k,1} = \sum_{i=0}^k \binom{k}{i} A_{i,1} \sqcup B_{k-i,1}$$

for $k = 0, \dots, p-1$. Iteration of this formula shows that $(A \sqcup B)_{k,f}$ (for arbitrary $k, f \in \mathbb{N}$ such that $k < p^f$) belongs to the vector space spanned by shuffle products with factors in the vector spaces $\mathcal{K}(A)$ and $\mathcal{K}(B)$ of dimension $\kappa(A)$ and $\kappa(B)$.

Christol's Theorem (Theorem 2.2) ends the proof. \square

4 The special shuffle-group

We call the group of units of the shuffle algebra $(\mathbb{K}[[X]], \sqcup)$ the *shuffle-group*. Its elements are given by the set $\mathbb{K}^* + X\mathbb{K}[[X]]$ underlying the multiplicative unit group. The shuffle-group is the direct product of the unit group \mathbb{K}^* of \mathbb{K} with the *special shuffle-group* $(1 + X\mathbb{K}[[X]], \sqcup)$.

The inverse in the shuffle group of $1 - A \in (1 + X\mathbb{K}[[X]], \sqcup)$ is given by

$$\sum_{n=0}^{\infty} A \sqcup^n = 1 + A + A \sqcup A + A \sqcup A \sqcup A + \dots$$

where $A \sqcup^0 = 1$ and $A \sqcup^{n+1} = A \sqcup A \sqcup^n$ for $n \geq 1$.

This shows in particular the identity $(1 - X) \sqcup (\sum_{n=0}^{\infty} n! X^n) = 1$. Invertible rational (analytical) power series have thus generally a transcendental (non-analytical) shuffle-inverse over the complex numbers.

Proposition 4.1. *The special shuffle-group $(1 + X\mathbb{K}[[X]], \sqcup)$ is isomorphic to an infinite-dimensional \mathbb{F}_p -vector-space if the field \mathbb{K} is of positive characteristic p .*

Proposition 4.1 shows that $(1 + X\mathbb{K}[[X]], \sqcup)$ is not isomorphic to the multiplicative group structure on $1 + X\mathbb{K}[[X]]$ if \mathbb{K} is of positive characteristic.

Proof of Proposition 4.1 Follows directly from the fact that \exp_1 is a group isomorphism between the \mathbb{F}_p -vector space \mathfrak{m} and the special shuffle group.

One can also give a direct proof by computing $A \sqcup^p$. □

Remark 4.2. *One can show that a rational fraction $A \in 1 + X\mathbb{C}[[X]]$ has a rational inverse for the shuffle-product if and only if $A = \frac{1}{1-\lambda X}$ with $\lambda \in \mathbb{C}$. (Compute $A \sqcup B = 1$ using the decomposition into simple fractions of the rational series A, B .)*

5 The exponential and the logarithm for exponential generating functions

Proposition 5.1. *For all natural numbers $j, k \geq 1$, the set $\{1, \dots, jk\}$ can be partitioned in exactly*

$$\frac{(jk)!}{(j!)^k k!}$$

different ways into k unordered disjoint subsets of j elements.

In particular, the rational number $(jk)!/((j!)^k k!)$ is an integer for all natural numbers j, k such that $j \geq 1$.

Proof The multinomial coefficient $(jk)!/(j!)^k$ counts the number of ways of partitioning $\{1, \dots, jk\}$ into k disjoint ordered subset of j elements. Dividing by $k!$ removes the order on these k subsets.

This proves that the formula defines an integer for all $j, k \geq 1$ and the integrality holds also obviously for $k = 0$ and $j \geq 1$. □

Proposition 5.2. *For any natural integer $k \in \mathbb{N}$, there exists polynomials $P_{k,n} \in \mathbb{N}[\alpha_1, \dots, \alpha_n]$ such that*

$$\frac{1}{k!} \left(\sum_{n=1}^{\infty} \alpha_n \frac{X^n}{n!} \right)^k = \sum_{n=0}^{\infty} P_{k,n}(\alpha_1, \alpha_2, \dots, \alpha_n) \frac{X^n}{n!} .$$

Proof The contribution of a monomial

$$\alpha_1^{j_1} \alpha_2^{j_2} \dots \alpha_s^{j_s} \frac{X^{\sum_{i=1}^s i j_i}}{(\sum_{i=1}^s i j_i)!}$$

with $j_1 + j_2 + \cdots + j_s = k$ to $(1/k!) (\sum_{n=1}^{\infty} \alpha_n X^n / n!)^k$ is given by

$$\begin{aligned} & \frac{1}{k!} \frac{k!}{(j_1)!(j_2)! \cdots (j_s)!} \frac{(\sum_{i=1}^s i j_i)!}{\prod_{i=1}^s (i!)^{j_i}} \\ &= \left(\prod_{i=1}^s \frac{(i j_i)!}{(i!)^{j_i} (j_i)!} \right) \frac{(\sum_{i=1}^s i j_i)!}{\prod_{i=1}^s (i j_i)!} \end{aligned}$$

and the last expression is a product of a natural integer by Proposition 5.1 and of a multinomial coefficient. It is thus a natural integer. \square

Corollary 5.3. For $a = \sum_{n=1}^{\infty} \alpha_n \frac{X^n}{n!}$ the formulae

$$\exp \left(\sum_{n=1}^{\infty} \alpha_n \frac{X^n}{n!} \right) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P_{k,n}(\alpha_1, \dots, \alpha_n) \frac{X^n}{n!}$$

and

$$\log \left(1 + \sum_{n=1}^{\infty} \alpha_n \frac{X^n}{n!} \right) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} (-1)^{n+1} (n-1)! P_{k,n}(\alpha_1, \dots, \alpha_n) \frac{X^n}{n!}$$

define the exponential function and the logarithm of an exponential generating series in $a \in \mathfrak{m}_{\mathcal{E}}$ respectively $1 + a \in 1 + \mathfrak{m}_{\mathcal{E}}$ over an arbitrary field \mathbb{K} . These functions are one-to-one and mutually reciprocal.

6 The logarithm as a p -homogeneous form over $\overline{\mathbb{F}}_p[[x]]$

Given a fixed prime number p , Proposition 4.1 shows that there exists polynomials $P_{p,n} \in \mathbb{N}[\alpha_0, \dots, \alpha_n]$ for $n \geq 1$ such that

$$\left(\sum_{n=0}^{\infty} \alpha_n X^n \right)^{\square p} = \alpha_0^p + p \sum_{n=1}^{\infty} P_{p,n}(\alpha_0, \dots, \alpha_n) X^n .$$

The polynomials $P_{p,n}$ are homogeneous of degree p with respect to the variables $\alpha_0, \dots, \alpha_n$ and we denote by

$$\mu_p \left(\sum_{n=0}^{\infty} \alpha_n X^n \right) = \sum_{n=1}^{\infty} P_{p,n}(\alpha_0, \dots, \alpha_n) X^n$$

the p -homogeneous form defined by the ordinary generating series of the polynomials $P_{p,1}, P_{p,2}, \dots$

Proposition 6.1. The restriction of μ_p to $1 + \mathfrak{m} \subset \overline{\mathbb{F}}_p[[X]]$ coincides with the function \log_1 .

Proof We have

$$\tau(\mu_p(1 + A)) = (1 + A) \sqcup^{p-1} \sqcup \tau(1 + A)$$

for A in \mathfrak{m} . This identity defines the restriction of the p -homogenous form μ_p to $1 + \mathfrak{m}$. The identity $(1 + A) \sqcup^{p-1} \sqcup (1 + A) = 1$ shows that the same equation

$$\tau(\log_!(1 + A)) = (1 + A) \sqcup^{p-1} \sqcup \tau(1 + A)$$

is also satisfied by the function $\log_!$ since τ corresponds to the differential operator d/dX of the associated exponential series and since the shuffle product corresponds to the ordinary product of exponential generating series.

Since both series $\mu_p(1 + A)$ and $\log_!(1 + A)$ are without constant term, the equality $\tau(\mu_p(1 + A)) = \tau(\log_!(1 + A))$ implies the equality $\mu_p(1 + A) = \log_!(1 + A)$. \square

7 Proofs

Proposition 7.1. *If A in $X\overline{\mathbb{F}}_p[[X]]$ is rational (respectively algebraic) then the formal power series $\log_!(1 + A)$ is rational (respectively algebraic).*

More precisely, we have

$$\|\log_!(1 + A)\| \leq 1 + \|1 + A\|^p$$

respectively

$$\kappa(\log_!(1 + A)) \leq 1 + 4(\kappa(1 + A))^p$$

for rational respectively algebraic A in $X\overline{\mathbb{F}}_p[[X]]$.

Proposition 7.2. *If A in $X\overline{\mathbb{F}}_p[[X]]$ is rational (respectively algebraic) then $\exp_!(A)$ is rational (respectively algebraic).*

More precisely, denoting by $q = p^e$ the cardinality of a finite field $\mathbb{F}_q \subset \overline{\mathbb{F}}_p$ containing all coefficients of A we have

$$\|\exp_!(A)\| \leq p^{q^{\|A\|}}$$

respectively

$$\kappa(\exp_!(A)) \leq q^{\kappa(A)} p^{q^{\kappa(A)}}$$

for rational respectively algebraic A in \mathfrak{m} .

Theorems 1.1, 1.3, 1.5 and 1.6 are now simple reformulations of Propositions 7.1 and 7.2.

Proof of Proposition 7.1 We have

$$\tau(\log_!(1 + A)) = (1 + A) \sqcup^{p-1} \sqcup \tau(A) .$$

This shows

$$\|\tau(\log_!(1+A))\| \leq \|1+A\|^{p-1} \|\tau(A)\| \leq \|1+A\|^p$$

and implies

$$\|\log_!(1+A)\| \leq 1 + \|1+A\|^p .$$

This settles the rational case.

For algebraic A we have similarly

$$\kappa(\tau(\log_!(1+A))) \leq (\kappa(1+A))^{p-1} \kappa(\tau(A)) \leq (\kappa(1+A))^{p-1} 2\kappa(A) \leq 2(\kappa(1+A))^p$$

showing

$$\kappa(\log_!(1+A)) \leq 1 + 2\kappa(\tau(\log_!(1+A))) \leq 1 + 4(\kappa(1+A))^p$$

and ending the proof. \square

Given a vector-space $\mathcal{V} \subset \mathbb{K}[[X]]$ containing \mathbb{K} , we denote by $\Gamma(\mathcal{V})$ the shuffle-subgroup generated by all elements of $\mathcal{V} \cap (1 + X\mathbb{K}[[X]])$.

Lemma 7.3. *Every element of a vector space $\mathcal{V} \subset \mathbb{K}[[X]]$ containing the constants \mathbb{K} can be written as a linear combination of elements in $\Gamma(\mathcal{V})$.*

Proof We have the identity

$$A = (1 - \epsilon(A) + A) + (\epsilon(A) - 1)$$

where $\epsilon(\sum_{n=0}^{\infty} \alpha_n X^n) = \alpha_0$ is the augmentation map and where $(1 - \epsilon(A) + A)$ and $(\epsilon(A) - 1)$ are both in $\mathbb{K}\Gamma(\mathcal{V})$ for $A \in \mathcal{V}$. \square

Proof of Proposition 7.2 for rational A Corollary 2.3 shows that we can work over a finite subfield $\mathbb{K} = \mathbb{F}_q$ of $\overline{\mathbb{F}}_p$ with $q = p^e$.

Given a rational series A in $\mathfrak{m} = X\mathbb{K}[[X]]$, we denote by Γ_A the shuffle-subgroup generated by all elements of the set

$$\left\{ \bigcup_{n=0}^{\infty} (\tau^n(A) + \mathbb{K}) \right\} \cap \{1 + X\mathbb{K}[[X]]\}.$$

The group Γ_A is an \mathbb{F}_p -vector space generated by at most $q^{\|A\|}$ elements. We denote by $\mathbb{K}[\Gamma_A]$ the subalgebra of dimension $\leq p^{q^{\|A\|}}$ in $\mathbb{K}[[X]]$ spanned by all elements of Γ_A . The identity

$$\tau(\exp_!(A)) = \exp_!(A) \sqcup \tau(A)$$

and the fact that the linear application τ is a derivation of $\mathbb{K}[[X]]$ show the inclusion

$$\tau^n(\exp_!(A)) \in \exp_!(A) \sqcup \mathbb{K}[\Gamma_A]$$

which ends the proof since the right-hand side is a \mathbb{K} -vector space of dimension at most $p^{q^{\|A\|}}$.

Proposition 7.4. *We have for every prime number p and for all natural integers j, k such that $j \geq 1$ the identity*

$$\frac{(jk)!}{(j!)^k k!} \equiv \frac{(pjk)!}{((pj)!)^k k!} \pmod{p}.$$

Proof The right-hand-side counts partitions of $\{1, \dots, pjk\}$ into k subsets of pj elements. Consider the action on such partitions obtained from the group generated by the jk cyclic permutations $(i, i + jk, i + 2jk, \dots, i + (p-1)jk)$ for $i = 1, \dots, jk$. Its fixpoints are in bijection with partitions of $\{1, \dots, jk\}$ into k subsets of j elements. \square

Corollary 7.5. *\exp_1 and \log_1 commute with the “Frobenius substitution”*

$$\varphi\left(\sum_{n=0}^{\infty} \alpha_n X^n\right) = \sum_{n=0}^{\infty} \alpha_n X^{pn}$$

for series in $X\overline{\mathbb{F}}_p[[X]]$, respectively in $1 + X\overline{\mathbb{F}}_p[[X]]$.

This implies

$$(\exp_1(A))_{0,f} = \exp_1(A_{0,f})$$

Lemma 7.6. *We have*

$$(B \sqcup C)_{0,1} = B_{0,1} \sqcup C_{0,1}$$

Proof Follows from the identity

$$\binom{pn}{k} \equiv 0 \pmod{p}$$

if $k \not\equiv 0 \pmod{p}$. \square

Proof of Proposition 7.2 for algebraic A We work again over a finite subfield $\mathbb{K} = \mathbb{F}_q \subset \overline{\mathbb{F}}_p$ containing all coefficients of A .

Let Γ_A denote the shuffle subgroup generated by all elements in

$$(\mathcal{K}(A) + \mathbb{K}) \cap (1 + X\mathbb{K}[[X]])$$

where $\mathcal{K}(A) = \sum_{k,f} \mathbb{K}A_{k,f}$. We denote by $\mathbb{K}[\Gamma_A] \subset (\mathbb{K}[[X]], \sqcup)$ the shuffle-subalgebra of dimension at most $p^{q^{\kappa(A)}}$ spanned by all elements of the group $\Gamma(A) \subset (1 + X\mathbb{K}[[X]], \sqcup)$.

Using the convention $A_{0,0} = A$, we have for $B \in \mathbb{K}[\Gamma(A)]$ and for k such that $0 \leq k < p$

$$(\exp_1(A_{0,f}) \sqcup B)_{k,1} = \left(\tau^k(\exp_1(A_{0,f}) \sqcup B)\right)_{0,1} =$$

$$\begin{aligned}
&= \left(\sum_{j=0}^k \binom{k}{j} \tau^j(\exp_!(A_{0,f})) \sqcup \tau^{k-j}(B) \right)_{0,1} = \\
&= \sum_{j=0}^k \binom{k}{j} (\tau^j(\exp_!(A_{0,f})))_{0,1} \sqcup B_{k-j,1}
\end{aligned}$$

where the last equality is due to Lemma 7.6 (and to the equality $(\tau^k(C))_{0,1} = C_{k,1}$ for $0 \leq k < p$).

An iterated application of the identity $\tau(\exp_!(A_{0,f})) = \exp_!(A_{0,f}) \sqcup \tau(A_{0,f})$ implies

$$(\tau^j(\exp_!(A_{0,f})))_{0,1} \in \exp_!(A_{0,f+1}) \sqcup \mathbb{K}[\Gamma(A)]$$

and shows the inclusion

$$(\exp_!(A))_{k,f} \in \exp_!(A_{0,f}) \sqcup \mathbb{K}[\Gamma_A]$$

for all k such that $0 \leq k < p^f$ and for $(k, f) = (0, 0)$. Since the vector space $\sum_f \mathbb{K}\exp_!(A_{0,f})$ is of dimension at most $q^{\kappa(A)}$ we have the inequality

$$\kappa(\exp_!(A)) \leq q^{\kappa(A)} p^{q^{\kappa(A)}}.$$

This ends the proof. \square

8 Power series in free non-commuting variables

This and the next section recall a few basic and well-known facts concerning (rational) power series in free non-commuting variables, see for instance [6], [3] or a similar book on the subject. We use however sometimes a different terminology, motivated by [2].

We denote by \mathcal{X}^* the free monoid on a finite set $\mathcal{X} = \{X_1, \dots, X_k\}$. We write 1 for the identity element and we use a boldface capital \mathbf{X} for a non-commutative monomial $\mathbf{X} = X_{i_1} X_{i_2} \cdots X_{i_l} \in \mathcal{X}^*$. We denote by

$$A = \sum_{\mathbf{X} \in \mathcal{X}^*} (A, \mathbf{X}) \mathbf{X} \in \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$$

a non-commutative formal power series where

$$\mathcal{X}^* \ni \mathbf{X} \longmapsto (A, \mathbf{X}) \in \mathbb{K}$$

stands for the coefficient function.

We denote by $\mathfrak{m} \subset \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ the maximal ideal consisting of formal power series without constant coefficient and by $\mathbb{K}^* + \mathfrak{m} = \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle \setminus \mathfrak{m}$ the unit-group of the algebra $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ consisting of all (multiplicatively) invertible elements in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$. The unit group is isomorphic

to the direct product $\mathbb{K}^* \times (1 + \mathfrak{m})$ where \mathbb{K}^* is the central subgroup consisting of non-zero constants and where $1 + \mathfrak{m}$ denotes the multiplicative subgroup given by the affine subspace formed by power series with constant coefficient 1. We have $(1 - A)^{-1} = 1 + \sum_{n=1}^{\infty} A^n$ for the multiplicative inverse $(1 - A)^{-1}$ of an element $1 - A \in 1 + \mathfrak{m}$.

8.1 The shuffle algebra

The *shuffle-product* $\mathbf{X} \sqcup \mathbf{X}'$ of two non-commutative monomials $\mathbf{X}, \mathbf{X}' \in \mathcal{X}^*$ of degrees $a = \deg(\mathbf{X})$ and $b = \deg(\mathbf{X}')$ (for the obvious grading given by $\deg(X_1) = \dots = \deg(X_k) = 1$) is the sum of all $\binom{a+b}{a}$ monomials of degree $a+b$ obtained by “shuffling” in all possible ways the linear factors (elements of \mathcal{X}) involved in \mathbf{X} with the linear factors of \mathbf{X}' . A monomial involved in $\mathbf{X} \sqcup \mathbf{X}'$ can be thought of as a monomial of degree $a+b$ whose linear factors are coloured by two colours with \mathbf{X} corresponding to the product of all linear factors of the first colour and \mathbf{X}' corresponding to the product of the remaining linear factors. The shuffle product $\mathbf{X} \sqcup \mathbf{X}'$ can also be recursively defined by $\mathbf{X} \sqcup 1 = 1 \sqcup \mathbf{X} = \mathbf{X}$ and

$$(\mathbf{X}X_s) \sqcup (\mathbf{X}'X_t) = (\mathbf{X} \sqcup (\mathbf{X}'X_t))X_s + ((\mathbf{X}X_s) \sqcup \mathbf{X}')X_t$$

where $X_s, X_t \in \mathcal{X} = \{X_1, \dots, X_k\}$ are monomials of degree 1.

Extending the shuffle-product in the obvious way to formal power series endows the vector space $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ with an associative and commutative algebra structure called the *shuffle-algebra*. In the case of one variable $X = X_1$ we recover the definition of Section 3.

The group $\mathrm{GL}_k(\mathbb{K})$ acts on the vector-space $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ by linear substitutions. This action induces an automorphism of the multiplicative (non-commutative) algebra-structure or of the (commutative) shuffle algebra-structure underlying the vector space $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$.

Substitution of all variables X_j of formal power series in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ by X yields a homomorphism of (shuffle-)algebras into the commutative (shuffle-)algebra $\mathbb{K}[[X]]$.

The commutative unit group (set of invertible elements for the shuffle-product) of the shuffle algebra, given by the set $\mathbb{K}^* + \mathfrak{m}$, is isomorphic to the direct product $\mathbb{K}^* \times (1 + \mathfrak{m})$ where $1 + \mathfrak{m}$ is endowed with the shuffle product. The inverse of an element $1 - A \in (1 + \mathfrak{m}, \sqcup)$ is given by $\sum_{n=0}^{\infty} A \sqcup^n = 1 + A + A \sqcup A + A \sqcup A \sqcup A + \dots$

The following result generalises Proposition 4.1:

Proposition 8.1. *Over a field of positive characteristic p , the subgroup $1 + \mathfrak{m}$ of the shuffle-group is an infinite-dimensional \mathbb{F}_p -vector space.*

Proof Contributions to a p -fold shuffle product $A_1 \sqcup A_2 \sqcup \dots \sqcup A_p$ are given by monomials with linear factors coloured by p colours $\{1, \dots, p\}$

keeping track of their “origin” with coefficients given by the product of the corresponding “monochromatic” coefficients in A_1, \dots, A_p . A permutation of the colours $\{1, \dots, p\}$ (and in particular, a cyclic permutation of all colours) leaves such a contribution invariant if $A_1 = \dots = A_p$. Forgetting the colours, coefficients of strictly positive degree in $A \sqcup^p$ are thus zero in characteristic p . \square

As in the one variable case, one can prove that

$$\frac{1}{k!} A \sqcup^k$$

is defined over an arbitrary field \mathbb{K} for $A \in \mathfrak{m}$. Indeed, monomials contributing to $A \sqcup^k$ can be considered as colored by k colours and the $k!$ possible colour-permutations yield identical contributions.

For $A \in \mathfrak{m}$, we denote by

$$\exp_!(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A \sqcup^n$$

the resulting exponential map from the Lie algebra \mathfrak{m} into the infinite-dimensional commutatif Lie group $(1 + \mathfrak{m}, \sqcup)$. Its reciprocal function is the as usual defined by

$$\log_!(1 + A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A \sqcup^n.$$

In the case of a field \mathbb{K} of positive characteristic p the function $\log_!$ is again given by the restriction to $1 + \mathfrak{m}$ of a p -homogeneous form μ_p .

The form μ_p has all its coefficients in \mathbb{N} and is defined by the equality

$$A \sqcup^p = (A, 1)^p + p\mu_p(A)$$

over an arbitrary field.

9 Rational formal power series

A formal power series A is *rational* if it belongs to the smallest subalgebra in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ which contains the free associative algebra $\mathbb{K}\langle X_1, \dots, X_k \rangle$ of non-commutative polynomials and intersects the multiplicative unit group of $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ in a subgroup.

Given a monomial $\mathbf{T} \in \mathcal{X}^*$, we denote by

$$\rho(\mathbf{T}) : \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle \longrightarrow \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$$

the linear application defined by

$$\rho(\mathbf{T})A = \sum_{\mathbf{X} \in \mathcal{X}^*} (A, \mathbf{X})\mathbf{X}$$

for $A = \sum_{\mathbf{X} \in \mathcal{X}^*} (A, \mathbf{X})\mathbf{X}$ in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$. The identity $\rho(\mathbf{T})(\rho(\mathbf{T}')A) = \rho(\mathbf{T}\mathbf{T}')A$ shows that we have a representation

$$\rho : \mathcal{X}^* \longrightarrow \text{End}(\mathbb{K}\langle\langle \mathcal{X} \rangle\rangle)$$

of the free monoid \mathcal{X}^* on \mathcal{X} . The *recursive closure* \overline{A} of a power series A is the vector-space spanned by its orbit $\rho(\mathcal{X}^*)A$ under $\rho(\mathcal{X}^*)$. We call the dimension $\dim(\overline{A})$ of \overline{A} the *complexity* of A .

We call a subspace $\mathcal{A} \subset \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ *recursively closed* if it contains the recursive closure of all its elements.

Rational series coincide with series of finite complexity by a Theorem of Schützenberger (cf. [3], Theorem 1 of page 22).

Remark 9.1. *In the case of one variable, the complexity $\dim(\overline{A})$ of a reduced non-zero rational fraction $A = \frac{f}{g}$ with $f \in \mathbb{K}[X]$ and $g \in 1 + X\mathbb{K}[X]$ equals $\dim(\overline{A}) = \max(1 + \deg(f), \deg(g))$.*

Remark 9.2. *The (generalised) Hankel matrix $H = H(A)$ of*

$$A = \sum_{\mathbf{X} \in \mathcal{X}^*} (A, \mathbf{X})\mathbf{X} \in \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$$

is the infinite matrix with rows and columns indexed by the free monoid \mathcal{X}^ of monomials and entries $H_{\mathbf{X}\mathbf{X}'} = (A, \mathbf{X}\mathbf{X}')$. The rank $\text{rank}(H)$ is given by the complexity $\dim(\overline{A})$ of A and \overline{A} corresponds to the row-span of H .*

Given subspace \mathcal{A}, \mathcal{B} of $\mathbb{K}\langle\langle \mathcal{X} \rangle\rangle$, we denote by $\mathcal{A} \sqcup \mathcal{B}$ the vector space spanned by all products $A \sqcup B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proposition 9.3. *We have the inclusion*

$$\overline{A \sqcup B} \subset \overline{A} \sqcup \overline{B}$$

for the shuffle product $A \sqcup B$ of $A, B \in \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$.

Corollary 9.4. *We have*

$$\dim(\overline{A \sqcup B}) \leq \dim(\overline{A}) \dim(\overline{B})$$

for the shuffle product $A \sqcup B$ of $A, B \in \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$.

In particular, shuffle products of rational elements in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ are rational.

Proof of Proposition 9.3 For $Y \in \overline{A}, Z \in \overline{B}$ and X in $\{X_1, \dots, X_k\}$, the recursive definition of the shuffle product given in Section 8.1 shows

$$\rho(X)(Y \sqcup Z) = (\rho(X)Y) \sqcup Z + Y \sqcup (\rho(X)Z) .$$

We have thus the inclusions

$$\rho(X)(Y \sqcup Z) \in \overline{A} \sqcup Z + Y \sqcup \overline{B} \subset \overline{A} \sqcup \overline{B}$$

which show that the vector space $\overline{A} \sqcup \overline{B}$ is recursively closed. Proposition 9.3 follows now from the inclusion $A \sqcup B \in \overline{A} \sqcup \overline{B}$. \square

Remark 9.5. *Similar arguments show that the set of rational elements in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ is also closed under the ordinary product (and multiplicative inversion of invertible series), Hadamard product and composition (where one considers $A \circ (B_1, \dots, B_k)$ with $A \in \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ and $B_1, \dots, B_k \in \mathfrak{m} \subset \mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$).*

Remark 9.6. *The shuffle inverse of a rational element in $\mathbb{K}^* + \mathfrak{m}$ is in general not rational in characteristic 0. An exception is given by geometric progressions $\frac{1}{1 - \sum_{j=1}^k \lambda_j X_j} = \sum_{n=0}^{\infty} \left(\sum_{j=1}^k \lambda_j X_j \right)^n$ since we have*

$$\frac{1}{1 - \sum_{j=1}^k \lambda_j X_j} \sqcup \frac{1}{1 - \sum_{j=1}^k \mu_j X_j} = \frac{1}{1 - \sum_{j=1}^k (\lambda_j + \mu_j) X_j}$$

corresponding to $e^{\lambda X} e^{\mu X} = e^{(\lambda + \mu)X}$ in the one-variable case.

There are no other such elements in $1 + \mathfrak{m} \subset \mathbb{K}[[X]]$, see Remark 4.2. I ignore if the maximal rational shuffle subgroup of $1 + \mathfrak{m} \subset \mathbb{C}\langle\langle X_1, \dots, X_k \rangle\rangle$ (defined as the set of all rational elements in $1 + \mathfrak{m}$ with rational inverse for the shuffle product) contains other elements if $k \geq 2$.

Remark 9.7. *Any finite set of rational elements in $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ over a field \mathbb{K} of positive characteristic is included in a unique minimal finite-dimensional recursively closed subspace of $\mathbb{K}\langle\langle X_1, \dots, X_k \rangle\rangle$ which intersects the shuffle group $(\mathbb{K}^* + \mathfrak{m}, \sqcup)$ in a subgroup.*

10 Main result for generating series in non-commuting variables

Theorem 10.1. *Let \mathbb{K} be a subfield of $\overline{\mathbb{F}_p}$. Given a non-commutative formal power series $A \in \mathfrak{m} \subset \mathbb{K}\langle\langle \mathcal{X} \rangle\rangle$, the following two assertions are equivalent:*

- (i) *A is rational.*
- (ii) *$\exp_1(A)$ is rational.*

More precisely, we have for a rational series A in \mathfrak{m} the inequalities

$$\dim \left(\overline{\log_1(1 + A)} \right) \leq 1 + (\dim(\overline{1 + A}))^p$$

and

$$\dim \left(\overline{\exp_1(A)} \right) \leq p^{q^{\dim(\bar{A})}}$$

where $q = p^e$ is the cardinality of a finite field \mathbb{F}_q containing all coefficients of A .

Proof The identity

$$\log_1(1 + A) = \sum_{X \in \mathcal{X}} \left((1 + A) \sqcup^{p-1} \sqcup \rho(X)A \right) X$$

and Corollary 9.4 show

$$\dim \left(\overline{\log_1(1 + A)} \right) \leq 1 + (\dim(\bar{1} + A))^p .$$

For the opposite direction we denote by $\mathbb{K} = \mathbb{F}_q$ a finite subfield of $\bar{\mathbb{F}}_p$ containing all coefficients of A . We have

$$\overline{\exp_1(A)} \subset \exp_1(A) \sqcup \mathbb{K}[\Gamma(A)]$$

where $\mathbb{K}[\Gamma(A)]$ is the shuffle subalgebra of dimension $\leq p^{q^{\dim(\bar{A})}}$ generated by all elements of the form

$$(\bar{A} + \mathbb{K}) \cap (1 + \mathfrak{m}) .$$

This implies the inequality

$$\dim \left(\overline{\exp_1(A)} \right) \leq p^{q^{\dim(\bar{A})}}$$

which ends the proof. \square

Acknowledgements I thank J-P. Allouche, M. Brion, A. Pantchichkine, T. Rivoal and B. Venkov for their interest in a preliminary version and helpful remarks.

References

- [1] J.-P. Allouche, J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press (2003).
- [2] R. Bacher, *Determinants related to Dirichlet characters modulo 2, 4 and 8 of binomial coefficients and the algebra of recurrence matrices*, *Int. J. of Alg. and Comp.*, vol. 18 No. 3 (2008), 535–566.
- [3] J. Berstel, C. Reutenauer, *Rational Series and Their Languages*, electronic book available at the author's website's.

- [4] G. Christol, *Ensembles presque périodiques k -reconnaisables*. Theoret. Comput. Sci. **9** (1979), 141–145.
- [5] L. Comtet, *Analyse combinatoire, Tome second*, Presses Universitaires de France, (1970).
- [6] R.P. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press (1999).

Roland BACHER
INSTITUT FOURIER
Laboratoire de Mathématiques
UMR 5582 (UJF-CNRS)
BP 74
38402 St Martin d'Hères Cedex (France)
e-mail: Roland.Bacher@ujf-grenoble.fr