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A family of generalized gamma convoluted variables

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Abstract This paper consists of three parts : in the first part, we describe a family of generalized gamma convoluted (abbreviated as GGC) variables. In the second part, we use this description to prove that several r.v.'s, related to the length of excursions away from 0 for a recurrent linear diffusion on \mathbb{R}_+ , are GGC. Finally, in the third part, we apply our results to the case of Bessel processes with dimension $d = 2(1 - \alpha)$ ($0 < d < 2$, or $0 < \alpha < 1$).

Key words : 60 J 25, 60 G 51, 60 E 07, 60 E 05.

0 Notation and Introduction

0.1 Let $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denote a Borel function such that :

$$\int_0^\infty \frac{l(z)}{z} dz < \infty \quad (0.1)$$

Without loss of generality, we assume that :

$$\int_0^\infty \frac{l(z)}{z} dz = 1 \quad (0.2)$$

With l , we associate a r.v. Y on \mathbb{R}_+ whose probability density f_Y is given by :

$$f_Y(u) = \int_0^\infty e^{-uz} l(z) dz \quad (u \geq 0) \quad (0.3)$$

Indeed, due to (0.2), we get :

$$\int_0^\infty f_Y(u) du = \int_0^\infty du \int_0^\infty e^{-uz} l(z) dz = \int_0^\infty \frac{l(z)}{z} dz = 1 \quad (0.4)$$

To emphasize the relation between Y and l , we shall (sometimes) write Y_l .

We denote by $\varphi_l \equiv \varphi_{Y_l}$ the Laplace transform of Y_l :

$$\begin{aligned} \varphi_l(\lambda) = \varphi_{Y_l}(\lambda) &= E(e^{-\lambda Y_l}) = \int_0^\infty e^{-\lambda u} f_{Y_l}(u) du \\ &= \int_0^\infty \frac{l(z)}{\lambda + z} dz \end{aligned} \quad (0.5)$$

Thus, since f_{Y_l} is the Laplace transform of l , φ_l is the Stieltjes transform of l .

0.2 A reminder about GGC variables

Let μ denote a positive, σ -finite measure on \mathbb{R}_+ . We recall (see [Bon]) that a positive r.v. Y is a GGC variable with Thorin measure μ if :

$$E(e^{-\lambda Y}) = \exp \left\{ - \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} \int_0^\infty e^{-xz} \mu(dz) \right\} \quad (\lambda \geq 0) \quad (0.6)$$

Such a r.v. is self-decomposable, hence infinitely divisible.

The GGC r.v.'s Y whose Thorin measure μ has a finite total mass, equal to m , are characterized by (see [JRY]) :

$$E(e^{-\lambda Y}) = \exp \left\{ -m \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} E(e^{-xG}) \right\} \quad (0.7)$$

where G is an \mathbb{R}_+ -valued r.v. such that $E(\log^+(1/G)) < \infty$.

Such a r.v. is a gamma- m mixture, i.e. it satisfies ¹ :

$$Y \stackrel{(\text{law})}{=} \gamma_m \cdot Z \quad (0.8)$$

where γ_m is a gamma variable with parameter m , independent from the \mathbb{R}_+ -valued variable Z . We note that any r.v. which is a gamma- m mixture is also a gamma- m' mixture, for any $m' > m$, since there is the identity :

$$\gamma_m \stackrel{(\text{law})}{=} \gamma_{m'} \cdot \beta_{m, m'-m} \quad (0.9)$$

where $\gamma_{m'}$ is a gamma variable with parameter m' and $\beta_{m, m'-m}$ is a beta variable with parameters $(m, m' - m)$ independent from $\gamma_{m'}$.

We also recall (see [Bon], p. 51) that the parameter m of a GGC r.v. Y , with Thorin measure with total mass m , may be obtained from the formula :

$$m = \sup \left\{ \delta \geq 0 ; \lim_{u \downarrow 0_+} \frac{f_Y(u)}{u^{\delta-1}} = 0 \right\} \quad (0.10)$$

1 A family of GGC variables

The aim of this part is to present a sufficient condition on l which implies that the associated variable Y_l is GGC.

Definition 1 A function l which satisfies (0.1) belongs to the class \mathcal{C} if there exist $a \geq 0$, $b > a$, $\sigma \geq 0$ and $\theta : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{+\infty\}$ a Borel, decreasing function, which is identically equal to $+\infty$ on $[0, a[$, such that :

$$l(z) = \exp \left\{ \sigma + \int_b^z \frac{\theta(y)}{y} dy \right\} \quad (1.1)$$

¹It would be more correct to say that : the law of such a r.v. is a gamma- m mixture ; however, such abuse is usual, and should not lead to confusion.

Of course, if (1.1) is satisfied with $a > 0$, then the function l is identically 0 on $[0, a[$. On the other hand, if l is identically 0 on $[0, a[$ and differentiable on $]a, \infty[$, then l belongs to the class \mathcal{C} if and only if the function :

$$y \rightarrow y (\log l)'(y) := \theta(y) \quad (1.2)$$

is decreasing on $]a, \infty[$.

The following properties are elementary :

- If $l \in \mathcal{C}$, then for every $u > 0$, $x \rightarrow l(ux) \in \mathcal{C}$ (1.3)

- If $l_1, l_2 \in \mathcal{C}$, then $l_1 \cdot l_2 \in \mathcal{C}$ (1.4)

- For every α real, $x \rightarrow x^\alpha \in \mathcal{C}$ (1.5)

- For every $k < 0$ and $\gamma \geq 0$, $x \rightarrow (x + \gamma)^k \in \mathcal{C}$ (1.6)

Theorem 2 *Let l which satisfies (0.2) and belongs to \mathcal{C} , and let Y_l denote the r.v. associated with l . Then :*

Y_l is a GGC r.v. whose Thorin measure μ has total mass m smaller than or equal to 1. In other terms, there exists a r.v. G taking values in $\overline{\mathbb{R}}_+$, and satisfying $E(\log^+(1/G)) < \infty$ and $m \leq 1$ such that :

$$E(e^{-\lambda Y_l}) = \exp \left\{ -m \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} E(e^{-xG}) \right\} \quad (\lambda \geq 0) \quad (1.7)$$

Proof of Theorem 2

1. It suffices to show that Y_l is GGC since, if so, then the total mass m of its Thorin measure equals, from (0.3) and (0.10) :

$$m = \sup \left\{ \delta \geq 0 ; \lim_{u \downarrow 0_+} \frac{1}{u^{\delta-1}} \int_0^\infty e^{-uz} l(z) dz = 0 \right\}$$

and, of course, $m \leq 1$ since, for $\delta = 1$:

$$\frac{1}{u^{\delta-1}} \int_0^\infty e^{-uz} l(z) dz = \int_0^\infty e^{-uz} l(z) dz \xrightarrow{u \downarrow 0_+} \int_0^\infty l(z) dz > 0$$

2. To show that Y_l is GGC, we shall use the following characterization (see [Bon], Th. 6.1.1, p. 90) of these r.v.'s :

Y is GGC if and only if its Laplace transform φ_Y is hyperbolically completely monotone, that is it satisfies : for every $u > 0$, the function H_u , defined by :

$$H_u(w) = \varphi_Y(uv) \cdot \varphi_Y\left(\frac{u}{v}\right), \quad \text{where } w = v + \frac{1}{v} \quad (1.8)$$

is a completely monotone function, i.e. it is the Laplace transform of a positive measure carried by \mathbb{R}_+ .

In our framework, this criterion becomes : for every $u > 0$, H_u is completely monotone with, from (0.5) :

$$H_u(w) = \int_0^\infty \int_0^\infty \frac{l(x)l(y)}{(x+uv)(y+\frac{u}{v})} dx dy \quad \left(w = v + \frac{1}{v} \right) \quad (1.9)$$

$$= \int_0^\infty \int_0^\infty \frac{l(ux)l(uy)}{(x+v)(y+\frac{1}{v})} dx dy \quad (1.10)$$

(after the change of variables $x = ux'$, $y = uy'$).

Our aim being to show that the hypothesis : $l \in \mathcal{C}$ implies that H_u is completely monotone, and since $x \rightarrow l(ux)$ belongs to \mathcal{C} if $l \in \mathcal{C}$ (from (1.3)), it suffices to see that the function H defined by :

$$H(w) := \int_0^\infty \int_0^\infty \frac{l(x)l(y)}{(x+v)(y+\frac{1}{v})} dx dy \quad \left(w = v + \frac{1}{v} \right) \quad (1.11)$$

is completely monotone.

3. We show that H , defined by (1.11), is completely monotone :

i) We write :

$$\begin{aligned} H(w) &= \int_0^\infty \int_0^\infty \frac{l(x)l(y)}{(x+v)(y+\frac{1}{v})} dx dy = \frac{1}{2} \int_0^\infty \int_0^\infty l(x)l(y) \left[\frac{1}{(x+v)(y+\frac{1}{v})} + \frac{1}{(x+\frac{1}{v})(y+v)} \right] dx dy \\ &\quad \text{(by symmetry)} \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty l(x)l(y) \left[\frac{x^2-1}{xy-1} \cdot \frac{1}{x^2+xw+1} + \frac{y^2-1}{xy-1} \cdot \frac{1}{y^2+yw+1} \right] dx dy \end{aligned} \quad (1.12)$$

(after reducing both reciprocals to the same denominator and decomposing into simple elements)

$$\begin{aligned} &= \frac{1}{2} \int_0^\infty \int_0^\infty l(x)l(y) dx dy \left[\frac{x^2-1}{xy-1} \int_0^\infty e^{-b(x^2+xw+1)} db + \frac{y^2-1}{xy-1} \int_0^\infty e^{-b(y^2+yw+1)} db \right] \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty l(x)l(y) dx dy \cdot \left[\frac{x^2-1}{xy-1} \cdot \frac{1}{x} \int_0^\infty e^{-bw-b(x+\frac{1}{x})} db + \frac{y^2-1}{xy-1} \cdot \frac{1}{y} \int_0^\infty e^{-bw-b(y+\frac{1}{y})} db \right] \end{aligned}$$

(after making the change of variables $bx = b'$, $by = b'$)

$$= \int_0^\infty e^{-bw} db \left(\int_0^\infty \int_0^\infty l(x)l(y) \frac{x^2-1}{(xy-1)x} e^{-b(x+\frac{1}{x})} dx dy \right) \quad (1.13)$$

after interverting the orders of integration.

We note that the preceding computation is a little formal : we have transformed an absolutely convergent integral in an integral which is no longer absolutely convergent ; however, this does not matter for our purpose, as we shall soon gather the different terms in another way.

ii) Thus, we need to show, from (1.13), that, for every $b \geq 0$:

$$I_b := \int_0^\infty \int_0^\infty l(x)l(y) \frac{x^2-1}{(xy-1)x} e^{-b(x+\frac{1}{x})} dx dy \geq 0 \quad (1.14)$$

• Let us show (1.14). For this purpose, we define the 4 domains :

$$\begin{aligned} \mathcal{N}_1 &= \left\{ 0 < x \leq 1, y > \frac{1}{x} \right\}, & \mathcal{N}_2 &= \left\{ x \geq 1, y < \frac{1}{x} \right\} \\ \mathcal{P}_1 &= \left\{ x \geq 1, y > \frac{1}{x} \right\}, & \mathcal{P}_2 &= \left\{ 0 < x \leq 1, y < \frac{1}{x} \right\} \end{aligned}$$

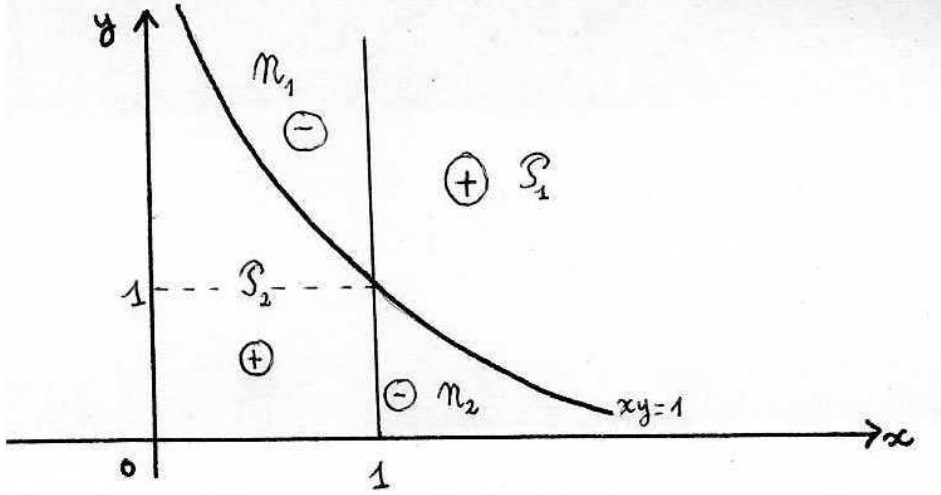


Figure 1

Let us define :

$$\psi(x, y) := l(x)l(y) \frac{x^2 - 1}{(xy - 1)x} e^{-b(x + \frac{1}{x})} \quad (1.15)$$

It is clear that ψ is negative on \mathcal{N}_1 and \mathcal{N}_2 and positive on \mathcal{P}_1 and \mathcal{P}_2 . We note :

$$N_i := \int \int_{\mathcal{N}_i} |\psi(x, y)| dx dy \quad (i = 1, 2)$$

$$P_i := \int \int_{\mathcal{P}_i} \psi(x, y) dx dy \quad (i = 1, 2)$$

• To prove (1.14) it suffices to see that : $N_i \leq P_i$ ($i = 1, 2$). To compute N_1 and P_2 ($\subset \{(x, y) \in \mathbb{R}_+^2; x \leq 1\}$), we make the change of variables for $x \in]0, 1]$, $t \geq 2$: $x = \frac{t - \sqrt{t^2 - 4}}{2}$;

so : $\frac{1}{x} = \frac{t + \sqrt{t^2 - 4}}{2}$, $x + \frac{1}{x} = t$, $\frac{x^2 - 1}{x^2} dx = dt$. We obtain :

$$\begin{aligned} N_1 &= \int_2^\infty dt \int_{\frac{t + \sqrt{t^2 - 4}}{2}}^\infty dy l\left(\frac{t - \sqrt{t^2 - 4}}{2}\right) l(y) \frac{e^{-bt}}{y - \frac{t + \sqrt{t^2 - 4}}{2}} \\ &= \int_2^\infty dt \int_0^\infty dz l\left(\frac{t - \sqrt{t^2 - 4}}{2}\right) l\left(\left(\frac{t + \sqrt{t^2 - 4}}{2}\right)(1 + z)\right) \frac{e^{-bt}}{z} \end{aligned} \quad (1.16)$$

(after making the change of variable $y = (1+z) \left(\frac{t+\sqrt{t^2-4}}{2} \right)$)

$$P_2 = \int_2^\infty dt \int_0^1 dz l \left(\frac{t-\sqrt{t^2-4}}{2} \right) l \left(\left(\frac{t+\sqrt{t^2-4}}{2} \right) (1-z) \right) \frac{e^{-bt}}{z} \quad (1.17)$$

To compute N_2 and P_1 ($\subset \{(x, y) \in \mathbb{R}_+^2; x \geq 1\}$) for $x \geq 1$, $t \geq 2$ we make the change of variable : $x = \frac{t+\sqrt{t^2-4}}{2}$. So $\frac{1}{x} = \frac{t-\sqrt{t^2-4}}{2}$, $x + \frac{1}{x} = t$ and $\frac{x^2-1}{x^2} dx = dt$. We obtain :

$$N_2 = \int_2^\infty dt \int_0^1 dz l \left(\frac{t+\sqrt{t^2-4}}{2} \right) l \left(\left(\frac{t-\sqrt{t^2-4}}{2} \right) (1-z) \right) \frac{e^{-bt}}{z} \quad (1.18)$$

$$P_1 = \int_2^\infty dt \int_0^\infty dz l \left(\frac{t+\sqrt{t^2-4}}{2} \right) l \left(\left(\frac{t-\sqrt{t^2-4}}{2} \right) (1+z) \right) \frac{e^{-bt}}{z} \quad (1.19)$$

We shall now use the hypothesis : l belongs to \mathcal{C} to show that :

$$P_1 \geq N_1 \quad \text{and} \quad P_2 \geq N_2$$

which will end the proof of our Theorem.

• Comparing (1.19) and (1.16), it suffices, to prove that $P_1 \geq N_1$ to show that :

$$l \left(\frac{t+\sqrt{t^2-4}}{2} \right) l \left(\left(\frac{t-\sqrt{t^2-4}}{2} \right) (1+z) \right) \geq l \left(\frac{t-\sqrt{t^2-4}}{2} \right) l \left(\left(\frac{t+\sqrt{t^2-4}}{2} \right) (1+z) \right)$$

i.e. : $l \left(\frac{1}{x} \right) l(cx) \geq l(x) \cdot l \left(\frac{c}{x} \right)$ (1.20)

with $x \leq 1$ and $c \geq 1$.

If $a \geq 1$ (a being featured in the definition of \mathcal{C}), the relation (1.20) is trivially satisfied since $l(x) = 0$ for $x \leq a$ (and $x \leq 1$).

We now examine the case $0 \leq a < 1$.

If $x \leq a$, the relation (1.20) is again trivially satisfied. Thus, let us assume that $1 \geq x \geq a$. Relation (1.20) is equivalent to

$$\log l \left(\frac{1}{x} \right) - \log l(x) \geq \log l \left(\frac{c}{x} \right) - \log(cx)$$

or also to :

$$\int_x^{1/x} \frac{\theta(y)}{y} dy - \int_{cx}^{c/x} \frac{\theta(y)}{y} dy \geq 0 \quad (1.21)$$

(since $\log l(x) = \sigma + \int_b^x \frac{\theta(y)}{y} dy$, from (1.1)).

Thus, (1.21) is equivalent to :

$$\int_x^{1/x} \frac{\theta(y)}{y} dy - c \int_x^{1/x} \frac{\theta(cy)}{cy} dy = \int_x^{1/x} \frac{\theta(y) - \theta(cy)}{y} dy \geq 0 \quad (1.22)$$

and (1.22) is satisfied since θ is decreasing (and $c \geq 1$). We have shown that $P_1 \geq N_1$.

We now show that $P_2 \geq N_2$:

This time, using (1.17) and (1.18) it suffices to show that :

$$l\left(\frac{t - \sqrt{t^2 - 4}}{2}\right) l\left(\frac{t + \sqrt{t^2 - 4}}{2}(1 - z)\right) \geq l\left(\frac{t + \sqrt{t^2 - 4}}{2}\right) l\left(\frac{t - \sqrt{t^2 - 4}}{2}(1 - z)\right)$$

or, equivalently :

$$l(x) l\left(\frac{c}{x}\right) \geq l\left(\frac{1}{x}\right) l(cx) \quad \text{with } x \leq 1 \text{ and } c \leq 1 \quad (1.23)$$

Relation (1.23) is trivial for $x \leq a$ (since $cx \leq a$ and $l(cx) = 0$). It remains to examine the case $x \geq a$, $a \leq 1$. Relation (1.23) is then equivalent to :

$$\int_{cx}^{\frac{c}{x}} \frac{\theta(y)}{y} dy - \int_x^{\frac{1}{x}} \frac{\theta(y)}{y} dy \geq 0, \quad \text{i.e.} \quad \int_x^{\frac{1}{x}} \frac{\theta(cy) - \theta(y)}{y} dy \geq 0.$$

The latter relation is obvious since θ is decreasing (and $c < 1$). This ends the proof of Theorem 2. \blacksquare

Remark 3 Recall (see (1.8) above) that a function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be hyperbolically completely monotone (HCM) if, for every $u > 0$, the function of w :

$$w + \frac{1}{v} = u \longrightarrow \varphi(wv)\varphi\left(\frac{u}{v}\right) \quad (\text{with } v \geq 0)$$

is completely monotone. Thus, from (0.5), our Theorem 2 may be stated as follows : if l belongs to \mathcal{C} then its Stieltjes transform is HCM.

2 Application to some r.v.'s related to recurrent linear diffusions

2.1 Our notation and hypotheses are now those of Salminen-Vallois-Yor ([SVY]) to which we refer the reader. $(X_t, t \geq 0)$ denotes a \mathbb{R}_+ -valued diffusion which is recurrent ; we denote its speed measure (assumed to have no atoms) by m and its scale function by S . $(L_t, t \geq 0)$ denotes the (continuous) local time at 0 and $(\tau_u, u \geq 0)$ its right-continuous inverse :

$$\tau_u := \inf\{t \geq 0 ; L_t > u\} \quad (2.1)$$

$(\tau_u, u \geq 0)$ is a subordinator whose Lévy measure admits a density (see [SVY]) which we shall denote by ν :

$$E(\exp -\lambda\tau_u) = \exp \left\{ -u \int_0^\infty (1 - e^{-\lambda x}) \nu(x) dx \right\} \quad (2.2)$$

In fact, ν may be expressed in the form :

$$\nu(x) = \int_0^\infty e^{-xz} K(dz) \quad (2.3)$$

where K - the Krein measure - (see Kotani-Watanabe [K.W], Knight [K]) satisfies :

$$\int_0^\infty \frac{K(dz)}{z(1+z)} < \infty \quad \text{and} \quad \int_0^\infty \frac{K(dz)}{z} = \infty \quad (2.4)$$

2.2 Let, for every $t \geq 0$:

$$g_t := \sup\{s \leq t ; X_s = 0\}, \quad d_t := \inf\{s \geq t ; X_s = 0\} \quad (2.5)$$

and denote by ϵ_p ($p > 0$) an exponentially distributed variable with parameter p , i.e. with density $f_{\epsilon_p}(u) = p e^{-pu} 1_{u \geq 0}$; ϵ_p is assumed to be independent from $(X_t, t \geq 0)$. We define :

$$Y_p^{(1)} := \epsilon_p - g_{\epsilon_p}, \quad Y_p^{(2)} := d_{\epsilon_p} - \epsilon_p \quad Y_p^{(3)} := d_{\epsilon_p} - g_{\epsilon_p} \quad (2.6)$$

It is shown in [SVY], Theorem 16, that for $i = 1, 2, 3$, $Y_p^{(i)}$ is infinitely divisible.

More precisely, concerning $Y_p^{(3)}$, it is shown that $Y_p^{(3)}$ is a gamma-2 mixture, which implies from Kristiansen (see [Kr]) that $Y_p^{(3)}$ is infinitely divisible.

The aim of the following Theorem 4 is to improve, if possible, the results we just recalled.

More precisely, we shall prove that, under certain hypotheses the r.v.'s $Y_p^{(2)}$ ($i = 1, 2, 3$) are GGC r.v.'s whose Thorin measures have total masses $m \leq 1$. Thus, these variables :

- are GGC, hence are self-decomposable, and a fortiori are infinitely divisible,
- are gamma- m mixtures, with $m \leq 1$, and not only gamma-2 mixtures (see (0.9)).

Theorem 4 *We assume that Krein's measure K (defined by (2.3)) admits a differentiable density k .*

1. *Assume that :*

$$\frac{k'}{k}(x) = \frac{1}{x} + \frac{\theta(p+x)}{p+x}, \quad \text{with } \theta \text{ decreasing,} \quad (2.7)$$

then $Y_p^{(1)}$ is a GGC r.v. whose Thorin measure is a subprobability.

2. *Assume that :*

$$\frac{k'}{k}(x) = \frac{1}{x+p} + \frac{\theta(x)}{x}, \quad \text{with } \theta \text{ decreasing} \quad (2.8)$$

then $Y_p^{(2)}$ is a GGC r.v. whose Thorin measure is a subprobability.

3. *Assume that :*

$$\begin{cases} \frac{k'}{k}(z) = \frac{\theta(z)}{z} & \text{for } z < p \\ \frac{k'(z) - k'(z-p)}{k(z) - k(z-p)} = \frac{\theta(z)}{z} & \text{for } z \geq p \end{cases} \quad (2.9)$$

with θ decreasing

Then $Y_p^{(3)}$ is a GGC r.v. whose Thorin measure is a subprobability.

Proof of Theorem 4

We denote by $f_{Y_p^{(i)}}$ the density of $Y_p^{(i)}$. From [SVY], p. 115, we have :

$$f_{Y_p^{(1)}}(u) = C_1(p) \int_p^\infty e^{-uz} \cdot \frac{k(z-p)}{z-p} dz \quad (2.10)$$

$$f_{Y_p^{(2)}}(u) = C_2(p) \int_0^\infty e^{-uz} \frac{k(z)}{z+p} dz \quad (2.11)$$

$$f_{Y_p^{(3)}}(u) = C_3(p) \int_0^\infty e^{-uz} (k(z) - 1_{\{z \geq p\}} k(z-p)) dz \quad (2.12)$$

where $C_i(p)$, $i = 1, 2, 3$ are three normalising constants. We shall now use Theorem 2 with, successively :

$$l^{(1)}(x) = C_1(p) \frac{k(x-p)}{x-p} 1_{x \geq p} \quad (2.13)$$

$$l^{(2)}(x) = C_2(p) \frac{k(x)}{x+p} \quad (2.14)$$

$$l^{(3)}(x) = C_3(p) (k(x) - 1_{x \geq p} k(x-p)) \quad (2.15)$$

We already note that, for $i = 1, 2, 3$, $\int_0^\infty \frac{l^{(i)}(x)}{x} dx < \infty$. Indeed :

$$\begin{aligned} \int_0^\infty \frac{l^{(1)}(x)}{x} dx &= C_1(p) \int_p^\infty \frac{k(x-p)}{x(x-p)} dx = C_1(p) \int_0^\infty \frac{k(x)}{x(x+p)} dx \\ &< \infty \quad (\text{from (2.4)}) \end{aligned}$$

$$\int_0^\infty \frac{l^{(2)}(x)}{x} dx = C_2(p) \int_0^\infty \frac{k(x)}{x(x+p)} dx < \infty \quad (\text{from (2.4)})$$

$$\begin{aligned} \int_0^\infty \frac{l^{(3)}(x)}{x} dx &= C_3(p) \int_0^\infty k(x) \left(\frac{1}{x} - \frac{1}{x+p} \right) dx \\ &= p C_3(p) \int_0^\infty \frac{k(x)}{x(x+p)} dx < \infty \quad (\text{from 2.4}) \end{aligned}$$

Finally, it remains to observe that hypothesis (2.7) (resp. (2.8), resp. (2.9)) implies that $l^{(1)} \in \mathcal{C}$ (resp. $l^{(2)} \in \mathcal{C}$, resp. $l^{(3)} \in \mathcal{C}$). ■

3 Application to recurrent Bessel processes

3.1 The notation is the same as in the preceding part, but, now $(X_t, t \geq 0)$ is a Bessel process with dimension $d = 2(1 - \alpha)$ with $0 < d < 2$, or equivalently $0 < \alpha < 1$.

Theorem 5 For any $\alpha \in]0, 1[$, for any $p > 0$, the r.v.'s

$$Y_p^{(1)} = \mathfrak{e}_p - g_{\mathfrak{e}_p}, \quad Y_p^{(2)} = d_{\mathfrak{e}_p} - \mathfrak{e}_p, \quad Y_p^{(3)} = d_{\mathfrak{e}_p} - g_{\mathfrak{e}_p}$$

are GGC r.v.'s whose Thorin measures have the same total mass : $1 - \alpha = \frac{d}{2} (< 1)$.

Proof of Theorem 5

We already note that, since :

$$\nu(a) = \int_0^\infty e^{-az} K(dz) \quad (\text{from (2.3)})$$

$$\text{and} \quad \nu(a) = \frac{1}{2^\alpha \Gamma(\alpha)} \frac{1}{a^{\alpha+1}} \quad (\text{from [D-M, RVY], p. 5})$$

then the density k of Krein's measure equals here :

$$k(a) = \frac{1}{2^\alpha \Gamma(\alpha) \Gamma(\alpha + 1)} a^\alpha \quad (a > 0) \quad (3.1)$$

1. We begin by proving Theorem 5 for the r.v. $Y^{(2)}$.

(To simplify the notation, we write $Y^{(2)}$ instead $Y_p^{(2)}$). To see that $Y^{(2)}$ is GGC, it suffices, from Theorem 2, to show that $l^{(2)} \in \mathcal{C}$ where here :

$$l^{(2)}(x) = C \frac{x^\alpha}{x+p} \quad (\text{from (3.1) and (2.14)}) \quad (3.2)$$

Thus :

$$x(\log l^{(2)})'(x) = \alpha - \frac{x}{x+p} = \alpha - 1 + \frac{p}{x+p}$$

is a decreasing function of x , hence $l^{(2)} \in \mathcal{C}$ from (1.2). It remains to see that the total mass of the Thorin measure of $Y^{(2)}$ equals $1 - \alpha$. Now, from (0.10), this total mass m equals :

$$\begin{aligned} m &:= \sup \left\{ \delta \geq 0 ; \lim_{u \downarrow 0+} \frac{1}{u^{\delta-1}} f_{Y^{(2)}}(u) = 0 \right\} \\ &= \sup \left\{ \delta \geq 0 ; \lim_{u \downarrow 0+} \frac{C}{u^{\delta-1}} \int_0^\infty e^{-ux} \frac{x^\alpha}{x+p} dx = 0 \right\} \end{aligned} \quad (3.3)$$

However, since the function $x \rightarrow \frac{x^\alpha}{x+p}$ decreases for x large enough and is equivalent to $x^{\alpha-1}$ when $x \rightarrow \infty$, the Tauberian Theorem implies :

$$f_{Y^{(2)}}(u) \underset{u \rightarrow 0}{\sim} \frac{C'}{u^\alpha} \quad (3.4)$$

It is then clear that (3.3) and (3.4) imply $m = 1 - \alpha$.

2. We now prove Theorem 5 for the r.v. $Y^{(1)}$.

For this purpose, we shall use a more direct method than relying on Theorem 2. Indeed, we have, from (0.5), (2.13) and (3.1) :

$$\begin{aligned} E(e^{-\lambda Y^{(1)}}) &= \int_0^\infty \frac{l^{(1)}(z)}{\lambda+z} dz = C \int_p^\infty \frac{1}{\lambda+z} (z-p)^{\alpha-1} dz \\ &= C \int_0^\infty \frac{1}{\lambda+p+z} z^{\alpha-1} dz = C \int_0^\infty z^{\alpha-1} dz \int_0^\infty e^{-(\lambda+p+z)u} du \\ &= C \int_0^\infty e^{-(\lambda+p)u} du \int_0^\infty e^{-zu} z^{\alpha-1} dz \\ &= C \Gamma(\alpha) \int_0^\infty e^{-(\lambda+p)u} \frac{du}{u^\alpha} = (\lambda+p)^{\alpha-1} C \Gamma(\alpha) \Gamma(1-\alpha) \\ &= \left(1 + \frac{\lambda}{p}\right)^{\alpha-1} \end{aligned} \quad (3.5)$$

since the Laplace transform $E(e^{-\lambda Y^{(1)}})$ equals 1 for $\lambda = 0$. Thus :

$$Y^{(1)} \stackrel{(\text{law})}{=} \frac{1}{p} \gamma_{1-\alpha} \text{ where } \gamma_{1-\alpha} \text{ is a gamma r.v. with parameter } 1 - \alpha, \text{ i.e.} \quad (3.6)$$

with density :

$$f_{\gamma_{1-\alpha}}(u) := \frac{e^{-u}}{\Gamma(1-\alpha)} u^{-\alpha} \mathbf{1}_{u \geq 0}$$

It follows clearly from (3.5) that :

$$\begin{aligned} E(e^{-\lambda Y^{(1)}}) &= \exp \left\{ -(1-\alpha) \log \left(1 + \frac{\lambda}{p} \right) \right\} \\ &= \exp \left\{ -(1-\alpha) \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} e^{-xp} \right\} \end{aligned} \quad (3.7)$$

Thus, from (0.7), formula (3.7) shows that $Y^{(1)}$ is a GGC variable with Thorin measure $(1-\alpha)\delta_p$.

3. We now prove Theorem 5 for the r.v. $Y_p^{(3)}$.

In fact, this result - $Y^{(3)}$ is a GGC variable whose Thorin measure has total mass equal to $1-\alpha$ - has already been proven in [BFRY] (with $p = 1$, but this involves no loss of generality). The proof we shall give now is a totally different one from that of [BFRY]. We also assume here, for simplicity, that $p = 1$ and we denote $Y^{(3)}$ instead of $Y_1^{(3)}$.

Following the arguments of the proof of Theorem 2, we need to show, from (1.16), (1.17), (1.18) and (1.19) that, for every $x \in [0, 1]$:

$$\begin{aligned} \Delta(x) &= \int_0^\infty \left\{ l\left(\frac{1}{x}\right) l(x(1+z)) - l(x) l\left(\frac{1}{x}(1+z)\right) \right\} \frac{dz}{z} \\ &+ \int_0^1 \left\{ l(x) l\left(\frac{1}{x}(1-z)\right) - l\left(\frac{1}{x}\right) l(x(1-z)) \right\} \frac{dz}{z} \geq 0 \end{aligned} \quad (3.8)$$

where the function $l(= l^{(3)})$ equals here, from (3.1) and (2.15) :

$$l(y) = y^\alpha - \mathbf{1}_{y \geq 1} (y-1)^\alpha \quad (y \geq 0) \quad (3.9)$$

Thus, we need to show (3.8). For this purpose, we need to compute the integrals featured in (3.8) hence, given (3.9) to discuss, owing to the positions of $x(1+z)$, $\frac{1}{x}(1+z)$, $\frac{1}{x}(1-z)$ and $x(1-z)$ with respect to 1. We consider the first integral in (3.8) for $x(1+z) \geq 1$ (hence, a

fortiori $\frac{1}{x}(1+z) \geq 1$ since $x \leq 1$). This first term equals :

$$\begin{aligned}
\Delta_1(x) &= \int_{\frac{1}{x}-1}^{\infty} \left\{ \left(\frac{1}{x^\alpha} - \left(\frac{1}{x} - 1 \right)^\alpha \right) (x^\alpha (1+z)^\alpha - (x(1+z) - 1)^\alpha) \right. \\
&\quad \left. - x^\alpha \left[\left(\frac{1}{x^\alpha} (1+z)^\alpha \right) - \left(\frac{1}{x} (1+z) - 1 \right)^\alpha \right] \right\} \frac{dz}{z} \\
&= \int_{\frac{1}{x}-1}^{\infty} (1 - (1-x)^\alpha) \left((1+z)^\alpha - \left(1+z - \frac{1}{x} \right)^\alpha \right) - ((1+z)^\alpha - (1+z-x)^\alpha) \frac{dz}{z} \\
&= \int_{\frac{1}{x}-1}^{\infty} \left\{ \left[(1+z-x)^\alpha - \left(1+z - \frac{1}{x} \right)^\alpha \right] - \left[(1-x)^\alpha (1+z)^\alpha - \left((1-x)(1+z) - \frac{1-x}{x} \right)^\alpha \right] \right\} \frac{dz}{z} \\
&:= \Delta_1^{(1)}(x) - \Delta_1^{(2)}(x).
\end{aligned}$$

Let us examine $\Delta_1^{(1)}(x)$:

$$\begin{aligned}
\Delta_1^{(1)}(x) &= \int_{\frac{1}{x}-1}^{\infty} \left[(1+z-x)^\alpha - \left(1+z - \frac{1}{x} \right)^\alpha \right] \frac{dz}{z} \\
&= \int_{\frac{1}{x}-1}^{\infty} \frac{dz}{z} \int_{1+z-\frac{1}{x}}^{1+z-x} \alpha u^{\alpha-1} du
\end{aligned}$$

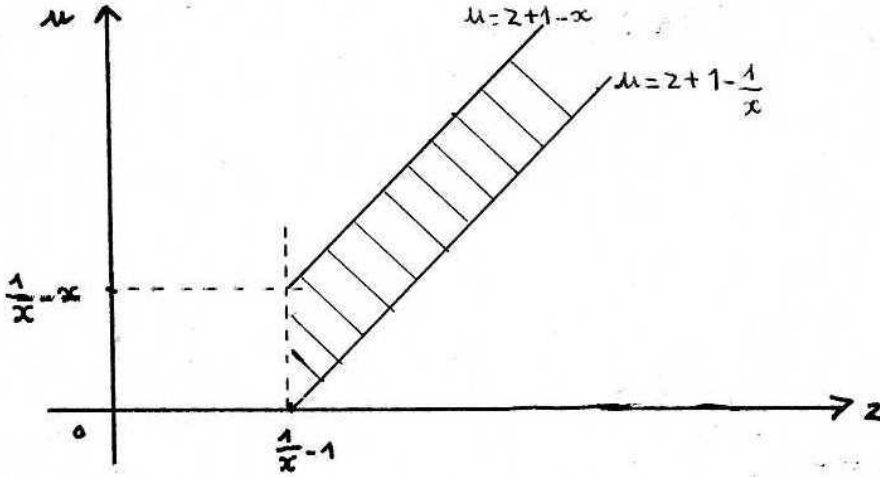


Figure 2

Now, we apply Fubini's Theorem :

$$\begin{aligned}
\Delta_1^{(1)}(x) &= \int_0^{\frac{1}{x}-1} \alpha u^{\alpha-1} du \int_{\frac{1}{x}-1}^{u+\frac{1}{x}-1} \frac{dz}{z} + \int_{\frac{1}{x}-1}^{\infty} \alpha u^{\alpha-1} du \int_{u+x-1}^{u+\frac{1}{x}-1} \frac{dz}{z} \\
&= \int_0^{\frac{1}{x}-1} \alpha u^{\alpha-1} \log \left(\frac{ux + 1 - x}{1 - x} \right) du + \int_{\frac{1}{x}-1}^{\infty} \alpha u^{\alpha-1} \log \left(\frac{u + \frac{1}{x} - 1}{u + x - 1} \right) du.
\end{aligned}$$

We compute thus each term of $\Delta(x)$ and we obtain, after some simple, although tedious, computations :

$$\begin{aligned}\Delta(x) &= \int_0^{\frac{1}{x}-1} \log\left(\frac{1}{1-x}\right) \cdot \alpha u^{\alpha-1} du + \int_0^{\frac{1}{x}-x} \log\left(\frac{ux+1-x}{1-x}\right) \alpha u^{\alpha-1} du \\ &+ \int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{u+\frac{1}{x}-1}{u+x-1}\right) \alpha u^{\alpha-1} du + \int_0^{x(1-x)} \log\left(\frac{1-x-u}{(1-x)^2}\right) \alpha u^{\alpha-1} du \\ &- \int_0^{\frac{1}{x}-1} \log\left(\frac{\frac{u}{1-x}+\frac{1}{x}-1}{\frac{1}{x}-1}\right) \alpha u^{\alpha-1} du - \int_{\frac{1}{x}-1}^{\infty} \log\left(\frac{\frac{u}{1-x}+\frac{1}{x}-1}{\frac{u}{1-x}-1}\right) \alpha u^{\alpha-1} du\end{aligned}\quad (3.10)$$

We note that, since $x \in [0, 1]$, we have :

$$x(1-x) \leq 1-x \leq \frac{1}{x}-1 \leq \frac{1}{x}-x$$

and that all the integrals found in (3.10) are positive. In (3.10) we shall gather the terms with opposite signs. For example, we have :

$$\begin{aligned}&\int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{u+\frac{1}{x}-1}{u+x-1}\right) \alpha u^{\alpha-1} du - \int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{\frac{u}{1-x}+\frac{1}{x}-1}{\frac{u}{1-x}-1}\right) \alpha u^{\alpha-1} du \\ &= \int_{\frac{1}{x}-x}^{\infty} \log\left(\frac{ux+(1-x)}{ux+(1-x)^2}\right) \alpha u^{\alpha-1} du\end{aligned}$$

and this last integral is positive since $(1-x)^2 \leq 1-x$. Gathering thus all the terms in $\Delta(x)$, we obtain :

$$\begin{aligned}\Delta(x) &= \int_0^{\infty} \log\left(\frac{ux+1-x}{ux+(1-x)^2}\right) \alpha u^{\alpha-1} du - \int_{\frac{1}{x}-1}^{\frac{1}{x}-x} \log\left(\frac{1-x}{x(u+x-1)}\right) \alpha u^{\alpha-1} du \\ &\quad \int_0^{x(1-x)} \log\left(\frac{1-x-u}{(1-x)^2}\right) \alpha u^{\alpha-1} du \\ &= \alpha(1-x)^\alpha \int_{\frac{1}{x}}^{\frac{1}{x}+1} v^{\alpha-1} \log\left(\frac{1}{x(v-1)}\right) \\ &\quad \left[\left(\frac{xv-1}{1+v-xv}\right)^{\alpha-1} \frac{x^{2-\alpha}}{(1+x-xv)^2} + \left(\frac{xv-1}{x(v-1)}\right)^{\alpha-1} \frac{1-x}{x(v-1)^2} - v^{\alpha-1} \right] dv\end{aligned}\quad (3.11)$$

after making the changes of variables :

$$\begin{aligned}\frac{ux+1-x}{ux+(1-x)^2} &= \frac{1}{x(v-1)} && \text{in the first integral of (3.11)} \\ u &= (1-x)v && \text{in the second integral of (3.11)} \\ \frac{1-x-u}{(1-x)^2} &= \frac{1}{x(v-1)} && \text{in the third integral of (3.11)}\end{aligned}$$

Thus, to conclude, it remains to show that, for every $v \in \left[\frac{1}{x}, \frac{1}{x}+1\right]$:

$$v^{\alpha-1} \leq \left(\frac{xv-1}{1+x-xv}\right)^{\alpha-1} \frac{1}{x^{\alpha-1}} \frac{x}{(1+x-xv)^2} + \left(\frac{xv-1}{x(v-1)}\right)^{\alpha-1} \frac{1-x}{x(v-1)^2}\quad (3.12)$$

or, equivalently that :

$$\left(\frac{xv-1}{xv}\right)^{1-\alpha} \leq \frac{x}{(1+x-xv)^{\alpha+1}} + \frac{1-x}{x} \frac{1}{(v-1)^{\alpha+1}} \quad (3.13)$$

Now, this last inequality is obvious ; indeed, since $f_1(v) := \left(\frac{xv-1}{xv}\right)^{1-\alpha}$ is increasing as well as $f_2(v) := \frac{x}{(1+x-xv)^{\alpha+1}}$ it suffices to verify that $f_1\left(\frac{1}{x}+1\right) \leq f_2\left(\frac{1}{x}\right)$. We have :

$$f_1\left(\frac{1}{x}+1\right) = \left(\frac{x}{1+x}\right)^{1-\alpha} \leq 1 \leq f_2\left(\frac{1}{x}\right) = \frac{1}{x^\alpha}$$

since $x \in [0, 1]$. This shows that $Y^{(3)}$ is GGC. Finally, it is not difficult to prove that the total mass of the Thorin measure equals $1 - \alpha$: this follows from the fact that since $l^{(3)}(x) = C(x^\alpha - 1_{x \geq 1}(x-1)^\alpha)$ then $l^{(3)}(x) \underset{x \rightarrow \infty}{\sim} Cx^{\alpha-1}$, hence, from the Tauberian Theorem, $f_{Y^{(3)}}(u) \underset{u \rightarrow 0}{\sim} \frac{C}{u^\alpha}$, and we finally use (0.10). ■

3.2 Description of the r.v.'s $\mathbb{G}_\alpha^{(i)}$ ($i = 1, 2, 3$; $0 < \alpha < 1$)

In the sequel, it will be convenient to assume that $p = 1$ and we write simply $Y^{(i)}$ for the r.v.'s $Y_1^{(i)}$ ($i = 1, 2, 3$). Theorem 5 implies, from (0.7), the existence of r.v.'s

$$\begin{aligned} &\mathbb{G}_\alpha^{(i)} \quad (i = 1, 2, 3 ; \alpha \in]0, 1[) \text{ such that } E(\log^+(1/\mathbb{G}_\alpha^{(i)})) < \infty \quad \text{and :} \\ &E(e^{-\lambda Y^{(i)}}) = \exp \left\{ -(1-\alpha) \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} E(e^{-x\mathbb{G}_\alpha^{(i)}}) \right\} \end{aligned} \quad (3.14)$$

The aim of this section is to identify the (laws of the) r.v.'s $\mathbb{G}_\alpha^{(i)}$ and to describe some of their properties.

i) The case $i = 1$

Formula (3.6) implies that the r.v. $\mathbb{G}_\alpha^{(1)}$ is a.s. equal to 1, i.e. its distribution is δ_1 , the Dirac measure at 1. In particular, this distribution does not depend on α .

ii) The case $i = 3$

In [BFRY] a complete study of the r.v.'s $\mathbb{G}_\alpha^{(3)}$ - denoted as \mathbb{G}_α in [BFRY] - has been undertaken. We refer the reader to [BFRY]. In particular, it is shown there that the density $f_{\mathbb{G}_\alpha^{(3)}}$ of $\mathbb{G}_\alpha^{(3)}$ equals :

$$f_{\mathbb{G}_\alpha^{(3)}}(u) = \frac{\alpha \sin \pi \alpha}{(1-\alpha)\pi} \frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha} - 2(1-u)^\alpha u^\alpha \cos(\pi \alpha) + 1} 1_{[0,1]}(u) \quad (3.15)$$

Thus, $\mathbb{G}_{1/2}^{(3)}$ is arc-sine distributed :

$$f_{\mathbb{G}_{1/2}^{(3)}}(u) = \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}} 1_{[0,1]}(u) \quad (3.16)$$

and the r.v.'s $\mathbb{G}_\alpha^{(3)}$ converge in law, as $\alpha \rightarrow 0$ and $\alpha \rightarrow 1$ respectively towards $\mathbb{G}_0^{(3)}$ and $\mathbb{G}_1^{(3)}$, where :

$$\mathbb{G}_0^{(3)} \stackrel{\text{(law)}}{=} \frac{1}{1 + \exp \pi C}, \quad \text{with } C \text{ a standard Cauchy r.v.} \quad (3.17)$$

$$\mathbb{G}_1^{(3)} \stackrel{\text{(law)}}{=} U, \quad \text{with } U \text{ uniform on } [0, 1] \quad (3.18)$$

iii) The case $i = 2$

Theorem 6 For every $\alpha \in]0, 1[$

$$1) \quad i) \quad Y^{(2)} \stackrel{\text{(law)}}{=} \boldsymbol{\epsilon} \cdot \frac{\gamma_{1-\alpha}}{\gamma_\alpha} \stackrel{\text{(law)}}{=} \boldsymbol{\epsilon} \frac{\beta_{1-\alpha, \alpha}}{1 - \beta_{1-\alpha, \alpha}} \quad (3.19)$$

where $\boldsymbol{\epsilon}, \gamma_{1-\alpha}, \gamma_\alpha$ are independent, with respective laws the standard exponential and the gamma distributions with respective parameters $(1-\alpha)$ and α , and where $\boldsymbol{\epsilon}$ and $\beta_{1-\alpha, \alpha}$ are independent with respective distributions the standard exponential and the beta distribution with parameters $(1-\alpha, \alpha)$.

$$ii) \quad E(e^{-\lambda Y^{(2)}}) = \frac{\lambda^\alpha - 1}{\lambda - 1} \quad (= \alpha \text{ if } \lambda = 1) \quad (\lambda \geq 0) \quad (3.20)$$

2) $Y^{(2)}$ is a gamma- $(1-\alpha)$ mixture, i.e. :

$$Y^{(2)} = \gamma_{1-\alpha} \cdot D_{1-\alpha}^{(2)} \quad (3.21)$$

where $\gamma_{1-\alpha}$ is a gamma $(1-\alpha)$ variable, independent from the positive r.v. $D_{1-\alpha}^{(2)}$. Furthermore :

$$D_{1-\alpha}^{(2)} \stackrel{\text{law}}{=} \frac{\boldsymbol{\epsilon}}{\gamma_\alpha} \quad (3.22)$$

$$E(e^{-\lambda D_{1-\alpha}^{(2)}}) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{\lambda + y} dy = \alpha \int_0^\infty \frac{e^{-\lambda y}}{(1+y)^{\alpha+1}} dy \quad (3.23)$$

The density $f_{D_{1-\alpha}^{(2)}}$ of $D_{1-\alpha}^{(2)}$ equals :

$$f_{D_{1-\alpha}^{(2)}}(u) = \frac{\alpha}{(1+u)^{\alpha+1}} 1_{[0, \infty[}(u) \quad (3.24)$$

3) i) The density $f_{\mathbb{G}_\alpha^{(2)}}$ of $\mathbb{G}_\alpha^{(2)}$ equals :

$$f_{\mathbb{G}_\alpha^{(2)}}(u) = \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \frac{u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1} 1_{[0, \infty[}(u) \quad (3.25)$$

ii) The r.v.'s $\mathbb{G}_\alpha^{(2)}$ are related to the r.v.'s $\mathbb{G}_\alpha^{(3)}$ via the identity in law :

$$\frac{\mathbb{G}_\alpha^{(2)}}{1 + \mathbb{G}_\alpha^{(2)}} \stackrel{\text{(law)}}{=} \mathbb{G}_\alpha^{(3)}, \quad \text{or, equivalently,} \quad \mathbb{G}_\alpha^{(2)} \stackrel{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1 - \mathbb{G}_\alpha^{(3)}} \quad (3.26)$$

$$iii) \quad \mathbb{G}_\alpha^{(2)} \stackrel{\text{(law)}}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}} \quad (3.27)$$

iv) As $\alpha \rightarrow 0$ and $\alpha \rightarrow 1$, $\mathbb{G}_\alpha^{(2)}$ converges in law towards, respectively

$$\mathbb{G}_0^{(2)} \stackrel{(\text{law})}{=} \exp \pi C \quad \text{and} \quad \mathbb{G}_1^{(2)} \stackrel{(\text{law})}{=} \frac{U}{1-U} \quad (3.28)$$

with C a standard Cauchy r.v. and U uniform on $[0, 1]$.

4) Let $\mu \in]0, 1[$ and T_μ denote the positive stable r.v. with index μ whose law is characterized by :

$$E(e^{-\lambda T_\mu}) = \exp(-\lambda^\mu) \quad (\lambda > 0)$$

Then :

$$\mathbb{G}_\alpha^{(2)} \stackrel{(\text{law})}{=} \left(\frac{T_{1-\alpha}}{T'_{1-\alpha}} \right)^{\frac{1-\alpha}{\alpha}} \quad (3.29)$$

where $T'_{1-\alpha}$ is an independent copy of $T_{1-\alpha}$.

ii) An equivalent way of writing (3.29) is :

$$\mathbb{G}_\alpha^{(2)} \stackrel{(\text{law})}{=} \left(\frac{M_{1-\alpha}}{M'_{1-\alpha}} \right)^{\frac{1}{\alpha}} \quad (3.30)$$

where $M_{1-\alpha}$ and $M'_{1-\alpha}$ are two independent Mittag-Leffler r.v.'s with parameter $1-\alpha$, whose common law is characterized by :

$$E(\exp \lambda M_{1-\alpha}) = \sum_{n \geq 0} \frac{\lambda^n}{\Gamma(1+n(1-\alpha))}, \quad E[M_{1-\alpha}^n] = \frac{\Gamma(n+1)}{\Gamma(1+n(1-\alpha))}$$

$$M_{1-\alpha} \stackrel{(\text{law})}{=} \left(\frac{1}{T_{1-\alpha}} \right)^{1-\alpha} \quad (3.31)$$

(see [CY], p.114, Exercise 4.19).

Proof of Theorem 6

1) i) We prove (3.19)

Denoting by $(R_t, t \geq 0)$ a Bessel process with dimension $2(1-\alpha)$ ($0 < \alpha < 1$) starting from 0, we have by scaling :

$$Y^{(2)} = d_\epsilon - \epsilon \stackrel{(\text{law})}{=} \epsilon(d_1 - 1) \stackrel{(\text{law})}{=} \epsilon \left(\frac{R_1^2}{2\gamma_\alpha} \right)$$

(see [BFRY]) where R_1^2 is the value of R_t^2 for $t = 1$. Hence :

$$Y^{(2)} \stackrel{(\text{law})}{=} \epsilon \frac{\gamma_{1-\alpha}}{\gamma_\alpha} = \epsilon \frac{\beta_{1-\alpha, \alpha}}{1 - \beta_{1-\alpha, \alpha}}$$

(from the classical "beta-gamma algebra").

ii) We prove (3.20)

We have, from (3.1) and (3.2) :

$$l^{(2)}(x) = \frac{\sin(\pi\alpha)}{\pi} \frac{x^\alpha}{1+x} \quad x \geq 0$$

(we note that $\int_0^\infty \frac{l^{(2)}(x)}{x} dx = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\sin(\pi\alpha)}{\pi} B(\alpha, 1-\alpha) = \frac{\sin(\pi\alpha)}{\pi} \Gamma(\alpha) \Gamma(1-\alpha) = 1$ (see [L], p. 3 and 13)). Hence from (0.3), $f_{Y^{(2)}}$, the density of $Y^{(2)}$, equals :

$$f_{Y^{(2)}}(u) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-ux} \frac{x^\alpha}{1+x} dx$$

(we might also have derived this formula from (3.19)).

iii) We now compute the Laplace transform of $Y^{(2)}$

$$\begin{aligned} E(e^{-\lambda Y^{(2)}}) &= \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-\lambda u} du \int_0^\infty e^{-ux} \frac{x^\alpha}{1+x} dx \\ &= \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{x^\alpha}{(1+x)(\lambda+x)} dx \\ &= \frac{1}{\lambda-1} \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty x^\alpha \left[\frac{1}{1+x} - \frac{1}{\lambda+x} \right] dx \\ &= \lim_{A \rightarrow \infty} \frac{1}{\lambda-1} \frac{\sin(\pi\alpha)}{\pi} \left[\int_0^A \frac{x^\alpha}{1+x} dx - \lambda^\alpha \int_0^{\frac{A}{\lambda}} \frac{x^\alpha}{1+x} dx \right] \\ &= \lim_{A \rightarrow \infty} \frac{1}{\lambda-1} \frac{\sin(\pi\alpha)}{\pi} \left[\int_0^A \left(x^{\alpha-1} - \frac{x^{\alpha-1}}{1+x} \right) dx - \lambda^\alpha \int_0^{\frac{A}{\lambda}} \left(x^{\alpha-1} - \frac{x^{\alpha-1}}{1+x} \right) dx \right] \\ &= \lim_{A \rightarrow \infty} \frac{1}{\lambda-1} \frac{\sin(\pi\alpha)}{\pi} \left[\frac{A^\alpha}{\alpha} - \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx - \frac{\lambda^\alpha}{\alpha} \left(\frac{A}{\lambda} \right)^\alpha + \lambda^\alpha \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx \right] \\ &= \frac{\lambda^\alpha - 1}{\lambda-1} \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\lambda^\alpha - 1}{\lambda-1} \frac{\sin(\pi\alpha)}{\pi} B(\alpha, 1-\alpha) \\ &= \frac{\lambda^\alpha - 1}{\lambda-1} \end{aligned}$$

(since (see [L], p. 3) $B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$).

2) Let us show (3.25)

By taking the logarithmic derivative of (3.20) :

$$E(e^{-\lambda Y^{(2)}}) = \frac{\lambda^\alpha - 1}{\lambda - 1} = \exp \left\{ -(1-\alpha) \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} E(e^{-x \mathbb{G}_\alpha^{(2)}}) \right\}$$

we obtain :

$$E \left[\frac{1}{\lambda + \mathbb{G}_\alpha^{(2)}} \right] = \frac{1}{1-\alpha} \left[\frac{1}{1-\lambda} - \frac{\alpha \lambda^{\alpha-1}}{\lambda^\alpha - 1} \right] \quad (3.32)$$

Thus, we have just computed the Stieltjes transform of the r.v. $\mathbb{G}_\alpha^{(2)}$. The inversion formula

for the Stieltjes transform (see [W], p. 345) leads us to :

$$\begin{aligned} f_{\mathbb{G}_\alpha^{(2)}}(u) &= \frac{1}{2i\pi(1-\alpha)} \lim_{\eta \rightarrow 0} \left[\frac{1}{1-\lambda(-u-i\eta)} - \frac{\alpha(-u-i\eta)^{\alpha-1}}{(-u-i\eta)^\alpha - 1} - \frac{1}{1-\lambda(-u+i\eta)} \right. \\ &\quad \left. + \frac{\alpha(-u+i\eta)^{\alpha-1}}{(-u+i\eta)^\alpha - 1} \right] \quad (u > 0) \\ &= \frac{-\alpha}{2i\pi(1-\alpha)} \left[\frac{-u^{\alpha-1} e^{-i\pi\alpha}}{u^\alpha e^{-i\pi\alpha} - 1} + \frac{u^{\alpha-1} e^{i\pi\alpha}}{u^\alpha e^{i\pi\alpha} - 1} \right] \quad (u > 0) \end{aligned}$$

(We note that, in the preceding limit, the contribution of the term $\frac{1}{1-\lambda}$ is 0).

$$\begin{aligned} &= \frac{-\alpha}{2i\pi(1-\alpha)} \left[\frac{-u^{2\alpha-1} + u^{\alpha-1} e^{-i\pi\alpha} + u^{2\alpha-1} - u^{\alpha-1} e^{i\pi\alpha}}{u^{2\alpha} - u^\alpha e^{i\pi\alpha} - u^\alpha e^{-i\pi\alpha} + 1} \right] \quad (u > 0) \\ &= \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \frac{u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1} 1_{(u>0)} \end{aligned}$$

3) We now show (3.26)

For every h Borel and positive, we have :

$$\begin{aligned} E \left[h \left(\frac{\mathbb{G}_\alpha^{(2)}}{1 + \mathbb{G}_\alpha^{(2)}} \right) \right] &= \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^\infty h \left(\frac{u}{1+u} \right) \frac{u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1} du \\ &\quad \text{(from (3.25))} \end{aligned}$$

Thus, making the change of variable $\frac{u}{1+u} = x$:

$$\begin{aligned} E \left[h \left(\frac{\mathbb{G}_\alpha^{(2)}}{1 + \mathbb{G}_\alpha^{(2)}} \right) \right] &= \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^1 h(x) \frac{dx}{(1-x)^2} \frac{\frac{x^{\alpha-1}}{(1-x)^{\alpha-1}}}{\frac{x^{2\alpha}}{(1-x)^{2\alpha}} - \frac{2 \cos(\pi\alpha) x^\alpha}{(1-x)^\alpha} + 1} \\ &= \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^1 h(x) \frac{x^{\alpha-1} (1-x)^{\alpha-1}}{x^{2\alpha} - 2x^\alpha (1-x)^\alpha \cos(\pi\alpha) + (1-x)^{2\alpha}} dx \\ &= E[h(\mathbb{G}_\alpha^{(3)})] \quad \text{from (3.15)} \end{aligned}$$

4) We now prove (3.27)

It is shown in [BFRY], p. 319, (1.27) that :

$$\mathbb{G}_\alpha^{(3)} \stackrel{\text{(law)}}{=} 1 - \mathbb{G}_\alpha^{(3)} \quad (3.33)$$

which is, indeed, obvious ! Thus, from (3.26) :

$$\mathbb{G}_\alpha^{(2)} \stackrel{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1 - \mathbb{G}_\alpha^{(3)}} \stackrel{\text{(law)}}{=} \frac{1 - \mathbb{G}_\alpha^{(3)}}{\mathbb{G}_\alpha^{(3)}} = \frac{1 + \mathbb{G}_\alpha^{(2)} - \mathbb{G}_\alpha^{(2)}}{1 + \mathbb{G}_\alpha^{(2)}} \stackrel{\text{(law)}}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}}$$

5) The relation (3.28) follows immediately from (3.26) and from (3.17) and (3.18).

6) We prove (3.29)

It is shown in [BFRY], p. 320, that :

$$\mathbb{G}_\alpha^{(3)} \stackrel{\text{(law)}}{=} \frac{(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}} + (T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}} \quad \text{and} \quad \mathbb{G}_\alpha^{(3)} \stackrel{\text{(law)}}{=} \frac{(M_{1-\alpha})^{\frac{1}{\alpha}}}{(M_{1-\alpha})^{\frac{1}{\alpha}} + (M'_{1-\alpha})^{\frac{1}{\alpha}}} \quad (3.34)$$

Thus, from (3.26) and (3.34) :

$$\mathbb{G}_\alpha^{(2)} \stackrel{\text{(law)}}{=} \frac{\mathbb{G}_\alpha^{(3)}}{1 - \mathbb{G}_\alpha^{(3)}} = \frac{\frac{(T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}} + (T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}}{\frac{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}{(T'_{1-\alpha})^{\frac{1-\alpha}{\alpha}} + (T_{1-\alpha})^{\frac{1-\alpha}{\alpha}}}} = \left(\frac{T_{1-\alpha}}{T'_{1-\alpha}} \right)^{\frac{1-\alpha}{\alpha}} \quad (3.35)$$

We note that (3.35) implies (3.27) and that (3.30) may be obtained from (3.35), in the same manner as (3.35).

7) We now prove point 2 of Theorem 6

The formula (3.21) $D_{1-\alpha}^{(2)} \stackrel{\text{(law)}}{=} \frac{\mathbf{e}}{\gamma_\alpha}$ is an immediate consequence of (3.19) :

$$Y^{(2)} \stackrel{\text{(law)}}{=} \mathbf{e} \cdot \frac{\gamma_{1-\alpha}}{\gamma_\alpha} \stackrel{\text{(law)}}{=} \gamma_{1-\alpha} D_{1-\alpha}^{(2)}$$

after observing that, in the latter formula, we may "simplify by $\gamma_{1-\alpha}$ " (see [C.Y] or [JRY], point 1.4.6 for a justification of this "simplification"). The value of the density of $D_{1-\alpha}^{(2)}$ which is given by (3.24) now follows easily from $D_{1-\alpha}^{(2)} \stackrel{\text{(law)}}{=} \frac{\mathbf{e}}{\gamma_\alpha}$. Finally, we have :

$$\begin{aligned} E(e^{-\lambda D_{1-\alpha}^{(2)}}) &= E(e^{-\lambda \frac{\mathbf{e}}{\gamma_\alpha}}) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty e^{-\lambda \frac{x}{y} - x - y} y^{\alpha-1} dx dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^\alpha dy \int_0^\infty e^{-z(\lambda+y)} dz \left(\text{after making the change of variable } \frac{x}{y} = z \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^\alpha}{(\lambda+y)} e^{-y} dy \end{aligned} \quad (3.36)$$

The formula :

$$E(e^{-\lambda D_{1-\alpha}^{(2)}}) = \alpha \int_0^\infty e^{-\lambda y} \frac{dy}{(1+y)^{\alpha+1}} \quad (3.37)$$

follows immediately from (3.24) and it is easy to verify that :

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{\lambda+y} dy = \alpha \int_0^\infty e^{-\lambda y} \frac{dy}{(1+y)^{\alpha+1}}$$

Indeed :

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{\lambda+y} dy &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^\alpha dy \int_0^\infty e^{-z(\lambda+y)} dz \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda z} dz \int_0^\infty e^{-y(1+z)} y^\alpha dy = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda z} \frac{dz}{(1+z)^{\alpha+1}} \\ &= \alpha \int_0^\infty e^{-\lambda z} \frac{dz}{(1+z)^{\alpha+1}} \end{aligned}$$

This ends the proof of Theorem 6.

Remark 7

1) From the relation $Y^{(2)} \stackrel{(\text{law})}{=} \gamma_{1-\alpha} D_{1-\alpha}^{(2)}$, we deduce :

$$\begin{aligned} E(e^{-\lambda Y^{(2)}}) &= E(e^{-\lambda \gamma_{1-\alpha} D_{1-\alpha}^{(2)}}) = E\left(\frac{1}{(1 + \lambda D_{1-\alpha}^{(2)})^{1-\alpha}}\right) \\ &= \alpha \int_0^\infty \left(\frac{1 + \lambda x}{1 + x}\right)^{\alpha-1} \frac{dx}{(1 + x)^2} \quad (\text{from (3.24)}) \\ &= \frac{\alpha}{\lambda - 1} \int_1^\lambda y^{\alpha-1} dy \quad \left(\text{after making the change of variable } \frac{1 + \lambda x}{1 + x} = y\right) \\ &= \frac{\lambda^\alpha - 1}{\lambda - 1} \end{aligned}$$

This is another way to obtain (3.20).

2) Here is now another way to obtain (3.23). It is clear, from (3.25) that $E(|\log \mathbb{G}_\alpha^{(2)}|) < \infty$ and, since $\mathbb{G}_\alpha^{(2)} \stackrel{(\text{law})}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}}$, that $E(\log \mathbb{G}_\alpha^{(2)}) = 0$. Thus, from Theorem 2.1, point *ii*) in [JRY], we have :

$$f_{Y^{(2)}}(u) = \frac{u^{-\alpha}}{\Gamma(1 - \alpha)} E(e^{-u D_{1-\alpha}^{(2)}})$$

(this is formula (2.7) in [JRY], with $t = 1 - \alpha$, $E(\log G) = 0$ and $G \stackrel{(\text{law})}{=} \frac{1}{G}$). Hence, since :

$$f_{Y^{(2)}}(u) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-ux} \frac{x^\alpha}{1 + x} dx = \frac{u^{-\alpha} \sin(\pi\alpha)}{\pi} \int_0^\infty e^{-y} \frac{y^\alpha}{u + y} dy$$

(after the change of variable $ux = y$), we obtain :

$$\begin{aligned} E(e^{-u D_{1-\alpha}^{(2)}}) &= \frac{\sin(\pi\alpha)}{\pi} \Gamma(1 - \alpha) \int_0^\infty e^{-y} \frac{y^\alpha}{u + y} dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^\alpha}{u + y} dy \end{aligned}$$

3) Furthermore, we remark that, from Theorem 2.1 of [JRY] :

$$f_{D_{1-\alpha}^{(2)}}(u) = u^{-\alpha-1} f_{D_{1-\alpha}^{(2)}}\left(\frac{1}{u}\right)$$

This formula follows also from (3.24).

4) Finally, we also observe, from Theorem 2.1 in [JRY], as a consequence of $\mathbb{G}_\alpha^{(2)} \stackrel{(\text{law})}{=} \frac{1}{\mathbb{G}_\alpha^{(2)}}$ and $E(\log \mathbb{G}_\alpha^{(2)}) = 0$, that :

$$f_{Y^{(2)}}(u) = E\left[\left(\frac{Y^{(2)}}{u}\right)^{\frac{\alpha}{2}} J_{-\alpha}\left(2\sqrt{u Y^{(2)}}\right)\right] \quad (3.38)$$

where $J_{-\alpha}$ denotes the Bessel function with index $(-\alpha)$.

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